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Classification of algebras of level two in the variety of nilpotent algebras and Leibniz algebras

James Francese [a], Abror Khudoyberdiyev [b,*], Bennett Rennier [c], Anastasia Voloshinov [d]

[a] Fullerton College, Fullerton, CA, 92832, USA
[b] National University of Uzbekistan, Institute of Mathematics, Tashkent, 100125, Uzbekistan
[c] University of Oklahoma, Norman, OK, 73019, USA
[d] Pomona College, Claremont, CA, 91711, USA

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ABSTRACT

This paper is devoted to the description of complex finite-dimensional algebras of level two. We obtain the classification of algebras of level two in the variety of Leibniz algebras. It is shown that, up to isomorphism, there exist three Leibniz algebras of level two, one of which is solvable, and two of which are nilpotent. Moreover, we describe all algebras of level two in the variety of nilpotent algebras.

1. Introduction and background

The theory of deformations and degenerations of algebras has its origins in certain formal relations between physical theories and has become a lively subject in algebraic and differential geometry, as well as noncommutative and nonassociative algebra. It was a very influential precept of Niels Bohr that a new physical theory, which is supposed to ontologically overlap with a previously accepted theory, should somehow yield the old theory as a limiting or special case [1]. This is a statement of his "correspondence principle", which is realized in quantum mechanics via the limit of the Moyal bracket:

$$[f, g] = \{f, g\} + \mathcal{O}(\hbar^2)$$

as $\hbar \to 0$. Here the bracket $\{\ ,\ \} = \partial^i\partial_i - \partial_i\partial^i$ is the Poisson bracket of classical mechanics, being a sum of commutators for first-order differential operators. In this way classical mechanics emerges, as a limiting case, from quantum mechanics for small values of $\hbar$. This realization motivated the modern deformation theory of algebras, which originated with Gerstenhaber [2] and others. This theory has powerful applications in the classification of algebraic varieties and the quantization theory of Poisson manifolds, where the physical meaning of deformation remains especially explicit [3,4]. There

* Corresponding author.
E-mail addresses: jamesafrancese@gmail.com (J. Francese), khabror@mail.ru (A. Khudoyberdiyev), bennett@brennier.com (B. Rennier), nastiav13@gmail.com (A. Voloshinov).

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we have a manifold with tangent space V equipped with a Poisson bivector $\mathfrak{p} \in V \wedge V^*$, and an algebra of observables $f, g \in C^\infty(V)$. We then define a product of these observables as a power series in a parameter $\hbar$:

$$(f \star g)(\mathfrak{p}) := f \cdot g + \hbar \mathfrak{p}^i_j \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x_j} + \frac{1}{2} \hbar^2 \mathfrak{p}^k_\ell \mathfrak{p}^i_j \frac{\partial^2 f}{\partial x^k \partial x^i} \frac{\partial^2 g}{\partial x_\ell \partial x_j} + \cdots$$

where we have made use of the summation convention. Defining the bracket

$$[f, g] := -i\hbar^{-1}(f \star g - g \star f),$$

in the ordinary quantum mechanical case for a single state (i.e. where all differential operators are first-order) we recover the above limit, relating the Moyal to the Poisson bracket. The full higher-order approach is currently being used in the study of formal quantum field theory [5,6], a version of Hilbert's 6th problem [7].

For this paper, we consider the case of a finite-dimensional algebra over a closed field k, with $k = \mathbf{C}$ being of special focus. In general, an algebra over k can be considered as an element $\mu \in \mathrm{Hom}(V \otimes V, V)$, where V is an n-dimensional k-vector space. Thus, in the purely abstract deformation theory we consider algebras μ, $\mu_t \in V \otimes k[[t]]$ related by a formal power series:

$$\mu_t = \mu + \sum_{i=1}^{\infty} t^i \varphi_i \quad \text{where} \quad \varphi_i \in \mathrm{Hom}(V \otimes V, V)$$

so that in the linear case $\mu_{t,1} = \mu + t\varphi_1$ we study algebras with multiplication differing by a 2-coboundary.[1] Kodaira and Spencer gave the original idea of infinitesimal deformations for complex analytic manifolds [8]. Most notably, they proved that infinitesimal deformations can be parametrized by a related cohomology group. In fact, cohomology detects deformation at all orders, and it is therefore unsurprising that one can develop the deformation theory in any abelian category [9].

Prior to the development of deformation theory, it had already been realized that the space and time symmetries of Newtonian mechanics were recovered in the $c \to \infty$ limit of special relativity, where c is the speed of light. In that case the Lie algebra of the Poincare group *degenerates* to the Lie algebra of the Galilean group, an observation first made by İnönü and Wigner in [10]. This is a geometric process, and can be defined purely in terms of the Zariski topology on $\mathrm{Hom}(V \otimes V, V)$.

In the finite-dimensional case over a field of characteristic zero, such degenerations can be described directly in terms of the singular limit of a linear group action. Let $\mathrm{Alg}_n(k)$ be the variety of n-dimensional algebras over k, and let $\lambda, \mu \in \mathrm{Alg}_n(k)$. Define an action on $\mathrm{Alg}_n(k)$ by means of

$$(g * \mu)(x, y) := g(\lambda(g^{-1}(x), g^{-1}(y))) \qquad \text{where} \quad g \in \mathrm{GL}_n(k), \quad x, y \in A$$

which just represents a change of basis for A as an algebra. Thus the orbit of the algebra (A, λ) under this action is given by

$$\mathrm{Orb}(A) := \{L \in \mathrm{Alg}_n(k) \mid L \simeq A\}.$$

Definition 1. An algebra (A, λ) is said to degenerate to the algebra (A, μ) if $\mathrm{Orb}(A, \mu) \subseteq \overline{\mathrm{Orb}(A, \lambda)}$. We write $\lambda \to \mu$ to denote this degeneration.

In the case $k = \mathbb{C}$, we have that $\lambda \to \mu$ if and only if there is a $g_t \in GL_n(\mathbb{C}(t))$ such that $\forall x, y \in A$,

$$\mu(x, y) = \lim_{t \to 0} g_t(\lambda(g_t^{-1}(x), g_t^{-1}(y))).$$

We call a degeneration $\lambda \to \mu$ *trivial* if $(A, \lambda) \simeq (A, \mu)$, and *direct* if it is non-trivial, and there is no algebra (A, ν) such that $\lambda \to \nu \to \mu$. If $\lambda \to \mu$, then λ is a non-trivial deformation of μ, thus it is common to pass from the degeneration theory to the deformation theory.

It is clear that every non-abelian algebra in $\mathrm{Alg}_n(\mathbf{C})$ degenerates non-trivially to the abelian algebra ab_n, but of course not all such degenerations will be direct; the distance of an algebra from ab_n, in terms of the degeneration theory, is given by its *level*.

Definition 2. The level of an algebra λ is the maximum length of a chain of direct degenerations to ab_n. We denote the level of an algebra by $\mathrm{lev}_n(\lambda)$.

Concerning algebras of level one, we have the following result proved by Khudoyberdiyev and Omirov [11].

Theorem 1. *Let A be an algebra of level one. Then A is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
p_n^- : &\quad e_1e_i = e_i, \quad e_ie_1 = -e_i, \quad 2 \le i \le n; \\
n_3^- \oplus \mathrm{ab}_{n-3} : &\quad e_1e_2 = e_3, \quad e_2e_1 = -e_3; \\
\lambda_2 \oplus \mathrm{ab}_{n-2} : &\quad e_1e_1 = e_2; \\
\nu_n(\alpha) : &\quad e_1e_1 = e_1, \quad e_1e_i = \alpha e_i, \quad e_ie_1 = (1-\alpha)e_i, \quad 2 \le i \le n.
\end{aligned}$$

The level two case, within the varieties of Lie, Jordan, and associative algebras, has been resolved by Khudoyberdiyev in [12]. In particular, that paper provides the following theorem.

[1] An abstract "Poisson bracket" on an algebra is always available in the case of a linear deformation, since we can define $\{x, y\} := \frac{1}{2}(\mu_{t,1}(x, y) - \mu_{t,1}(y, x))$.

Theorem 2. *Let G be a Lie algebra of level two. Then G is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
n_{5,1}\oplus \mathrm{ab}_{n-5}: &\quad e_1e_3=e_5, \quad e_2e_4=e_5, \quad 2\le i\le n;\\
n_{5,2}\oplus \mathrm{ab}_{n-5}: &\quad e_1e_2=e_4, \quad e_1e_3=e_5;\\
r_2\oplus \mathrm{ab}_{n-2}: &\quad e_1e_1=e_2;\\
g_{n,1}(\alpha): &\quad e_1e_2=\alpha e_2, \quad e_1e_i=e_i, \quad 3\le i\le n, \alpha\in \mathbf{C}/\{0,1\};\\
g_{n,2}: &\quad e_1e_2=e_2+e_3, \quad e_1e_i, \quad 3\le i\le n.
\end{aligned}$$

It is still desirable to obtain a complete classification of level two algebras. One step is to ask about the existence of level two algebras in other varieties. Since $n_{5,1}$ and $n_{5,2}$ are Lie, we may well ask if there are any non-Lie *Leibniz* algebras of level two.

Definition 3 (*[13]*)**.** A (right) Leibniz algebra is a non-associative algebra such that for all $x, y, z \in L$, the following identity holds:

$$x(yz)=(xy)z-(xz)y.$$

This is a natural generalization of Lie algebras, in that an antisymmetric Leibniz algebra is Lie. Note that a left Leibniz algebra is defined by identity $(xy)z = x(yz) - y(xz)$.

Degenerations of Lie and Leibniz algebras were the subject of numerous papers, see for instance [14–18] and references given therein, and their research continues actively. In particular, in [19,20] some irreducible components of Leibniz algebras are found.

In this paper, we extend Theorem 2 to identify all non-Lie Leibniz algebras of level two; we find that two of these are nilpotent and one is solvable. We then proceed to classify all n-dimensional nilpotent algebras of level two, and find that these are all Leibniz.

2. Main results

Our first main result is the classification of Leibniz algebras of level two.

Theorem 3. *Let L be a n-dimensional non-Lie Leibniz algebra of level two. Then L is isomorphic one of the following three algebras:*

$$\begin{aligned}
L_4(\alpha)\oplus \mathrm{ab}_{n-3}: &\quad e_1e_1=e_3, \quad e_2e_1=e_3, \quad e_2e_2=\alpha e_3;\\
L_5\oplus \mathrm{ab}_{n-3}: &\quad e_1e_1=e_3, \quad e_1e_2=e_3, \quad e_2e_1=e_3;\\
r_n: &\quad e_ie_1=e_i, \quad 2\le i\le n.
\end{aligned}$$

Together with the nilpotent Lie algebras of level two identified in [12], our other main result identifies these four algebras as the only nilpotent algebras of level two.

Theorem 4. *Any finite-dimensional nilpotent algebra of level two is isomorphic to one of the following algebras:*

$$\begin{aligned}
n_{5,1}\oplus \mathrm{ab}_{n-5}: &\quad e_1e_3=e_5, \quad e_2e_1=e_5;\\
n_{5,2}\oplus \mathrm{ab}_{n-5}: &\quad e_1e_2=e_4, \quad e_1e_3=e_5;\\
L_4(\alpha)\oplus \mathrm{ab}_{n-3}: &\quad e_1e_1=e_3, \quad e_2e_2=\alpha e_3, \quad e_1e_2=e_3;\\
L_5\oplus \mathrm{ab}_{n-3}: &\quad e_1e_1=e_3, \quad e_1e_2=e_3, \quad e_2e_1=e_3.
\end{aligned}$$

It is now natural to ask if any algebra of level two is a direct sum of two level one algebras. The following examples give us a negative answer to this question.

Example 1. The algebras

$$\begin{aligned}
n_3^-\oplus\lambda_2=\{x_1,x_2,x_3,x_4,x_5\}: &\quad x_1x_1=x_2, \quad x_3x_4=x_5, \quad x_4x_3=-x_5;\\
\lambda_2\oplus\lambda_2=\{x_1,x_2,x_3,x_4\}: &\quad x_1x_1=x_2, \quad x_3x_3=x_4;\\
\lambda_2\oplus p_n^-=\{x_1,x_2,x_3,x_4,\dots,x_n\}: &\quad x_1x_1=x_2, \quad x_ix_3=x_i, \quad x_3x_i=-x_i, \quad 4\le i\le n.
\end{aligned}$$

via the family of matrices

$$\begin{cases}
g_t^{-1}(x_1)=t(x_1+x_3),\\
g_t^{-1}(x_2)=\frac{t}{2}(x_1+x_4),\\
g_t^{-1}(x_3)=t^2x_2,\\
g_t^{-1}(x_4)=t^2x_1,\\
g_t^{-1}(x_5)=t(x_5+x_2),
\end{cases}
\quad
\begin{cases}
g_t^{-1}(x_1)=tx_1,\\
g_t^{-1}(x_2)=t(x_1+x_3),\\
g_t^{-1}(x_3)=t^2x_2,\\
g_t^{-1}(x_4)=t(x_2+x_4),
\end{cases}
\quad
\begin{cases}
g_t^{-1}(x_1)=t(x_1+x_4),\\
g_t^{-1}(x_2)=t(\frac{1}{2}x_1+x_3),\\
g_t^{-1}(x_3)=t^2x_2,\\
g_t^{-1}(x_4)=t(x_4+\frac{1}{2}x_2),\\
g_t^{-1}(x_i)=tx_i, \qquad 5\le i\le n,
\end{cases}$$

degenerate to the algebras $L_4(\frac{1}{4})\oplus \mathrm{ab}_2$, $L_5\oplus \mathrm{ab}_1$ and $L_4(\frac{1}{4})\oplus \mathrm{ab}_{n-3}$ respectively.

Since the algebras $L_4(\alpha)$ and L_5 are not algebras of level one, we deduce that the level of the algebras $n_3^- \oplus \lambda_2$, $\lambda_2 \oplus \lambda_2$ and $\lambda_2 \oplus p_n^-$ must be greater than two.

Now let L be a n-dimensional complex algebra and $\{e_1, e_2, \dots, e_n\}$ be a basis of L. The multiplication on the algebra L is defined by the products of the basis elements; namely, by the products

$$e_i e_j = \sum_{k=1}^{n} \gamma_{i,j}^k e_k,$$

where $\gamma_{i,j}^k$ are the structural constants.

We first prove a very useful lemma, which will allow us to immediately conclude a degeneration to either $L_4(\alpha)$ or L_5 based on a multiplication table of a certain form.

Lemma 1. *Suppose L is an n-dimensional algebra and let $\{e_1, e_2, \dots, e_n\}$ be a basis of L. If there exist distinct i, j, k such that*

$$(\gamma_{i,i}^k, \gamma_{i,j}^k, \gamma_{j,i}^k, \gamma_{j,j}^k) \notin \{(0, \beta, -\beta, 0), (\delta, \beta, \beta, \frac{\beta^2}{\delta})\} \quad \text{where} \quad \delta \neq 0,$$

then $L \to L_4(\alpha)$ or $L \to L_5$.

Proof. Without loss of generality, we may assume $i = 1, j = 2$, and $k = 3$. We see that if we take the degeneration

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = t^{-2}e_i \quad 3 \le i \le n,$$

then we have following nontrivial products

$$e_1e_1 = \gamma_{1,1}^3 e_3 + \sum_{s=4}^{n} \gamma_{1,1}^s e_s, \qquad e_1e_2 = \gamma_{1,2}^3 e_3 + \sum_{s=4}^{n} \gamma_{1,2}^s e_s,$$
$$e_2e_1 = \gamma_{2,1}^3 e_3 + \sum_{s=4}^{n} \gamma_{2,1}^s e_s, \qquad e_2e_2 = \gamma_{2,2}^3 e_3 + \sum_{s=4}^{n} \gamma_{2,2}^s e_s.$$

Furthermore, if we take the additional degeneration

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_3) = t^{-2}e_3, \quad g_t(e_i) = t^{-1}e_i \quad 4 \le i \le n,$$

then we have an algebra with the following multiplication

$$e_1e_1 = \gamma_{1,1}^3 e_3, \quad e_1e_2 = \gamma_{1,2}^3 e_3, \quad e_2e_1 = \gamma_{2,1}^3 e_3, \quad e_2e_2 = \gamma_{2,2}^3 e_3.$$

We see that this algebra is nilpotent and also non-Lie as $(\gamma_{1,1}^3, \gamma_{1,2}^3, \gamma_{2,1}^3, \gamma_{2,2}^3) \neq (0, \beta, -\beta, 0)$. Moreover, since $(\gamma_{1,1}^3, \gamma_{1,2}^3, \gamma_{2,1}^3, \gamma_{2,2}^3) \neq (\delta, \beta, \beta, \frac{\beta^2}{\delta})$, we conclude that L is not isomorphic to the algebra λ_2. Due to the classification of three dimensional nilpotent Leibniz algebras [21], we conclude that this algebra is isomorphic to either $L_4(\alpha)$ or L_5. □

2.1. Classification of algebras of level two in the variety of Leibniz algebras

Let L be a Leibniz algebra and let $x \in L$. We define $\varphi_x : L \to L$ to be the linear operator where $\varphi_x(y) = yx + xy$. We see that by applying the Leibniz identity we get the following two equations:

$$z\varphi_x(y) = z(yx + xy) = z(yx) + z(xy) = (zy)x - (zx)y + (zx)y - (zy)x = 0,$$
$$z(xx) = (zx)x - (zx)x = 0.$$

This proves that both $\varphi_x(y)$ and xx are in the right annihilator for any $x \in L$.

In this section we will examine the matrix representation of φ_x on a case-by-case basis in order to prove Theorem 3.

Proposition 1. *Let L be a n-dimensional non-Lie Leibniz algebra which is not of level one (i.e. $L \not\to \lambda_2$). Then L degenerates to one of the following three algebras:*

$$L_4(\alpha) \oplus \mathrm{ab}_{n-3} : \quad e_1e_1 = e_3, \quad e_2e_1 = e_3, \quad e_2e_2 = \alpha e_3;$$
$$L_5 \oplus \mathrm{ab}_{n-3} : \quad e_1e_1 = e_3, \quad e_1e_2 = e_3, \quad e_2e_1 = e_3;$$
$$r_n : \quad e_ie_1 = e_i, \quad 2 \le i \le n.$$

Proof. Since L is a non-Lie Leibniz algebra, we know that there exists an element $x \in L$ such that $xx \neq 0$. Suppose that $xx = \alpha x$ for some constant α. Then, by the Leibniz identity, we have that

$$\alpha^2 x = \alpha xx = x(\alpha x) = x(xx) = (xx)x - (xx)x = 0$$

which means that $\alpha = 0$. This is a contradiction, though, as $xx \neq 0$. Thus, it must be that xx is linearly independent from x. Using this, we can form a basis $\{e_1, e_2, \dots, e_n\}$, where $e_1 = x$ and $e_2 = xx$.

We define the linear operator $\varphi = \varphi_x$ and let $(\alpha_{i,j})$ be its matrix form. Thus, we have that

$$\varphi = \begin{bmatrix} 0 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \dots & \alpha_{1,n} \\ 2 & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \dots & \alpha_{2,n} \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} & \dots & \alpha_{3,n} \\ 0 & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} & \dots & \alpha_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n,2} & \alpha_{n,3} & \alpha_{n,4} & \dots & \alpha_{n,n} \end{bmatrix}$$

Suppose that $\alpha_{j,k} \neq 0$ for some $1, j, k$ distinct and $k \geq 3$. This means that we have the following products

$$e_1e_1 = e_2, \qquad e_je_j = \gamma_{j,j}^k e_k + \sum_{s=1, s\neq k}^{n} \gamma_{j,j}^s e_s,$$

$$e_1e_j = \gamma_{1,j}^k e_k + \sum_{s=1, s\neq k}^{n} \gamma_{1,j}^s e_s, \qquad e_je_1 = (\alpha_{j,k} - \gamma_{1,j}^k)e_k + \sum_{s=1, s\neq k}^{n} \gamma_{j,1}^s e_s.$$

Since $\gamma_{1,1}^k = 0$ and $\gamma_{1,j}^k \neq -(\alpha_{j,k} - \gamma_{1,j}^k)$, we can apply Lemma 1 on the indices $1, j, k$ to see that $L \to L_4(\alpha)$ or L_5. Now we focus the case where $\alpha_{j,k} = 0$ for $1, j, k$ distinct and $k \geq 3$. This gives us the following matrix representation of φ:

$$\varphi = \begin{bmatrix} 0 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \dots & \alpha_{1,n} \\ 2 & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \dots & \alpha_{2,n} \\ 0 & 0 & \alpha_{3,3} & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha_{4,4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{n,n} \end{bmatrix}$$

Suppose that $\alpha_{2,i} \neq 0$ for some $i \neq 1, 2$. Without loss of generality, let $i = 3$. Then if we take the change of basis $e_3' = e_1 - \frac{2}{\alpha_{2,3}}e_3$, we have that

$$\begin{aligned}\varphi(e_3') &= \varphi(e_1 - \frac{2}{\alpha_{2,3}}e_3) = \varphi(e_1) - \frac{2}{\alpha_{2,3}}\varphi(e_3) = 2e_2 - \frac{2}{\alpha_{2,3}}(\alpha_{1,3}e_1 + \alpha_{2,3}e_2 + \alpha_{3,3}e_3) \\ &= -\frac{2\alpha_{1,3}}{\alpha_{2,3}}e_1 - \frac{2\alpha_{3,3}}{\alpha_{2,3}}e_3 = -\frac{2\alpha_{1,3}}{\alpha_{2,3}}e_1 - \alpha_{3,3}(e_1 - e_3') = -\left(\frac{2\alpha_{1,3}}{\alpha_{2,3}} - \alpha_{3,3}\right)e_1 + \alpha_{3,3}e_3'.\end{aligned}$$

Thus, we may assume that $\alpha_{2,3} = 0$. Since $i = 3$ was arbitrary, we can therefore assume that $\alpha_{2,i} = 0$ for $3 \leq i \leq n$. This means that the matrix representation of φ has the form:

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \dots & \alpha_{1n} \\ 2 & \alpha_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha_{44} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{nn} \end{bmatrix}$$

Now suppose there exists $j \geq 3$ such that $\alpha_{2,2} \neq \alpha_{j,j}$. Then we see that if we take the basis change $e_j' = e_2 + e_j$, we have that

$$\begin{aligned}\varphi(e_j') = \varphi(e_2) + \varphi(e_j) &= (\alpha_{1,2} + \alpha_{1,j})e_1 + \alpha_{2,2}e_2 + \alpha_{j,j}e_j \\ &= (\alpha_{1,2} + \alpha_{1,j})e_1 + (\alpha_{2,2} - \alpha_{j,j})e_2 + \alpha_{j,j}e_j'\end{aligned}$$

By reapplying our previous argument, we see that it must be that either $L \to L_4(\alpha)$, L_5 or $\alpha_{2,2} - \alpha_{j,j} = 0$. Thus, we have that $\alpha = \alpha_{2,2} = \alpha_{i,i}$ for $3 \leq i \leq n$.

Now suppose that $\alpha \neq 0$. We can then take the basis change $e_i' = \varphi(e_i)$ for $2 \leq i \leq n$. This means that $e_2, e_3, \dots, e_n \in \mathrm{Ann}_R(L)$ and thus that $\varphi(e_i) = e_ie_1$ for $2 \leq i \leq n$. Moreover, if $\alpha_{1,i} \neq 0$ for some $2 \leq i \leq n$, then we would have that

$$\alpha_{1i}e_1 = \varphi(e_i) - \alpha e_i \in \mathrm{Ann}_R(L)$$

which is a contradiction to the fact that $e_1e_1 = e_2$. Therefore, we have an algebra L with the following multiplication table:

$$e_1e_1 = e_2, \qquad e_ie_1 = \alpha e_i, \ (\alpha \neq 0), \qquad 2 \leq i \leq n.$$

If we then take the basis transformation $e_1' = \frac{1}{\alpha}(e_1 - \frac{1}{\alpha}e_2)$, we see that this is exactly the algebra r_n.

Suppose then that $\alpha = 0$. Again, we see that if $\alpha_{1,i} \neq 0$ for some $2 \leq i \leq n$, then we would have that $\alpha_{1,i}e_1 = \varphi(e_i) \in \mathrm{Ann}_R(L)$, which is a contradiction. This means that φ has the following matrix representation:

$$\varphi = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We now consider following products

$$e_2e_2 = 0, \quad e_3e_2 = 0, \quad e_2e_3 = \gamma_{2,3}^2 e_2 + \gamma_{2,3}^3 e_3 + \sum_{s=4}^{n} \gamma_{2,3}^s e_s, \quad e_3e_3 = \gamma_{3,3}^2 e_2 + \gamma_{3,3}^3 e_3 + \sum_{s=4}^{n} \gamma_{3,3}^s e_s.$$

If there exist $\gamma_{2,3}^s \neq 0$, for some $4 \leq s \leq n$, then by Lemma 1, we have that $L \to L_4(\alpha), L_5$. Thus, we assume that $\gamma_{2,3}^s = 0$ for $4 \leq s \leq n$. Therefore, $e_2e_3 = \gamma_{2,3}^2 e_2 + \gamma_{2,3}^3 e_3$. If $\gamma_{2,3}^3 \neq 0$, then we have that $\gamma_{2,3}^3 e_3 = e_2e_3 - \gamma_{2,3}^2 e_2 \in \mathrm{Ann}_R(L)$. This means that $e_2e_3 = 0$, a contradiction as $\gamma_{2,3}^3 \neq 0$. Hence, $e_2e_3 = \gamma_{2,3}^2 e_2$.

Assume that $\gamma_{2,3}^2 \neq 0$. If we take the change of basis $e_2' = e_2 - Ae_1$, then we have that

$$e_2'e_2' = (e_2 - Ae_1)(e_2 - Ae_1) = A^2 e_1e_1 = A^2 e_2 = A^3 e_1 + A^2 e_2',$$

$$e_2'e_3 = (e_2 - Ae_1)e_3 = \gamma_{2,3}^2 e_2 - Ae_1e_3 = A(\gamma_{2,3}^2 - \gamma_{1,3}^1 - A\gamma_{1,3}^2)e_1 + (\gamma_{2,3}^2 - A\gamma_{1,3}^2)e_2' - A\sum_{s=3}^{n} \gamma_{1,3}^s e_s,$$

$$e_3e_2' = e_3(e_2 - Ae_1) = Ae_1e_3 = A(\gamma_{1,3}^1 + A\gamma_{1,3}^2)e_1 + A\gamma_{1,3}^2 e_2' + A\sum_{s=3}^{n} \gamma_{1,3}^s e_s,$$

$$e_3e_3 = \gamma_{3,3}^1 e_1 + \gamma_{3,3}^2 e_1 + \sum_{s=3}^{n} \gamma_{3,3}^s e_s.$$

Since $\gamma_{2,3}^2 \neq 0$, we see that we can always choose A so that $\gamma_{2,3}^2 - \gamma_{1,3}^1 - A\gamma_{1,3}^2 \neq \gamma_{1,3}^1 + A\gamma_{1,3}^2$. Hence, using Lemma 1 we obtain that $L \to L_4(\alpha), L_5$. Thus, it must be that $\gamma_{2,3}^2 = 0$, which means that $e_2e_3 = 0$. Since e_3 was arbitrary, we may assume that $e_2e_i = 0$ for $i \neq 1, 2$ and thus we have that $e_2 \in \mathrm{Ann}(L)$.

Therefore, we have the following multiplication table

$$e_1e_1 = e_2, \quad e_2e_i = e_ie_2 = 0, \quad 1 \leq i \leq n,$$

$$e_1e_i = -e_ie_1 = \sum_{s=1}^{n} \gamma_{1,i}^s e_s, \quad 3 \leq i \leq n,$$

$$e_ie_j = \sum_{s=1}^{n} \gamma_{i,j}^s e_s, \quad 3 \leq i, j \leq n.$$

Using the Leibniz identity

$$0 = (e_1e_1)e_i - (e_1e_i)e_1 + e_1(e_ie_1) = e_2e_i - \left(\sum_{s=1}^{n} \gamma_{1,i}^s e_s\right) e_1 - e_1 \left(\sum_{s=1}^{n} \gamma_{1,i}^s e_s\right) = -2\gamma_{1,i}^1 e_2$$

we obtain that $\gamma_{1,i}^1 = 0$ for $3 \leq i \leq n$. Furthermore, if $\gamma_{1,i}^2 \neq 0$ or $\gamma_{i,i}^2 \neq 0$ for some $3 \leq i \leq n$, then by Lemma 1, we have that $L \to L_4(\alpha), L_5$. Therefore, we may assume $\gamma_{1,i}^2 = 0$ and $\gamma_{i,i}^2 = 0$ for any $3 \leq i \leq n$.

Moreover, if $\gamma_{i,j}^2 \neq 0$ for some i, j then taking the basis change $e_1' = e_1 + e_j$, we obtain that

$$e_1e_1 = e_2 + \sum_{i=3}^{n} (*)e_i, \quad e_ie_1 = \gamma_{i,j}^2 e_2 + \sum_{i=3}^{n} (*)e_i,$$

$$e_ie_1 = \gamma_{j,i}^2 e_2 + \sum_{i=3}^{n} (*)e_i, \quad e_ie_i = \sum_{i=3}^{n} (*)e_i.$$

If we apply Lemma 1 on the indices $1, i, 2$, we see that $L \to L_4(\alpha), L_5$. Therefore we can suppose $\gamma_{i,j}^2 = 0$ for any $3 \leq i, j \leq n$.

By applying the Leibniz identity once again we see that

$$0=(e_ie_j)e_1-(e_ie_1)e_j-e_i(e_je_1)=\gamma_{i,j}^1e_2+\sum_{i=1,i\neq 2}^{n}(*)e_i,$$

which means that $\gamma_{i,j}^1=0$ for $3\leq i,j\leq n$.

Therefore, we obtain that any Leibniz algebras L either degenerates to $L_4(\alpha)$, L_5 or r_n or is a decomposed algebra with ideals $M_1=\{e_1,e_2\}$ and $M_2=\{e_3,e_4,\ldots,e_n\}$. If M_2 were trivial, then our only nontrivial multiplication would be $e_1e_1=e_2$. This is a contradiction, as we assumed that $L\not\simeq\lambda_2$. Thus, M_2 is not trivial, and therefore M_2 degenerates to an algebra of level one: λ_2, n_3^- or p_n^-. This means that L degenerates to $\lambda_2\oplus\lambda_2$, $\lambda_2\oplus n_3^-$, or $\lambda_2\oplus p_n^-$. By Example 1, we conclude that L degenerates to either $L_4(\alpha)$ or L_5. □

Theorem 5. *Let L be an n-dimensional Leibniz algebra of level two. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$L_4(\alpha)\oplus \mathrm{ab}_{n-3},\quad L_5\oplus \mathrm{ab}_{n-3},\quad r_n,\quad \alpha\in\mathbb{C}$$

Proof. Due to Theorem 3, it is sufficient to prove that these algebras do not degenerate to each other. To facilitate this, we compute the dimensions of right annihilator and derivations of these algebras and we call upon the following table:

$$\begin{aligned}&\dim \mathrm{Ann}_R(L_5)=n-2, && \dim \mathrm{Der}(L_4(0))=n^2-3n+4,\\ &\dim \mathrm{Ann}_R(L_4(\alpha))=n-2,\ \alpha\neq 0, && \dim \mathrm{Der}(r_n)=(n-1)^2=n^2-2n+1,\\ &\dim \mathrm{Ann}_R(r_n)=n-1.\end{aligned}$$

We first note that $L_4(\alpha)$ and L_5 cannot degenerate to r_n, as $L_4(\alpha)$ and L_5 are nilpotent and r_n is not. We also see that r_n does not degenerate to $L_4(\alpha)$ $(\alpha\neq 0)$ or L_5, as dimension of right annihilator of r_n is more than dimensions of right annihilators of $L_4(\alpha)$ $(\alpha\neq 0)$ and L_5. Additionally, since for $n\geq 4$, the dimension of derivations of r_n is more than the dimension of derivations of $L_4(0)$, we have that $r_n\not\to L_4(0)$. Lastly, we see that $L_4(\alpha)\not\to L_5$ and that $L_5\not\to L_4(\alpha)$ by the following paper [14]. □

Remark 1. We note that in the context of left Leibniz algebras, the following algebra replaces the algebra r_n as a Leibniz algebra of level two:

$$\ell_n:\qquad e_1e_i=e_i,\qquad 2\leq i\leq n.$$

2.2. Nilpotent algebras of level two

Working in the variety of n-dimensional nilpotent algebras $\mathrm{Nil}_n(\mathbb{C})$ will allow us to exclude certain products from our multiplication tables, in particular all products of the form $xy=x$.

Theorem 6. *Any n-dimensional* $(n\geq 5)$ *nilpotent algebra of level two is isomorphic to one of the following algebras:*

$$\begin{array}{rllll} \mathfrak{n}_{5,1}: & e_1e_2=e_5, & e_3e_4=e_5, & e_2e_1=-e_5, & e_4e_3=-e_5;\\ \mathfrak{n}_{5,2}: & e_1e_2=e_4, & e_1e_3=e_5, & e_2e_1=-e_4, & e_3e_1=-e_5;\\ L_4(\alpha): & e_1e_1=e_3, & e_2e_2=\alpha e_3, & e_1e_2=e_3; & \\ L_5: & e_1e_1=e_3, & e_1e_2=e_3, & e_2e_1=e_3. & \end{array}$$

Proof. Our overall strategy is to look separately at antisymmetric and non-antisymmetric cases, and then at the way products fall into the square of the algebra A^2.

Case 1. First, we assume that $A\in\mathrm{Nil}_n(\mathbb{C})$ is non-antisymmetric.

Case 1.1. Assume that $\dim(A^2)=1$, then we assume that $A^2=\{e_n\}$ and have the following multiplication

$$A:\begin{cases} e_1e_1=e_n, & e_ie_j=\alpha_{i,j}e_n,\quad 2\leq i,j\leq n-1,\\ e_1e_j=\alpha_{1,j}e_n, & e_je_1=\alpha_{j,1}e_n,\quad 2\leq j\leq n-1.\end{cases}$$

If there exist i such that $|\alpha_{1i}-\alpha_{i1}|+|\alpha_{ii}-\alpha_{1i}\alpha_{i1}|\neq 0$, then by Lemma 1 we obtain that L degenerates to $L_4(\alpha)$ or L_5. Now let $|\alpha_{1i}-\alpha_{i1}|+|\alpha_{ii}-\alpha_{1i}\alpha_{i1}|=0$ for $1\leq i\leq n$. Making the change of basis

$$e_1'=e_1,\quad e_i'=e_i+\alpha_{1,i}e_1,\qquad 2\leq i\leq n-1$$

the multiplication of A simply becomes

$$(*)\qquad e_1e_1=e_n,\quad e_1e_i=e_ie_1=e_ie_i=0,\quad e_ie_j=\alpha_{ij}e_n,\qquad 2\leq i,j\leq n-1.$$

We consider now the subalgebra $M : \{e_2, \dots, e_n\}$. Note that M cannot be abelian, since otherwise $(*)$ becomes an algebra λ_2, which is level one. Thus, we have the following two subcases, which will complete the case $\dim(A^2) = 1$.

Case 1.1.1. Assume that M is Lie. Since M is also not abelian, we are free to choose $\alpha_{2,3} = 1$, $\alpha_{3,2} = -1$. Taking the degeneration

$$\begin{cases} g_t(e_1) = t^{-2}e_n, & g_t(e_2) = 2t^{-1}e_2 - t^{-2}e_n, \\ g_t(e_3) = t^{-1}e_1 - t^{-2}e_n, & g_t(e_n) = t^{-2}e_3, \\ g_t(e_i) = t^{-2}e_i, & 4 \le i \le n-1, \end{cases}$$

we obtain that A degenerates to $L_4(\frac{1}{4})$.

Case 1.1.2. Assume that M is non-Lie. Then we may assume $e_2e_2 \neq 0$, moreover $e_2e_2 = e_n$. Taking the degeneration

$$g_t(e_1) = e_1, \quad g_t(e_2) = e_2, \quad g_t(e_n) = e_n, \quad g_t(e_i) = t^{-1}e_i, \quad 3 \le i \le n-1,$$

we obtain that A degenerates to L_5.

Case 1.2. We now assume that $\dim(A^2) \ge 2$. Let $A = \{e_1, \dots, e_n\}$, and $A^2 = \{e_{k+1}, \dots, e_n\}$. We consider five logically exhaustive cases in which the products e_ie_1, e_1e_i fall in the square A^2 in different ways. It is obvious that we may always assume $e_1e_1 = e_{k+1}$.

Case 1.2.1. Assume $e_ie_1 \in span\{e_{k+1}\}$ for $1 \le i \le k$, and let $e_1e_2 \notin span\{e_{k+1}\}$, so that we have the multiplication:

$$e_1e_1 = e_{k+1}, \quad e_1e_2 = e_{k+2}, \quad e_ie_1 = \gamma_{i,1}^{k+1}e_{k+1}, \quad 2 \le i \le k,$$

$$e_ie_j = \sum_{\ell=k+1}^{n} \gamma_{i,j}^{\ell}e_\ell, \qquad 2 \le i,\ j \le k.$$

Applying Lemma 1 for the elements $(\gamma_{1,1}^{k+2}, \gamma_{1,2}^{k+2}, \gamma_{2,1}^{k+2}, \gamma_{2,2}^{k+2}) = (0, 0, 1, \gamma_{2,2}^{k+2})$ we obtain that A degenerates to $L_4(\alpha)$ or L_5.

Case 1.2.2. Let there exist i such that $e_ie_1 \notin span\{e_{k+1}\}$. Without loss of generality, we may put $e_2e_1 = e_{k+2}$, so that our products are

$$e_1e_1 = e_{k+1}, \qquad e_2e_1 = e_{k+2}, \qquad e_ie_1 = \sum_{\ell=k+1}^{n} \gamma_{i,1}^{\ell}e_\ell, \qquad 3 \le i \le k,$$

$$e_1e_i = \sum_{\ell=k+1}^{n} \gamma_{1,i}^{\ell}e_\ell, \quad 2 \le i \le k, \qquad e_ie_j = \sum_{\ell=k+1}^{n} \gamma_{i,j}^{\ell}e_\ell, \quad 2 \le i,\ j \le k.$$

If $\gamma_{1,2}^{k+2} \neq -1$, then applying Lemma 1 for the $(\gamma_{1,1}^{k+2}, \gamma_{1,2}^{k+2}, \gamma_{2,1}^{k+2}, \gamma_{2,2}^{k+2}) = (0, \gamma_{1,2}^{k+2}, 1, \gamma_{2,2}^{k+2})$ we obtain that L degenerates to $L_4(\alpha)$ or L_5.

If $\gamma_{1,2}^{k+2} = -1$, then taking the change of basis

$$e_2' = e_2 + \eta e_1,\ e_{k+2}' = e_{k+2} + \eta e_{k+1},\ e_i' = e_i,\ 1 \le i (i \neq 2, k+2) \le n,$$

we obtain that

$$e_1'e_1' = e_{k+1}', \qquad e_2'e_1' = e_{k+2}', \qquad e_1'e_2' = (2\eta + \gamma_{1,2}^{k+1})e_{k+1}' - e_{k+2}' + \sum_{\ell=k+3}^{n} \gamma_{1,i}^{\ell}e_\ell,$$

$$e_i'e_1' = \sum_{\ell=k+1}^{n} \gamma_{i,1}^{\ell}e_\ell', \qquad e_1'e_i' = \sum_{\ell=k+1}^{n} \gamma_{1,i}^{\ell}e_\ell', \qquad 3 \le i \le k,$$

$$e_i'e_j' = \sum_{\ell=k+1}^{n} \gamma_{i,j}^{\ell}e_\ell' \qquad 2 \le i,\ j \le k.$$

Taking the value of η such that $\gamma_{1,2}^{k+1'} = 2\eta + \gamma_{1,2}^{k+1} \neq 0$ we apply the Lemma 1 for the elements $(\gamma_{1,1}^{k+1'}, \gamma_{1,2}^{k+1}, \gamma_{2,1}^{k+1'}, \gamma_{2,2}^{k+2'})$ $= (1, \gamma_{1,2}^{k+1'}, 0, \gamma_{2,2}^{k+2'})$ and obtain that A degenerates to $L_4(\alpha)$ or L_5.

Case 1.2.3. Now suppose that $e_ie_1, e_1e_i \in span\{e_{k+1}\}$ for $1 \le i \le k$ and there exist j such that $e_je_j \notin span\{e_{k+1}\}$. In this case without loss of generality, we may suppose $e_2e_2 = e_{k+2}$, so that our multiplication becomes

$$e_1e_1 = e_{k+1},\ e_2e_2 = e_{k+2},\ e_ie_1 = \gamma_{i,1}^{k+1}e_{k+1},\ e_1e_i = \gamma_{1,i}^{k+1}e_{k+1}, \quad 2 \le i \le k,$$

$$e_ie_j = \sum_{\ell=k+1}^{n} \gamma_{i,j}^{\ell} e_\ell, \quad 3 \le i, \quad j \le k.$$

If $(\gamma_{1,2}^{k+1}, \gamma_{2,1}^{k+1}) \neq (0,0)$ then applying Lemma 1 for the $(\gamma_{1,1}^{k+1}, \gamma_{1,2}^{k+1}, \gamma_{2,1}^{k+1}, \gamma_{2,2}^{k+1}) = (1, \gamma_{1,2}^{k+1}, \gamma_{2,1}^{k+1}, 0)$ we obtain that A degenerates to $L_4(\alpha)$ or L_5.

If $\gamma_{1,2}^{k+1} = \gamma_{2,1}^{k+1} = 0$, then taking the degeneration

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \qquad g_t(e_i) = t^{-2}e_i, \quad 3 \le i \le n,$$

we degenerate to the algebra

$$\lambda_2 \oplus \lambda_2 : e_1e_1 = e_{k+1}, \quad e_2e_2 = e_{k+2}.$$

By Example 1, we obtain that algebra $\lambda_2 \oplus \lambda_2$ degenerates to the algebra L_5.

Case 1.2.4. Now we suppose that $e_ie_i, e_1e_i, e_ie_1 \in span\{e_{k+1}\}$ for $1 \le i \le k$. Let there exist i, j $(2 \le i, j \le k)$ such that $e_ie_j \notin span\{e_{k+1}\}$. Without loss of generality we can suppose $i = 2, j = 3$ moreover, if $e_2e_3 + e_3e_2 \neq 0$, then taking $e_2' = e_2 + e_3$, we get that $e_2'e_2' = e_2e_2 + e_2e_3 + e_3e_2 + e_3e_3 \notin \{e_{k+1}\}$ which have the situation of Case 1.2.3. Therefore, we may suppose

$$e_1e_1 = e_{k+1}, \ e_1e_2 = \alpha_{1,2}e_{k+1}, \ e_2e_1 = \alpha_{2,1}e_{k+1}, \ e_2e_2 = \alpha_{2,2}e_{k+1},$$

$$e_1e_3 = \alpha_{1,3}e_{k+1}, \ e_3e_1 = \alpha_{3,1}e_{k+1}, \ e_3e_3 = \alpha_{3,3}e_{k+1},$$

$$e_2e_3 = e_{k+2}, \ e_3e_2 = -e_{k+2}.$$

Now applying the degeneration

$$\begin{cases} g_t(e_1) = t^{-2}e_1, & g_t(e_2) = t^{-3}e_2, \\ g_t(e_3) = t^{-3}e_3, & g_t(e_{k+1}) = t^{-4}e_{k+1}, \\ g_t(e_i) = t^{-5}e_i, & 4 \le i \le n. \end{cases}$$

we obtain the products

$$e_1e_1 = e_{k+1}, \quad e_2e_3 = e_{k+2}, \quad e_3e_2 = -e_{k+2}.$$

which by Example 1 degenerates to the algebra L_5.

Case 1.2.5. Now we suppose that $e_ie_j \in span\{e_{k+1}\}$ for all i, j $(1 \le i, j \le k)$. Thus we have

$$e_1e_1 = e_{k+1}, \quad e_ie_j = \alpha_{i,j}e_{k+1}.$$

Since $\dim(A^2) \ge 2$, then we have that there exist i $(2 \le i \le k)$, such that e_ie_{k+1} or $e_{k+1}e_i$ is non zero, which we can suppose as e_{k+2}.

Let $i = 1$, then we have $e_1e_1 = e_{k+1}$, $e_1e_{k+1} = e_{k+2}$.

If $e_{k+1}e_1 + e_1e_{k+1} \neq 0$, then using the Lemma 1 for the basis elements $\{e_1, e_{k+1}, e_{k+2}\}$ we have that A degenerates to the algebra $L_4(\alpha)$ or L_5.

If $e_{k+1}e_1 = -e_1e_{k+1} = -e_{k+2}$ then making the change $e_{k+1}' = e_{k+1} - e_{k+2}$, we have that

$$e_1e_1 = e_{k+1} + e_{k+2}, \quad e_1e_{k+1} = e_{k+2}, \quad e_{k+1}e_1 = -e_{k+2}.$$

Again applying Lemma 1 for the basis elements $\{e_1, e_{k+1}, e_{k+2}\}$, i.e. $(\gamma_{1,1}^{k+2}, \gamma_{1,k+1}^{k+2}, \gamma_{k+1,1}^{k+2}, \gamma_{k+1,k+1}^{k+2}) = (1, 1, -1, \gamma_{k+1,k+1}^{k+2})$ we have that A degenerates to the algebra $L_4(\alpha)$ or L_5.

Let $i \neq 1$, then we can suppose $i = 2$ and we have $e_1e_1 = e_{k+1}$, $e_2e_{k+1} = e_{k+2}$.

Similarly to the case $i = 1$ if $e_{k+1}e_2 + e_2e_{k+1} \neq 0$, then using Lemma 1 for the basis elements $\{e_2, e_{k+1}, e_{k+2}\}$ we have that A degenerates to the algebra $L_4(\alpha)$ or L_5.

If $e_{k+1}e_2 = -e_2e_{k+1} = -e_{k+2}$ then making the change $e_{k+1}' = e_{k+1} - e_{k+2}$, we have that

$$e_1e_1 = e_{k+1} + e_{k+2}, \quad e_1e_{k+1} = e_{k+2}, \quad e_{k+1}e_1 = -e_{k+2}.$$

Again applying Lemma 1 for the basis elements $\{e_1, e_{k+1}, e_{k+2}\}$, i.e. $(\gamma_{1,1}^{k+2}, \gamma_{1,k+1}^{k+2}, \gamma_{k+1,1}^{k+2}, \gamma_{k+1,k+1}^{k+2}) = (1, 1, -1, \gamma_{k+1,k+1}^{k+2})$ we have that A degenerates to the algebra $L_4(\alpha)$ or L_5.

Case 2. Let $A \in \mathrm{Nil}_n(\mathbb{C})$ be antisymmetric. It should be noted that if $\dim(A^2) = 1$, then we have that $A^3 = 0$. Thus, A is a Lie algebra. In [12] it is shown that any nilpotent Lie algebra with condition $\dim(A^2) = 1$, $A^3 = 0$ degenerates to algebra $n_{5,1}$.

Therefore, we consider case $\dim(A^2) \ge 2$. Assume that $\{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ be a basis of A, and $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ be a basis of A^2.

Then, without loss of generality, we can assume $e_1e_2 = e_{k+1}$, $e_2e_1 = -e_{k+1}$.

Below, we show that it may always be assumed

$$e_1e_2 = e_4, \ e_1e_3 = e_5.$$

- Let there exists i_0 such that $e_1e_{i_0} \notin span\langle x_{k+1}\rangle$. Then taking
$$e'_1 = e_1,\ e'_2 = e_2,\ e'_3 = e_{i_0},\ e'_4 = e_{k+1},\ e'_5 = e_1e_{i_0}$$
we obtain $e'_1e'_2 = e'_4,\ e'_1e'_3 = e'_5$.
- Let $e_1e_i \in span\{e_{k+1}\}$ for all $3 \le i \le k$ and there exists some i_0 such that $e_2e_{i_0} \notin span\langle x_{k+1}\rangle$. According to symmetrically of e_1 and e_2, similarly to the previous case we can choose a basis $\{e'_1, e'_2, \dots, e'_n\}$ with condition $e'_1e'_2 = e'_4,\ e'_1e'_3 = e'_5$.
- Let $x_1x_i,\ x_2x_i \in span\{e_{k+1}\}$ for all $3 \le i \le k$. We set $e_1e_i = \alpha_ie_{k+1}$ and $e_2e_i = \beta_ix_{k+1}$. Let e_{i_0} and e_{j_0} be generators of A such that $e_{i_0}e_{j_0} \notin span\{e_{k+1}\}$. Since $\dim(A^2) \ge 2$ one can assume $e_{i_0}e_{j_0} = e_{k+2}$.
Putting
$$e'_1 = e_1 + Ae_{i_0},\ e'_2 = e_2,\ e'_3 = e_{j_0},\ e'_4 = (1 - A\beta_{i_0})e_{k+1},\ e'_5 = Ae_{k+2} + \alpha_{i_0}e_{k+1}$$
with $A(1 - A\beta_{i_0}) \neq 0$, we deduce $e'_1e'_2 = e'_4,\ e'_1e'_3 = e'_5$.
- Let $e_ie_j \in span\{e_{k+1}\}$ for all $1 \le i, j \le k$. Then for some i_0 we have $e_{i_0}e_{k+1} \neq 0$. Without loss of generality, one can assume $e_1e_{k+1} = e_{k+2}$.
  - If $k \ge 3$, then setting
  $$e'_1 = e_1,\ e'_2 = e_2,\ e'_3 = e_3 + e_{k+1},\ e'_4 = e_{k+1},\ e'_5 = e_{k+2} + \alpha_{1,3}e_{k+1},$$
  we obtain $e'_1e'_2 = e'_4,\ e'_1e'_3 = e'_5$.
  - If $k = 2$, then we have $e_1e_2 = e_3,\ e_1e_3 = e_4$. It is not difficult to obtain that $e_1e_4 = e_5$ or $e_2e_3 = e_5$ (because of $n \ge 5$). Indeed, taking
  $$e'_1 = e_1,\ e'_2 = e_2,\ e'_3 = e_4,\ e'_4 = e_3,\ e'_5 = e_5$$
  in the case of $e_1e_4 = e_5$ and
  $$e'_1 = -e_3,\ e'_2 = e_1,\ e'_3 = e_4,\ e'_4 = e_2,\ e'_5 = e_5$$
  in the case of $e_2e_3 = e_5$, we derive the products $e_1e_2 = e_4,\ e_1e_3 = e_5$.

Thus, there exists a basis $\{e_1, e_2, e_3, \dots, e_n\}$ of A with the products
$$e_1e_2 = e_4,\ e_1e_3 = e_5.$$
Note that A degenerates to the algebra with multiplication:
$$e_1e_2 = e_4,\ e_1e_3 = e_5,\ e_2e_3 = \gamma_4x_4 + \gamma_4x_5$$
via the following degeneration:
$$g_t : \begin{cases} g_t(e_1) = t^{-2}e_1, & g_t(e_2) = t^{-2}e_2, \quad g_t(e_3) = t^{-2}e_3, \\ g_t(e_4) = t^{-4}e_4, & g_t(e_5) = t^{-4}e_5, \quad g_t(e_i) = t^{-3}e_i, \quad 6 \le i \le n. \end{cases}$$
From the change of basis $e'_2 = e_2 - \gamma_5e_1$, $e'_3 = e_3 + \gamma_4e_1$, we obtain that this algebra is isomorphic to $n_{5,2} \oplus \mathrm{ab}_{n-5}$. □

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