P_1 FINITE ELEMENT METHODS FOR AN ELLIPTIC STATE-CONSTRAINED DISTRIBUTED OPTIMAL CONTROL PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We investigate two P_1 finite element methods for an elliptic state-constrained distributed optimal control problem with Neumann boundary conditions on general polygonal domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider the following optimal control problem (cf. [20]):

$$\text{Find } (\bar{y}, \bar{u}) = \underset{(y,u)\in\mathbb{K}_g}{\operatorname{argmin}} \left[\frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Omega)}\right], \tag{1.1}$$

where $(y, u) \in H^1(\Omega) \times L^2(\Omega)$ belongs to $\mathbb{K}_g$ if and only if

$$\int_\Omega \nabla y \cdot \nabla w dx + \int_\Omega y w dx = \int_\Omega u w dx + \int_{\partial\Omega} g w ds \qquad \forall\, w \in H^1(\Omega), \tag{1.2}$$

$$y \le \psi \qquad \text{a.e. in } \Omega. \tag{1.3}$$

Remark 1.1. Throughout this paper we follow the standard notation for differential operators, functions spaces and norms that can be found for example in [22, 1, 14].

We assume that

(i) y_d belongs to $L^2(\Omega)$ and β is a positive constant,
(ii) $g = \dfrac{\partial \zeta_g}{\partial n}$ for some $\zeta_g \in H^4(\Omega)$,
(iii) ψ belongs to $H^3_{loc}(\Omega) \cap W^{2,\infty}_{loc}(\Omega)$ such that $\dfrac{\partial \psi}{\partial n} > g$ on $\partial\Omega$.

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We can reformulate the optimal control problem in terms of $\bar{y}$ alone. To this end, we introduce the affine subspace V_g of $H^1(\Omega)$ defined by

$$V_g = \Big\{ y \in H^1(\Omega) : \text{ there exists } u \in L^2(\Omega) \text{ such that} \\ \int_\Omega \nabla y \cdot \nabla w dx + \int_\Omega yw dx = \int_\Omega uw dx + \int_{\partial\Omega} gw ds \quad \forall\, w \in H^1(\Omega) \Big\}.$$

In the homogeneous case where $g = 0$, we will denote the linear subspace V_0 of $H^1(\Omega)$ by V.

Remark 1.2. The constraint (1.2) in the definition of V_g is the weak form of the following boundary value problem:

$$-\Delta y + y = u \text{ in } \Omega \quad \text{and} \quad \partial y/\partial n = g \text{ on } \partial\Omega,$$

where Δy is understood in the sense of distributions, and $g \in H^{-\frac{1}{2}}(\partial\Omega)$ is understood as the normal trace of $\nabla y \in H(\text{div}, \Omega)$. Therefore an alternative definition of V_g is given by

$$V_g = \left\{ y \in H^1(\Omega) : \mathscr{L} y \in L^2(\Omega) \text{ and } \partial y/\partial n = g \text{ on } \partial\Omega \right\},$$

where $\mathscr{L} y = -\Delta y + y$ defines an isomorphism from V_g onto $L^2(\Omega)$.

Due to elliptic regularity [33, 23, 41], V_g is an affine subspace of $H^{1+\alpha}(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$, where $\alpha = 1$ if Ω is convex, and

$$\|z\|_{H^{1+\alpha}(\Omega)} \le C_\Omega \big[\|\mathscr{L} z\|_{L^2(\Omega)} + \|\zeta_g\|_{H^2(\Omega)} \big] \qquad \forall\, z \in V_g. \tag{1.4}$$

Note that V_g is also an affine subspace of $H^2_{loc}(\Omega)$ by interior elliptic regularity.

It follows from (1.4) and the Sobolev inequality [1] that $V_g \subset C(\bar{\Omega})$ and we can reformulate the minimization problem (1.1)–(1.3) as follows:

$$\text{Find } \bar{y} = \underset{y \in K_g}{\operatorname{argmin}} \left[\frac{1}{2} \|y - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2} \|\mathscr{L} y\|^2_{L^2(\Omega)} \right], \tag{1.5}$$

where

$$K_g = \{ v \in V_g : v \le \psi \text{ in } \Omega \}. \tag{1.6}$$

Our goal is to develop P_1 finite element methods (FEMs) for (1.5)–(1.6).

FEMs for elliptic distributed optimal control problems with pointwise state constraints have been studied by many authors (cf. [24, 42, 34, 36, 40, 37, 4, 50, 32, 16, 21, 9, 17, 43, 13, 15, 11, 10, 18, 12] and the references therein). In [18], a C^0 interior penalty method for the optimal control problem (1.1)–(1.3) on convex domains with the homogeneous boundary condition ($g = 0$) was analyzed by the tools developed in [15]. In [11], theoretical and numerical results for two P_1 FEMs for a state-constrained elliptic distributed optimal control problem with Dirichlet boundary conditions were obtained for general polygonal/polyhedral domains, where the analysis extended the framework in [15]. In this paper, we will extend the results in [11] to (1.5)–(1.6). We note the convergence results in [11, 10] and the current paper are the first ones for nonconvex and nonsmooth domains.

The remainder of the paper is organized as follows. In the next section, we recall some results regarding the continuous problem (1.5)–(1.6), and we present two discrete problems in Section 3. Preliminary estimates for the convergence analysis are gathered in Section 4, followed by the convergence analysis of the FEMs in Sections 5 and 6. We present numerical results in Section 7 that corroborate the theoretical results and end with some concluding remarks in Section 8.

We will use C (with or without subscript) to denote a generic positive constant independent of the mesh size. To avoid the proliferation of constants, we will also use the notation $A \lesssim B$ to denote the inequality $A \leq (\text{constant})B$, where the hidden constant is independent of the mesh size. The notation $A \approx B$ is equivalent to the statement that $A \lesssim B$ and $B \lesssim A$.

2. The Continuous Problem

In this section we will collect information on the continuous problem (1.5)–(1.6). From here on we use $(\cdot,\cdot)$ to denote the inner product for $L^2(\Omega)$ (or $[L^2(\Omega)]^2$).

Let $\bar{z} = \bar{y} - \zeta_g$. We can rewrite (1.5)–(1.6) as

$$\bar{z} = \operatorname*{argmin}_{z\in K} \Big[\frac{1}{2}\|z - (y_d - \zeta_g)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\mathscr{L}(z+\zeta_g)\|^2_{L^2(\Omega)}\Big], \tag{2.1}$$

where

$$K = \{v \in V : v \leq \psi - \zeta_g \text{ in } \Omega\}. \tag{2.2}$$

Since V is a Hilbert space under the inner product

$$(\!(y,z)\!) = (y,z) + (\mathscr{L}y, \mathscr{L}z),$$

it follows from the standard theory [39, 26, 38] that (2.1)–(2.2) (and hence (1.5)–(1.6)) has a unique solution characterized by the variational inequality

$$(\bar{y} - y_d, y - \bar{y}) + \beta(\mathscr{L}\bar{y}, \mathscr{L}(y-\bar{y})) \geq 0 \quad \forall\, y \in K_g. \tag{2.3}$$

Interior Regularity of $\bar{y}$

By the interior regularity results for fourth order variational inequalities in [29, 30, 19], we have

$$\bar{y} \in H^3_{loc}(\Omega) \cap W^{2,\infty}_{loc}(\Omega). \tag{2.4}$$

Lagrange Multiplier μ

Recall that $V \subset C(\bar{\Omega})$. Let $\phi \in C^\infty(\Omega) \cap V$ be nonnegative. Then $y = \bar{y} - \phi \in K_g$ and we have, by (2.3),

$$(\bar{y} - y_d, \phi) + \beta(\mathscr{L}\bar{y}, \mathscr{L}\phi) \leq 0. \tag{2.5}$$

Since $C^\infty(\Omega)\cap V$ is dense in $C(\bar{\Omega})$, it follows from (2.5) and the Riesz representation theorem [45, 46, 28] that

$$(\bar{y} - y_d, z) + \beta(\mathscr{L}\bar{y}, \mathscr{L}z) = \int_{\bar{\Omega}} z\, d\mu \qquad \forall\, z \in V, \tag{2.6}$$

where μ is a nonpositive finite Borel measure on $\bar{\Omega}$.

Let $\mathfrak{A} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$ be the active set for the constraint (1.3). Under the assumption $\partial\psi/\partial n > g$, we have (cf. [18, Appendix])

$$\mathfrak{A} \text{ is a compact subset of } \Omega. \tag{2.7}$$

For any $z \in V$ whose support is disjoint from $\mathfrak{A}$, $\pm\epsilon z + \bar{y}$ belong to K_g for sufficiently small ϵ. Therefore, by (2.3), we have

$$(\bar{y} - y_d, z) + \beta(\mathscr{L}\bar{y}, \mathscr{L}z) = 0 \tag{2.8}$$

for all $z \in V$ such that $\mathrm{supp}(z) \cap \mathfrak{A} = \emptyset$. Hence, in view of (2.6),

$$\mu \text{ is a nonpositive finite Borel measure supported on } \mathfrak{A}, \tag{2.9}$$

which is equivalent to

$$\int_\Omega (\bar{y} - \psi) d\mu = 0. \tag{2.10}$$

Remark 2.1. The conditions (2.6), (2.9) and (2.10) are the Karush-Kuhn-Tucker (KKT) conditions that characterize the solution of (1.5)–(1.6).

Let Φ belong to $C_c^\infty(\Omega)$ (the space of C^∞ functions with compact supports in Ω) such that $\Phi = 1$ in an open neighborhood of the compact subset $\mathfrak{A}$ of Ω. Given any $z \in V$, We have, by (2.6) and (2.9),

$$\begin{aligned}\int_\Omega z\, d\mu = \int_{\bar{\Omega}} z\Phi\, d\mu \\ &= (\bar{y} - y_d, z\Phi) + \beta(\mathscr{L}\bar{y}, \mathscr{L}(z\Phi)) \\ &= (\bar{y} - y_d, z\Phi) + \beta(\mathscr{L}\bar{y}, -\Delta(z\Phi) + (z\Phi)) \\ &= (\bar{y} - y_d, z\Phi) + \beta(\nabla(\mathscr{L}\bar{y}), \nabla(z\Phi)) + \beta(\mathscr{L}\bar{y}, z\Phi),\end{aligned}$$

where the integration by parts is justified by (2.4) and the fact that z belongs to $H^2_{loc}(\Omega)$. It follows that

$$\Big|\int_\Omega z\, d\mu\Big| \le C\|z\|_{H^1(G)} \qquad \forall\, z \in V, \tag{2.11}$$

where G is an open neighborhood of the support of Φ such that $G \subset\subset \Omega$ (i.e., the closure of G is a compact subset of Ω).

Given any $z \in H^1(\Omega)$, we can construct a sequence $z_n \in V$ such that $\|z_n - z\|_{H^1(G)} \to 0$ as $n \to \infty$. (In fact we can choose z_n from $C_c^\infty(\Omega)$.) In view of (2.11), $\lim_{n\to\infty} \int_\Omega z_n d\mu$ is independent of the choices of z_n. We can therefore define

$$\langle \mu, z\rangle = \lim_{n\to\infty} \int_\Omega z_n d\mu \qquad \forall\, z \in H^1(\Omega). \tag{2.12}$$

Note that $\langle \mu, z\rangle = \int_\Omega z\, d\mu$ for $z \in V$ because we can take $z_n = z$ for all n in (2.12).

It follows from (2.11) and (2.12) that

$$|\langle \mu, z\rangle| \le C\|z\|_{H^1(\Omega)} \qquad \forall\, z \in H^1(\Omega). \tag{2.13}$$

Regularity of $\bar{u}$

In view of (2.13), we can define the adjoint state $\bar{p} \in H^1(\Omega)$ by

$$(\nabla \bar{p}, \nabla z) + (\bar{p}, z) = (\bar{y} - y_d, z) - \langle \mu, z \rangle \qquad \forall\, z \in H^1(\Omega).$$

It then follows from the definition of V (cf. Remark 1.2 with $g = 0$) that

$$(\bar{p}, \mathscr{L} z) = (\bar{y} - y_d, z) - \int_\Omega z\, d\mu \qquad \forall\, z \in V. \tag{2.14}$$

Comparing (2.6) and (2.14), we find

$$(\bar{p} - \beta \mathscr{L} \bar{y}, \mathscr{L} z) = 0 \qquad \forall\, z \in V,$$

and hence, since $\mathscr{L} : V \longrightarrow L^2(\Omega)$ is an isomorphism,

$$\bar{u} = \mathscr{L} \bar{y} = \beta^{-1} \bar{p} \in H^1(\Omega). \tag{2.15}$$

Global Regularity of $\bar{y}$

According to (1.4), we have

$$\bar{y} \in H^{1+\alpha}(\Omega), \tag{2.16}$$

where α belongs to $(\frac{1}{2}, 1]$ in general. In the case where Ω is convex, the constraint (1.2) and the regularity of $\bar{u}$ in (2.15) imply that $1 < \alpha \leq 2$ (cf. [33, Chapter 5] and [23, Section 18]). The assumption $\zeta_g \in H^4(\Omega)$ ensures that the Neumann boundary condition does not interfere with the higher regularity for convex domains.

3. The Discrete Problems

Let $\mathscr{T}_h$ be a regular triangulation of Ω and $V_h \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathscr{T}_h$. The diameter of $T \in \mathscr{T}_h$ is denoted by h_T and $h = \max_{T \in \mathscr{T}_h} h_T$ is the mesh parameter.

3.1. **The First P_1 Finite Element Method.** The first P_1 FEM is to find

$$\bar{y}_h = \underset{y_h \in K_h}{\operatorname{argmin}} \Big[\frac{1}{2} \| y_h - y_d \|^2_{L^2(\Omega)} + \frac{\beta}{2} (\mathscr{L}_{h,g} y_h, \mathscr{L}_{h,g} y_h) \Big], \tag{3.1}$$

where

$$K_h = \{ v_h \in V_h : v_h \leq I_h \psi \}, \tag{3.2}$$

and $I_h : C(\bar{\Omega}) \longrightarrow V_h$ is the nodal interpolation operator. In other words, the discrete constraint is only imposed at the vertices of $\mathscr{T}_h$. The affine map $\mathscr{L}_{h,g} : H^1(\Omega) \longrightarrow V_h$ is defined by

$$(\mathscr{L}_{h,g} w, v_h) = (\nabla w, \nabla v_h) + (w, v_h) - \int_{\partial\Omega} g v_h ds \quad \forall\, v_h \in V_h. \tag{3.3}$$

Remark 3.1. The P_1 FEM defined by (3.1) and (3.2) is identical to the method in [42], but our convergence analysis in Section 5 is completely different. In particular our convergence results do not require Ω to be convex and we also have error estimates in $L^\infty(\Omega)$.

Notice that

$$\mathscr{L}_{h,g} z = Q_h \mathscr{L} z \qquad \forall\, z \in V_g, \tag{3.4}$$

where $Q_h : L^2(\Omega) \to V_h$ is the $L^2(\Omega)$ orthogonal projection. This is true, since

$$(\mathscr{L}_{h,g} z, v_h) = (\nabla z, \nabla v_h) + (z, v_h) - \int_{\partial\Omega} g v_h\, ds = (\mathscr{L} z, v_h) = (Q_h \mathscr{L} z, v_h) \qquad \forall\, v_h \in V_h,$$

by Remark 1.2 and (3.3).

In the case where $g = 0$, the affine map $\mathscr{L}_{h,0}$ becomes a linear map that will be denoted simply by $\mathscr{L}_h$, i.e., $\mathscr{L}_h : H^1(\Omega) \longrightarrow V_h$ satisfies

$$(\mathscr{L}_h w, v_h) = (\nabla w, \nabla v_h) + (w, v_h) \qquad \forall\, w \in H^1(\Omega), v_h \in V_h. \tag{3.5}$$

We have a useful relation

$$\mathscr{L}_{h,g} v_1 - \mathscr{L}_{h,g} v_2 = \mathscr{L}_h (v_1 - v_2) \qquad \forall\, v_1, v_2 \in H^1(\Omega) \tag{3.6}$$

that follows immediately from (3.3) and (3.5).

Using (3.6) and a standard computation, we can characterize the unique solution $\bar{y}_h \in K_h$ of (3.1)–(3.2) by the following discrete variational inequality:

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g} \bar{y}_h, \mathscr{L}_h (y_h - \bar{y}_h)) \ge 0 \qquad \forall\, y_h \in K_h. \tag{3.7}$$

3.2. **The Second P_1 Finite Element Method.** To construct the second P_1 FEM, we first introduce another inner product $(\cdot,\cdot)_h$ defined by

$$(v_h, w_h)_h = \sum_{p \in \mathfrak{V}_h} \Big(\sum_{T \in \mathscr{T}_p} \frac{|T|}{3} \Big) v_h(p) w_h(p) \quad \forall\, v_h, w_h \in V_h, \tag{3.8}$$

where $\mathfrak{V}_h$ is the set of the vertices in the triangulation $\mathscr{T}_h$, $\mathscr{T}_p$ denotes the collection of all elements that have p as a common vertex, and $|T|$ is the area of T.

The second P_1 FEM is to find

$$\bar{y}_h = \underset{y_h \in K_h}{\operatorname{argmin}} \Big[\frac{1}{2} \|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2} (\tilde{\mathscr{L}}_{h,g} y_h, \tilde{\mathscr{L}}_{h,g} y_h)_h \Big], \tag{3.9}$$

where K_h is defined in (3.2), and the affine map $\tilde{\mathscr{L}}_{h,g} : H^1(\Omega) \longrightarrow V_h$ is given by

$$(\tilde{\mathscr{L}}_{h,g} w, v_h)_h = (\nabla w, \nabla v_h) + (w, v_h) - \int_{\partial\Omega} g v_h ds \quad \forall\, v_h \in V_h. \tag{3.10}$$

As before, we will denote $\tilde{\mathscr{L}}_{h,g}$ by $\tilde{\mathscr{L}}_h$ when $g = 0$, i.e., $\tilde{\mathscr{L}}_h : H^1(\Omega) \longrightarrow V_h$ satisfies

$$(\tilde{\mathscr{L}}_h w, v_h)_h = (\nabla w, \nabla v_h) + (w, v_h) \qquad \forall\, v_h \in V_h. \tag{3.11}$$

Then we again have

$$\tilde{\mathscr{L}}_{h,g} v_1 - \tilde{\mathscr{L}}_{h,g} v_2 = \tilde{\mathscr{L}}_h (v_1 - v_2) \qquad \forall\, v_1, v_2 \in H^1(\Omega), \tag{3.12}$$

and the unique solution $\bar{y}_h \in K_h$ of (3.9) can be characterized by the following discrete variational inequality:

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta(\tilde{\mathscr{L}}_{h,g} \bar{y}_h, \tilde{\mathscr{L}}_h (y_h - \bar{y}_h))_h \ge 0 \qquad \forall\, y_h \in K_h. \tag{3.13}$$

Remark 3.2. The P_1 FEM defined by (3.9) and its counterpart in [11] are new methods for elliptic distributed optimal control problems with pointwise state constraints. The motivation for introducing these methods is the fact that, unlike traditional P_1 FEMs (such as the P_1 FEM from Section 3.1), the system matrices for FEMs with mass lumping are readily available because the mass matrix for the inner product $(\cdot,\cdot)_h$ is diagonal. Therefore it is straightforward to solve the discrete variational inequalities by a primal-dual active algorithm [5, 6, 35] that converges superlinearly.

4. Preliminary Estimates

In this section we derive some estimates that will be used in the convergence analysis in Sections 5 and 6. We assume that $\mathscr{T}_h$ is either quasi-uniform [22, 14] or graded around the reentrant corners [33, 31, 2, 8].

4.1. **The Interpolation Operator I_h.** We summarize here some estimates regarding the nodal interpolation operator that we need in the convergence analysis. They follow from (1.4) and the standard error estimates of the nodal interpolation operator I_h in [33, 22, 3, 25, 14]:

$$\|z-I_hz\|_{L^2(\Omega)}+h|z-I_hz|_{H^1(\Omega)}+h\|z-I_hz\|_{L^\infty(\Omega)}\lesssim h^{1+\tau}(\|\mathscr{L}z\|_{L^2(\Omega)}+\|\zeta_g\|_{H^2(\Omega)}) \tag{4.1}$$

for all $z\in V_g$, where

$$\tau=\begin{cases}\alpha & \text{if } \mathscr{T}_h \text{ is quasi-uniform,}\\ 1 & \text{if } \mathscr{T}_h \text{ is graded around the reentrant corners.}\end{cases} \tag{4.2}$$

In particular, we have

$$\|z-I_hz\|_{L^2(\Omega)}+h|z-I_hz|_{H^1(\Omega)}+h\|z-I_hz\|_{L^\infty(\Omega)}\lesssim h^{1+\tau}\|\mathscr{L}z\|_{L^2(\Omega)}\qquad\forall\, z\in V. \tag{4.3}$$

Let $\phi\in H^2(\Omega)$ be arbitrary. We have, by (3.5), standard inverse and interpolation error estimates [22, 14],

$$\begin{aligned}(\mathscr{L}_h(\phi-I_h\phi),v_h)&=(\nabla(\phi-I_h\phi),\nabla v_h)+(\phi-I_h\phi,v_h)\\ &\le\|\phi-I_h\phi\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}\lesssim h|\phi|_{H^2(\Omega)}\|v_h\|_{H^1(\Omega)}\lesssim|\phi|_{H^2(\Omega)}\|v_h\|_{L^2(\Omega)},\end{aligned}$$

and hence

$$\|\mathscr{L}_h(\phi-I_h\phi)\|_{L^2(\Omega)}\lesssim|\phi|_{H^2(\Omega)}\qquad\forall\phi\in H^2(\Omega). \tag{4.4}$$

We conclude by using (3.4) and (4.4) that

$$\begin{aligned}\|\mathscr{L}_h(I_h\phi)\|_{L^2(\Omega)}&\le\|\mathscr{L}_h(I_h\phi-\phi)\|_{L^2(\Omega)}+\|Q_h\mathscr{L}\phi\|_{L^2(\Omega)},\\ &\lesssim|\phi|_{H^2(\Omega)}+\|\mathscr{L}\phi\|_{L^2(\Omega)}\lesssim\|\phi\|_{H^2(\Omega)}\qquad\forall\,\phi\in H^2(\Omega)\cap V.\end{aligned} \tag{4.5}$$

4.2. **The Operator E_h.** The operator $E_h:V_h\longrightarrow V$ is defined by

$$\mathscr{L}E_hv_h=\mathscr{L}_hv_h\qquad\forall\,v_h\in V_h, \tag{4.6}$$

or equivalently

$$(\nabla E_hv_h,\nabla w)+(E_hv_h,w)=(\mathscr{L}_hv_h,w)\qquad\forall\,w\in H^1(\Omega). \tag{4.7}$$

Due to the interior elliptic regularity (cf. [27]), E_hv_h belongs to $H^2_{loc}(\Omega)$ and

$$\|E_hv_h\|_{H^2(G)}\le C_G\|\mathscr{L}_hv_h\|_{L^2(\Omega)} \tag{4.8}$$

for any open set $G \subset\subset \Omega$.

Comparing (3.5) and (4.7), we see that $v_h \in V_h$ is the $H^1(\Omega)$ orthogonal projection of $E_h v_h \in V$. It then follows from (4.3) and (4.6) that

$$\begin{aligned} \|E_h v_h - v_h\|_{H^1(\Omega)} &= \inf_{w_h \in V_h} \|E_h v_h - w_h\|_{H^1(\Omega)} \\ &\le \|E_h v_h - I_h E_h v_h\|_{H^1(\Omega)} \lesssim h^\tau \|\mathscr{L} E_h v_h\|_{L^2(\Omega)} = h^\tau \|\mathscr{L}_h v_h\|_{L^2(\Omega)}. \end{aligned} \tag{4.9}$$

Furthermore, by a standard duality argument, we get

$$\|E_h v_h - v_h\|_{L^2(\Omega)} \lesssim h^{2\tau} \|\mathscr{L}_h v_h\|_{L^2(\Omega)}. \tag{4.10}$$

Combining (4.8), (4.9) and the local error estimate in [49, Theorem 9.1], we also have

$$|v_h - E_h v_h|_{H^1(G(\mathfrak{A}))} \lesssim h \|\mathscr{L}_h v_h\|_{L^2(\Omega)} \qquad \forall\, v_h \in V_h, \tag{4.11}$$

where $G(\mathfrak{A}) \subset\subset \Omega$ is an open neighborhood of the active set $\mathfrak{A}$.

According to (1.4) (with $\zeta_g = 0$) and the Sobolev inequality, we have

$$\|z\|_{L^\infty(\Omega)} + \|z\|_{H^1(\Omega)} \le C_\Omega \|\mathscr{L} z\|_{L^2(\Omega)} \qquad \forall\, z \in V. \tag{4.12}$$

We can use the operator E_h to obtain a discrete analog of (4.12).

Lemma 4.1. *There exists a positive constant C independent of h such that*

$$\|v_h\|_{L^\infty(\Omega)} + \|v_h\|_{H^1(\Omega)} \le C \|\mathscr{L}_h v_h\|_{L^2(\Omega)} \quad \forall\, v_h \in V_h.$$

Proof. Since $v_h \in V_h$ is the $H^1(\Omega)$ orthogonal projection of $E_h v_h$, we have, by (4.6) and (4.12),

$$\|v_h\|_{H^1(\Omega)} \le \|E_h v_h\|_{H^1(\Omega)} \lesssim \|\mathscr{L} E_h v_h\|_{L^2(\Omega)} = \|\mathscr{L}_h v_h\|_{L^2(\Omega)}.$$

Observe that we have a discrete Sobolev inequality [14, Lemma 4.9.2]

$$\|v_h\|_{L^\infty(\Omega)} \lesssim (1 + |\ln h|)^{\frac{1}{2}} \|v_h\|_{H^1(\Omega)} \qquad \forall\, v_h \in V_h, \tag{4.13}$$

which together with (4.3), (4.6), (4.9) and (4.13) implies

$$\begin{aligned} \|v_h - E_h v_h\|_{L^\infty(\Omega)} &\le \|v_h - I_h E_h v_h\|_{L^\infty(\Omega)} + \|I_h E_h v_h - E_h v_h\|_{L^\infty(\Omega)} \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} \|v_h - I_h E_h v_h\|_{H^1(\Omega)} + h^\tau \|\mathscr{L} E_h v_h\|_{L^2(\Omega)} \\ &\le (1 + |\ln h|)^{\frac{1}{2}} (\|v_h - E_h v_h\|_{H^1(\Omega)} + \|E_h v_h - I_h E_h v_h\|_{H^1(\Omega)}) + h^\tau \|\mathscr{L}_h v_h\|_{L^2(\Omega)} \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} h^\tau \|\mathscr{L}_h v_h\|_{L^2(\Omega)}. \end{aligned}$$

On the other hand, we have, by (4.6) and (4.12),

$$\|E_h v_h\|_{L^\infty(\Omega)} \lesssim \|\mathscr{L} E_h v_h\|_{L^2(\Omega)} = \|\mathscr{L}_h v_h\|_{L_2(\Omega)}.$$

The estimate for $\|v_h\|_{L^\infty(\Omega)}$ follows from these two estimates. ∎

4.3. **The Operator** R_h**.** The Riesz projection $R_h : H^1(\Omega) \longrightarrow V_h$ is defined by

$$(\nabla R_h v, \nabla v_h) + (R_h v, v_h) = (\nabla v, \nabla v_h) + (v, v_h) \qquad \forall\, v_h \in V_h. \tag{4.14}$$

It follows immediately from (3.5) and (4.14) that

$$\mathscr{L}_h R_h z = \mathscr{L}_h z \qquad \forall\, z \in H^1(\Omega), \tag{4.15}$$

and hence also, in view of (3.6),

$$\mathscr{L}_{h,g} R_h z = \mathscr{L}_{h,g} z \qquad \forall\, z \in H^1(\Omega). \tag{4.16}$$

Note that (3.4) and (4.16) imply

$$\mathscr{L}_{h,g} R_h z = Q_h \mathscr{L} z \qquad \forall\, z \in V_g. \tag{4.17}$$

Similarly, we have, by (3.11), (3.12) and (4.14),

$$\tilde{\mathscr{L}}_{h,g} R_h z = \tilde{\mathscr{L}}_{h,g} z \qquad \forall\, z \in H^1(\Omega). \tag{4.18}$$

As in (4.9) and (4.10), we have the following standard error estimates:

$$\|\bar{y} - R_h \bar{y}\|_{H^1(\Omega)} \le C h^{\tau}, \tag{4.19}$$

$$\|\bar{y} - R_h \bar{y}\|_{L^2(\Omega)} \le C h^{2\tau}. \tag{4.20}$$

Combining the interior regularity (2.4) and the L^2 error estimate (4.20) with the local error estimate in [49, Theorem 10.1] , we have

$$\|\bar{y} - R_h \bar{y}\|_{L^\infty(G(\mathfrak{A}))} \lesssim |\ln h| h^2 + h^{2\tau}. \tag{4.21}$$

Finally, it follows from (4.1), (4.13) and (4.19) that

$$\begin{aligned} \|\bar{y} - R_h \bar{y}\|_{L^\infty(\Omega)} &\le \|\bar{y} - I_h \bar{y}\|_{L^\infty(\Omega)} + \|I_h \bar{y} - R_h \bar{y}\|_{L^\infty(\Omega)} \\ &\lesssim h^{\tau} + (1 + |\ln h|)^{\frac{1}{2}} \|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)} \\ &\lesssim h^{\tau} + (1 + |\ln h|)^{\frac{1}{2}} \big[\|I_h \bar{y} - \bar{y}\|_{H^1(\Omega)} + \|\bar{y} - R_h \bar{y}\|_{H^1(\Omega)}\big] \\ &\lesssim (1 + |\ln h|)^{\frac{1}{2}} h^{\tau}, \end{aligned} \tag{4.22}$$

and hence

$$\lim_{h \to 0} \|\bar{y} - R_h \bar{y}\|_{L^\infty(\Omega)} = 0. \tag{4.23}$$

5. Convergence Analysis of the First P_1 Finite Element Method

We will use the mesh dependent norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = (v, v) + \beta(\mathscr{L}_h v, \mathscr{L}_h v). \tag{5.1}$$

5.1. **An Abstract Error Estimate.** Let $\bar{y} \in K_g$ be the solution to (1.5)–(1.6), $\bar{y}_h \in K_h$ be the solution to the discrete problem (3.1)–(3.2), and $y_h \in K_h$ be arbitrary.

It follows from (3.6), (3.7) and (5.1) that

$$\begin{aligned}\|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}_h, y_h - \bar{y}_h) + \beta(\mathscr{L}_h(y_h - \bar{y}_h), \mathscr{L}_h(y_h - \bar{y}_h)) \\ &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta(\mathscr{L}_h(y_h - \bar{y}), \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\quad + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_h \bar{y}, \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\quad - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\mathscr{L}_h \bar{y}_h, \mathscr{L}_h(y_h - \bar{y}_h)) \\ &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta(\mathscr{L}_h(y_h - \bar{y}), \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\quad + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g} \bar{y}, \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\quad - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\mathscr{L}_{h,g} \bar{y}_h, \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\le \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + \big[(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g} \bar{y}, \mathscr{L}_h(y_h - \bar{y}_h))\big].\end{aligned} \tag{5.2}$$

Remark 5.1. The derivation of (5.2) is the only place where we use the fact that $\bar{y}_h$ is the solution to (3.1)–(3.2). The relation (5.3), the estimate (5.4) and Lemma 5.1 below actually hold for any $\bar{y}_h \in V_h$.

Using (2.6), (3.4) and (4.6), we can rewrite the second term on the last line of (5.2) as

$$\begin{aligned}&(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g} \bar{y}, \mathscr{L}_h(y_h - \bar{y}_h)) \\ &\qquad = (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) \\ &\qquad\quad + \big[(\bar{y} - y_d, E_h(y_h - \bar{y}_h)) + \beta(\mathscr{L} \bar{y}, \mathscr{L} E_h(y_h - \bar{y}_h))\big] \\ &\qquad = (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) + \int_\Omega E_h(y_h - \bar{y}_h) d\mu,\end{aligned} \tag{5.3}$$

and we have, by (4.10),

$$(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) \le C h^{2\tau} \|\mathscr{L}_h(y_h - \bar{y}_h)\|_{L^2(\Omega)}. \tag{5.4}$$

The next Lemma will give a bound on the last term of the right-hand side of (5.3).

Lemma 5.1. *We have*

$$\int_\Omega E_h(y_h - \bar{y}_h) d\mu \lesssim h \|\mathscr{L}_h(y_h - \bar{y}_h)\|_{L^2(\Omega)} + h^2 + \|y_h - I_h \bar{y}\|_{L^\infty(\mathfrak{A})} \qquad \forall\, \bar{y}_h, y_h \in K_h,$$

where $\mathfrak{A} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$ *is the active set for the constraint* (1.3).

Proof. We begin with the estimate

$$\begin{aligned}\int_\Omega E_h(y_h - \bar{y}_h) d\mu &= \int_\Omega \big[E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)\big] d\mu + \int_\Omega (I_h \psi - \bar{y}_h) d\mu \\ &\quad + \int_\Omega I_h(\bar{y} - \psi) d\mu + \int_\Omega (y_h - I_h \bar{y}) d\mu \\ &\le \int_\Omega \big[E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)\big] d\mu + \int_\Omega I_h(\bar{y} - \psi) d\mu \\ &\quad + \int_\Omega (y_h - I_h \bar{y}) d\mu\end{aligned} \tag{5.5}$$

that follows from (2.9) and (3.2).

We can bound the terms on the right-hand side of (5.5) in the following way:

$$\int_\Omega \big[E_h(y_h-\bar{y}_h)-(y_h-\bar{y}_h)\big]d\mu \lesssim \|E_h(y_h-\bar{y}_h)-(y_h-\bar{y}_h)\|_{H^1(G(\mathfrak{A}))} \lesssim h\|\mathscr{L}_h(y_h-\bar{y}_h)\|_{L^2(\Omega)} \tag{5.6}$$

by (2.9), (2.13) and (4.11);

$$\int_\Omega I_h(\bar{y}-\psi)d\mu = \int_{\mathfrak{A}} \big[I_h(\bar{y}-\psi)-(\bar{y}-\psi)\big]d\mu \lesssim \|I_h(\bar{y}-\psi)-(\bar{y}-\psi)\|_{L^\infty(\mathfrak{A})} \le Ch^2 \tag{5.7}$$

by (2.9), (2.10) and the fact that $\psi, \bar{y} \in W^{2,\infty}_{loc}(\Omega)$; and

$$\int_\Omega (y_h - I_h\bar{y})d\mu \lesssim \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})} \tag{5.8}$$

by (2.9). □

Putting (5.2)–(5.4) and Lemma 5.1 together, we find

$$\begin{aligned}\|y_h-\bar{y}_h\|_h^2 &\lesssim \|y_h-\bar{y}\|_h\|y_h-\bar{y}_h\|_h + h^{2\tau}\|\mathscr{L}_h(y_h-\bar{y}_h)\|_{L^2(\Omega)} \\ &\quad + h\|\mathscr{L}_h(y_h-\bar{y}_h)\|_{L^2(\Omega)} + h^2 + \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})} \\ &\lesssim \big(\|y_h-\bar{y}\|_h + h\big)\|y_h-\bar{y}_h\|_h + h^2 + \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})},\end{aligned}$$

which together with the inequality of arithmetic and geometric means implies

$$\|y_h-\bar{y}_h\|_h \lesssim \|y_h-\bar{y}\|_h + h + \|y_h-I_h\bar{y}\|^{\frac12}_{L^\infty(\mathfrak{A})} \qquad \forall\, y_h \in K_h. \tag{5.9}$$

Finally by applying the triangle inequality twice, we conclude from (5.9) that

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_h &\le \|\bar{y}-y_h\|_h + \|y_h-\bar{y}_h\|_h \\ &\lesssim \|\bar{y}-y_h\|_h + h + \|y_h-I_h\bar{y}\|^{\frac12}_{L^\infty(\mathfrak{A})} \\ &\lesssim \|\bar{y}-y_h\|_h + h + \|\bar{y}-y_h\|^{\frac12}_{L^\infty(\mathfrak{A})} + \|\bar{y}-I_h\bar{y}\|^{\frac12}_{L^\infty(\mathfrak{A})} \\ &\lesssim \|\bar{y}-y_h\|_h + h + \|\bar{y}-y_h\|^{\frac12}_{L^\infty(\mathfrak{A})} \qquad \forall\, y_h \in V_h,\end{aligned}$$

where we have also used the interior regularity $\bar{y} \in W^{2,\infty}_{loc}(\Omega)$. It follows that

$$\|\bar{y}-\bar{y}_h\|_h \lesssim h + \inf_{y_h\in K_h}\big[\|\bar{y}-y_h\|_h + \|\bar{y}-y_h\|^{\frac12}_{L^\infty(\mathfrak{A})}\big]. \tag{5.10}$$

Remark 5.2. The abstract error estimate (5.10) implies that $\|\bar{y}-\bar{y}_h\|_h$ is uniformly bounded with respect to h. Indeed, let c be a sufficiently large positive number so that $\zeta_g - c < \psi$ on $\bar{\Omega}$. Then $\bar{y}-\zeta_g+c$ belongs to V and $y_h = I_h(\zeta_g - c)$ belongs to K_h. We obtain from (5.10) that

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_h &\lesssim 1 + \|\bar{y}-I_h(\zeta_g-c)\|_h + \|\bar{y}-I_h(\zeta_g-c)\|^{\frac12}_{L_\infty(\Omega)} \\ &\lesssim 1 + \|\bar{y}-I_h(\zeta_g-c)\|_{L^2(\Omega)} + \beta\|\mathscr{L}_h(\bar{y}-I_h(\zeta_g-c))\|_{L^2(\Omega)} + \|\bar{y}-I_h(\zeta_g-c)\|^{\frac12}_{L_\infty(\Omega)}\end{aligned}$$

and the right-hand side is uniformly bounded with respect to h because

$$\begin{aligned}\|\mathscr{L}_h(\bar{y}-I_h(\zeta_g-c))\|_{L^2(\Omega)} &\le \|\mathscr{L}_h(\bar{y}-\zeta_g+c)\|_{L^2(\Omega)}+\|\mathscr{L}_h(\zeta_g-c-I_h(\zeta_g-c))\|_{L^2(\Omega)}\\ &\lesssim \|\mathscr{L}(\bar{y}-\zeta_g+c)\|_{L^2(\Omega)}+|\zeta_g|_{H^2(\Omega)}\end{aligned}$$

by (3.4) (applied to the case where $g=0$) and (4.4).

5.2. **Concrete Error Estimates.** We can obtain concrete error estimates from (5.10) by producing $y_h\in K_h$ that is an accurate approximation of $\bar{y}$.

Lemma 5.2. *For h sufficiently small, there exists $y_h\in K_h$ such that*

$$\|y_h-\bar{y}\|_h+\|y_h-\bar{y}\|_{L^\infty(\mathfrak{A})}^{\frac12}\le C(|\ln h|^{\frac12}h+h^\tau),$$

where the positive constant C is independent of h.

Proof. Let $\epsilon_h=\|\bar{y}-R_h\bar{y}\|_{L^\infty(G(\mathfrak{A}))}$. It follows from (4.21) that

$$\epsilon_h\lesssim|\ln h|h^2+h^{2\tau}. \tag{5.11}$$

We claim that

$$y_h=R_h\bar{y}-\epsilon_h I_h\phi \tag{5.12}$$

belongs to K_h for $h\ll 1$, where $\phi\in C_c^\infty(\Omega)$ is nonnegative and $\phi=1$ on $G(\mathfrak{A})$.

Indeed, since $\psi-\bar{y}\ge\delta>0$ on $\Omega\setminus G(\mathfrak{A})$, by the definition of y_h in (5.12) we have

$$y_h\le R_h\bar{y}=\bar{y}+(R_h\bar{y}-\bar{y})\le\psi-\delta+(R_h\bar{y}-\bar{y})\qquad\text{on }\Omega\setminus G(\mathfrak{A}),$$

and therefore, by (4.23),

$$y_h(p)<\psi(p)\quad\text{ for all vertices }p\in\Omega\setminus G(\mathfrak{A})$$

if h is sufficiently small. On the other hand, we can use (5.11), (5.12) and the fact that $\phi=1$ on $G(\mathfrak{A})$ to get

$$y_h=\bar{y}+(R_h\bar{y}-\bar{y})-\epsilon_h\le\bar{y}\le\psi\qquad\text{on }G(\mathfrak{A}),$$

and therefore

$$y_h(p)\le\psi(p)\qquad\text{for all vertices }p\in G(\mathfrak{A}).$$

So y_h belongs to K_h. Moreover, we have

$$\begin{aligned}\|\bar{y}-y_h\|_h^2&=\|\bar{y}-y_h\|_{L^2(\Omega)}^2+\beta\|\mathscr{L}_h(\bar{y}-y_h)\|_{L^2(\Omega)}^2\\ &\lesssim\|R_h\bar{y}-\bar{y}\|_{L^2(\Omega)}^2+\|\epsilon_h I_h\phi\|_{L^2(\Omega)}^2+\|\epsilon_h\mathscr{L}_h I_h\phi\|_{L^2(\Omega)}^2\lesssim h^{4\tau}+\epsilon_h^2\lesssim|\ln h|^2h^4+h^{4\tau}\end{aligned}$$

by (4.5), (4.15), (4.20) and (5.11); and

$$\|y_h-\bar{y}\|_{L^\infty(\mathfrak{A})}\le\|R_h\bar{y}-\bar{y}\|_{L^\infty(\mathfrak{A})}+\|\epsilon_h I_h\phi\|_{L^\infty(\mathfrak{A})}\lesssim|\ln h|h^2+h^{2\tau}$$

by (4.21) and (5.11).

Putting these together, we finally reach

$$\|y_h-\bar{y}\|_h+\|y_h-\bar{y}\|_{L^\infty(\mathfrak{A})}^{\frac12}\lesssim|\ln h|^{\frac12}h+h^\tau.$$

□

The following theorem presents a concrete error estimate for the first P_1 FEM.

Theorem 5.1. *Suppose* $(\bar{y}, \bar{u}) \in \mathbb{K}_g$ *is the solution of* (1.1)–(1.3), $\bar{y}_h \in K_h$ *is the solution of* (3.1)–(3.2), *and* $\bar{u}_h = \mathscr{L}_{h,g}\bar{y}_h$. *Then, we have*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau), \tag{5.13}$$

where the positive constant C *is independent of* h.

Proof. For h sufficiently small, we have by (5.1), (5.10), and Lemma 5.2,

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\mathscr{L}_h(\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} \lesssim |\ln h|^{\frac{1}{2}}h + h^\tau. \tag{5.14}$$

It follows from (2.15), (3.4), (3.6) and (5.14) that

$$\begin{aligned} \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} &= \|\mathscr{L}_{h,g}\bar{y}_h - \mathscr{L}\bar{y}\|_{L^2(\Omega)} \\ &\leq \|\mathscr{L}_{h,g}\bar{y}_h - \mathscr{L}_{h,g}\bar{y}\|_{L^2(\Omega)} + \|\mathscr{L}_{h,g}\bar{y} - \mathscr{L}\bar{y}\|_{L^2(\Omega)} \\ &\leq \|\mathscr{L}_h(\bar{y}_h - \bar{y})\|_{L^2(\Omega)} + \|Q_h\mathscr{L}\bar{y} - \mathscr{L}\bar{y}\|_{L^2(\Omega)} \\ &\lesssim |\ln h|^{\frac{1}{2}}h + h^\tau, \end{aligned} \tag{5.15}$$

where we have also used the standard estimate

$$\|Q_h w - w\|_{L^2(\Omega)} \leq Ch|w|_{H^1(\Omega)} \qquad \forall\, w \in H^1(\Omega). \tag{5.16}$$

Next, since

$$\|R_h\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq \|\mathscr{L}_h(R_h\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} = \|\mathscr{L}_h(\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} \lesssim |\ln h|^{\frac{1}{2}}h + h^\tau$$

by Lemma 4.1, (4.15) and (5.14), and

$$\|\bar{y} - R_h\bar{y}\|_{H^1(\Omega)} \leq Ch^\tau$$

by (4.19), we have

$$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq \|\bar{y} - R_h\bar{y}\|_{H^1(\Omega)} + \|R_h\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \lesssim |\ln h|^{\frac{1}{2}}h + h^\tau.$$

The estimate (5.13) is also valid for h bounded away from 0 because the left-hand side of (5.13) is uniformly bounded with respect to h. The uniform boundedness of $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ follows immediately from Lemma 4.1 and Remark 5.2, and from (5.15) we find

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq \|\mathscr{L}_h(\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} + \|Q_h\mathscr{L}\bar{y} - \mathscr{L}\bar{y}\|_{L^2(\Omega)} \leq \|\bar{y} - \bar{y}_h\|_h + \|\mathscr{L}\bar{y}\|_{L^2(\Omega)},$$

which together with Remark 5.2 implies the uniform boundedness of $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$. □

We also have the following L^∞ error estimate that indicates, up to a term of magnitude $\mathcal{O}(|\ln h|^{\frac{1}{2}}h + h^\tau)$, the L^∞ error for the optimal control problem is the same as the L^∞ error for the P_1 FEM for a second order elliptic boundary value problem.

Theorem 5.2. *Suppose* $\bar{y} \in K_g$ *is the solution of* (1.5)–(1.6) *and* $\bar{y}_h \in K_h$ *is the solution of* (3.1)–(3.2). *Then we have*

$$\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau) + \|\bar{y} - R_h\bar{y}\|_{L^\infty(\Omega)},$$

where the positive constant C *is independent of* h, *Proof.* The theorem follows from the triangle inequality, Lemma 4.1, (4.15) and (5.14):

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_{L^\infty(\Omega)} &\le \|\bar{y}-R_h\bar{y}\|_{L^\infty(\Omega)} + \|R_h\bar{y}-\bar{y}_h\|_{L^\infty(\Omega)}\\ &\le \|\bar{y}-R_h\bar{y}\|_{L^\infty(\Omega)} + C\|\mathscr{L}_h(R_h\bar{y}-\bar{y}_h)\|_{L^2(\Omega)}\\ &= \|\bar{y}-R_h\bar{y}\|_{L^\infty(\Omega)} + \|\mathscr{L}_h(\bar{y}-\bar{y}_h)\|_{L^2(\Omega)}\\ &\le \|\bar{y}-R_h\bar{y}\|_{L^\infty(\Omega)} + C(|\ln h|^{\frac{1}{2}}h + h^\tau).\end{aligned}$$

□

6. Convergence of the Second P_1 Finite Element Method

We will use the following mesh dependent norm

$$|\!|\!|v|\!|\!|_h^2 = (v,v) + \beta(\tilde{\mathscr{L}}_h v, \tilde{\mathscr{L}}_h v)_h \tag{6.1}$$

in the analysis of the second P_1 FEM, which relies on the results for the first P_1 FEM in Section 5 and the relation between $\mathscr{L}_{h,g}$ and $\tilde{\mathscr{L}}_{h,g}$.

6.1. Relations between $\mathscr{L}_{h,g}$ and $\tilde{\mathscr{L}}_{h,g}$.

It is clear from the definition of the two discrete operators (3.3) and (3.10) that, for any $w \in H^1(\Omega)$, we have

$$(\mathscr{L}_{h,g}w, v_h) = (\tilde{\mathscr{L}}_{h,g}w, v_h)_h \qquad \forall\, v_h \in V_h, \tag{6.2}$$

and in particular,

$$(\mathscr{L}_h w, v_h) = (\tilde{\mathscr{L}}_h w, v_h)_h \qquad \forall\, w \in H^1(\Omega), v_h \in V_h. \tag{6.3}$$

One can easily verify that

$$(v_h, v_h)_h \approx (v_h, v_h) \qquad \forall\, v_h \in V_h, \tag{6.4}$$

and by a property of mass lumping (cf. [44, 48]), we have

$$|(v_h, w_h) - (v_h, w_h)_h| \lesssim \Big(\sum_{T\in\mathscr{T}_h} h_T^2 |v_h|_{H^1(T)}^2\Big)^{\frac{1}{2}} \|w_h\|_{L^2(\Omega)} \qquad \forall\, v_h, w_h \in V_h. \tag{6.5}$$

Using (6.2) and (6.5), we find

$$\begin{aligned}|(\tilde{\mathscr{L}}_{h,g}w - \mathscr{L}_{h,g}w, w_h)_h| &= |(\mathscr{L}_{h,g}w, w_h) - (\mathscr{L}_{h,g}w, w_h)_h|\\ &\lesssim h|\mathscr{L}_{h,g}w|_{H^1(\Omega)}\|w_h\|_{L^2(\Omega)} \qquad \forall\, w \in H^1(\Omega), w_h \in V_h,\end{aligned} \tag{6.6}$$

and so by (6.4),

$$\|\tilde{\mathscr{L}}_{h,g}w - \mathscr{L}_{h,g}w\|_{L^2(\Omega)} \lesssim h|\mathscr{L}_{h,g}w|_{H^1(\Omega)} \qquad \forall\, w \in H^1(\Omega). \tag{6.7}$$

It is also easy to show that

$$(\mathscr{L}_{h,g}w, \mathscr{L}_{h,g}w) \lesssim (\tilde{\mathscr{L}}_{h,g}w, \tilde{\mathscr{L}}_{h,g}w)_h \qquad \forall\, w \in H^1(\Omega). \tag{6.8}$$

6.2. **An Abstract Error Estimate.** We will need the following estimate regarding $\mathscr{L}_h$ and $\tilde{\mathscr{L}}_h$ to derive an abstract error estimate for the second P_1 FEM.

From (3.4), (6.1), (6.6) and (6.8), we have

$$\begin{aligned} |(\tilde{\mathscr{L}}_{h,g}\bar{y} - \mathscr{L}_{h,g}\bar{y}, \tilde{\mathscr{L}}_h(y_h - \bar{y}_h))_h| &\lesssim h|\mathscr{L}_{h,g}\bar{y}|_{H^1(\Omega)}\|\tilde{\mathscr{L}}_h(y_h - \bar{y}_h)\|_{L^2(\Omega)} \\ &\lesssim h|Q_h\mathscr{L}\bar{y}|_{H^1(\Omega)}|\!|\!|y_h - \bar{y}_h|\!|\!|_h \\ &\lesssim h|\mathscr{L}\bar{y}|_{H^1(\Omega)}|\!|\!|y_h - \bar{y}_h|\!|\!|_h, \end{aligned} \tag{6.9}$$

where we have used the estimate (cf. [7, 47])

$$|Q_h w|_{H^1(\Omega)} \lesssim |w|_{H^1(\Omega)} \qquad \forall\, w \in H^1(\Omega). \tag{6.10}$$

Using (3.12), (3.13), (6.1), (6.3) and (6.9), we may proceed as in (5.2) to obtain

$$\begin{aligned} |\!|\!|y_h - \bar{y}_h|\!|\!|_h^2 &\le |\!|\!|y_h - \bar{y}|\!|\!|_h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\tilde{\mathscr{L}}_{h,g}(\bar{y}), \tilde{\mathscr{L}}_h(y_h - \bar{y}_h))_h \\ &= |\!|\!|y_h - \bar{y}|\!|\!|_h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g}\bar{y}, \tilde{\mathscr{L}}_h(y_h - \bar{y}_h))_h \\ &\qquad + \beta(\tilde{\mathscr{L}}_{h,g}\bar{y} - \mathscr{L}_{h,g}\bar{y}, \tilde{\mathscr{L}}_h(y_h - \bar{y}_h))_h \\ &\lesssim |\!|\!|y_h - \bar{y}|\!|\!|_h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + \big[(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g}\bar{y}, \mathscr{L}_h(y_h - \bar{y}_h))\big] \\ &\qquad + h|\mathscr{L}\bar{y}|_{H^1(\Omega)}|\!|\!|y_h - \bar{y}_h|\!|\!|_h. \end{aligned} \tag{6.11}$$

Notice that since the term $(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\mathscr{L}_{h,g}\bar{y}, \mathscr{L}_h(y_h - \bar{y}_h))$ appearing in the last inequality of (6.11) is identical to the last term that appears in (5.2), we can directly apply the estimates (5.3), (5.4) and Lemma 5.1 from Section 5.1 (cf. Remark 5.1).

Continuing from (6.11), we find

$$\begin{aligned} |\!|\!|y_h - \bar{y}_h|\!|\!|_h^2 &\lesssim |\!|\!|y_h - \bar{y}|\!|\!|_h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + (h\|\mathscr{L}_h(y_h - \bar{y}_h)\|_{L^2(\Omega)} + h^2 + \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})}) \\ &\qquad + h|\!|\!|y_h - \bar{y}_h|\!|\!|_h \\ &\lesssim |\!|\!|y_h - \bar{y}|\!|\!|_h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + h|\!|\!|y_h - \bar{y}_h|\!|\!|_h + h^2 + \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})}, \end{aligned}$$

which together with the inequality of arithmetic and geometric means implies

$$|\!|\!|y_h - \bar{y}_h|\!|\!|_h \lesssim |\!|\!|y_h - \bar{y}|\!|\!|_h + h + \|y_h - I_h\bar{y}\|_{L^\infty(\mathfrak{A})}^{\frac{1}{2}} \qquad \forall\, y_h \in K_h.$$

So by the triangle inequality, we arrive at

$$|\!|\!|\bar{y} - \bar{y}_h|\!|\!|_h \lesssim h + \inf_{y_h \in K_h}\Big[|\!|\!|y_h - \bar{y}|\!|\!|_h + \|y_h - \bar{y}\|_{L^\infty(\mathfrak{A})}^{\frac{1}{2}}\Big]. \tag{6.12}$$

6.3. **Concrete Error Estimates.** Let $y_h \in K_h$ be defined by (5.12). Then, by using (4.18), (6.4) and (6.7), one can show that Lemma 5.2 also holds with $\|y_h - \bar{y}\|_h$ replaced by $|\!|\!|y_h - \bar{y}|\!|\!|_h$. That is, for h sufficiently small, y_h satisfies

$$|\!|\!|y_h - \bar{y}|\!|\!|_h + \|y_h - \bar{y}\|_{L^\infty(\mathfrak{A})}^{\frac{1}{2}} \lesssim |\ln h|^{\frac{1}{2}}h + h^\tau.$$

Therefore it follows from (6.12) that

$$|\!|\!|\bar{y} - \bar{y}_h|\!|\!|_h \lesssim |\ln h|^{\frac{1}{2}}h + h^\tau, \tag{6.13}$$

and we have the following concrete error estimates.

Theorem 6.1. *Suppose* $(\bar{y},\bar{u}) \in \mathbb{K}_g$ *is the solution of* (1.1)–(1.3), $\bar{y}_h \in K_h$ *is the solution of* (3.9) *and* $\bar{u}_h = \tilde{\mathscr{L}}_{h,g}\bar{y}_h$. *Then we have*

$$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)} + \|\bar{y}-\bar{y}_h\|_{H^1(\Omega)} \le C(|\ln h|^{\frac{1}{2}}h + h^{\tau}),$$

where the positive constant C *is independent of* h.

Proof. We have, by (2.15), (3.4), (5.16), (6.1), (6.4), (6.7), (6.10) and (6.13),

$$\begin{aligned}\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)} &= \|\mathscr{L}\bar{y} - \tilde{\mathscr{L}}_{h,g}\bar{y}_h\|_{L^2(\Omega)}\\ &\le \|\mathscr{L}\bar{y} - \mathscr{L}_{h,g}\bar{y}\|_{L^2(\Omega)} + \|\mathscr{L}_{h,g}\bar{y} - \tilde{\mathscr{L}}_{h,g}\bar{y}\|_{L^2(\Omega)} + \|\tilde{\mathscr{L}}_{h,g}\bar{y} - \tilde{\mathscr{L}}_{h,g}\bar{y}_h\|_{L^2(\Omega)}\\ &= \|\mathscr{L}\bar{y} - Q_h\mathscr{L}\bar{y}\|_{L^2(\Omega)} + \|\mathscr{L}_{h,g}\bar{y} - \tilde{\mathscr{L}}_{h,g}\bar{y}\|_{L^2(\Omega)} + \|\tilde{\mathscr{L}}_h(\bar{y}-\bar{y}_h)\|_{L^2(\Omega)}\\ &\lesssim h|\mathscr{L}\bar{y}|_{H^1(\Omega)} + h|\mathscr{L}\bar{y}|_{H^1(\Omega)} + \|\!|\bar{y}-y_h|\!\|_h\\ &\lesssim |\ln h|^{\frac{1}{2}}h + h^{\tau}.\end{aligned}$$

Next, it follows from Lemma 4.1, (4.15) , (6.1), (6.8) and (6.13) that

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)} &\le \|\bar{y}-R_h\bar{y}\|_{H^1(\Omega)} + \|R_h\bar{y}-\bar{y}_h\|_{H^1(\Omega)}\\ &\lesssim h^{\tau} + \|\mathscr{L}_h(R_h\bar{y}-\bar{y}_h)\|_{L^2(\Omega)}\\ &= h^{\tau} + \|\mathscr{L}_h(\bar{y}-\bar{y}_h)\|_{L^2(\Omega)} \lesssim h^{\tau} + \|\!|\bar{y}-\bar{y}_h|\!\|_h \lesssim |\ln h|^{\frac{1}{2}}h + h^{\tau}.\end{aligned}$$

□

We also have the following L^∞ error estimate as we did for the first P_1 FEM. The proof proceeds as in Theorem 5.2 but by additionally using (6.8) and (6.13).

Theorem 6.2. *Suppose* $\bar{y} \in K_g$ *is the solution of* (1.5)–(1.6) *and* $\bar{y}_h \in K_h$ *is the solution of* (3.9). *We have*

$$\|\bar{y}-\bar{y}_h\|_{L^\infty(\Omega)} \le C(|\ln h|^{\frac{1}{2}}h + h^{\tau}) + \|\bar{y}-R_h\bar{y}\|_{L^\infty(\Omega)},$$

where the positive constant C *is independent of* h.

7. Numerical Results

In this section, we report numerical results that corroborate the theory and illustrate the performance of the two P_1 FEMs. We solved the discrete problem for the first P_1 FEM by using the MATLAB quadprog M-function, and we solved the discrete problem for the second P_1 FEM by a primal-dual active set algorithm [5, 6, 35]. The approximate optimal state and optimal control on the k-th level mesh are denoted by $\bar{y}_k$ and $\bar{u}_k$ respectively.

In the first two examples, we consider convex domains with the homogeneous Neumann boundary condition. Nonhomogeneous boundary conditions are treated in the other two examples. Since the results for the two FEMs are very similar, for brevity we only report the results for the second P_1 FEM after the first example.

Example 7.1. In this example Ω is the pentagon (cf. Figure 7.1) with vertices $(0.5, 0)$, $(0, 0.5)$, $(-0.5, 0.5)$, $(-0.5,-0.5)$ and $(0.5,-0.5)$. Following [18, Section 6, Example 3], we choose $y_d(x) = 2-|x|^2$, $\psi(x) = 1.85 + (x_1 + 0.25)^4 + (x_2 + 0.25)^4$, $\beta = 0.001$ and $g = 0$. Since we do not know the exact solution $(\bar{y}, \bar{u})$ of this problem, we report the errors between consecutive approximations in Tables 7.1 and 7.2.

k	$\|\bar{y}_{k+1} - \bar{y}_k\|_{L^2(\Omega)}$	rate	$\|\bar{y}_{k+1} - \bar{y}_k\|_{H^1(\Omega)}$	rate	$\|\bar{y}_{k+1} - \bar{y}_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}_{k+1} - \bar{u}_k\|_{L^2(\Omega)}$	rate
0	2.04e-02		1.36e-01		7.45e-02		6.92e-01	
1	1.08e-02	0.92	6.45e-02	1.08	4.37e-02	0.77	3.28e-01	1.08
2	3.01e-03	1.84	4.58e-02	0.49	9.77e-03.	2.16	2.67e-01	0.30
3	1.15e-03	1.39	2.50e-02	0.87	3.95e-03	1.31	1.10e-01	1.28
4	2.92e-04	1.98	1.28e-02	0.97	1.24e-03	1.67	3.29e-02	1.74
5	6.54e-05	2.16	6.43e-03	0.99	3.83e-04	1.69	1.43e-02	1.20
6	1.81e-05	1.85	3.23e-03	0.99	1.09e-04	1.82	4.18e-03	1.78
7	4.08e-06	2.15	1.62e-03	1.00	3.40e-05	1.68	1.36e-03	1.62

TABLE 7.1. Results for the first P_1 FEM on uniform meshes for Example 7.1

k	$\|\bar{y}_{k+1} - \bar{y}_k\|_{L^2(\Omega)}$	rate	$\|\bar{y}_{k+1} - \bar{y}_k\|_{H^1(\Omega)}$	rate	$\|\bar{y}_{k+1} - \bar{y}_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}_{k+1} - \bar{u}_k\|_{L^2(\Omega)}$	rate
0	2.95e-02		2.23e-01		1.19e-01		4.29e-01	
1	1.32e-02	1.16	9.34e-02	1.25	5.56e-02	1.10	3.09e-01	0.47
2	3.19e-03	2.05	5.42e-02	0.79	6.94e-03	3.00	2.50e-01	0.31
3	8.32e-04	1.94	2.65e-02	1.03	3.17e-03	1.13	9.00e-02	1.48
4	2.20e-04	1.92	1.29e-02	1.04	8.45e-04	1.91	2.85e-02	1.66
5	3.62e-05	2.61	6.45e-03	1.00	2.58e-04	1.71	1.36e-02	1.07
6	1.08e-05	1.75	3.23e-03	1.00	7.91e-05	1.70	4.02e-03	1.76
7	3.29e-06	1.71	1.62e-03	1.00	3.24e-05	1.29	1.30e-03	1.63

TABLE 7.2. Results for the second P_1 FEM on uniform meshes for Example 7.1

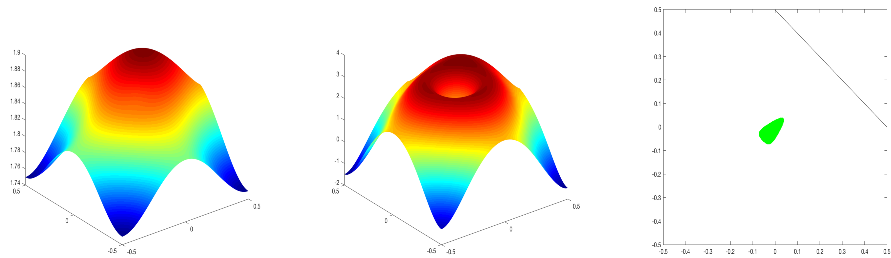
For both FEMs, we observe $\mathcal{O}(h)$ convergence for the approximation of $\bar{y}$ in the H^1 semi-norm which agrees with Theorems 5.1 and 6.1. The convergence rates for the approximations of $\bar{y}$ and $\bar{u}$ in the L^2-norm are better than the estimates in Theorems 5.1 and 6.1, and the convergence for the approximation of $\bar{y}$ in the L^∞ norm is also better than the estimates in Theorems 5.2 and 6.2. These higher convergence rates are consistent with the fact that the optimal state $\bar{y}$ (and hence the optimal control $\bar{u}$) has higher interior and global regularities since Ω is convex and the free boundary $\partial\mathfrak{A}$ is sufficiently smooth.

The graphs of $\bar{y}_8$ and $\bar{u}_8$ and the active set obtained by the second P_1 FEM are displayed in Figure 7.1. All of them match the ones obtained in [18] by a quadratic C^0 interior penalty method.

Example 7.2. In this example $\Omega = (-4, 4)^2$ and we construct the exact solution $\bar{y}$ as in [11, Section 7, Example 1] but modify it in a way that $\bar{y}$ satisfies the homogeneous Neumann boundary condition.

We construct $\bar{y}(x)$ in the following way:

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \le 1, \\ v(|x|) + [1 + \phi(|x|)]w(|x|) & \text{if } 1 \le |x| \le 3, \\ w(x) & \text{if } |x| \ge 3, \end{cases}$$



(a) State (b) Control (c) Active Set

FIGURE 7.1. State, control and active set for Example 7.1

where

$$v(t) = (t^2-1)\Big(1-\frac{t-1}{2}\Big)^4 + \frac{1}{4}(t-1)^2(t-3)^4,$$
$$\phi(t) = \Big[1+4\Big(\frac{t-1}{2}\Big)+10\Big(\frac{t-1}{2}\Big)^2+20\Big(\frac{t-1}{2}\Big)^3\Big]\Big(1-\frac{t-1}{2}\Big)^4,$$
$$w(x) = 2\cos\Big(\frac{\pi}{8}(x_1+4)\Big)\cos\Big(\frac{\pi}{8}(x_2+4)\Big).$$

The control $\bar{u}$ is then equal to $-\Delta\bar{y}+\bar{y}$. Now we choose $\psi(x) = |x|^2-1$, $\beta = 1$, and

$$y_d(x) = \begin{cases} \beta\Delta^2\bar{y} - 2\beta\Delta\bar{y} + \beta\bar{y} + \bar{y} & \text{if } |x| \geq 1, \\ \beta\Delta^2\bar{y} - 2\beta\Delta\bar{y} + \beta\bar{y} + \bar{y} + 1 & \text{if } |x| \leq 1. \end{cases}$$

By construction, such choices of ψ, β, y_d and $\bar{y}$ satisfy the KKT conditions (cf. Remark 2.1) with the measure μ in (2.6) defined by

$$\int_\Omega z\,d\mu = -42\int_{\partial\mathfrak{A}} z\,ds - \int_{\mathfrak{A}} z\,dx \qquad \forall\, z \in V, \tag{7.1}$$

and the active set $\mathfrak{A}$ is the closed disc with radius 1 centered at the origin.

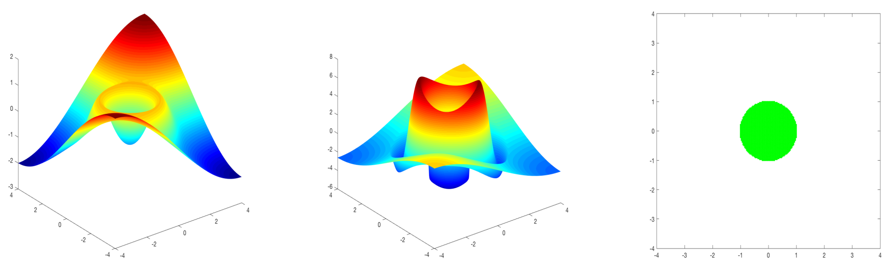
k	$\|\bar{y}-y_k\|_{L^2(\Omega)}$	rate	$\|\bar{y}-y_k\|_{H^1(\Omega)}$	rate	$\|I_k\bar{y}-y_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}-u_k\|_{L^2(\Omega)}$	rate
0	8.13e+00		8.01e+00		1.38e+00		1.51e+01	
1	9.84e+00	-0.28	9.72e+00	-0.28	2.58e+00	-0.90	1.64e+01	-0.12
2	1.05e+01	-0.09	1.19e+01	-0.29	2.61e+00	-0.02	1.80e+01	-0.13
3	1.40e+00	2.91	4.63e+00	1.36	3.45e-01	2.92	1.20e+01	0.58
4	2.99e-01	2.22	2.35e+00	0.98	9.76e-02	1.82	5.03e+00	1.25
5	8.28e-02	1.85	1.19e+00	0.99	3.48e-02	1.49	1.67e+00	1.59
6	2.89e-02	1.52	5.92e-01	1.00	1.37e-02	1.35	5.33e-01	1.64
7	9.14e-03	1.66	2.95e-01	1.00	3.68e-03	1.89	1.83e-01	1.54
8	3.11e-03	1.56	1.48e-01	1.00	1.13e-03	1.70	6.13e-02	1.58

TABLE 7.3. Results for the second P_1 FEM on uniform meshes for Example 7.2

The results for the second P_1 FEM on uniform meshes are reported in Table 7.3, where we use I_k to denote the nodal interpolation operator onto the finite element space associated with the k-th level mesh. The reduction rate of $\|I_k\bar{y} - y_k\|_{L^\infty(\Omega)}$ represents the order of convergence of $\bar{y}$ in the L^∞-norm.

The $\mathcal{O}(h)$ convergence of the approximation of $\bar{y}$ agrees with Theorem 6.1. The convergence rates for the approximations of $\bar{y}$ in L^2 and L^∞ norms and for the approximation of $\bar{u}$ in the L^2 norm are better than those predicted by Theorem 6.1 and Theorem 6.2. These higher convergence rates are consistent with the higher regularity enjoyed by $\bar{y}$ and $\bar{u}$.

The graphs of $\bar{y}_8$ and $\bar{u}_8$ and the active set obtained by the second P_1 FEM are displayed in Figure 7.2. The active set has clearly been correctly captured.



(a) State (b) Control (c) Active Set

FIGURE 7.2. State, control and active set for Example 7.2

Example 7.3. This example is a modification of Example 7.2 so that the exact solution has non-homogeneous Neumann boundary condition. We take $\Omega = (-4,4)^2$, $\beta = 1$, $q(x) = x_1$, $\psi^\star = \psi + q$,

$$y_d^\star = y_d + (1+\beta)q, \quad \bar{y}^\star = \bar{y} + q \quad \text{and} \quad \bar{u}^\star = \bar{u} + q,$$

where ψ, y_d, $\bar{y}$ and $\bar{u}$ are identical to the ones in Example 7.2.

Then $\bar{y}^\star$ is the exact solution of the following slightly more general problem:

$$\bar{y}^\star = \underset{y \in K_g^\star}{\operatorname{argmin}} \Big[\frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\mathscr{L}y\|^2_{L^2(\Omega)} - \beta\int_{\partial\Omega} gy\,ds\Big], \tag{7.2}$$

where $g = \partial q/\partial n$ and

$$K_g^\star = \{v \in V_g : v \le \psi^\star \text{ in } \Omega\}. \tag{7.3}$$

Indeed, we have $\bar{y}^\star \le \psi^\star$, $\mathfrak{A}^\star = \mathfrak{A}$ (the active set from Example 7.2), and since q is harmonic,

$$(\bar{y}^\star - y_d^\star, z) + \beta(\mathscr{L}\bar{y}^*, \mathscr{L}z) - \int_{\partial\Omega} qz\,ds = \int_\Omega z\,d\mu_* \qquad \forall\, z \in V,$$

where $\mu_* = \mu$ is defined in (7.1). Therefore the KKT conditions for (7.2) are satisfied.

Remark 7.1. Note that (7.2) is identical to (1.5) when $g = 0$. For nonhomogeneous Neumann boundary conditions, the more general cost functional in (7.2) facilitates the construction of an exact solution from the exact solution of the corresponding problem with the homogeneous Neumann boundary condition.

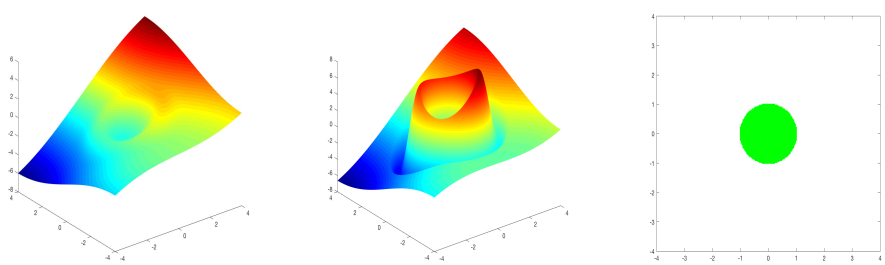
We can solve (7.2) by a straightforward modification of the P_1 FEMs in Sections 3.1 and 3.2, where the additional term $\beta \int_{\partial\Omega} g y_h ds$ is included in the cost functionals in (3.1) and (3.9). It is easy to check that the error estimates in Sections 5 and 6 remain valid.

The numerical results for the second P_1 FEM on uniform meshes are given in Table 7.4. The performance is similar to what we observed in Example 7.2. This is not surprising since the difference between the exact solutions of Example 7.2 and Example 7.3 is just the linear polynomial q.

k	$\|\bar{y}^\star - y_k\|_{L^2(\Omega)}$	rate	$\|\bar{y}^\star - y_k\|_{H^1(\Omega)}$	rate	$\|I_k\bar{y}^\star - y_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}^\star - u_k\|_{L^2(\Omega)}$	rate
0	4.39e+01		2.42e+01		1.21e+01		3.83e+01	
1	8.61e+00	2.35	8.62e+00	1.49	2.99e+00	2.02	1.78e+01	1.10
2	9.34e+00	-0.12	1.22e+01	-0.50	2.56e+00	0.22	1.75e+01	0.03
3	1.44e+00	2.70	4.63e+00	1.40	3.45e-01	2.89	1.20e+01	0.54
4	3.32e-01	2.12	2.40e+00	0.95	1.76e-01	0.97	4.93e+00	1.28
5	8.48e-02	1.97	1.19e+00	1.01	4.93e-02	1.84	1.63e+00	1.59
6	2.35e-02	1.85	5.92e-01	1.00	1.36e-02	1.86	5.23e-01	1.64
7	5.83e-03	2.01	2.96e-01	1.00	3.24e-03	2.07	1.80e-01	1.54
8	2.66e-03	1.13	1.48e-01	1.00	8.03e-04	2.01	6.06e-02	1.57

TABLE 7.4. Results for the second P_1 FEM on uniform meshes for Example 7.3

The graphs of $\bar{y}_8$ and $\bar{u}_8$ and the active set obtained by the second P_1 FEM are displayed in Figure 7.3. The relations $\bar{y}^\star = \bar{y} + q$ and $\bar{u}^\star = \bar{u} + q$ can be observed by comparing Figure 7.2 and Figure 7.3. The active set has also been correctly captured.



(a) State (b) Control (c) Active Set

FIGURE 7.3. State, control and active set for Example 7.3

Example 7.4. In this example, we use the L-shaped domain $\Omega = (-8,8)^2 \backslash \big([0,8] \times [-8,0]\big)$ (cf. Figure 7.4) and solve the minimization problem (7.2) with a nonhomogeneous Neumann boundary condition.

First of all, let $a = (-4,4)$ and take ψ and y_d to be the functions from Example 2. If we use $\psi_a(x) = \psi(x-a)$ and $y_d^a(x) = y_d(x-a)$ as the input with $\beta = 1$, then the exact solution of (1.1)–(1.3) with $g = 0$ will be $\bar{y}_a(x) = \bar{y}(x-a)$ and $\bar{u}_a(x) = \bar{u}(x-a)$, where $(\bar{y}, \bar{u})$ is the exact solution of Example 2. Furthermore, the active set in this case will simply be the shift of the active set of Example 2 by a.

Let the harmonic function q in polar coordinates be defined by

$$q(r,\theta) = r^{\frac{2}{3}} \cos(2\theta/3)$$

and take

$$\psi^\star = \psi_a + 4q \quad \text{and} \quad y_d^\star = y_d^a + (1+\beta)4q.$$

As in Example 7.3, the exact solution of (7.2) with $g = 4(\partial q/\partial n)$ is then given by

$$(\bar{y}^\star, \bar{u}^\star) = (\bar{y}_a + 4q, \bar{u}_a + 4q).$$

Note that the singularity due to the reentrant corner is captured by q.

In Table 7.5, we report results for the second P_1 FEM on uniform meshes. In this case, the estimates in Theorems 6.1 and 6.2 hold with $\tau = 2/3$, and the reduction in the order of convergence (compared to previous examples) is noticeable except for the L^2-error of the control.

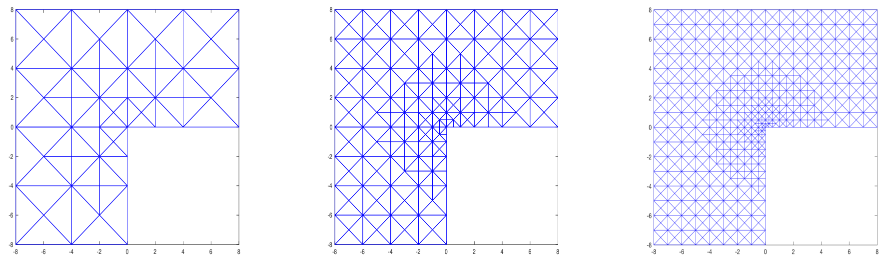
k	$\|\bar{y}^\star - y_k\|_{L^2(\Omega)}$	rate	$\lvert\bar{y}^\star - y_k\rvert_{H^1(\Omega)}$	rate	$\|I_k\bar{y}^\star - y_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}^\star - u_k\|_{L^2(\Omega)}$	rate
0	4.62e+01		1.86e+01		1.14e+01		4.19e+01	
1	1.19e+01	1.96	1.16e+01	0.67	2.80e+00	2.03	1.94e+01.	1.11
2	9.79e+00	0.28	1.31e+01	-0.17	2.56e+00	0.13	1.78e+01	0.12
3	1.72e+00	2.51	5.25e+00	1.32	4.62e-01	2.47	1.20e+01	0.56
4	4.57e-01	1.91	2.79e+00	0.91	2.67e-01	0.79	4.94e+00	1.28
5	1.40e-01	1.70	1.47e+00	0.93	1.60e-01	0.74	1.64e+00	1.59
6	4.79e-02	1.55	7.95e-01	0.88	9.92e-02	0.69	5.24e-01	1.64
7	1.68e-02	1.52	4.43e-01	0.84	6.18e-02	0.68	1.80e-01	1.54
8	6.55e-03	1.35	2.53e-01	0.80	3.88e-02	0.67	6.08e-02	1.57

Table 7.5. Results for the second P_1 FEM on uniform meshes for Example 7.4

We have also run the same numerical example on graded meshes of the L-shaped domain. The graded meshes are generated by the refinement procedure in [31], and they are depicted in Figure 7.4. The results are presented in Table 7.6. The observed improvement in the convergence rates agrees with Theorems 6.1 and 6.2, since τ is improved to 1 for graded meshes (cf. (4.2)).

k	$\|\bar{y}^\star - y_k\|_{L^2(\Omega)}$	rate	$\lvert\bar{y}^\star - y_k\rvert_{H^1(\Omega)}$	rate	$\|I_k\bar{y}^\star - y_k\|_{L^\infty(\Omega)}$	rate	$\|\bar{u}^\star - u_k\|_{L^2(\Omega)}$	rate
0	1.96e+01		1.40e+01		5.36e+00		2.30e+01	
1	9.77e+00	1.01	1.07e+01	0.38	3.02e+00	0.83	1.76e+01	0.39
2	4.86e+00	1.01	8.28e+00	0.37	1.98e+00	0.61	1.38e+01	0.35
3	7.87e-01	2.63	3.47e+00	1.26.	3.45e-01	2.52	6.76e+00	1.03
4	2.09e-01	1.92	1.61e+00	1.11	1.51e-01	1.19	2.91e+00	1.21
5	6.11e-02	1.77	7.70e-01	1.06	4.00e-02	1.92	1.05e+00	1.47
6	1.36e-02	2.16	3.80e-01	1.02	1.08e-02	1.89	4.40e-01	1.26
7	2.93e-03	2.22	1.89e-01	1.01	4.25e-03	1.34	1.74e-01	1.34
8	9.89e-04	1.57	9.46e-02	1.00	2.46e-03	0.79	5.98e-02	1.54

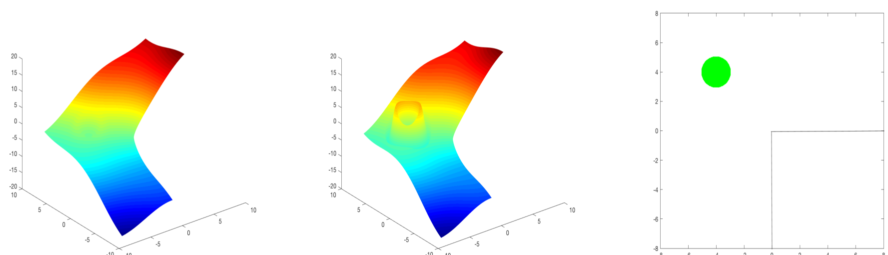
Table 7.6. Results for the second P_1 FEM on graded meshes for Example 7.4



(a) k=1 (b) k=2 (c) k=3

Figure 7.4. Graded meshes on the L-shaped domain with grading parameter 0.6

The graphs of $\bar{y}_8$ and $\bar{u}_8$ and the active set are displayed in Figure 7.5. Again the active set has been correctly captured.



(a) State (b) Control (c) Active Set

Figure 7.5. State, control and active set for Example 7.4

8. Concluding Remarks

The P_1 FEMs from Sections 3.1 and 3.2 can also be applied to the optimal control problem (1.1)–(1.3) on a three dimensional polyhedral domain. This was carried out in [11] for the Dirichlet boundary condition.

The analysis of the P_1 FEMs are considerably simpler under the condition that the active set is a compact subset of Ω. In the Dirichlet case, this condition is satisfied in any dimension as long as the (pointwise) constraint for the state is separated from the boundary condition of the state. In the Neumann case, this condition is implied by our assumption $\partial\psi/\partial n > g$ for two dimensional domains. Unfortunately the arguments in [18, Appendix] do not extend immediately to three dimensions. This is the reason that the three dimensional case is not addressed in this paper.

Higher order FEMs are advantageous when $\bar{y}$ enjoys additional regularities (cf. [18]). Therefore it will be interesting to extend the approach in this paper to higher order FEMs based on discontinuous Galerkin discretizations of the constraint (1.2), for which mass lumping is not required.

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