Mathematical Models and Methods in Applied Sciences
(2020)

DOI: 10.1142/S0218202520500037

On the wave turbulence theory for stratified flows in the ocean

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Received 18 January 2019
Revised 26 September 2019
Accepted 18 October 2019
Published 19 December 2019
Communicated by M. Lewin

After the pioneering work of Garrett and Munk, the statistics of oceanic internal gravity waves has become a central subject of research in oceanography. The time evolution of the spectral energy of internal waves in the ocean can be described by a near-resonance wave turbulence equation, of quantum Boltzmann type. In this work, we provide the first rigorous mathematical study for the equation by showing the global existence and uniqueness of strong solutions.

Keywords: Wave (weak) turbulence theory; quantum Boltzmann equation; wave–wave interactions; stratified fluids; oceanography; near-resonance.

AMS Subject Classification: 35B05, 35B60, 82C40

1. Introduction

The study of wave turbulence has obtained spectacular success in the understanding of spectral energy transfer processes in plasmas, oceans and planetary atmospheres. Wave–wave interactions in continuously stratified fluids have been a fascinating

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subject of intensive research in the last few decades. In particular, the observation of a nearly universal internal-wave energy spectrum in the ocean, first described by Garrett and Munk (cf. Refs. 22, 23 and 11), plays a very important role in understanding such wave–wave interactions. The existence of a universal spectrum is generally perceived to be the result of nonlinear interactions of waves with different wavenumbers. As the nonlinearity of the underlying primitive equations is quadratic, waves interact in triads (cf. Ref. 64). Furthermore, since the linear internal wave dispersion relation can satisfy a three-wave resonance condition, resonant triads are expected to dominate the dynamics for weak nonlinearity (cf. Ref. 43).

Resonant wave interactions can be characterized by Zakharov kinetic equations (cf. Refs. 71, 46, 42, 10, 69 and 68). The equations describe, under the assumption of weak nonlinearity, the spectral energy transfer on the resonant manifold, which is a set of wave vectors k, k_1, k_2 satisfying

$$k = k_1 + k_2, \quad \omega_k = \omega_{k_1} + \omega_{k_2}, \tag{1.1}$$

where the frequency ω is given by the dispersion relation between the wave frequency ω and the wave number k. However, it is known that exact resonances defined by $\omega_k = \omega_{k_1} + \omega_{k_2}$ do not capture some important physical effects, such as energy transfer to non-propagating wave modes with zero frequency, corresponding to generation of anisotropic coherent structures,[3,4,7,15,16,25,33–35,41,51,65,66] see also Refs. 18 and 44 for analytical arguments on reduced isotropic models. Some authors have included more physics by allowing near-resonant interactions (cf. Refs. 13, 32, 39, 36, 40, 37, 38, 47, 58, 53 and 54), defined as

$$k = k_1 + k_2, \quad |\omega_k - \omega_{k_1} - \omega_{k_2}| < \theta(f, k), \tag{1.2}$$

where θ accounts for broadening of the resonant surfaces and depends on the wave density f and the wave number k. When near resonances are included in the dynamics, numerical studies have demonstrated the formation of the anisotropic, non-propagating wave modes in dispersive wave systems relevant to geophysical flows (cf. Refs. 12, 27, 32, 55, 57, 58, 59 and 60).

We consider in this paper the following near-resonance turbulence kinetic equation for internal wave interactions in the open ocean (cf. Refs. 13, 36, 40, 37 and 39),

$$\partial_t f(t,k) + \mu_k f(t,k) = Q[f](t,k), \quad f(0,k) = f_0(k), \tag{1.3}$$

in which $f(t,k)$ is the nonnegative wave density at wave number $k \in \mathbb{R}^d$, $d \geq 2$. As proposed by Zakharov in Ref. 69 water wave models must include the term $\mu_k f = 2\nu|k|^2 f$ for viscous damping effects, with ν the viscosity coefficient.

This model equation consist in a kinetic three-wave interaction modeled by an interaction (or collision) operator given by the nonlocal form

$$Q[f](k) = \iint_{\mathbb{R}^{2d}} \left[R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f]\right] dk_1 dk_2, \tag{1.4}$$

with

$$R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\mathcal{L}_f(\omega_k - \omega_{k_1} - \omega_{k_2})(f_1 f_2 - f f_1 - f f_2), \tag{1.5}$$

with the short-hand notation $f = f(t,k)$ and $f_j = f(t,k_j)$. The singular measure given by the Dirac delta function $\delta(\cdot)$ ensures that interactions are between triads with

$$k = k_1 + k_2. \tag{1.6}$$

The transition probability factor or collision kernel V_{k,k_1,k_2} under consideration is of the form (cf. Refs. 37, 13, 40, 39 and 36)

$$V_{k,k_1,k_2} = \mathfrak{C}(|k||k_1||k_2|)^{\frac{1}{2}}, \tag{1.7}$$

with $\mathfrak{C}$ is some physical constant.

Next, we consider the dispersion law

$$\omega_k = \sqrt{F^2 + \frac{g^2}{\rho_0^2 N^2}\frac{|k|^2}{m^2}}, \tag{1.8}$$

where F is the Coriolis parameter, N is the (Brunt–Vaisala) buoyancy frequency, In addition, the parameter m is the reference vertical wave number determined from observations, g is the gravitational constant, ρ_0 is the reference value for density, or equivalently

$$\omega_k = \sqrt{\lambda_1 + \lambda_2 |k|^2}, \quad \text{for } \lambda_1 = F^2, \quad \text{and} \quad \lambda_2 = \frac{1}{m^2}\left(\frac{g}{\rho_0 N}\right)^2 = \frac{1}{k_z^2}, \tag{1.9}$$

where k_z Cartesian vertical wave number and $m = k_z g(\rho_0 N)^{-1}$. In the absence of the Coriolis force, i.e. $F = 0$, the dispersion relation becomes

$$\omega_k = \frac{|k|}{m} \approx \frac{|k|}{k_z}. \tag{1.10}$$

The operator $\mathcal{L}_f$ is defined as

$$\mathcal{L}_f(\zeta) = \frac{\Gamma_{k,k_1,k_2}}{\zeta^2 + \Gamma_{k,k_1,k_2}^2}, \tag{1.11}$$

with the condition that

$$\lim_{\Gamma_{k,k_1,k_2} \to 0} \mathcal{L}_f(\zeta) = \pi\delta(\zeta).$$

Thus, when Γ_{k,k_1,k_2} tends to 0, (1.4) becomes the following exact resonance collision operator (cf. Refs. 69, 68 and 26):

$$Q_e[f](k) = \pi \iint_{\mathbb{R}^{2d}} \left[\tilde{R}_{k,k_1,k_2}[f] - \tilde{R}_{k_1,k,k_2}[f] - \tilde{R}_{k_2,k,k_1}[f]\right] dk_1 dk_2, \tag{1.12}$$

with

$$\tilde{R}_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\delta(\omega_k-\omega_{k_1}-\omega_{k_2})(f_1f_2-ff_1-ff_2).$$

Without loss of generality, one could ignore the constant π in the collision operator $Q_e[f]$ since it can be absorbed in the time variable.

Moreover, the resonance broadening frequency Γ_{k,k_1,k_2} may be written

$$\Gamma_{k,k_1,k_2} = \gamma_k + \gamma_{k_1} + \gamma_{k_2}, \tag{1.13}$$

where γ_k is computed in Ref. 36 using a one-loop diagram approximation:

$$\gamma_k \backsim \mathfrak{c}|k|^2\int_{\mathbb{R}_+}|k|^2|f(t,|k|)|d|k|,$$

and $\mathfrak{c}$ is a physical constant, which can be normalized to be 1. Approximating the integral

$$\int_{\mathbb{R}_+}|k|^2|f(t,|k|)|d|k| \approx \int_{\mathbb{R}^3} f(t,k)dk,$$

we obtain a formula for γ_k that will be used throughout the paper

$$\gamma_k = |k|^2\int_{\mathbb{R}^3} f(t,k)dk. \tag{1.14}$$

The above formulation of γ_k indicate the broadening resonance width θ defined in (1.2). Note that the formulation of Γ_{k,k_1,k_2} is given

$$\Gamma_{k,k_1,k_2} = (|k|^2+|k_1|^2+|k_2|^2)\int_{\mathbb{R}^3} f(t,k)dk. \tag{1.15}$$

Observe that

$$\sqrt{n}\Gamma_{k,k_1,k_2} \le |\omega_k-\omega_{k_1}-\omega_{k_2}| \le \sqrt{n+1}\Gamma_{k,k_1,k_2}, \quad n\in\mathbb{N},$$

then

$$\frac{1}{(n+2)\Gamma_{k,k_1,k_2}} \le \mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2}) \le \frac{1}{(n+1)\Gamma_{k,k_1,k_2}},$$

in other words, function $\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2})$ is mostly concentrated in the interval where

$$|\omega_k-\omega_{k_1}-\omega_{k_2}| \le \Gamma_{k,k_1,k_2}. \tag{1.16}$$

In other words, the resonance width θ is proportional to Γ_{k,k_1,k_2}, which depends on f and k.

This fact will be used in the proof of Propositions 2.3, 2.1 and 3.1.

In the field of wave turbulence, the most commonly used asymptotical analysis to derive the kinetic equation (1.3)–(1.6) is statistical closure of the infinite hierarchy of cumulants, in the weakly nonlinear and long-time limits (see, for example, the review by Newell and Rumpf Ref. 48). Evolution of higher-order cumulants can be interpreted as a modification of the wave frequency, with real part corresponding to a frequency shift and with imaginary part corresponding to resonance broadening.

A Feynman-Dyson diagrammatic approach may also be used, adapted for turbulence in fluids by Wyld,[67] for more general classical systems by Martin *et al.*,[56] and for Hamiltonian nonlinear wave fields by Zakharov and Lvov.[70] In the context of acoustic turbulence, Lvov *et al.*[36] considered a one-loop approximation to the resonance broadening, the form of which is the one to be adopted in our study.

It is noted that wave turbulence equation (1.3) shares a similar structure with the quantum Boltzmann equation describing the evolution of the excitations in thermal cloud Bose–Einstein condensate systems (cf. Refs. 21, 29, 30, 31, 45 and 72). Our recent progress on the classical Boltzmann equation (cf. Refs. 8, 19, 20 and 63) and the quantum Boltzmann equation (cf. Refs. 2, 14, 17, 24, 28, 50, 49, 62, 52 and 61) has shed some light on the open question of building a rigorous mathematical study for (1.3). Different from the quantum Boltzmann cases (cf. Refs. 62, 2 and 14), which could be considered as the exact resonance case (1.12) with

$$\omega_k = \omega_{k_1} + \omega_{k_2},$$

the energy of solutions for the near-resonance kinetic equation (1.3) is not conserved. The underlying shallow-water equations conserve a cubic energy, and the flow restricted to exact resonances conserves the quadratic part of the total energy.[66] However, conservation of the quadratic energy no longer holds when near resonant three-wave interactions are included in the dynamics.

We also split Q as the sum of their positive and negative parts, referred to as a gain and a loss operators, respectively,

$$Q[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f], \tag{1.17}$$

as is done with the classical Boltzmann operator for binary elastic interactions. Here, the gain operator is also defined by the positive contributions in the total rate of change in time of the collisional form $Q(f)(t,k)$

$$\begin{aligned} Q_{\text{gain}}[f] = & \iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k-k_1-k_2)\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2}) f_1 f_2 dk_1 dk_2 \\ & + 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k_1,k,k_2}|^2 \delta(k_1-k-k_2)\mathcal{L}_f(\omega_{k_1}-\omega_k-\omega_{k_2}) \\ & \times (f f_1 + f_1 f_2) dk_1 dk_2 \end{aligned} \tag{1.18}$$

and the loss operator models the negative contributions in the total rate of change in time of the same collisional form $Q(f)(t,k)$

$$Q_{\text{loss}}[f] = f\vartheta[f], \tag{1.19}$$

with $\vartheta[f]$ being the collision frequency or attenuation coefficient, defined by

$$\begin{aligned}\vartheta[f](k) = 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2})f_1dk_1dk_2\\ +2\iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)\mathcal{L}_f(\omega_{k_1}-\omega_{k}-\omega_{k_2})f_2dk_1dk_2.\end{aligned} \tag{1.20}$$

Inspired by recent work by Alonso and two of the authors of this paper[2] on the quantum Boltzmann equation for cold bosonic gases, whose equation can also be derived by diagrammatic techniques, we present here the existence and uniqueness solution to a Cauchy problem associated to the model (1.3)–(1.11)

The strategy consists in finding a suitable convex, positive cone, time invariant subspace S_T of the Banach space $L^1_N(\mathbb{R}^d)$, for which the weak turbulence equation has a unique strong solution, where this Banach space has norms defined by the Nth moment as the expectation of the Nth-power of the dispersion relation, that is for any given density g,

$$L^1_N(\mathbb{R}^d) := \left\{g\in L^1(\mathbb{R}^d),\ \text{s.t. } \|g\|_{L^1_N} := \mathfrak{M}_N[g] = \int_{\mathbb{R}^d}\omega_k^N g(k)dk<\infty\right\}, \tag{1.21}$$

in which we recall the dispersion relation $\omega_k = \sqrt{\lambda_1+\lambda_2|k|^2}$ as defined in (1.9). Notice that when g is positive, both $\mathfrak{M}_n[g]$ and $\|g\|_{L^1_n}$ are equivalent. Hence, the construction of such invariant subspace S_T depend on the control of higher order moments defined as follows.

Our solution are global and unique in $L^1_N(\mathbb{R}^d)$ to (1.3), that is the satisfy

$$\partial_t f(t,k) = Q_{\text{gain}}[f](t,k) - f(t,k)\vartheta[f](t,k) - 2\nu|k|^2 f, \quad f(0,k) = f_0(k)\in S_T. \tag{1.22}$$

A fundamental tool to accomplish our goal is to prove that there exists a differential equation of the following type, for the moments of the solution f of (1.22):

$$\frac{d}{dt}\mathfrak{M}_N[f] \le C_1\mathfrak{M}_{N+1}[f] - C_2\mathfrak{M}_{N+2}[f],$$

for some positive constants C_1, C_2, which leads to

$$\frac{d}{dt}\mathfrak{M}_N[f] \le C_3\mathfrak{M}_N[f],$$

with C_3 being a positive constant. The above inequality then yields an exponential bound on the Nth moment of f

$$\mathfrak{M}_N[f] \le Ce^{C'T}.$$

In order to do that, estimates on Q_{gain} and Q_{loss} are provided in Propositions 2.3 and 2.1. The proofs of these estimates are based on careful bounds of $\mathcal{L}_f$ and Γ_{k,k,k_1}, that reduces to bounding the 0th moment of f, $\mathfrak{M}_0[f](t)$, from below by $e^{-(2\nu R_0^2+4R_0)t}\|f_0\chi_{R_0}\|_{L^1}$, where χ_{R_0} is the characteristic function of the ball $B(O,R_0)$ centered at the origin with radius R_0 so that the quantity $\|f_0\chi_{R_0}\|_{L^1}>0$.

Finally, on any arbitrary fixed time interval $[0,T]$, we construct the solution of (1.22) within a time-dependent invariant set $\mathcal{S}_T$, based on the exponential in time upper bound of $\mathfrak{M}_N[f]$ and the lower bound of $\mathfrak{M}_0[f]$.

More specifically, we define first the following two constants, C^* and C_*, for any given any R_0 by

$$C_* := \frac{C_0(\lambda_1,\lambda_2)\big(1+e^{(4\nu R_0^2+8R_0)T}\big)}{\|f_0(k)\chi_{R_0}\|_{L^1}}, \quad \text{and} \quad C^* := 4\nu R_0^2+8R_0. \tag{1.23}$$

The specific value of R_0 will be determined later to secure the conditions to obtain a time invariant region.

Hence, for any number $R^*>0$, $R_*>1$, moment order N, and time $t>0$, we define the convex positive cone S_T as a subset on L^1_N given by

$$\mathcal{S}_T := \Bigg\{ f\in L^1_{N+3}(\mathbb{R}^d) : \mathbf{S1)} f\geq 0; \mathbf{S2)}\|f\|_{L^1_{N+3}} \leq c_0(t) := (2R_*+1)e^{C_* t}; \\ \mathbf{S3)}\|f\|_{L^1} \geq c_1(t) := \frac{R^* e^{-C^* t}}{2}\Bigg\}, \tag{1.24}$$

where the $c_0(t)$ is an increasing function and $c_1(t)$ is a decreasing function, so $S_t\subset S_{t'}$ for $0\leq t\leq t'\leq T$

Our main result is as follows.

Theorem 1.1. *Let $N>0$, and let $f_0(k)\in\mathcal{S}_0\cap B_*(O,R_*)\backslash\overline{B_*(O,R^*)}$ for some $R^*>R_*>0$, where $B_*(O,R^*), B_*(O,R_*)$ is the ball centered at O with radius R^*, R_* of $L^1_{N+3}(\mathbb{R}^d)$.*

Then the weak turbulence equation (1.3) *has a unique strong solution $f(t,k)$ so that*

$$0\leq f(t,k)\in C\big([0,T); L^1_N(\mathbb{R}^d)\big)\cap C^1\big((0,T); L^1_N(\mathbb{R}^d)\big). \tag{1.25}$$

Moreover, $f(t,k)\in\mathcal{S}_T$ for all $t\in[0,T)$.

Since T can be chosen arbitrarily large, the weak turbulence equation (1.3) *has a unique global solution for all time $t>0$.*

The proof of Theorem 1.1 relies on the following abstract Ordinary Differential Equations theorem in Banach spaces, which provides a framework to developed the existence and uniqueness theory to space homogeneous Boltzmann type equations ranging from the classical Boltzmann equation for binary interaction, to nonlocal kinetic model for rods alignment, to quantum kinetic theory of bosonic cold gases.[1,2,6,9]

Applied to the initial value problem (1.3)–(1.16), the framework is given by the following abstract existence and uniqueness theorem in Banach spaces along the lines proposed by Bressan in the unpublished notes,[9] whose application to the classical Boltzmann theory for hard potential and integrable angular cross section has been recently completed in Ref. 1, as follows.

Let $E := (E, \|\cdot\|)$ be a Banach space of real functions on $\mathbb{R}^d$, $(F, \|\cdot\|_*)$ be a Banach subspace of E satisfying $\|u\| \le \|u\|_* \ \forall\, u \in F$. Denote by $B(O,r)$, $B_*(O,r)$ the balls centered at O with radius $r > 0$ with respect to the norm $\|\cdot\|$ and $\|\cdot\|_*$. Suppose that there exists a function $|\cdot|_*$ from F to $\mathbb{R}$ such that

$$|u|_* \le \|u\|_*, \quad \forall\, u \in F, \quad |u+v|_* \le |u|_* + |v|_*, \quad \forall\, u,v \in F,$$

$$\lambda|u|_* = |\lambda u|_*, \quad \forall\, u \in F, \lambda \in \mathbb{R}_+,$$

where C is some positive constant.

Theorem 1.2. *Let $[0,T]$ be a time interval, and $\mathcal{S}_t$, $(t \in [0,T])$, be a class of bounded and closed subset of Fsatisfying $S_t \subset S_{t'}$ for $0 \le t \le t'$ and containing only nonnegative functions and*

$$|u|_* = \|u\|_*, \quad \forall\, u \in \mathcal{S}_T.$$

Moreover, for any sequence $\{u_n\}$ in S_T,

$$\text{If } u_n \ge 0, \|u_n\|_* \le C, \quad \lim_{n\to\infty} \|u_n - u\| = 0, \quad \text{then} \quad \lim_{n\to\infty} \|u_n - u\|_* = 0, \tag{1.26}$$

Set $R_* > R^* > 0$ and suppose $\mathcal{Q} : \mathcal{S}_T \to E$ is an operator satisfying the following properties: There exist $R_0, C_*, C^* > 0$ such that

($\mathfrak{A}$) *Hölder continuity condition*

$$\|Q[u] - Q[v]\| \le C\|u - v\|^\beta, \quad \beta \in (0,1), \quad \forall\, u,v \in \mathcal{S}_T.$$

($\mathfrak{B}$) *Sub-tangent condition*

For an element u in $\mathcal{S}_T$, there exists $\xi_u > 0$ such that for $0 < \xi < \xi_u$, there exists z in $B(u + \xi\mathcal{Q}[u], \delta) \cap \mathcal{S}_T \backslash \{u + \xi\mathcal{Q}[u]\}$ for δ small enough. Moreover,

$$\begin{aligned} |z - u|_* &\le \frac{C_*\xi}{2}\|u\|_*, \\ \chi_{R_0}\frac{z-u}{\xi} &\ge -\frac{C^*\chi_{R_0}}{2}u, \end{aligned} \tag{1.27}$$

where χ_{R_0} is the characteristic function of the ball $B_{\mathbb{R}^d}(0, R_0)$ of $\mathbb{R}^d$.

($\mathfrak{C}$) *One-side Lipschitz condition*

$$[Q[u] - Q[v], u - v] \le C\|u - v\|, \quad \forall\, u,v \in \mathcal{S}_T,$$

where

$$[\varphi, \phi] := \lim_{h\to 0^-} h^{-1}(\|\phi + h\varphi\| - \|\phi\|).$$

Moreover, $\mathcal{S}_T \cap B\big(0, \frac{R^* e^{-C^*T}}{2}\big) = \emptyset$ and $\mathcal{S}_T \subset B(0, (2R_* + 1)e^{C_*T})$.

Then the equation

$$\partial_t u = Q[u] \quad \text{on } [0,T) \times E, \quad u(0) = u_0 \in \mathcal{S}_0 \cap B_*(O, R_*) \backslash \overline{B_*(O, R^*)}, \tag{1.28}$$

has a unique solution

$$u \in C^1((0,T),E) \cap C([0,T),\mathcal{S}_T).$$

We end this introduction by giving the structure of the paper. In Sec. 2, we provide an *a priori* estimate on the L^1_N norm of the solution. The Hölder continuity of the collision operator will be established in Sec. 3. The proof of Theorem 1.1 is given in Sec. 4. The proof of Theorem 1.2 is given in Sec. 5.

Throughout the paper, we normally denote by C, C' universal constants that may vary from line to line.

2. *A Priori* Estimate

In this section, we shall derive uniform estimates on the Nth moment of f.

2.1. *Preliminaries*

The following lemma represents the weak formulation for the collision operator.

Lemma 2.1. *There holds*

$$\int_{\mathbb{R}^d} Q[f](t,k)\varphi(k)dk = \iiint_{\mathbb{R}^{3d}} R_{k,k_1,k_2}[f][\varphi(k) - \varphi(k_1) - \varphi(k_2)]dkdk_1dk_2$$

for any test functions φ so that the integrals are well-defined.

Proof. By definition, the integral of the product of $Q[f]$ and φ is written

$$\int_{\mathbb{R}^d} Q[f](t,k)\varphi(k)dk = \iiint_{\mathbb{R}^{3d}} [R_{k,k_1,k_2} - R_{k_1,k,k_2} - R_{k_2,k,k_1}]\varphi(k)dkdk_1dk_2.$$

By employing the change of variables $k \leftrightarrow k_1$, $k \leftrightarrow k_2$ in the first integral on the right, the lemma then follows. □

In this paper, we also need the following Hölder-type inequality.

Lemma 2.2. *For $N > n > p$, and $g \geq 0$ there holds*

$$\mathfrak{M}_n[g] \leq \mathfrak{M}_p^{\frac{N-n}{N-p}}[g]\mathfrak{M}_N^{\frac{n-p}{N-p}}[g], \tag{2.1}$$

where g is such that all of the integrals are well-defined.

Proof. The lemma follows from the definition of $\mathfrak{M}_n$ and the following Hölder inequality:

$$\int_{\mathbb{R}^d} g(k)\omega_k^n dk \leq \left(\int_{\mathbb{R}^d} g(k)\omega_k^p dk\right)^{\frac{N-n}{N-p}} \left(\int_{\mathbb{R}^d} g(k)\omega_k^N dk\right)^{\frac{n-p}{N-p}}.$$ □

2.2. *Estimate of the collision operator*

The main result of this subsection is the following estimate on the gain part of the collision operator $Q[g]$ as defined in (1.17) and (1.18).

Lemma 2.3. *Let $N \geq 0$. For any positive function $g \in L^1_{N+1}$, there exists a constant $\mathcal{C}C(\lambda_1, \lambda_2, N)$, depending on λ_1, λ_2, N, such that the following holds*:

$$\int_{\mathbb{R}^d} Q_{\text{gain}}[g](k)\omega_k^N \, dk \leq \frac{\mathcal{C}(\lambda_1, \lambda_2, N)\mathfrak{M}_{N+1}[g]}{\mathfrak{M}_0[g]}. \tag{2.2}$$

Remark 2.1. The proof below is based on the fact that the resonance broadening width θ defined in (1.2) is chosen proportional to

$$(|k|^2 + |k_1|^2 + |k_2|^2) \int_{\mathbb{R}^3} f(t,k) dk,$$

as discussed in the introduction.

Proof. By the same argument used to obtain the weak formulation proved in Lemma 2.1, the following identity holds true:

$$\int_{\mathbb{R}^d} Q[g](k)\omega_k^N dk = \iiint_{\mathbb{R}^{3d}} \tilde{R}_{k,k_1,k_2}[g]\left[\omega_k^N - \omega_{k_1}^N - \omega_{k_2}^N\right] dk dk_1 dk_2,$$

where

$$\tilde{R}_{k,k_1,k_2}[g] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\mathcal{L}(\omega_k - \omega_{k_1} - \omega_{k_2})(g_1 g_2 + g g_1 + g g_2),$$

and the integration of the gain term in multiplying with the test function ω_k^N is then

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q_{\text{gain}}[g](k)\omega_k^N \, dk \\
&\quad = C \iiint_{\mathbb{R}^{3d}} \delta(k - k_1 - k_2) \frac{\mathfrak{M}_0[g](|k|^2 + |k_1|^2 + |k_2|^2)|k||k_1||k_2|}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \mathfrak{M}_0[g]^2(|k|^2 + |k_1|^2 + |k_2|^2)^2} \\
&\qquad \times g_1 g_2 \omega_k^N dk dk_1 dk_2 \\
&\qquad + C \iiint_{\mathbb{R}^{3d}} \delta(k_1 - k - k_2) \frac{\mathfrak{M}_0[g](|k|^2 + |k_1|^2 + |k_2|^2)|k||k_1||k_2|}{(\omega_{k_1} - \omega_k - \omega_{k_2})^2 + \mathfrak{M}_0[g]^2(|k|^2 + |k_1|^2 + |k_2|^2)^2} \\
&\qquad \times (g g_1 + g_1 g_2)\omega_k^N dk dk_1 dk_2,
\end{aligned}$$

which by the change of variable $(k, k_1) \to (k_1, k)$ in the second integral, whose Jacobian is 1, could be expressed as

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q_{\text{gain}}[g](k)\omega_k^N \, dk \\
&\quad = C \iiint_{\mathbb{R}^{3d}} \delta(k - k_1 - k_2) \frac{\mathfrak{M}_0[g](|k|^2 + |k_1|^2 + |k_2|^2)|k||k_1||k_2|}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \mathfrak{M}_0[g]^2(|k|^2 + |k_1|^2 + |k_2|^2)^2} \\
&\qquad \times g_1 g_2 \omega_k^N dk dk_1 dk_2
\end{aligned}$$

$$+C\iiint_{\mathbb{R}^{3d}}\delta(k-k_1-k_2)\frac{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)|k||k_1||k_2|}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\mathfrak{M}_0[g]^2(|k|^2+|k_1|^2+|k_2|^2)^2}$$
$$\times(gg_1+gg_2)\omega_{k_1}^N dkdk_1dk_2.$$

By the symmetry of k_1 and k_2 in the second integral,

$$\int_{\mathbb{R}^d}Q_{\text{gain}}[g](k)\omega_k^N\,dk$$
$$=C\iiint_{\mathbb{R}^{3d}}\delta(k-k_1-k_2)\frac{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)|k||k_1||k_2|}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\mathfrak{M}_0[g]^2(|k|^2+|k_1|^2+|k_2|^2)^2}$$
$$\times g_1g_2\omega_k^N dkdk_1dk_2$$
$$+C\iiint_{\mathbb{R}^{3d}}\delta(k-k_1-k_2)\frac{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)|k||k_1||k_2|}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\mathfrak{M}_0[g]^2(|k|^2+|k_1|^2+|k_2|^2)^2}$$
$$\times gg_1\left[\omega_{k_1}^N+\omega_{k_2}^N\right]dkdk_1dk_2.$$

Let us now look at the fractional term in the above integral

$$K:=\frac{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)|k||k_1||k_2|}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\mathfrak{M}_0[g]^2(|k|^2+|k_1|^2+|k_2|^2)^2}.$$

Since the denominator $(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\mathfrak{M}_0^2(|k|^2+|k_1|^2+|k_2|^2)^2$ is greater than $\mathfrak{M}_0[g]^2(|k|^2+|k_1|^2+|k_2|^2)^2$, the whole fraction can be bounded as

$$K\le\frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)},$$

which leads to the following:

$$\int_{\mathbb{R}^d}Q_{\text{gain}}[g](k)\omega_k^N\,dk$$
$$\le C\iiint_{\mathbb{R}^{3d}}\delta(k-k_1-k_2)\frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)}g_1g_2\omega_k^N dkdk_1dk_2$$
$$+C\iiint_{\mathbb{R}^{3d}}\delta(k-k_1-k_2)\frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)}$$
$$\times gg_1\left[\omega_{k_1}^N+\omega_{k_2}^N\right]dkdk_1dk_2,$$

which can be rewritten in the following equivalent form, with the right-hand side being the sum of I_1 and I_2:

$$\int_{\mathbb{R}^d}Q_{\text{gain}}[g](k)\omega_k^N\,dk\le I_1+I_2, \tag{2.3}$$

where

$$\begin{aligned} I_1 &:= C \iiint_{\mathbb{R}^{3d}} \delta(k-k_1-k_2)\frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)} \\ &\quad\times\ g_1 g_2 \omega_k^N dk dk_1 dk_2, \\ I_2 &:= C \iiint_{\mathbb{R}^{3d}} \delta(k-k_1-k_2)\frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)} \\ &\quad\times\ g g_1 \big[\omega_{k_1}^N+\omega_{k_2}^N\big] dk dk_1 dk_2. \end{aligned} \tag{2.4}$$

Let us first estimate I_1. By the resonant condition $k=k_1+k_2$, we have

$$\begin{aligned} \omega_k &= \sqrt{\lambda_1+\lambda_2|k|^2} \le \sqrt{\lambda_1+\lambda_2(|k_1|+|k_2|)^2} \\ &\qquad < 2\sqrt{\lambda_1+\lambda_2|k_1|^2}+2\sqrt{\lambda_1+\lambda_2|k_2|^2} = 2\omega_{k_1}+2\omega_{k_2}, \end{aligned}$$

which, thanks to the Cauchy–Schwarz inequality, leads to

$$\omega_k^N \le C(\lambda_1,\lambda_2,N)\big(\omega_{k_1}^N+\omega_{k_2}^N\big),$$

where $C(\lambda_1,\lambda_2,N)$ is some constant depending on λ_1,λ_2,N.

Thus, we obtain

$$\begin{aligned} I_1 &\le C(\lambda_1,\lambda_2,N) \iiint_{\mathbb{R}^{3d}} \delta(k-k_1-k_2) \\ &\qquad \times \frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)} g_1 g_2 \big[\omega_{k_1}^N+\omega_{k_2}^N\big] dk dk_1 dk_2. \end{aligned}$$

Taking into account the definition of the Dirac function $\delta(k-k_1-k_2)$ the above integral on $\mathbb{R}^{3d}$ can be reduced to an integral on $\mathbb{R}^{2d}$ only

$$I_1 \le C(\lambda_1,\lambda_2,N) \iint_{\mathbb{R}^{2d}} \frac{|k_1+k_2||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2+|k_1|^2+|k_2|^2)} g_1 g_2 \big[\omega_{k_1}^N+\omega_{k_2}^N\big] dk_1 dk_2.$$

Due to the inequality $|k_1+k_2|^2+|k_1|^2+|k_2|^2 \ge 2|k_1||k_2|$, the kernel of the above integral can be bounded as

$$\frac{|k_1+k_2||k_1||k_2|}{|k_1+k_2|^2+|k_1|^2+|k_2|^2} \le \frac{|k_1+k_2|}{2} \le \frac{|k_1|+|k_2|}{2},$$

yielding

$$I_1 \le \frac{C(\lambda_1,\lambda_2,N)}{\mathfrak{M}_0[g]} \iint_{\mathbb{R}^{2d}} (|k_1|+|k_2|) g_1 g_2 \big[\omega_{k_1}^N+\omega_{k_2}^N\big] dk_1 dk_2.$$

Observing that

$$|k_1| \le \frac{\omega_{k_1}}{\sqrt{\lambda_2}}, \quad |k_2| \le \frac{\omega_{k_2}}{\sqrt{\lambda_2}},$$

we can bound

$$(|k_1|+|k_2|)\big[\omega_{k_1}^N+\omega_{k_2}^N\big] \le C(\omega_{k_1}+\omega_{k_2})\big[\omega_{k_1}^N+\omega_{k_2}^N\big] \le C\big[\omega_{k_1}^{N+1}+\omega_{k_2}^{N+1}\big],$$

which yields the following estimate on I_1 in terms of the functional defined in (1.21):

$$\begin{aligned} I_1 &\le \frac{C(\lambda_1,\lambda_2,N)}{\mathfrak{M}_0[g]} \iint_{\mathbb{R}^{2d}} g_1 g_2 \left[\omega_{k_1}^{N+1} + \omega_{k_2}^{N+1}\right] dk_1 dk_2 \\ &\le \frac{C}{\mathfrak{M}_0[g]} \mathfrak{M}_{N+1}[g]. \end{aligned} \tag{2.5}$$

Let us now estimate I_2. Using the resonant condition $k_2 = k - k_1$, we obtain the following relation between ω_{k_2} and ω_k, ω_{k_1}:

$$\begin{aligned} \omega_{k_2} = \sqrt{\lambda_1 + \lambda_2 |k_2|^2} &< \sqrt{\lambda_1 + \lambda_2(|k_1| + |k|)^2} \\ &\le 2\sqrt{\lambda_1 + \lambda_2|k|^2} + 2\sqrt{\lambda_1 + \lambda_2|k_1|^2} = 2\omega_k + 2\omega_{k_1}, \end{aligned}$$

which, by the Cauchy–Schwarz inequality, leads to

$$\omega_{k_2}^N \le C(\lambda_1,\lambda_2,N)\left(\omega_k^N + \omega_{k_1}^N\right),$$

where C is some universal positive constant.

Thus, we obtain

$$\begin{aligned} I_2 \le C(\lambda_1,\lambda_2,N) &\iiint_{\mathbb{R}^{3d}} \delta(k - k_1 - k_2) \frac{|k||k_1||k_2|}{\mathfrak{M}_0[g](|k|^2 + |k_1|^2 + |k_2|^2)} \\ &\times g g_1 \left[\omega_k^N + \omega_{k_1}^N\right] dk dk_1 dk_2. \end{aligned}$$

By the definition of the Dirac function $\delta(k - k_1 - k_2)$, we can reduce the above triple integral into an integral on $\mathbb{R}^{2d}$ only

$$I_2 \le C(\lambda_1,\lambda_2,N) \iint_{\mathbb{R}^{2d}} \frac{|k||k_1||k - k_1|}{\mathfrak{M}_0[g](|k|^2 + |k_1|^2 + |k_2|^2)} g g_1 \left[\omega_k^N + \omega_{k_1}^N\right] dk dk_1.$$

It is straightforward from Cauchy–Schwarz inequality that $|k|^2 + |k_1|^2 + |k - k_1|^2 \ge 2|k_1||k|$, yielding the following estimate on the kernel of the above integral:

$$\frac{|k||k_1||k - k_1|}{|k|^2 + |k_1|^2 + |k - k_1|^2} \le \frac{|k - k_1|}{2} \le \frac{|k| + |k_1|}{2},$$

which implies the following bound on I_2

$$I_2 \le \frac{C(\lambda_1,\lambda_2,N)}{\mathfrak{M}_0[g]} \iint_{\mathbb{R}^{2d}} (|k| + |k_1|) g g_1 \left[\omega_k^N + \omega_{k_1}^N\right] dk dk_1.$$

The same argument used to estimate I_1 can now be applied again, that leads to a similar bound on I_2

$$\begin{aligned} I_2 &\le \frac{C(\lambda_1,\lambda_2,N)}{\mathfrak{M}_0[g]} \iint_{\mathbb{R}^{2d}} g g_1 \left[\omega_k^{N+1} + \omega_{k_1}^{N+1}\right] dk dk_1 \\ &\le \frac{C(\lambda_1,\lambda_2,N)}{\mathfrak{M}_0[g]} \mathfrak{M}_{N+1}[g]. \end{aligned} \tag{2.6}$$

Combining (2.3)–(2.6), we get (2.2) so the conclusion of Lemma 2.3 follows. □

2.3. *Lower bound of the solution (the choice of R_0)*

Proposition 2.1. *For any initial data $f_0 \geq 0$ and $f_0 \in L^1(\mathbb{R}^3)$. Suppose that $f \in L^1(\mathbb{R}^d)$ is a positive, strong solution of* (1.3), *then*

$$Q[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f] \geq -Q_{\text{loss}}[f] \geq -4|k|f, \tag{2.7}$$

pointwise in k and f satisfies the following lower bound:

$$f(t,k) \geq f_0(k)e^{-(2\nu|k|^2+4|k|)t}, \tag{2.8}$$

which implies

$$\|f(t,k)\chi_{R_0}\|_{L^1} \geq \tilde{\mathfrak{M}}_0(t) := e^{-(2\nu R_0^2+4R_0)t}\|f_0(k)\chi_{R_0}\|_{L^1}, \tag{2.9}$$

where χ_{R_0} is the characteristic function of the ball $B_{\mathbb{R}^d}(O,R_0)$ in $\mathbb{R}^d$, R_0 is any positive constant.

Proof. Let us first recall the formulation of $Q[f]$

$$\begin{aligned} Q[f] = {} & \iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2})(f_1f_2-2ff_1)dk_1dk_2 \\ & + 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)\mathcal{L}_f(\omega_{k_1}-\omega_k-\omega_{k_2}) \\ & \times(-ff_2+ff_1+f_1f_2)dk_1dk_2. \end{aligned}$$

and in order to get (2.8), we will work with

$$Q[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f],$$

where the formulation of $Q_{\text{loss}}[f]$

$$\begin{aligned} -Q_{\text{loss}}[f] &= -2f\int_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k_2})f_1dk_1dk_2 \\ &\quad -2f\int_{\mathbb{R}^d\times\mathbb{R}^d} |V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)\mathcal{L}_f(\omega_{k_1}-\omega_k-\omega_{k_2})f_2dk_1dk_2 \\ &=: -\mathcal{I}_1 - \mathcal{I}_2. \end{aligned} \tag{2.10}$$

In order to get the lower bound (2.7), we discard the gain operator defined in (1.18) and estimate from below the loss part.

Let us estimate the double integral $\mathcal{I}_1$, which can be reduced to an integral on $\mathbb{R}^d$ by taking into account the definition of $\delta(k-k_1-k_2)$ as follows:

$$\mathcal{I}_1 := 2f\int_{\mathbb{R}^d} |V_{k,k_1,k-k_1}|^2\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k-k_1})f_1dk_1.$$

By the definition of $V_{k,k_1,k-k_1}$, $\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k-k_1})$, Γ_{k,k_1,k_2}, and the inequality

$$(\omega_k-\omega_{k_1}-\omega_{k-k_1})^2 + \Gamma^2_{k,k_1,k-k_1} \geq \Gamma^2_{k,k_1,k-k_1},$$

we obtain the following inequality on the kernel of $\mathcal{I}_1$:

$$\begin{aligned}|V_{k,k_1,k-k_1}|^2\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k-k_1}) &= \frac{|k||k_1||k-k_1|\Gamma_{k,k_1,k-k_1}}{(\omega_k-\omega_{k_1}-\omega_{k-k_1})^2+\Gamma^2_{k,k_1,k-k_1}}\\ &\le \frac{|k||k_1||k-k_1|}{\Gamma_{k,k_1,k-k_1}}\\ &\le \frac{|k||k_1||k-k_1|}{\mathfrak{M}_0[f](|k|^2+|k_1|^2+|k-k_1|^2)}.\end{aligned}$$

By the positivity of $|k|^2$ and the Cauchy–Schwarz inequality, the following holds true:

$$|k|^2+|k_1|^2+|k-k_1|^2 \ge |k_1|^2+|k-k_1|^2 \ge 2|k_1||k-k_1|,$$

which implies

$$|V_{k,k_1,k-k_1}|^2\mathcal{L}_f(\omega_k-\omega_{k_1}-\omega_{k-k_1}) \le \frac{2|k|}{\mathfrak{M}_0[f]}.$$

As a result, we have the following estimate on $\mathcal{I}_1$:

$$\mathcal{I}_1 \le \frac{2|k|f\int_{\mathbb{R}^d} f_1 dk_1}{\mathfrak{M}_0[f]} \le 2|k|f. \tag{2.11}$$

$\mathcal{I}_2$ can be estimated in a similar way. We can reduce $\mathcal{I}_2$ to an integral on $\mathbb{R}^d$ by taking into account the definition of $\delta(k_1-k-k_2)$ as follows:

$$\mathcal{I}_2 := f\int_{\mathbb{R}^d}|V_{k+k_2,k,k_2}|^2\mathcal{L}_f(\omega_{k+k_2}-\omega_k-\omega_{k_2})f_2dk_2.$$

Taking into account the definite of V_{k+k_2,k,k_2}, $\mathcal{L}_f(\omega_{k+k_2}-\omega_k-\omega_{k_2})$, Γ_{k+k_2,k,k_2}, and the inequality

$$(\omega_{k+k_2}-\omega_k-\omega_{k_2})^2+\Gamma^2_{k+k_2,k,k_2} \ge \Gamma^2_{k+k_2,k,k_2},$$

the following estimate on the kernel of $\mathcal{I}_2$ can be obtained:

$$\begin{aligned}|V_{k+k_2,k,k_2}|^2\mathcal{L}_f(\omega_{k+k_2}-\omega_k-\omega_{k_2}) &= \frac{|k+k_2||k||k_2|\Gamma_{k+k_2,k,k_2}}{(\omega_{k+k_2}-\omega_k-\omega_{k_2})^2+\Gamma^2_{k+k_2,k,k_2}}\\ &\le \frac{|k+k_2||k||k_2|}{\mathfrak{M}_0[f](|k+k_2|^2+|k|^2+|k_2|^2)}.\end{aligned}$$

Using the positivity of $|k|^2$ and the Cauchy–Schwarz inequality, we find

$$|k+k_2|^2+|k|^2+|k_2|^2 \ge |k+k_2|^2+|k_2|^2 \ge 2|k+k_2||k_2|,$$

which implies

$$|V_{k+k_2,k,k_2}|^2\mathcal{L}_f(\omega_{k+k_2}-\omega_k-\omega_{k_2}) \le \frac{2|k|}{\mathfrak{M}_0[f]}.$$

We then obtain the following estimate on $\mathcal{I}_2$:

$$\mathcal{I}_2 \le \frac{2|k|f\int_{\mathbb{R}^d} f_2 dk_2}{\mathfrak{M}_0[f]} = 2|k|f. \tag{2.12}$$

Combining (2.10)–(2.12) yields

$$Q[f] \geq -4|k|f. \tag{2.13}$$

By plugging the above inequality into (1.3), we obtain a differential inequality on f

$$\partial_t f - Q[f] - 2\nu|k|^2 f \geq \partial_t f + (2\nu|k|^2 + 4|k|)f \geq 0.$$

A Gronwall inequality argument applied to the above differential inequality leads to

$$f(t,k) \geq f_0(k)e^{-(2\nu|k|^2+4|k|)t},$$

and so (2.8) holds.

Multiplying both sides of the above inequality with χ_{R_0} is the characteristic function of the ball $B_{\mathbb{R}^d}(O,R_0)$ in $\mathbb{R}^d$, and taking the integral with respect to k on $\mathbb{R}^d$, yield

$$\begin{aligned}\|f\chi_{R_0}\|_1 &\geq \int_{\mathbb{R}^d} \chi_{R_0} f(t,k)dk \\ &\geq \int_{\mathbb{R}^d} \chi_{R_0} f_0(k) e^{-(2\nu|k|^2+4|k|)t} dk \\ &\geq e^{-(2\nu R_0^2+4R_0)t} \int_{\mathbb{R}^d} \chi_{R_0} f_0(k)dk \geq \|f_0\chi_{R_0}\|_1,\end{aligned}$$

and so (2.9) holds true. The proof of Proposition 2.1 is completed. □

2.4. *Weighted L^1_N $(N \geq 0)$ estimates*

For a given function g, let us recall the Nth moment of g

$$\mathfrak{M}_N[g] = \int_{\mathbb{R}^d} \omega_k^N g(k)dk.$$

Proposition 2.2. *Let $N \geq 0$. Suppose that $f_0(k)$ is a nonnegative initial data satisfying*

$$\int_{\mathbb{R}^d} f_0(k)\omega_k^N\, dk < \infty,$$

and that nonnegative solutions $f(t,k)$ of (1.3) satisfies

$$\mathfrak{M}_0[f](t) \geq \tilde{\mathfrak{M}}_0(t) = e^{-(2\nu R_0^2+4R_0)t}\|f_0(k)\chi_{R_0}\|_{L^1} > 0,$$

where $\tilde{\mathfrak{M}}_0(t)$ is the quantity considered in Proposition 2.1.

Then, there exists a positive constant $C_0(\lambda_1,\lambda_2)$ is a constant depending on λ_1,λ_2 and independent of N such that

$$\begin{aligned}&\mathfrak{M}_{N+1}[Q[f]](t) - 2\nu\mathfrak{M}_N[|k|^2 f](t) \\ &\quad = \int_{\mathbb{R}^d} Q[f](t,k)\omega_k^{N+1}\, dk - 2\nu\int_{\mathbb{R}^d} |k|^2 f(t,k)\omega_k^N\, dk \\ &\quad \leq C_0(\lambda_1,\lambda_2)\left(1 + \frac{e^{(4\nu R_0^2+8R_0)t}}{\|f_0(k)\chi_{R_0}\|^2_{L^1}}\right)\int_{\mathbb{R}^d} f(t,k)\omega_k^N\, dk,\end{aligned} \tag{2.14}$$

which implies that nonnegative solutions $f(t,k)$ *of* (1.3), *with* $f(0,k) = f_0(k)$, *satisfy*

$$\begin{aligned}\mathfrak{M}_N[f](t) &= \int_{\mathbb{R}^d} f(t,k)\omega_k^N\,dk \\ &\le e^{\mathcal{C}(\lambda_1,\lambda_2)\left(t+\frac{e^{(4\nu R_0^2+8R_0)t}}{(4\nu R_0^2+8R_0)\|f_0(k)\chi_{R_0}\|_{L^1}^2}\right)}\int_{\mathbb{R}^d} f_0(k)\omega_k^N\,dk, \qquad (2.15)\end{aligned}$$

where $\mathcal{C}(\lambda_1,\lambda_2)$ *is a constant depending on* λ_1, λ_2.

Remark 2.2. Note that (2.14) says that the Nth moment of f only depends on the Nth moment of the initial data and the parameter R_0 defined in Proposition 2.1.

Proof of Proposition 2.2. Using $\varphi = \omega_k^N$ as a test function in (1.3), we obtain

$$\begin{aligned}\frac{d}{dt}\mathfrak{M}_N[f] + 2\nu\mathfrak{M}_N[|k|^2 f] &= \frac{d}{dt}\int_{\mathbb{R}^d} f(t,k)\omega_k^N dk + 2\nu\int_{\mathbb{R}^d}|k|^2 f(t,k)\omega_k^N\,dk \\ &= \int_{\mathbb{R}^d} Q[f](t,k)\omega_k^N dk.\end{aligned}$$

As a direct consequence of Lemma 2.3, the following inequality holds true:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d} f(t,k)\omega_k^N dk + 2\nu\int_{\mathbb{R}^d}|k|^2 f(t,k)\omega_k^N\,dk \\ &\quad\le \frac{C}{\mathfrak{M}_0[f]}\mathfrak{M}_{N+1}[f(t)] = \frac{C}{\mathfrak{M}_0[f]}\int_{\mathbb{R}^d} f(t,k)\omega_k^{N+1}\,dk. \qquad (2.16)\end{aligned}$$

Notice that

$$|k|^2 = \frac{\omega_k^2 - \lambda_1}{\lambda_2},$$

we get the following moment equation:

$$\frac{d}{dt}\mathfrak{M}_N[f(t)] + \frac{2\nu}{\lambda_2}\mathfrak{M}_{N+2}[f(t)] - \frac{2\nu\lambda_1}{\lambda_2}\mathfrak{M}_N[f(t)] \le \frac{C}{\mathfrak{M}_0[f]}\mathfrak{M}_{N+1}[f(t)].$$

Using the fact that

$$\mathfrak{M}_0[f] \ge e^{-(2\nu R_0^2+4R_0)T}\|f_0(k)\chi_{R_0}\|_{L^1},$$

we deduce from (2.16)

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d} f(t,k)\omega_k^N dk + 2\nu\int_{\mathbb{R}^d}|k|^2 f(t,k)\omega_k^N\,dk \\ &\quad\le \frac{C}{\mathfrak{M}_0[f]}\mathfrak{M}_{N+1}[f(t)] \le \frac{Ce^{(2\nu R_0^2+4R_0)T}}{\|f_0(k)\chi_{R_0}\|_{L^1}}\int_{\mathbb{R}^d} f(t,k)\omega_k^{N+1}\,dk.\end{aligned}$$

Now since

$$\begin{aligned}&\frac{Ce^{(2\nu R_0^2+4R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}}\omega_k^{N+1}-2\nu|k|^2\omega_k^N\\&\quad=(\lambda_1+\lambda_2|k|^2)^{\frac{N}{2}}\left(\frac{Ce^{(2\nu R_0^2+4R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}}(\lambda_1+\lambda_2|k|^2)^{\frac{1}{2}}-2\nu|k|^2\right),\end{aligned}$$

and observing that $\frac{Ce^{(2\nu R_0^2+4R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}}(\lambda_1+\lambda_2|k|^2)^{\frac{1}{2}}-2\nu|k|^2$ is bounded uniformly by some constant $\mathcal{C}(\lambda_1,\lambda_1)\big(1+\frac{e^{(4\nu R_0^2+8R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}^2}\big)$, we can bound

$$\frac{C}{\tilde{\mathfrak{M}}_0(t)}\omega_k^{N+1}-2\nu|k|^2\omega_k^N\leq\mathcal{C}(\lambda_1,\lambda_2)\left(1+\frac{e^{(4\nu R_0^2+8R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}^2}\right)(\lambda_1+\lambda_2|k|^2)^{\frac{N}{2}}.$$

The above estimate means that the difference

$$\begin{aligned}&\frac{Ce^{(2\nu R_0^2+4R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}}\int_{\mathbb{R}^d}f(t,k)\omega_k^{N+1}\,dk-2\nu\int_{\mathbb{R}^d}|k|^2f(t,k)\omega_k^N\,dk\\&\quad=\int_{\mathbb{R}^d}f(t,k)\left(\frac{Ce^{(2\nu R_0^2+4R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}}\omega_k^{N+1}-2\nu|k|^2\omega_k^N\right)dk,\end{aligned}$$

is smaller than $\mathcal{C}(\lambda_1,\lambda_2)\big(1+\frac{e^{(4\nu R_0^2+8R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}^2}\big)\int_{\mathbb{R}^d}f(t,k)\omega_k^N\,dk$, which immediately leads to

$$\frac{d}{dt}\int_{\mathbb{R}^d}f(t,k)\omega_k^N\,dk\leq\mathcal{C}(\lambda_1,\lambda_2)\left(1+\frac{e^{(4\nu R_0^2+8R_0)t}}{\|f_0(k)\chi_{R_0}\|_{L^1}^2}\right)\int_{\mathbb{R}^d}f(t,k)\omega_k^N\,dk.$$

Inequality (2.15) then follows as a consequence of the above inequality. □

3. Holder Estimates for $Q[f]$

In this section, we study the Hölder continuity of the collision operator $Q[f]$ with respect to weighted L^1_N norm.

Proposition 3.1. *Let $M,N\geq 0$, and let V_M be any bounded subset of $L^1_{N+2}(\mathbb{R}^d)$, with the L^1_{N+2} norms bounded from above by M and the L^1 norms bounded from below by M'. Then, there exists a constant $C_{M,M',N}$, depending on M,M',N, so that*

$$\begin{aligned}\|Q[g]-Q[h]\|_{L^1_N}&\leq\left(\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}+\frac{C}{\mathfrak{M}_0[|g|]}\right)\|g-h\|_{L^1_N}^{\frac{1}{2}}\\&\leq C_{M,M'}\|g-h\|_{L^1_N}^{\frac{1}{2}}\end{aligned}\tag{3.1}$$

for all $g,h\in V_M$.

We first prove the following lemma.

Lemma 3.1. *Let $M, N > 0$, and let V_M be any bounded subset of $L^1(\mathbb{R}^d) \cap L^1_{N+1}(\mathbb{R}^d)$, with the L^1_{N+2} norms bounded from above by M and the L^1 norms bounded from below by M'. Then, there exists a constant $C_{M,M'}$, depending on M, M', so that*

$$\begin{aligned}\|Q[g]-Q[h]\|_{L^1_N} &\le \left(\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}+\frac{C}{\mathfrak{M}_0[|g|]}\right)\|g-h\|_{L^1_{N+1}}\\ &\le C_{M,M'}\|g-h\|_{L^1_{N+1}} \end{aligned} \tag{3.2}$$

for all $g, h \in V_M$.

Proof. We first compute the difference between $Q[g]$ and $Q[h]$

$$Q[g]-Q[h] = \iint_{\mathbb{R}^{2d}} \left[R_{k,k_1,k_2}[g]-R_{k,k_1,k_2}[h]-2(R_{k_1,k,k_2}[g]-R_{k_1,k,k_2}[h])\right]dk_1dk_2,$$

whose L^1_N-norm is

$$\begin{aligned}\|Q[g]-Q[h]\|_{L^1_N} &= \int_{\mathbb{R}^d}\omega_k^N|Q[g](k)-Q[h](k)|dk\\ &\le \iiint_{\mathbb{R}^{3d}}\omega_k^N|R_{k,k_1,k_2}[g]-R_{k,k_1,k_2}[h]|\,dkdk_1dk_2\\ &\quad +2\iiint_{\mathbb{R}^{3d}}\omega_k^N|R_{k_1,k,k_2}[g]-R_{k_1,k,k_2}[h]|dkdk_1dk_2\\ &= \iiint_{\mathbb{R}^{3d}}|R_{k,k_1,k_2}[g]-R_{k,k_1,k_2}[h]|\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2.\end{aligned}$$

Recalling that

$$R_{k,k_1,k_2}[g] = C|V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\mathcal{L}_g(\omega_k-\omega_{k_1}-\omega_{k_2})(g_1g_2-gg_1-gg_2),$$

we find the following estimate on $\|Q[g]-Q[h]\|_{L^1_N}$:

$$\|Q[g]-Q[h]\|_{L^1_N} \le \mathbb{J}_1+\mathbb{J}_2, \tag{3.3}$$

where

$$\begin{aligned}\mathbb{J}_1 &:= \iiint_{\mathbb{R}^{3d}}|V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)|\mathcal{L}_g(\omega_k-\omega_{k_1}-\omega_{k_2})g_1g_2\\ &\quad -\mathcal{L}_h(\omega_k-\omega_{k_1}-\omega_{k_2})h_1h_2|\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2,\\ \mathbb{J}_2 &:= 2\iiint_{\mathbb{R}^{3d}}|V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)|\mathcal{L}_g(\omega_{k_1}-\omega_k-\omega_{k_2})gg_2\\ &\quad -\mathcal{L}_h(\omega_{k_1}-\omega_k-\omega_{k_2})hh_2|\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2.\end{aligned} \tag{3.4}$$

Let us now split the proof into two steps.

Step 1: Estimating $\mathbb{J}_1$. Define the quantity inside the triple integral of $\mathbb{J}_1$ after dropping $\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)$ to be J_1

$$J_1 := |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)|\mathcal{L}_g(\omega_k-\omega_{k_1}-\omega_{k_2})g_1g_2-\mathcal{L}_h(\omega_k-\omega_{k_1}-\omega_{k_2})h_1h_2|,$$

which, by the triangle inequality, can be bounded as

$$J_1 \le |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)\mathcal{L}_g(\omega_k-\omega_{k_1}-\omega_{k_2})|g_1g_2-h_1h_2| + |V_{k,k_1,k_2}|^2\delta(k-k_1-k_2)|\mathcal{L}_g(\omega_k-\omega_{k_1}-\omega_{k_2})-\mathcal{L}_h(\omega_k-\omega_{k_1}-\omega_{k_2})||h_1h_2|.$$

Define the two terms on the right-hand side of the above inequality to be J_{11} and J_{12}, respectively.

Let us now study J_{11} in details. Using the definition of $\mathcal{L}_g$ and the triangle inequality

$$|g_1g_2-h_1h_2| \le |g_1||g_2-h_2| + |h_2||g_1-h_1|,$$

yields the following estimate on J_{11}:

$$J_{11} \le C|k||k_1||k_2|\delta(k-k_1-k_2)\frac{\Gamma_{g,k,k_1,k_2}}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\Gamma^2_{g,k,k_1,k_2}}|g_1||g_2-h_2| + C|k||k_1||k_2|\delta(k-k_1-k_2)\frac{\Gamma_{g,k,k_1,k_2}}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\Gamma^2_{g,k,k_1,k_2}}|h_2||g_1-h_1|.$$

By the inequality

$$(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\Gamma^2_{g,k,k_1,k_2} \ge \Gamma^2_{g,k,k_1,k_2},$$

we can bound J_{11} as

$$J_{11} \le C|k||k_1||k_2|\delta(k-k_1-k_2)\frac{1}{\Gamma_{g,k,k_1,k_2}}|g_1||g_2-h_2| + C|k||k_1||k_2|\delta(k-k_1-k_2)\frac{1}{\Gamma_{g,k,k_1,k_2}}|h_2||g_1-h_1|.$$

The right-hand side of the above inequality can be estimated by employing the following Cauchy–Schwarz inequality:

$$\begin{aligned}\Gamma_{g,k,k_1,k_2} &= \mathfrak{M}_0[|g|](|k|^2+|k_1|^2+|k_2|^2)\\ &\ge \mathfrak{M}_0[|g|](|k_1|^2+|k_2|^2) \ge 2\mathfrak{M}_0[|g|]|k_1||k_2|,\end{aligned}$$

where we have just used the lower bound of $\mathfrak{M}_0[|g|]$, yielding

$$J_{11} \le \frac{C}{\mathfrak{M}_0[|g|]}|k|\delta(k-k_1-k_2)|g_1||g_2-h_2| + \frac{C}{\mathfrak{M}_0[|g|]}|k|\delta(k-k_1-k_2)|h_2||g_1-h_1|.$$

Multiplying the above inequality with $(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N)$ and integrating in k, k_1 and k_2 lead to

$$\begin{aligned}&\iiint_{\mathbb{R}^{3d}} J_{11}(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N)dkdk_1dk_2\\ &\quad\le \iiint_{\mathbb{R}^{3d}} \frac{C}{\mathfrak{M}_0[|g|]}||k|\delta(k-k_1-k_2)[|g_1||g_2-h_2|+|h_2||g_1-h_1|]\\ &\quad\quad\times(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N)dkdk_1dk_2.\end{aligned}$$

Using the resonant condition $k = k_1 + k_2$, we reduce the triple integral on the right-hand side to a double integral

$$\iiint_{\mathbb{R}^3} J_{11}\big(\omega_k^N + \omega_{k_1}^N + \omega_{k_2}^N\big) dk dk_1 dk_2 \le \frac{C}{\mathfrak{M}_0[|g|]} \iint_{\mathbb{R}^{2d}} |k_1 + k_2|[|g_1||g_2 - h_2| + |h_2||g_1 - h_1|]\big(\omega_{k_1}^N + \omega_{k_2}^N\big) dk_1 dk_2,$$

where we have just used the inequality

$$\omega_{k_1+k_2}^N \le C\omega_{k_1}^N + C\omega_{k_2}^N,$$

proved in Proposition 2.3, to bound the sum $\omega_k^N + \omega_{k_1}^N + \omega_{k_2}^N$ by $C\big(\omega_{k_1}^N + \omega_{k_2}^N\big)$.

Observing that

$$|k_1 + k_2|\big(\omega_{k_1}^N + \omega_{k_2}^N\big) \le (|k_1| + |k_2|)\big(\omega_{k_1}^N + \omega_{k_2}^N\big) \le C\big(\omega_{k_1}^{N+1} + \omega_{k_2}^{N+1}\big),$$

we find

$$\iiint_{\mathbb{R}^{3d}} J_{11}\big(\omega_k^N + \omega_{k_1}^N + \omega_{k_2}^N\big) dk dk_1 dk_2 \le \frac{C}{\mathfrak{M}_0[|g|]} \iint_{\mathbb{R}^{2d}} [|g_1||g_2 - h_2| + |h_2||g_1 - h_1|]\big(\omega_{k_1}^{N+1} + \omega_{k_2}^{N+1}\big) dk_1 dk_2,$$

which immediately leads to

$$\begin{aligned} &\iiint_{\mathbb{R}^{3d}} J_{11}\big(\omega_k^N + \omega_{k_1}^N + \omega_{k_2}^N\big) dk dk_1 dk_2 \\ &\quad\le \frac{C}{\mathfrak{M}_0[|g|]} \|g - h\|_{L^1_{N+1}} \big(\|g\|_{L^1} + \|g\|_{L^1_{N+1}} + \|h\|_{L^1} + \|h\|_{L^1_{N+1}}\big) \\ &\quad\le \frac{C}{\mathfrak{M}_0[|g|]} \|g - h\|_{L^1_{N+1}} \big(\|g\|_{L^1_{N+1}} + \|h\|_{L^1_{N+1}}\big). \end{aligned} \tag{3.5}$$

Now, let us look at J_{12}, which can be written as

$$\begin{aligned} J_{12} &= C|k||k_1||k_2|\delta(k - k_1 - k_2)|h_1 h_2| \\ &\quad\times \left| \frac{\Gamma_{g,k,k_1,k_2}[(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{h,k,k_1,k_2}] - \Gamma_{h,k,k_1,k_2}[(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{g,k,k_1,k_2}]}{[(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{g,k,k_1,k_2}][(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{h,k,k_1,k_2}]} \right| \\ &= C|k||k_1||k_2|\delta(k - k_1 - k_2)|h_1 h_2| \\ &\quad\times \frac{|(\omega_k - \omega_{k_1} - \omega_{k_2})^2 - \Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}||\Gamma_{g,k,k_1,k_2} - \Gamma_{h,k,k_1,k_2}|}{[(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{g,k,k_1,k_2}][(\omega_k - \omega_{k_1} - \omega_{k_2})^2 + \Gamma^2_{h,k,k_1,k_2}]}. \end{aligned}$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}&\left[(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\Gamma^2_{g,k,k_1,k_2}\right]\left[(\omega_k-\omega_{k_1}-\omega_{k_2})^2+\Gamma^2_{h,k,k_1,k_2}\right]\\&\quad\geq|(\omega_k-\omega_{k_1}-\omega_{k_2})^2-\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}||(\omega_k-\omega_{k_1}-\omega_{k_2})^2\\&\qquad+\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}|\\&\quad\geq|(\omega_k-\omega_{k_1}-\omega_{k_2})^2-\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}|\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2},\end{aligned}$$

from which, we obtain the following estimate on J_{12}:

$$J_{12}\leq C|k||k_1||k_2||h_1h_2|\delta(k-k_1-k_2)\frac{|\Gamma_{g,k,k_1,k_2}-\Gamma_{h,k,k_1,k_2}|}{\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}}.$$

The numerator of the fraction on the right-hand side has the following interesting property:

$$|\Gamma_{g,k,k_1,k_2}-\Gamma_{h,k,k_1,k_2}|=C|(k^2+k_1^2+k_2^2)\mathfrak{M}_0[|g|-|h|]|,$$

which can be bounded as follows:

$$|\Gamma_{g,k,k_1,k_2}-\Gamma_{h,k,k_1,k_2}|\leq C(k^2+k_1^2+k_2^2)\|g-h\|_{L^1},$$

yielding an upper bound on J_{12}

$$J_{12}\leq C|k||k_1||k_2||h_1h_2|\delta(k-k_1-k_2)\frac{\|g-h\|_{L^1},}{(k^2+k_1^2+k_2^2)\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}.$$

By the Cauchy–Schwarz inequality

$$k^2+k_1^2+k_2^2\geq k_1^2+k_2^2\geq 2|k_1||k_2|,$$

and the lower bound on $\mathfrak{M}_0[|g|]$ and $\mathfrak{M}_0[|h|]$, the following estimate on J_{12} then follows:

$$J_{12}\leq\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}|k||h_1h_2|\delta(k-k_1-k_2)\|g-h\|_{L^1}.$$

Multiplying the above inequality with $\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)$ and integrate in k, k_1 and k_2, the same argument used to deduce (3.5) leads to

$$\iiint_{\mathbb{R}^{3d}}J_{12}\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2\leq\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}\|g-h\|_{L^1_{N+1}}.\quad(3.6)$$

Note that C is a constant depending on $\left(\|g\|_{L^1_{N+1}}+\|h\|_{L^1_{N+1}}\right)$. Combining (3.5) and (3.6) yields

$$\mathbb{J}_1\leq\left(\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}+\frac{C}{\mathfrak{M}_0[|g|]}\right)\|g-h\|_{L^1_{N+1}},\quad(3.7)$$

where C is a constant depending on $\left(\|g\|_{L^1_{N+1}}+\|h\|_{L^1_{N+1}}\right)$.

Step 2: Estimating $\mathbb{J}_2$. The proof of estimating $\mathbb{J}_2$ follows exactly the same argument used in Step 1. As a consequence, we omit some details and give only the main estimates in the sequel. First, define the quantity inside the triple integral of $\mathbb{J}_2$ after dropping $\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)$ to be J_2

$$J_2:=|V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)|\mathcal{L}_g(\omega_{k_1}-\omega_k-\omega_{k_2})gg_2-\mathcal{L}_h(\omega_{k_1}-\omega_k-\omega_{k_2})hh_2|,$$

which, by the triangle inequality, can be bounded as

$$\begin{aligned}J_2&\le|V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)\mathcal{L}_g(\omega_{k_1}-\omega_k-\omega_{k_2})|gg_2-hh_2|\\&\quad+|V_{k_1,k,k_2}|^2\delta(k_1-k-k_2)|\mathcal{L}_g(\omega_{k_1}-\omega_k-\omega_{k_2})-\mathcal{L}_h(\omega_{k_1}-\omega_k-\omega_{k_2})||hh_2|.\end{aligned}$$

We set the two terms on the right-hand side of the above inequality to be J_{21} and J_{22}, respectively.

The following estimate on J_{21} is a direct consequence of the triangle inequality:

$$\begin{aligned}J_{21}&\le|k||k_1||k_2|\delta(k_1-k-k_2)\frac{\Gamma_{g,k,k_1,k_2}}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2+\Gamma_{g,k,k_1,k_2}^2}|g||g_2-h_2|\\&\quad+C|k||k_1||k_2|\delta(k_1-k-k_2)\frac{\Gamma_{g,k,k_1,k_2}}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2+\Gamma_{g,k,k_1,k_2}^2}|h_2||g-h|.\end{aligned}$$

The same argument used in Step 1 can be employed, implying the following estimate on J_{21}:

$$J_{21}\le\frac{C}{\mathfrak{M}_0[|g|]}|k_1|\delta(k-k_1-k_2)|g||g_2-h_2|+\frac{C}{\mathfrak{M}_0[|g|]}|k_1|\delta(k-k_1-k_2)|h_2||g-h|.$$

Multiplying the above inequality with $\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)$ and integrate in k, k_1 and k_2 yields

$$C\iiint_{\mathbb{R}^{3d}}J_{21}\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2\le C\left(\|g-h\|_{L^1}+\|g-h\|_{L^1_{N+1}}\right),\quad(3.8)$$

where C is a constant depending on $\left(\|g\|_{L^1}+\|g\|_{L^1_{N+1}}+\|h\|_{L^1}+\|h\|_{L^1_{N+1}}\right)$.

Now, similar to J_{12}, J_{22} can be bounded as

$$J_{22}\le C|k||k_1||k_2||hh_2|\delta(k_1-k-k_2)\frac{|\Gamma_{g,k,k_1,k_2}-\Gamma_{h,k,k_1,k_2}|}{\Gamma_{g,k,k_1,k_2}\Gamma_{h,k,k_1,k_2}}.$$

The same argument used in Step 1 can be applied and the following estimate on J_{22} then follows:

$$J_{22}\le|k||hh_2|\delta(k-k_1-k_2)\|g-h\|_{L^1}.$$

Multiplying the above inequality with $\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)$ and integrate in k, k_1 and k_2, we obtain

$$\begin{aligned}&\iiint_{\mathbb{R}^{3d}}J_{22}\left(\omega_k^N+\omega_{k_1}^N+\omega_{k_2}^N\right)dkdk_1dk_2\\&\quad\le\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]}\left(\|g-h\|_{L^1}+\|g-h\|_{L^1_{N+1}}\right),\end{aligned}\quad(3.9)$$

where C is a constant depending on $\left(\|g\|_{L^1_{N+1}}+\|h\|_{L^1_{N+1}}\right)$.

Combining (3.8) and (3.9) yields

$$\begin{aligned}\mathbb{J}_2 &\le \left(\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]} + \frac{C}{\mathfrak{M}_0[|g|]}\right)(\|g-h\|_{L^1} + \|g-h\|_{L^1_{N+1}}) \\ &\le \left(\frac{C}{\mathfrak{M}_0[|g|]\mathfrak{M}_0[|h|]} + \frac{C}{\mathfrak{M}_0[|g|]}\right)\|g-h\|_{L^1_{N+1}}.\end{aligned} \tag{3.10}$$

Putting the two estimates (3.7) and (3.10) together with (3.3) and (3.4), the conclusion of the lemma then follows. □

Proof of Proposition 3.1. The proposition now follows straightforwardly from the previous lemma. Indeed, we recall the interpolation inequality (see Lemma 2.2):

$$\|g\|_{L^1_n} \le \|g\|_{L^1_p}^{\frac{q-n}{q-p}} \|g\|_{L^1_q}^{\frac{n-p}{q-p}}$$

for $q > n > p$. Together with the boundedness of g, h in $L^1_1 \cap L^1_{N+2}$, we obtain

$$\|g-h\|_{L^1_{N+1}} \le \|g-h\|_{L^1_N}^{\frac{1}{2}} \|g-h\|_{L^1_{N+2}}^{\frac{1}{2}} \le C_M \|g-h\|_{L^1_N}^{\frac{1}{2}}.$$

Lemma 3.1 yields

$$\|Q[g]-Q[h]\|_{L^1_N} \le C_{M,M',N}\|g-h\|_{L^1_N}^{\frac{1}{2}},$$

which holds for all $N \ge 0$. The proposition follows. □

4. Proof of Theorem 1.1

We shall apply Theorem 1.2 for (1.3), which reads

$$\partial_t f = \widetilde{Q}[f], \quad \widetilde{Q}[f] := Q[f] - 2\nu|k|^2 f.$$

Fix an $N > 1$. We choose the Banach spaces $E = L^1_N(\mathbb{R}^d)$, $F = L^1_{N+3}(\mathbb{R}^d)$, endowed with the norms

$$\|f\|_E := \|f\|_{L^1_N}, \quad \|f\|_* := \|f\|_{L^1_{N+3}}.$$

We also define

$$|f|_* := \mathfrak{M}_{N+3}[f],$$

then

$$|f|_* \le \|f\|_*, \quad \forall f \in F, \quad |f+g|_* \le |f|_* + |g|_*, \quad \forall f, g \in F,$$
$$\lambda|f|_* = |\lambda f|_*, \quad \forall f \in F, \lambda \in \mathbb{R}_+,$$

and

$$|f|_* = \|f\|_{L^1_{N+3}}, \quad \forall f \in \mathcal{S}_T.$$

Moreover, condition (1.26) is automatically satisfied due to the Lebesgue dominated convergence theorem and Theorem 1.2.7.[5]

Clearly, $\mathcal{S}_T$ is a bounded and closed set with respect to the norm $\|\cdot\|_*$. By Proposition 2.2, for $f_0 \in \mathcal{S}_0 \subset \mathcal{S}_T$, solutions to (1.3) will remain in $\mathcal{S}_T$. Thus, it suffices to verify the three conditions $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$ of Theorem 1.2, then Theorem 1.1 is a consequence of Theorem 1.2. Notice that continuity condition $(\mathfrak{A})$ follows directly from Proposition 3.1, we therefore only need to verify $(\mathfrak{B})$ and $(\mathfrak{C})$.

4.1. *Condition* $(\mathfrak{B})$: *Subtangent condition*

Let f be an arbitrary element of the set $\mathcal{S}_T$. It suffices to prove the following claim: for all $\epsilon > 0$, there exists h_* depending on f and ϵ such that

$$B(f + h\widetilde{Q}[f], h\epsilon) \cap \mathcal{S}_T \neq \emptyset, \quad 0 < h < h_*. \tag{4.1}$$

For $R > 0$, let $\chi_R(k)$ be the characteristic function of the ball $B(0, R)$, and set

$$w_R := f + h\widetilde{Q}[f_R], \quad f_R(k) = \chi_R(k) f(k), \tag{4.2}$$

recalling $\widetilde{Q}[g] = Q[g] - 2\nu|k|^2 g$. We shall prove that for all $R > 0$, there exists an h_R so that w_R belongs to $\mathcal{S}_T$, for all $0 < h \le h_R$. It is clear that $w_R \in L^1(\mathbb{R}^d) \cap L^1_{N+3}(\mathbb{R}^d)$.

We now check the conditions **S1**, **S2** and **S3** in (1.24).

Condition (S1): Positivity of the set $\mathcal{S}_T$. Note that one can write $Q[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f]$, with $Q_{\text{gain}}[f] \ge 0$ and $Q_{\text{loss}}[f] = fQ_-[f]$. Since f_R is compactly supported, it is clear that $\chi_R Q_-[f_R]$ is bounded by a universal positive constant $4R$, computed in Proposition 2.1. Hence,

$$\begin{aligned} w_R &= f + h(Q[f_R] - 2\nu|k|^2 f_R) \\ &\ge f - hf_R(4R + 2\nu R^2), \end{aligned}$$

which is nonnegative, for sufficiently small h; precisely, $h < \frac{h_R}{2} := \frac{1}{2(4R+2\nu R^2)}$.

Suppose that $R > R_0$ are chosen large enough such that

$$\|\chi_R u_0\|_* > \|\chi_{R_0} u_0\|_* > R^*.$$

Let us check (1.27) for $R_0 < R$. By Proposition 2.1

$$\chi_{R_0} \frac{w_R - f}{h} = \chi_{R_0} \tilde{Q}[f_R] \ge -(4R_0 + \nu R_0^2) f_{R_0}. \tag{4.3}$$

Moreover

$$|w_R - f|_* = h|Q[f_R] - 2\nu|k|^2 f_R|_* \le C_0 \|f_R\|_*,$$

where the last inequality follows from Proposition 2.2. That leads to

$$|w_R - f|_* \le \frac{C(\lambda_1, \lambda_2) e^{(2\nu R_0^2 + 4R_0)T}}{\|f_0(k)\chi_{R_0}\|_{L^1}} \|f\|_*. \tag{4.4}$$

with $\frac{C_0(\lambda_1,\lambda_2) e^{(2\nu R_0^2 + 4R_0)T}}{\|f_0(k)\chi_{R_0}\|_{L^1}}$ computed in Proposition 2.2.

Condition (S2): Upper bound of the set $\mathcal{S}_T$. Since

$$\|f\|_* < (2R_* + 1)e^{C_* T},$$

and

$$\lim_{h\to 0} \|f - w_R\|_* = 0,$$

we can choose h_* small enough such that for $0 < h < h_*$

$$\|w_R\|_* < (2R_* + 1)e^{C_* T}.$$

Condition (S3): Lower bound of the set $\mathcal{S}_T$. Since

$$\|f\|_* > R^* e^{-C^* T}/2,$$

and

$$\lim_{h\to 0} \|f - w_R\|_* = 0,$$

we can choose h_* small enough such that

$$\|w_R\|_* > R^* e^{-C^* T}/2.$$

This proves the claim (4.1), and hence condition $(\mathfrak{A})$ is verified.

4.2. *Condition* $(\mathfrak{C})$: *One side Lipschitz condition*

By the Lebesgue's dominated convergence theorem, we have that

$$\begin{aligned}[\varphi, \phi] &= \lim_{h\to 0^-} h^{-1}(\|\phi + h\varphi\|_E - \|\phi\|_E) \\ &= \lim_{h\to 0^-} h^{-1} \int_{\mathbb{R}^d} (|\phi + h\varphi| - |\phi|)(\omega_k + \omega_k^N) dk \\ &\le \int_{\mathbb{R}^d} \varphi(k)\mathrm{sign}(\phi(k))\big(\omega_k + \omega_k^N\big) dk.\end{aligned}$$

Hence, recalling $\widetilde{Q}[f] = Q[f] - 2\nu|k|^2 f$, we estimate

$$\begin{aligned}\big[\widetilde{Q}[f] - \widetilde{Q}[g], f - g\big] &\le \int_{\mathbb{R}^d} [\widetilde{Q}[f](k) - \widetilde{Q}[g](k)]\mathrm{sign}((f-g)(k))\omega_k^N dk \\ &\le \|Q[f] - Q[g]\|_E - 2\nu\||k|^2(f-g)\|_E.\end{aligned}$$

Using Lemma 3.1 and recalling $\|\cdot\|_E = \|\cdot\|_{L^1_N}$, we have

$$\|Q[f] - Q[g]\|_E \le C_N \|f - g\|_{L^1_N}.$$

Since $C|k|^N - 2\nu|k|^{N+2}$ is always bounded by $C'|k|^N$ for $C' > 0$, we obtain

$$\big[\widetilde{Q}[f] - \widetilde{Q}[g], f - g\big] \le C_N \|f - g\|_E.$$

The condition $(\mathfrak{C})$ follows. The proof of Theorem 1.1 is complete.

5. Proof of Theorem 1.2

The proof is divided into four parts.

Part 1: According to our assumption, $\mathcal{S}_T$ is bounded by a constant C_S in the norm $\|\cdot\|$, due to the Hölder continuity property of $\mathcal{Q}[u]$,

$$\|\mathcal{Q}[u]\| \leq C_{\mathcal{Q}}, \quad \forall\, u \in \mathcal{S}_T.$$

By our assumption, for an element u in $\mathcal{S}_0 \subset \mathcal{S}_T$, there exists $\xi_u > 0$ such that for $0 < \xi < \xi_u$,

$$B(u + \xi\mathcal{Q}[u], \delta) \cap \mathcal{S}_T \backslash \{u + \xi\mathcal{Q}[u]\} \neq \emptyset,$$

for δ small enough.

For a fixed u and $\epsilon > 0$, there exists $\xi > 0$ such that $\|u - v\| \leq (C_{\mathcal{Q}} + 1)\xi$ then $\|Q(u) - Q(v)\| \leq \frac{\epsilon}{2}$. Let z be in $B\big(u + \xi Q[u], \frac{\epsilon\xi}{2}\big) \cap \mathcal{S}_T \backslash \{u + \xi\mathcal{Q}[u]\}$ satisfying

$$\left|\frac{z-u}{\xi}\right|_* \leq \frac{C_*}{2}\|u\|_*, \quad \chi_{R_0}\frac{z-u}{\xi} \geq -\chi_{R_0}\frac{C^*}{2}u,$$

and define

$$t \mapsto \Theta(t) = u + \frac{t(z-u)}{\xi}, \quad t \in [0, \xi].$$

Now, we also have the following lower bound on Θ:

$$\begin{aligned}\chi_{R_0}\Theta(t) &= \chi_{R_0}\left(u + \frac{t(z-u)}{\xi}\right)\\ &\geq \chi_{R_0}\left(1 - \frac{tC^*}{2}\right)u \\ &\geq \chi_{R_0}e^{-tC^*}\Theta(0),\end{aligned} \tag{5.1}$$

for ξ and $0 \leq t \leq \xi \leq \frac{\log 2}{C^*}$.

Hence

$$\|\chi_{R_0}\Theta(t)\|_* > \frac{R^* e^{-C^* t}}{2}. \tag{5.2}$$

We also have that

$$\begin{aligned}\|\Theta(t)\|_* = |\Theta(t)|_* &= \left|u + \frac{t(z-u)}{\xi}\right|_* \\ &\leq |u|_* + \left|\frac{t(z-u)}{\xi}\right|_* \leq |u|_* + |u|_*\frac{tC_*}{2} \\ &= \|\Theta(0)\|_*\left(1 + \frac{tC_*}{2}\right).\end{aligned}$$

We then obtain

$$\|\Theta(t)\|_* \leq (\|\Theta(0)\|_* + 1)e^{C_* t} - 1 < (2R_* + 1)e^{C_* t}. \tag{5.3}$$

Therefore, Θ maps $[0,\xi]$ into $\mathcal{S}_T$. It is straightforward that

$$\|\Theta(t)-u\| \le \left\|\frac{t(z-u)}{\xi}\right\| \le \xi\|\mathcal{Q}[u]\| + \frac{\epsilon\xi}{2} < (C_{\mathcal{Q}}+1)\xi,$$

which implies

$$\|\mathcal{Q}[\Theta(t)]-\mathcal{Q}[u]\| \le \frac{\epsilon}{2}, \quad \forall\, t\in[0,\xi].$$

Combining the above inequality and the fact that

$$\|\dot{\Theta}(t)-\mathcal{Q}[u]\| = \left\|\frac{z-u}{\xi}-\mathcal{Q}[u]\right\| \le \frac{\epsilon}{2},$$

we obtain

$$\|\dot{\Theta}(t)-\mathcal{Q}[\Theta(t)]\| \le \epsilon, \quad \forall\, t\in[0,\xi]. \tag{5.4}$$

Part 2: Let Θ be a solution to (5.4) on $[0,\xi]$ constructed in Part 1. Using the procedure of Part 1, we assume that Θ can be extended to the interval $[\tau,\tau+\tau']$.

The same arguments that lead to (5.3) imply

$$\|\Theta(\tau+t)\|_* \le \big((\|\Theta(\tau)\|_*+1)e^{C_*t}-1\big), \quad t\in[0,\tau'].$$

Combining the above inequality with (5.3) yields

$$\begin{aligned}\|\Theta(\tau+t)\|_* &\le \big(\big(\|\Theta(0)\|_*+1\big)e^{C_*\tau}-1+1\big)e^{C_*t}-1\\ &\le \big(\|\Theta(0)\|_*+1\big)e^{C_*(\tau+t)}-1\\ &< (2R_*+1)e^{C_*(\tau+t)},\end{aligned} \tag{5.5}$$

where the last inequality follows from the fact that $R_*\ge 1$.

Similar, we also have

$$\chi_{R_0}\Theta(\tau+t) \ge \chi_{R_0}e^{-(\tau+t)C^*}\Theta(0), \tag{5.6}$$

which implies

$$\|\chi_{R_0}\Theta(\tau+t)\|_* > \frac{R^*e^{-C^*(\tau+t)}}{2}. \tag{5.7}$$

Part 3: From Part 1, there exists a solution Θ to Eq. (5.4) on an interval $[0,\xi]$. Now, we have the following procedure:

- Step 1: Suppose that we can construct the solution Θ of (5.4) on $[0,\tau]$ $(\tau<T)$, where $\Theta(0)\in\mathcal{S}_0\cap B_*\big(O,R_*\big)\backslash\overline{B_*\big(O,R^*\big)}$. Since due to Part 2 $\Theta(\tau)\in\mathcal{S}_\tau$, by the same process as in Part 1 and by (5.3), (5.1), (5.2), (5.5)–(5.7) the solution Θ could be extended to $[\tau,\tau+h_\tau]$ where $\tau+h_\tau\le T$.

- Step 2: Suppose that we can construct the solution Θ of (5.4) on a series of intervals $[0,\tau_1]$, $[\tau_1,\tau_2],\ldots,[\tau_n,\tau_{n+1}],\ldots$. Since the increasing sequence $\{\tau_n\}$ is bounded by T, it has a limit, noted by τ. Moreover

$$\begin{aligned} \|\Theta(t)\|_* &\leq (\|\Theta(0)\|_*+1)e^{C_*t}-1 < (2R_*+1)e^{C_*t}, \quad \forall t\in[0,\tau), \\ \chi_{R_0}\Theta(t) &\geq \chi_{R_0}e^{-tC^*}\Theta(0), \quad \forall t\in[0,\tau), \end{aligned} \tag{5.8}$$

and

$$\|\chi_{R_0}\Theta(t)\|_* > \frac{R^*e^{-C^*t}}{2}, \quad \forall t\in[0,\tau). \tag{5.9}$$

Recall that $\|\mathcal{Q}(\Theta)\|$ is bounded by $C_{\mathcal{Q}}$ on $[\tau_n,\tau_{n+1}]$ for all $n\in\mathbb{N}$, then $\|\dot{\Theta}\|$ is bounded by $\epsilon+C_{\mathcal{Q}}$ on $[0,\tau)$. As a consequence, $\Theta(\tau)$ can be defined to be the limit of $\Theta(\tau_n)$ with respect to the norm $\|\cdot\|$. That, together with (1.26) and the fact that $\mathcal{S}_\tau$ is closed with respect to $\|\cdot\|_*$, implies that Θ is a solution of (5.4) on $[0,\tau]$. In addition (5.8) and (5.9) also hold true on $[0,\tau]$.

As a consequence, if the solution Θ can be defined on $[0,T_0)$, $T_0<T$, it could be extended to $[0,T_0]$. Now, we suppose that $[0,T_0]$ is the maximal closed interval that Θ could be defined, by Steps 1 and 2. Θ could be extended to a larger interval $[T_0,T_0+T_h]$, which means that $T=T_0$ and Θ is defined on the whole interval $[0,T]$.

Part 4: Finally, let us consider a sequence of solution $\{u^\epsilon\}$ to (5.4) on $[0,T]$. We will prove that this is a Cauchy sequence. Let $\{u^\epsilon\}$ and $\{v^\epsilon\}$ be two sequences of solutions to (5.4) on $[0,T]$. We note that u^ϵ and v^ϵ are affine functions on $[0,T]$. Moreover by the one-side Lipschitz condition

$$\begin{aligned} \frac{d}{dt}\|u^\epsilon(t)-v^\epsilon(t)\| &= \left[u^\epsilon(t)-v^\epsilon(t),\dot{u}^\epsilon(t)-\dot{v}^\epsilon(t)\right] \\ &\leq \left[u^\epsilon(t)-v^\epsilon(t),\mathcal{Q}[u^\epsilon(t)]-\mathcal{Q}[v^\epsilon(t)]\right]+2\epsilon \\ &\leq C\|u^\epsilon(t)-v^\epsilon(t)\|+2\epsilon, \end{aligned}$$

for a.e. $t\in[0,T]$, which leads to

$$\|u^\epsilon(t)-v^\epsilon(t)\| \leq 2\epsilon\frac{e^{LT}}{L}.$$

By letting ϵ tend to 0, $u^\epsilon\to u$ uniformly on $[0,T]$. It is straightforward that u is a solution to (1.28).

Acknowledgments

This work has been partially supported by the NSF grants DMS 143064 and RNMS (Ki-Net) DMS-1107444, DMS (Ki-Net) 1107291. M.-B Tran is partially supported by the NSF Grants DMS-1814149 and DMS-1854453.

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