Journal of Functional Analysis 278 (2020) 108290

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Sharp regularity for the integrability of elliptic structures

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A R T I C L E I N F O

Article history:
Received 18 November 2018
Accepted 27 July 2019
Available online 2 August 2019
Communicated by Guido De Philippis

MSC:
primary 58A30
secondary 53C15

Keywords:
Elliptic structures
Vector fields
Zygmund spaces

A B S T R A C T

As part of his celebrated Complex Frobenius Theorem, Nirenberg showed that given a smooth elliptic structure (on a smooth manifold), the manifold is locally diffeomorphic to an open subset of $\mathbb{R}^r \times \mathbb{C}^n$ (for some r and n) in such a way that the structure is locally the span of $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$; where $\mathbb{R}^r \times \mathbb{C}^n$ has coordinates $(t_1, \ldots, t_r, z_1, \ldots, z_n)$. In this paper, we give optimal regularity for the coordinate charts which achieve this realization. Namely, if the manifold has Zygmund regularity of order $s+2$ and the structure has Zygmund regularity of order $s+1$ (for some $s > 0$), then the coordinate charts may be taken to have Zygmund regularity of order $s+2$. We do this by generalizing Malgrange's proof of the Newlander-Nirenberg Theorem to this setting.

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[1] The author was partially supported by National Science Foundation Grant Nos. 1401671 and 1764265.

https://doi.org/10.1016/j.jfa.2019.108290

1. Introduction

Fix $s \in (0, \infty] \cup \{\omega\}$ and let M be a $\mathscr{C}^{s+2}$ manifold, where $\mathscr{C}^s$ denotes the Zygmund space[2] of order s, $\mathscr{C}^\infty$ denotes C^∞, $\mathscr{C}^\omega$ denotes the space of real analytic functions, and we use the convention $\infty + 2 = \infty + 1 = \infty$ and $\omega + 2 = \omega + 1 = \omega$. Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ complex elliptic structure on M; in particular, $\mathscr{L}$ is a complex sub-bundle of $\mathbb{C}TM$, is formally integrable, and $\mathscr{L}$ satisfies $\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta} = \mathbb{C}T_\zeta M$, $\forall \zeta \in M$. See Sections 2 and 3 for the full definitions. Set $n + r := \dim \mathscr{L}_\zeta$ and $r := \dim \mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta}$ (by hypothesis, n and r do not depend on ζ; see Section 3). For a Banach space $\mathscr{V}$, let $B_{\mathscr{V}}(\delta)$ denote the ball of radius $\delta > 0$, centered at 0, in $\mathscr{V}$. The main theorem of this paper is:

Theorem 1.1. *For all $\zeta \in M$, there exists an open neighborhood $V \subseteq M$ of ζ and a $\mathscr{C}^{s+2}$ diffeomorphism $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V$ such that $\forall (t, z) \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$:*

$$\operatorname{span}_{\mathbb{C}} \left\{ \left(\Phi_* \frac{\partial}{\partial t_k} \right) (\Phi(t, z)), \left(\Phi_* \frac{\partial}{\partial \bar{z}_j} \right) (\Phi(t, z)) : 1 \le k \le r, 1 \le j \le n \right\} = \mathscr{L}_{\Phi(t,z)}.$$

Here, we have given $\mathbb{R}^r \times \mathbb{C}^n$ coordinates $(t_1, \dots, t_r, z_1, \dots, z_n)$.

See Theorem 4.18 for a more abstract version of Theorem 1.1.

When $s = \omega$, Theorem 1.1 is classical. When $s = \infty$, Theorem 1.1 is a result of Nirenberg [7]; and the goal of this paper is to achieve the sharp regularity for Φ in terms of the regularity of M and $\mathscr{L}$. When $r = 0$, $\mathscr{L}$ is a complex structure, and Theorem 1.1 was proved by Malgrange [5]–in this case, the result gives the sharp regularity for the Newlander-Nirenberg Theorem [8].[3] One standard way to prove results like Theorem 1.1 for $r > 0$ is to reduce the claim to the setting of $r = 0$, and apply the Newlander-Nirenberg Theorem, where sharp regularity is known due to Malgrange's result. Unfortunately, this reduction loses a derivative (i.e., only proves Theorem 1.1 with Φ a $\mathscr{C}^{s+1}$ diffeomorphism). Instead, we proceed by adapting Malgrange's proof to directly prove Theorem 1.1.

This paper is outlined as follows. In Section 2 we introduce the basic function spaces we need. In Section 3 we give all the relevant (standard) definitions for bundles and structures. In Section 4 we define a category of manifolds in which our results are naturally stated: this is the category of manifolds endowed with an "elliptic" structure. This category contains both real and complex manifolds as *full* sub-categories. We use this to state a more abstract version of our main result (Theorem 4.18). In Section 7 we state and prove the main technical result of this paper. As discussed in Section 1.2, with

[2] For non-integer exponents, the Zygmund space agrees with the classical Hölder space. More precisely, for $m \in \mathbb{N}$, $a \in (0, 1)$, the Zygmund space $\mathscr{C}^{m+a}$ is locally the same as the Hölder space $C^{m,a}$–see Remark 2.3. However, for $a \in \{0, 1\}$, these spaces differ: $C^{m+1,0} \subsetneq C^{m,1} \subsetneq \mathscr{C}^{m+1}$.

[3] Another proof of the case $r = 0$ was later given by Webster [18]. Both [5] and [18] state results for Hölder spaces and avoid integer exponents. As is well-known, and described in the case $r = 0$ of Theorem 1.1, the results extend to integer exponents by using Zygmund spaces.

future applications in mind we keep careful track of what all the constants in Section 7 depend on. This is the heart of this paper. In Section 8 we prove the main result; i.e., Theorem 1.1 and more generally Theorem 4.18.

1.1. Some further comments

Results like Theorem 1.1 (in the smooth case, $s = \infty$) were introduced by Nirenberg to prove his more general Complex Frobenius Theorem [7]. There, one starts with a C^∞ formally integrable structure $\mathscr{L}$ on M (see Section 3). The classical (real) Frobenius Theorem applies to the essentially real sub-bundle $\mathscr{L}+\overline{\mathscr{L}}$ to foliate the ambient manifold into leaves, and $\mathscr{L}$ is an elliptic structure on each leaf. Then one can apply a result like Theorem 1.1[4] to each leaf. In this way, one can achieve a result which has the real Frobenius theorem, the Newlander-Nirenberg Theorem, and the integrability of elliptic structures as special cases (at least in the smooth setting).

In Theorem 1.1, the coordinate chart Φ is one derivative better than the bundle $\mathscr{L}$ (i.e., Φ is $\mathscr{C}^{s+2}$, while $\mathscr{L}$ is $\mathscr{C}^{s+1}$). This is the best one can hope for, since the hypotheses of Theorem 1.1 are invariant under $\mathscr{C}^{s+2}$ diffeomorphisms. However, even in the classical real Frobenius theorem, one cannot obtain appropriate coordinate charts which are one derivative better than the underlying vector fields: see [2, Example 4.5] for a very simple example involving only one vector field. Thus, we restrict attention to the setting of Theorem 1.1 (which does not involve any kind of foliation) because this seems to be a natural generality in which we can achieve this level of regularity.

As mentioned above, one common way of proving results like Theorem 1.1 is to reduce them to the Newlander-Nirenberg theorem; though this reduction unnecessarily costs a derivative. One can do this without losing a derivative by assuming the existence of some sufficiently regular vector fields which commute. This is the approach taken in [3] where results are proved for Lipschitz bundles. With our approach, we do not need to assume the existence of such vector fields (and in fact, their existence is a consequence of our result). It is possible that the methods of this paper combined with the methods of [3] could be used to prove results like the ones in that paper, without assuming the existence of such commuting vector fields.

1.2. A main motivation

A simple consequence of Theorem 1.1 is the following:

Corollary 1.2. *Fix $s \in (0,\infty] \cup \{\omega\}$ and let M be a $\mathscr{C}^{s+2}$ manifold. Let $L_1, \ldots, L_m$ be $\mathscr{C}^{s+1}$ complex vector fields on M and $X_1, \ldots, X_q$ be $\mathscr{C}^{s+1}$ real vector fields on M. Suppose:*

[4] One needs a version of Theorem 1.1 with a parameter, which can be achieved with a similar proof in the smooth case.

- *For all* $\zeta \in M$*,*

$$\mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\} = \mathbb{C}T_\zeta M.$$

- *For all* $\zeta \in M$*,* $1 \leq j, j_1, j_2 \leq m$*,* $1 \leq k, k_1, k_2 \leq q$*,*

$$\begin{aligned}&[L_{j_1}, L_{j_2}](\zeta), [L_j, X_k](\zeta), [X_{k_1}, X_{k_2}](\zeta)\\ &\quad\in \mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\}.\end{aligned}$$

- *For all* $\zeta \in M$*,*

$$\begin{aligned}&\mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\}\\ &\bigcap \mathrm{span}_{\mathbb{C}}\left\{\overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\}\\ &= \mathrm{span}_{\mathbb{C}}\left\{X_1(\zeta), \ldots, X_q(\zeta)\right\}.\end{aligned}$$

- *The map* $\zeta \mapsto \dim \mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\}$ *is constant in* ζ*.*

Set $n + r := \dim \mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\}$ *(which does not depend on* ζ *by hypothesis) and set* $r := \dim \mathrm{span}_{\mathbb{C}}\left\{X_1(\zeta), \ldots, X_q(\zeta)\right\}$ *(which also does not depend on* ζ*–see Lemma 3.6). Then,* $\forall \zeta \in M$*, there exists a neighborhood* V *of* ζ *and a* $\mathscr{C}^{s+2}$ *diffeomorphism* $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V$ *such that* $\forall \xi \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$

$$\begin{aligned}&\mathrm{span}_{\mathbb{C}}\left\{\Phi^* L_1(\xi), \ldots, \Phi^* L_m(\xi), \Phi^* X_1(\xi), \ldots, \Phi^* X_q(\xi)\right\}\\ &= \mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\},\end{aligned}$$

where we have given $\mathbb{R}^r \times \mathbb{C}^n$ *coordinates* $(t_1, \ldots, t_r, z_1, \ldots, z_n)$*.*

Proof. Apply Theorem 1.1 (see, also, Theorem 4.18) to the bundle

$$\mathscr{L}_\zeta := \mathrm{span}_{\mathbb{C}}\left\{L_1(\zeta), \ldots, L_m(\zeta), X_1(\zeta), \ldots, X_q(\zeta)\right\};$$

$\mathscr{L}$ is easily seen to be a $\mathscr{C}^{s+1}$ elliptic structure on M. See Section 3 for this terminology. □

We now consider a harder question. Let M be a C^2 manifold, and let $L_1, \ldots, L_m$ be C^1 complex vector fields on M and $X_1, \ldots, X_q$ be C^1 real vector fields on M.

Question 1.3. Fix $\zeta \in M$ and $s \in (0, \infty] \cup \{\omega\}$. When is there a neighborhood V of ζ and a C^2 diffeomorphism $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V$ such that $\Phi^* L_1, \ldots, \Phi^* L_m, \Phi^* X_1, \ldots, \Phi^* X_q$ are $\mathscr{C}^{s+1}$ vector fields on $B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ and

$$\begin{aligned}&\operatorname{span}_{\mathbb{C}}\{\Phi^*L_1(\xi),\ldots,\Phi^*L_m(\xi),\Phi^*X_1(\xi),\ldots,\Phi^*X_q(\xi)\}\\&=\operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r},\frac{\partial}{\partial \overline{z}_1},\ldots,\frac{\partial}{\partial \overline{z}_n}\right\}.\end{aligned}$$

When the vector fields are already known to be $\mathscr{C}^{s+1}$, Question 1.3 is answered by Corollary 1.2. But Question 1.3 asks more: it asks when one can pick the coordinate system Φ so that the vector fields are *more regular* than they were originally. It is not always possible to do this, but in a companion paper [14] we give necessary and sufficient conditions under which it is possible (for $s\in(1,\infty]\cup\{\omega\}$). By answering this question in a quantitative way we provide scaling maps adapted to sub-Riemannian geometries, which strengthen and generalize previous results in the case $m=0$ (i.e., all the vector fields are real) by Nagel, Stein, and Wainger [9], Tao and Wright [17], and the author [11]. The case when $m=0$ was covered in the series [10,12,13].

The case when $q=0$ of Question 1.3 is particularly interesting. In this case, the coordinate chart Φ can be thought of as a holomorphic coordinate system. When one turns to the quantitative theory discussed above, this allows us to create holomorphic analogs of the sub-Riemannian scaling maps introduced by Nagel, Stein, and Wainger [9]. In this way we can study sub-Riemannian geometries on complex manifolds, which are adapted to the complex structure. We call these sub-Hermitian geometries.

The main technical result of this paper (Theorem 7.3) is a key step in developing the theory in the companion work [14]. Because of this, it is important for our future applications that we keep track of the dependance various constants in Theorem 7.3. For this purpose we introduce several function spaces and definitions that we would not otherwise have to. This makes the statement of Theorem 7.3 a bit more involved than it would have to be to prove the main results of this paper; though, other than some bookkeeping, the proof is no more difficult. Because of its quantitative nature, it is possible Theorem 7.3 will be more useful in future applications than the "main results" of this paper.

2. Function spaces

Let $\Omega\subset\mathbb{R}^n$ be a connected, open set (we will almost always be considering the case when Ω is a ball in $\mathbb{R}^n$). We have the following classical spaces of functions on Ω:

$$\begin{aligned}&C(\Omega)=C^0(\Omega):=\{f:\Omega\to\mathbb{C}\bigm| f\text{ is continuous and bounded}\},\\&\|f\|_{C(\Omega)}=\|f\|_{C^0(\Omega)}:=\sup_{x\in\Omega}|f(x)|.\end{aligned}$$

For $m\in\mathbb{N}$, (we use the convention $0\in\mathbb{N}$)

$$C^m(\Omega):=\{f\in C(\Omega)\bigm|\partial_x^\alpha f\in C(\Omega),\forall|\alpha|\leq m\},\quad \|f\|_{C^m(\Omega)}:=\sum_{|\alpha|\leq m}\|\partial_x^\alpha f\|_{C(\Omega)}.$$

Next we define the classical Hölder spaces. For $s\in[0,1]$,

$$\|f\|_{C^{0,s}(\Omega)} := \|f\|_{C(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-s}|f(x)-f(y)|,$$
$$C^{0,s}(\Omega) := \{f \in C(\Omega) : \|f\|_{C^{0,s}(\Omega)} < \infty\}. \tag{2.1}$$

For $m \in \mathbb{N}$, $s \in [0,1]$,

$$\|f\|_{C^{m,s}} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{C^{0,s}}, \quad C^{m,s}(\Omega) := \{f \in C^m(\Omega) : \|f\|_{C^{m,s}(\Omega)} < \infty\}.$$

Next, we turn to the classical Zygmund spaces. Given $h \in \mathbb{R}^n$ define $\Omega_h := \{x \in \mathbb{R}^n : x, x+h, x+2h \in \Omega\}$. For $s \in (0,1]$ set

$$\|f\|_{\mathscr{C}^s(\Omega)} := \|f\|_{C^{0,s/2}(\Omega)} + \sup_{\substack{0\neq h\in\mathbb{R}^n \\ x\in\Omega_h}} |h|^{-s}|f(x+2h)-2f(x+h)+f(x)|,$$
$$\mathscr{C}^s(\Omega) := \{f \in C(\Omega) : \|f\|_{\mathscr{C}^s(\Omega)} < \infty\}.$$

For $m \in \mathbb{N}$, $s \in (0,1]$, set

$$\|f\|_{\mathscr{C}^{m+s}(\Omega)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{\mathscr{C}^s(\Omega)}, \quad \mathscr{C}^{m+s}(\Omega) := \{f \in C^m(\Omega) : \|f\|_{\mathscr{C}^{m+s}(\Omega)} < \infty\}.$$

We set

$$\mathscr{C}^\infty(\Omega) := \bigcap_{s>0} \mathscr{C}^s(\Omega), \quad C^\infty(\Omega) := \bigcap_{m\in\mathbb{N}} C^m(\Omega).$$

It is straightforward to verify that for a ball B, $\mathscr{C}^\infty(B) = C^\infty(B)$.

Finally, we let $\mathscr{C}^\omega(\Omega)$ be the space of real analytic functions on Ω.

If $\mathscr{V}$ is a Banach space, we define the same spaces taking values in $\mathscr{V}$ in the obvious way, and denote these spaces by $C(\Omega;\mathscr{V})$, $C^m(\Omega;\mathscr{V})$, $C^{m,s}(\Omega;\mathscr{V})$, $\mathscr{C}^s(\Omega;\mathscr{V})$, and $\mathscr{C}^\omega(\Omega;\mathscr{V})$. Given a complex vector field X on Ω, we identify $X = \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j}$ with the function $(a_1,\ldots,a_n) : \Omega \to \mathbb{C}^n$. It therefore makes sense to consider quantities like $\|X\|_{\mathscr{C}^s(\Omega;\mathbb{C}^n)}$. When $\mathscr{V}$ is clear from context, we sometimes suppress it and write, e.g., $\|f\|_{\mathscr{C}^s(\Omega)}$ instead of $\|f\|_{\mathscr{C}^s(\Omega;\mathscr{V})}$ for readability considerations.

Remark 2.1. The term $\|f\|_{C^{0,s/2}}$ in the definition of $\|f\|_{\mathscr{C}^s}$ is somewhat unusual, and is usually replaced by $\|f\|_{C^0}$. However, if Ω is a bounded Lipschitz domain these two choices yield equivalent norms: this is a simple consequence of [16, Theorem 1.118 (i)]. The definition we have chosen is somewhat more convenient to work with.

Definition 2.2. For $s \in (0,\infty] \cup \{\omega\}$, we say $f \in \mathscr{C}^s_{\mathrm{loc}}(\Omega)$ if $\forall x \in \Omega$, there exists an open ball $B \subseteq \Omega$, centered at x, with $f\big|_B \in \mathscr{C}^s(B)$.

Remark 2.3. If Ω is a bounded Lipschitz domain, $m \in \mathbb{N}$, $s \in (0,1)$, the spaces $C^{m,s}(\Omega)$ and $\mathscr{C}^{m+s}(\Omega)$ are the same–see [16, Theorem 1.118 (i)]. However, if $s \in \{0,1\}$, these spaces differ. As a consequence for *any* open set $\Omega \subseteq \mathbb{R}^n$, for $m \in \mathbb{N}$, $s \in (0,1)$, we have $\mathscr{C}^{m+s}_{\mathrm{loc}}(\Omega)$ equals the space of functions which are locally in $C^{m,s}$.

Remark 2.4. $\mathscr{C}^\omega$ and $\mathscr{C}^\omega_{\mathrm{loc}}$ denote the same thing. However, for $s \in (0,\infty]$, $\mathscr{C}^s$ and $\mathscr{C}^s_{\mathrm{loc}}$ are not the same. Since for any ball B we have $\mathscr{C}^\infty(B) = C^\infty(B)$, the space $\mathscr{C}^\infty_{\mathrm{loc}}(\Omega)$ corresponds with the usual space of smooth functions on Ω.

2.1. Manifolds

In this paper we use $\mathscr{C}^s$ manifolds; the definition is exactly what one would expect, though a little care is needed due to the subtleties of Zygmund spaces.[5] We present the relevant (standard) definitions here.

Definition 2.5. Let $U_1 \subseteq \mathbb{R}^{n_1}$ and $U_2 \subseteq \mathbb{R}^{n_2}$ be open sets. For $s \in (0,\infty] \cup \{\omega\}$, we say $f : U_1 \to U_2$ is a $\mathscr{C}^s_{\mathrm{loc}}$ map if $f \in \mathscr{C}^s_{\mathrm{loc}}(U_1; \mathbb{R}^{n_2})$.

Lemma 2.6. *Let $U_1 \subseteq \mathbb{R}^{n_1}$, $U_2 \subseteq \mathbb{R}^{n_2}$, and $U_3 \subseteq \mathbb{R}^{n_3}$ be open sets. For $s_1 \in (0,\infty] \cup \{\omega\}$, $s_2 \geq s_1$, $s_2 \in (1,\infty] \cup \{\omega\}$, if $f_1 : U_1 \to U_2$ is a $\mathscr{C}^{s_1}_{\mathrm{loc}}$ map and $f_2 : U_2 \to U_3$ is a $\mathscr{C}^{s_2}_{\mathrm{loc}}$ map, then $f_2 \circ f_1 : U_1 \to U_3$ is a $\mathscr{C}^{s_1}_{\mathrm{loc}}$ map.*

Proof. When $s_1 \in \{\infty, \omega\}$, the result is obvious. For $s_1 \in (0,\infty)$, because the notion of being a $\mathscr{C}^s_{\mathrm{loc}}$ map is local, it suffices to check $f_1 \circ f_2$ is in $\mathscr{C}^{s_1}$ on sufficiently small balls. This is described in Lemma 5.3, below. □

Lemma 2.7. *For $s \in (1,\infty] \cup \{\omega\}$ if $f : U_1 \to U_2$ is a $\mathscr{C}^s_{\mathrm{loc}}$ map which is also a C^1 diffeomorphism, then $f^{-1} : U_2 \to U_1$ is a $\mathscr{C}^s_{\mathrm{loc}}$ map.*

Proof. For $s \in \{\infty, \omega\}$ this is standard. For $s \in (1,\infty)$ it suffices to check f^{-1} is in $\mathscr{C}^s$ when restricted to sufficiently small balls, because the result is local. This is described in Lemma 5.4, below. □

Definition 2.8. Fix $s \in (1,\infty] \cup \{\omega\}$ and let M be a topological space. We say $\{(\phi_\alpha, V_\alpha) : \alpha \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) is a $\mathscr{C}^s$ atlas of dimension n if $\{V_\alpha : \alpha \in \mathcal{I}\}$ is an open cover for M, $\phi_\alpha : V_\alpha \to U_\alpha$ is a homeomorphism where $U_\alpha \subseteq \mathbb{R}^n$ is open, and $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(V_\beta \cap V_\alpha) \to U_\beta$ is a $\mathscr{C}^s_{\mathrm{loc}}$ map.

[5] For example, one must define the Zygmund maps in the right way to ensure that the composition of two Zygmund maps is again a Zygmund map.

Definition 2.9. For $s \in (1, \infty] \cup \{\omega\}$ a $\mathscr{C}^s$ manifold of dimension n is a paracompact[6] topological space M endowed with a $\mathscr{C}^s$ atlas of dimension n.

Remark 2.10. Let $U \subseteq \mathbb{R}^n$ be an open set. U is naturally a $\mathscr{C}^\omega$ manifold of dimension n; where we take the atlas consisting of a single coordinate chart (namely, the identity map $U \to U$). We henceforth give open sets this manifold structure.

Remark 2.11. In particular, a $\mathscr{C}^s$ manifold is a C^m manifold, for any $m < s$. In light of Remark 2.4, $\mathscr{C}^\infty$ and C^∞ manifolds are the same.

Definition 2.12. For $s \in (0, \infty] \cup \{\omega\}$ and let M and N be $\mathscr{C}^{s+1}$ manifolds with $\mathscr{C}^{s+1}$ atlases $\{(\phi_\alpha, V_\alpha)\}$ and $\{(\psi_\beta, W_\beta)\}$, respectively. We say $f : M \to N$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ map if $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ map, $\forall \alpha, \beta$.

Lemma 2.13. *For $s \in (0, \infty] \cup \{\omega\}$, suppose M_1, M_2, and M_3 are $\mathscr{C}^{s+1}$ manifolds and $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ are $\mathscr{C}^{s+1}_{\mathrm{loc}}$ maps. Then, $f_2 \circ f_1 : M_1 \to M_3$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ map.*

Proof. This follows from Lemma 2.6. □

Lemma 2.14. *Suppose $s \in (0, \infty] \cup \{\omega\}$, M_1 and M_2 are $\mathscr{C}^{s+1}$ manifolds, and $f : M_1 \to M_2$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ map which is also a C^1 diffeomorphism. Then $f^{-1} : M_2 \to M_1$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ map.*

Proof. This follows from Lemma 2.7. □

Definition 2.15. Suppose $s \in (0, \infty] \cup \{\omega\}$, and M_1 and M_2 are $\mathscr{C}^{s+1}$ manifolds. We say $f : M_1 \to M_2$ is a $\mathscr{C}^{s+1}$ diffeomorphism if $f : M_1 \to M_2$ is invertible and $f : M_1 \to M_2$ and $f^{-1} : M_2 \to M_1$ are $\mathscr{C}^{s+1}_{\mathrm{loc}}$ maps.

Remark 2.16. For $s \in (0, \infty] \cup \{\omega\}$, if M is a $\mathscr{C}^{s+1}$ manifold with $\mathscr{C}^{s+1}$ atlas $\{(\phi_\alpha, V_\alpha)\}$, as described in Definition 2.8, then the maps $\phi_\alpha : V_\alpha \to U_\alpha$ are $\mathscr{C}^{s+1}$ diffeomorphisms, where U_α is given the $\mathscr{C}^\omega$ manifold structure described in Remark 2.10. This follows from Lemma 2.14.

Because a $\mathscr{C}^{s+1}$ manifold is a C^1 manifold, it makes sense to talk about vector fields on such a manifold.

Definition 2.17. For $s \in (0, \infty] \cup \{\omega\}$ let M be a $\mathscr{C}^{s+1}$ manifold of dimension n with $\mathscr{C}^{s+1}$ atlas $\{(\phi_\alpha, V_\alpha)\}$; here $\phi_\alpha : V_\alpha \to U_\alpha$ is a $\mathscr{C}^{s+1}$ diffeomorphism and $U_\alpha \subseteq \mathbb{R}^n$ is open. We say a vector field X on M is a $\mathscr{C}^s$ vector field if $(\phi_\alpha)_* X \in \mathscr{C}^s_{\mathrm{loc}}(U_\alpha; \mathbb{R}^n)$, $\forall \alpha$.

[6] We do not use paracompactness in this paper, so the reader who wishes to define manifolds without requiring paracompactness is free to do this throughout this paper.

3. Bundles

In this section, we include the standard definitions we use concerning bundles. In the smooth case, these definitions are contained in [15,1], and we follow these sources. Fix $s \in (0, \infty] \cup \{\omega\}$, and let M be a $\mathscr{C}^{s+2}$ manifold. We let $\mathbb{C}TM$ denote the complexified tangent bundle of M: $\mathbb{C}TM := TM \otimes_{\mathbb{R}} \mathbb{C}$ (see Appendix A for some comments on the complexification of real vector spaces).

Definition 3.1. A $\mathscr{C}^{s+1}$ sub-bundle $\mathscr{L}$ of $\mathbb{C}TM$ of rank $m \in \mathbb{N}$ is a disjoint union

$$\mathscr{L} = \bigcup_{\zeta \in M} \mathscr{L}_\zeta \subseteq \mathbb{C}TM$$

such that:

- $\forall \zeta \in M$, $\mathscr{L}_\zeta$ is an m-dimensional vector subspace of $\mathbb{C}T_\zeta M$.
- $\forall \zeta_0 \in M$, there exists an open neighborhood $U \subseteq M$ of ζ_0 and a finite collection of complex $\mathscr{C}^{s+1}$ vector fields $L_1, \ldots, L_K$ on U, such that $\forall \zeta \in U$,

$$\mathrm{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_K(\zeta)\} = \mathscr{L}_\zeta.$$

Definition 3.2. For a $\mathscr{C}^{s+1}$ sub-bundle $\mathscr{L}$ of $\mathbb{C}TM$, we define $\overline{\mathscr{L}}$ by $\overline{\mathscr{L}_\zeta} = \{\overline{z} : z \in \mathscr{L}_\zeta\}$. It is easy to see that $\overline{\mathscr{L}}$ is a $\mathscr{C}^{s+1}$ sub-bundle of $\mathbb{C}TM$.

Definition 3.3. Let $W \subseteq M$ be open, L a complex vector field on W, and $\mathscr{L}$ a $\mathscr{C}^{s+1}$ sub-bundle of $\mathbb{C}TM$. We say L is a section of $\mathscr{L}$ over W if $\forall \zeta \in W$, $L(\zeta) \in \mathscr{L}_\zeta$. We say L is a $\mathscr{C}^{s+1}$ section of $\mathscr{L}$ over W is if L is a section of $\mathscr{L}$ over W and L is a $\mathscr{C}^{s+1}$ complex vector field on W.

Definition 3.4. Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ sub-bundle of $\mathbb{C}TM$. We say $\mathscr{L}$ is a $\mathscr{C}^{s+1}$ formally integrable structure if the following holds. For all $W \subseteq M$ open, and all $\mathscr{C}^{s+1}$ sections L_1 and L_2 of $\mathscr{L}$ over W, we have $[L_1, L_2]$ is a section of $\mathscr{L}$ over W.

Definition 3.5. Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ formally integrable structure on M. We say $\mathscr{L}$ is a $\mathscr{C}^{s+1}$ elliptic structure if $\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta} = \mathbb{C}T_\zeta M$, $\forall \zeta \in M$.

Lemma 3.6. *Let $\mathscr{L}$ be an elliptic structure on M. Then, the map $\zeta \mapsto \dim(\mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta})$ is constant, $M \to \mathbb{N}$.*

Proof. By Lemma A.1, $\dim(\mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta}) = 2\dim(\mathscr{L}_\zeta) - \dim(\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta})$. The definition of a sub-bundle implies $\zeta \mapsto \dim(\mathscr{L}_\zeta)$ is constant, and the definition of an elliptic structure implies $\dim(\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta}) = \dim \mathbb{C}T_\zeta M = \dim M$, $\forall \zeta \in M$. The result follows. □

Let $\mathscr{L}$ be an elliptic structure on M. Set $r := \dim(\mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta})$ and $n + r := \dim(\mathscr{L}_\zeta)$. By the definition of a sub-bundle and Lemma 3.6, n and r are constant in ζ.

Definition 3.7. Let $\mathscr{L}$ be a elliptic structure on M and let n and r be as above. We say $\mathscr{L}$ is an elliptic structure of dimension (r, n).

Remark 3.8. Let $\mathscr{L}$ be an elliptic structure of dimension (r, n). Then, $\dim M = \dim \mathbb{C}T_\zeta M = \dim(\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta}) = 2n + r$, where in the last equality we have used Lemma A.1.

4. E-manifolds

It is convenient to state our results in a category of manifolds which contain real manifolds and complex manifolds as full sub-categories. We define these manifolds here, and call them E-manifolds.[7]

Remark 4.1. "E" in the name E-manifolds stands for "elliptic". Indeed, using the terminology of [15, Definition I.2.3], a complex manifold is a manifold endowed with a complex structure, a CR-manifold is a manifold endowed with a CR structure, and (as we will see in Theorem 4.18) an E-manifold is a manifold endowed with an elliptic structure; see Definition 4.16. Unfortunately, the name "elliptic manifold" is already taken by an unrelated concept.

Definition 4.2. Let $U_1 \subseteq \mathbb{R}^{r_1} \times \mathbb{C}^{n_1}$ and $U_2 \subseteq \mathbb{R}^{r_2} \times \mathbb{C}^{n_2}$ be open sets. We give $\mathbb{R}^{r_1} \times \mathbb{C}^{n_1}$ coordinates (t, z) and $\mathbb{R}^{r_2} \times \mathbb{C}^{n_2}$ coordinates (u, w). We say a C^1 map $f : U_1 \to U_2$ is an E-map if

$$
\begin{aligned}
&df(t,z)\frac{\partial}{\partial t_k}, df(t,z)\frac{\partial}{\partial \overline{z}_j} \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{r_2}}, \frac{\partial}{\partial \overline{w}_1}, \ldots, \frac{\partial}{\partial \overline{w}_{n_2}} \right\}, \\
&\quad \forall (t,z) \in U_1, 1 \le k \le r_1, 1 \le j \le n_1.
\end{aligned}
$$

For $s \in (0, \infty] \cup \{\omega\}$, we say $f : U_1 \to U_2$ is a $\mathscr{C}^s_{\mathrm{loc}}$ E-map if it is an E-map which is also a $\mathscr{C}^s_{\mathrm{loc}}$ map.

Remark 4.3. Suppose $U_1, U_2 \subseteq \mathbb{R}^r \times \mathbb{C}^n$ and $f : U_1 \to U_2$ is an E-map which is also a C^1-diffeomorphism. Then, $f^{-1} : U_2 \to U_1$ is an E-map.

Remark 4.4. Note that when $r_1 = r_2 = 0$, if $U_1 \subseteq \mathbb{R}^0 \times \mathbb{C}^{n_1} \cong \mathbb{C}^{n_1}$, $U_2 \subseteq \mathbb{R}^0 \times \mathbb{C}^{n_2} \cong \mathbb{C}^{n_2}$, then $f : U_1 \to U_2$ is an E-map if and only if it is holomorphic.

[7] The manifold structure we discuss here is well-known to experts, but we could not find a name for the category of such manifolds, and decided to call them E-manifolds for lack of a better name.

Lemma 4.5. *Let $U_1 \subseteq \mathbb{R}^{r_1} \times \mathbb{C}^{n_1}$, $U_2 \subseteq \mathbb{R}^{r_2} \times \mathbb{C}^{n_2}$, and $U_3 \subseteq \mathbb{R}^{r_3} \times \mathbb{C}^{n_3}$ be open sets, and let $s \in (0,\infty] \cup \{\omega\}$. Suppose $f_1 : U_1 \to U_2$ and $f_2 : U_2 \to U_3$ are $\mathscr{C}^{s+1}_{\text{loc}}$ E-maps. Then $f_2 \circ f_1 : U_1 \to U_3$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map.*

Proof. That $f_2 \circ f_1$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ map follows from Lemma 2.6. That it is an E-map follows from the chain rule. □

Definition 4.6. Let M be a topological space and fix $n, r \in \mathbb{N}$, $s \in (1,\infty] \cup \{\omega\}$. We say $\{(\phi_\alpha, V_\alpha) : \alpha \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) is a $\mathscr{C}^s$ E-atlas of dimension (r,n) if $\{V_\alpha : \alpha \in \mathcal{I}\}$ is an open cover for M, $\phi_\alpha : V_\alpha \to U_\alpha$ is a homeomorphism where $U_\alpha \subseteq \mathbb{R}^r \times \mathbb{C}^n$ is open, and $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(V_\beta \cap V_\alpha) \to U_\beta$ is a $\mathscr{C}^s_{\text{loc}}$ E-map, $\forall \alpha, \beta$.

Definition 4.7. A $\mathscr{C}^s$ E-manifold M of dimension (r,n) is a paracompact[8] topological space M endowed with a $\mathscr{C}^s$ E-atlas of dimension (r,n).

Remark 4.8. One may analogously define C^m E-manifolds in the obvious way. C^∞ E-manifolds and $\mathscr{C}^\infty$ E-manifolds are the same.

Definition 4.9. For $s \in (0,\infty] \cup \{\omega\}$, let M and N be $\mathscr{C}^{s+1}$ E-manifolds with $\mathscr{C}^{s+1}$ E-atlases $\{(\phi_\alpha, V_\alpha)\}$ and $\{(\psi_\beta, W_\beta)\}$, respectively. We say $f : M \to N$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map if $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map, $\forall \alpha, \beta$.

Lemma 4.10. *For $s \in (0,\infty] \cup \{\omega\}$, let M_1, M_2, and M_3 be $\mathscr{C}^{s+1}$ E-manifolds and $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be $\mathscr{C}^{s+1}_{\text{loc}}$ E-maps. Then, $f_2 \circ f_1 : M_1 \to M_3$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map.*

Proof. This follows from Lemma 4.5. □

Lemma 4.11. *For $s \in (0,\infty] \cup \{\omega\}$, let M_1 and M_2 be $\mathscr{C}^{s+1}$ E-manifolds and let $f : M_1 \to M_2$ be a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map which is also a C^1 diffeomorphism. Then, $f^{-1} : M_2 \to M_1$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ E-map.*

Proof. That $f^{-1} : M_2 \to M_1$ is a $\mathscr{C}^{s+1}_{\text{loc}}$ map follows from Lemma 2.14. That $f^{-1} : M_2 \to M_1$ is an E-map follows from Remark 4.3. □

Definition 4.12. Suppose $s \in (0,\infty] \cup \{\omega\}$, M_1 and M_2 are $\mathscr{C}^{s+1}$ E-manifolds. We say $f : M_1 \to M_2$ is a $\mathscr{C}^{s+1}$ E-diffeomorphism if $f : M_1 \to M_2$ is invertible and $f : M_1 \to M_2$ and $f^{-1} : M_2 \to M_1$ are $\mathscr{C}^{s+1}_{\text{loc}}$ E-maps.

Remark 4.13. For $s > 1$, the category of $\mathscr{C}^s$ E-manifolds, whose objects are $\mathscr{C}^s$ E-manifolds and morphisms are $\mathscr{C}^s_{\text{loc}}$ E-maps, contains both $\mathscr{C}^s$ real manifolds and complex

[8] We do not use paracompactness in this paper; so the reader who does not require that manifolds be paracompact is free to do so in this paper.

manifolds as full subcategories. The real manifolds of dimension r are those with E-dimension $(r, 0)$, while the complex manifolds of complex dimension n are those with E-dimension $(0, n)$. That complex manifolds (with morphisms given by holomorphic maps) embed as a *full* subcategory follows from Remark 4.4. The isomorphisms in the category of $\mathscr{C}^s$ E-manifolds are the $\mathscr{C}^s$ E-diffeomorphisms.

Remark 4.14. Note that open subsets of $\mathbb{R}^r \times \mathbb{C}^n$ are $\mathscr{C}^\omega$ E-manifolds of dimension (r, n), by using the atlas consisting of one coordinate chart (the identity map). Henceforth, we give such sets this E-manifold structure.

Remark 4.15. An E-manifold of dimension (r, n) has an underlying manifold structure of dimension $2n + r$, and it therefore makes sense to talk about any of the usual objects on manifolds with respect to an E-manifold.

For $s \in (0, \infty] \cup \{\omega\}$, on a $\mathscr{C}^{s+2}$ E-manifold M of dimension (r, n), there is a naturally associated $\mathscr{C}^{s+1}$ elliptic structure on M of dimension (r, n) defined as follows. Let (ϕ_α, V_α) be an E-atlas for M. For $\zeta \in M$, we have $\zeta \in V_\alpha$ for some α. We set:

$$\begin{aligned}\mathscr{L}_\zeta := \operatorname{span}_{\mathbb{C}} \Big\{ & d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta)) \frac{\partial}{\partial t_1}, \ldots, d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta)) \frac{\partial}{\partial t_r}, \\ & d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta)) \frac{\partial}{\partial \overline{z}_1}, \ldots, d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta)) \frac{\partial}{\partial \overline{z}_n} \Big\}.\end{aligned}$$

It is straightforward to check that $\mathscr{L}_\zeta \subseteq \mathbb{C}T_\zeta M$ is well-defined[9] and $\mathscr{L} = \bigcup_{\zeta \in M} \mathscr{L}_\zeta$ is a $\mathscr{C}^{s+1}$ elliptic structure on M of dimension (r, n).

Definition 4.16. We call $\mathscr{L}$ the elliptic structure associated to the E-manifold M.

Lemma 4.17. *Suppose M and $\widehat{M}$ are $\mathscr{C}^{s+2}$ E-manifolds with associated elliptic structures $\mathscr{L}$ and $\widehat{\mathscr{L}}$. Then a $\mathscr{C}^{s+2}_{\mathrm{loc}}$ map $f : M \to \widehat{M}$ is a $\mathscr{C}^{s+2}_{\mathrm{loc}}$ E-map if and only if $df(\zeta)\mathscr{L}_\zeta \subseteq \widehat{\mathscr{L}}_{f(\zeta)}$, $\forall \zeta \in M$.*

Proof. This follows immediately from the definitions. □

The main result of this paper (Theorem 1.1) can be rephrased as follows.

Theorem 4.18. *Let $s \in (0, \infty] \cup \{\omega\}$ and let M be a $\mathscr{C}^{s+2}$ manifold. For each $\zeta \in M$, let $\mathscr{L}_\zeta$ be a vector subspace of $\mathbb{C}T_\zeta M$, and let $\mathscr{L} = \bigcup_{\zeta \in M} \mathscr{L}_\zeta$. The following are equivalent:*

(i) *There is a $\mathscr{C}^{s+2}$ E-manifold structure on M, compatible with its $\mathscr{C}^{s+2}$ structure, such that $\mathscr{L}$ is the $\mathscr{C}^{s+1}$ elliptic structure associated to M.*

[9] I.e., $\mathscr{L}_\zeta$ does not depend on which α we pick with $\zeta \in V_\alpha$.

(ii) *$\mathscr{L}$ is a $\mathscr{C}^{s+1}$ elliptic structure.*

Moreover, under these conditions, the E-manifold structure given in (i) is unique in the sense that if M is given another $\mathscr{C}^{s+2}$ E-manifold structure, compatible with its $\mathscr{C}^{s+2}$ structure, with respect to which $\mathscr{L}$ is the associated elliptic sub-bundle, then the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ E-diffeomorphism, between these two $\mathscr{C}^{s+2}$ E-manifold structures on M.

This paper is devoted to proving Theorem 4.18.

Remark 4.19. In Theorem 4.18, following standard terminology, we have used the word "structure" in two different ways. When we speak of a $\mathscr{C}^{s+2}$ E-manifold "structure" on M we mean the equivalence class of $\mathscr{C}^{s+2}$ E-atlases (where two atlases on M are equivalent if the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ E-diffeomorphism). When we speak of an elliptic "structure," we are referring to Definition 3.5. This double use of terminology is justified by Theorem 4.18 which shows that giving an E-manifold structure is equivalent to giving an elliptic structure.

Remark 4.20. When $s \in \{\infty, \omega\}$, Theorem 4.18 is well-known. Our proof yields these cases as simple corollaries, so we include them.

Remark 4.21. In the special case $\mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta} = \{0\}, \forall \zeta \in M$, Theorem 4.18 is the Newlander-Nirenberg Theorem [8], with sharp regularity as proved by Malgrange [5]. In this case E-manifolds are complex manifolds–see Remark 4.13.

5. Function spaces revisited

In this section we present some basic properties of the function spaces introduced in Section 2. Fix $\Omega \subseteq \mathbb{R}^n$ an open set.

Proposition 5.1. *For $s \in (0, \infty] \cup \{\omega\}$, $\mathscr{C}^s(\Omega)$ is an algebra: if $f, g \in \mathscr{C}^s(\Omega)$, then $fg \in \mathscr{C}^s(\Omega)$. Moreover, for $s \in (0, \infty)$ and $f, g \in \mathscr{C}^s(\Omega)$,*

$$\|fg\|_{\mathscr{C}^s(\Omega)} \leq C_s \|f\|_{\mathscr{C}^s(\Omega)} \|g\|_{\mathscr{C}^s(\Omega)}.$$

For $s \in (0, \infty] \cup \{\omega\}$, these spaces have multiplicative inverses for functions which are bounded away from zero: if $f \in \mathscr{C}^s(\Omega)$ with $\inf_x |f(x)| \geq c_0 > 0$, then $f(x)^{-1} = \frac{1}{f(x)} \in \mathscr{C}^s(\Omega)$. Moreover if $s \in (0, \infty)$ and $\inf_x |f(x)| \geq c_0 > 0$ then

$$\|f(x)^{-1}\|_{\mathscr{C}^s(\Omega)} \leq C,$$

where C can be chosen to depend only on s, n, c_0, and an upper bound for $\|f\|_{\mathscr{C}^s(\Omega)}$.

Proof. For $s = \omega$, this is standard. For $s \in (0, \infty]$, this is standard and contained in [10, Proposition 8.3]. $\square$

Remark 5.2. For $s \in (0, \infty] \cup \{\omega\}$, suppose $A \in \mathscr{C}^s(\Omega; \mathbb{M}^{k\times k})$ is such that $\inf_{t\in\Omega} |\det A(t)| > 0$. Then it follows that $A^{-1} \in \mathscr{C}^s(\Omega; \mathbb{M}^{k\times k})$. Indeed, this follows from Proposition 5.1 using the cofactor representation of A^{-1}. When $s \in (0, \infty)$, $\|A^{-1}\|_{\mathscr{C}^s(\Omega)}$ can be bounded in terms of s, k, n, a lower bound for $\inf_{t\in\Omega} |\det A(t)| > 0$, and an upper bound for $\|A\|_{\mathscr{C}^s(\Omega)}$.

Lemma 5.3. *Let $D_1, D_2 > 0$, $s_1 \in (0, \infty)$, $s_2 \geq s_1$, $s_2 \in (1, \infty)$, $f \in \mathscr{C}^{s_1}(B_{\mathbb{R}^n}(D_1))$, $g \in \mathscr{C}^{s_2}(B_{\mathbb{R}^m}(D_2); \mathbb{R}^n)$ with $g(B_{\mathbb{R}^m}(D_2)) \subseteq B_{\mathbb{R}^n}(D_1)$. Then, $f \circ g \in \mathscr{C}^{s_1}(B_{\mathbb{R}^m}(D_2))$ and $\|f \circ g\|_{\mathscr{C}^{s_1}(B_{\mathbb{R}^m}(D_2))} \leq C\|f\|_{\mathscr{C}^{s_1}(B_{\mathbb{R}^n}(D_1))}$, where C can be chosen to depend only on s_1, s_2, D_1, D_2, m, n, and an upper bound for $\|g\|_{\mathscr{C}^{s_2}(B_{\mathbb{R}^m}(D_2))}$.*

Proof. This is standard and proved in [12]. $\square$

Lemma 5.4. *Fix $s \in (1, \infty)$, $D_1, D_2 > 0$. Suppose $H \in \mathscr{C}^s(B_{\mathbb{R}^n}(D_1); \mathbb{R}^n)$ is such that $B_{\mathbb{R}^n}(D_2) \subseteq H(B_{\mathbb{R}^n}(D_1))$, $H : B_{\mathbb{R}^n}(D_1) \to H(B_{\mathbb{R}^n}(D_1))$ is a homeomorphism, and $\inf_{t\in B_{\mathbb{R}^n}(D_1)} |\det dH(t)| \geq c_0 > 0$. Then $H^{-1} \in \mathscr{C}^s(B_{\mathbb{R}^n}(D_2); \mathbb{R}^n)$, with $\|H^{-1}\|_{\mathscr{C}^s(B_{\mathbb{R}^n}(D_2);\mathbb{R}^n)} \leq C$, where C can be chosen to depend only on n, s, D_1, D_2, c_0, and an upper bound for $\|H\|_{\mathscr{C}^s(B_{\mathbb{R}^n}(D_1);\mathbb{R}^n)}$.*

Proof. This is standard and proved in [12]. $\square$

5.1. Spaces of real analytic functions

For the proofs that follow, it is convenient to introduce two, closely related, Banach spaces of real analytic functions. For $s > 0$, we define $\mathscr{A}^{n,s}$ to be the space of those $f \in C(B_{\mathbb{R}^n}(s))$ such that $f(t) = \sum_{\alpha\in\mathbb{N}^n} \frac{c_\alpha}{\alpha!} t^\alpha$, $\forall t \in B_{\mathbb{R}^n}(s)$, where

$$\|f\|_{\mathscr{A}^{n,s}} := \sum_{\alpha\in\mathbb{N}^n} \frac{|c_\alpha|}{\alpha!} s^{|\alpha|} < \infty.$$

We now turn to the other Banach space of real analytic functions we use. Let $\Omega \subset \mathbb{C}^N$ be a bounded, open set, and let $m \in \mathbb{N}$. We set

$$\mathscr{O}^m_b(\Omega) := \big\{ f : \Omega \to \mathbb{C}^m \,\big|\, f \text{ is holomorphic and } f \text{ extends to a continuous function } \mathscr{E}(f) \in C(\overline{\Omega}) \big\}.$$

With the norm

$$\|f\|_{\mathscr{O}^m_b(\Omega)} := \|\mathscr{E}(f)\|_{C(\Omega)},$$

$\mathscr{O}_b^m(\Omega)$ is a Banach space. We set, for $\eta > 0$,

$$\mathscr{B}_\eta^{N,m} := \left\{ f : B_{\mathbb{R}^N}(\eta) \to \mathbb{C}^m \;\middle|\; f \text{ is real analytic and extends to a holomorphic function } \mathscr{E}(f) \in \mathscr{O}_b^m(B_{\mathbb{C}^N}(\eta)) \right\}.$$

With the norm

$$\|f\|_{\mathscr{B}_\eta^{N,m}} := \|\mathscr{E}(f)\|_{\mathscr{O}_b^m(B_{\mathbb{C}^N}(\eta))},$$

$\mathscr{B}_\eta^{N,m}$ is a Banach space. Sometimes we wish to replace $\mathbb{C}^m$ in the above definitions with a more general complex Banach space $\mathscr{V}$. We write this space as $\mathscr{B}_\eta^{N,\mathscr{V}}$ and define the norm in the obvious way.

Lemma 5.5. *Let $\mathscr{V}$ be a Banach space. Then $\mathscr{A}^{n,\eta}(\mathscr{V}) \subseteq \mathscr{B}_\eta^{n,\mathscr{V}}$ and $\|f\|_{\mathscr{B}_\eta^{n,\mathscr{V}}} \leq \|f\|_{\mathscr{A}^{n,\eta}(\mathscr{V})}$.*

Proof. This follows immediately from the definitions. □

Lemma 5.6. *Let $\mathscr{Y}$ be a Banach algebra. Then $\mathscr{A}^{n,s}(\mathscr{Y})$ and $\mathscr{B}_\eta^{N,\mathscr{Y}}$ are Banach algebras. Indeed, if $\mathscr{V}$ denotes either of these spaces, then if $f, g \in \mathscr{V}$, we have $fg \in \mathscr{V}$ and $\|fg\|_{\mathscr{V}} \leq \|f\|_{\mathscr{V}} \|g\|_{\mathscr{V}}$.*

Proof. This follows easily from the definitions. □

Lemma 5.7. *Fix $0 < \eta_1 < \eta_2$. If $f \in \mathscr{A}^{n,\eta_2}$ with $f(0) = 0$, then*

$$\|f\|_{\mathscr{A}^{n,\eta_1}} \leq \frac{\eta_1}{\eta_2} \|f\|_{\mathscr{A}^{n,\eta_2}}. \tag{5.1}$$

Similarly, if $f \in \mathscr{B}_{\eta_2}^{N,m}$ with $f(0) = 0$, then

$$\|f\|_{\mathscr{B}_{\eta_1}^{N,m}} \leq \frac{\eta_1}{\eta_2} \|f\|_{\mathscr{B}_{\eta_2}^{N,m}}. \tag{5.2}$$

The same results hold (with the same proofs) for functions taking values in Banach spaces.

Proof. Suppose $f \in \mathscr{A}^{n,\eta_2}$ with $f(0) = 0$. Then, $f(t) = \sum_{|\alpha|>0} c_\alpha t^\alpha$ with $\|f\|_{\mathscr{A}^{n,\eta_2}} = \sum_{|\alpha|>0} |c_\alpha| \eta_2^{|\alpha|}$. We have

$$\|f\|_{\mathscr{A}^{n,\eta_1}} = \sum_{|\alpha|>0} |c_\alpha| \eta_1^{|\alpha|} \leq \frac{\eta_1}{\eta_2} \sum_{|\alpha|>0} |c_\alpha| \eta_2^{|\alpha|} = \frac{\eta_1}{\eta_2} \|f\|_{\mathscr{A}^{n,\eta_2}},$$

completing the proof of (5.1).

Let $g \in \mathscr{O}_b^m(B_{\mathbb{C}}(\eta_2))$ with $g(0) = 0$. We claim

$$\|g\|_{\mathscr{O}_b^m(B^1(\eta_1))} \leq \frac{\eta_1}{\eta_2}\|g\|_{\mathscr{O}_b^m(B_{\mathbb{C}}(\eta_2))}. \tag{5.3}$$

Indeed, we may write $g(z) = zg_1(z)$, where $g_1 \in \mathscr{O}_b^m(B_{\mathbb{C}}(\eta_2))$. We have, by the Maximum Modulus Principle:

$$\begin{aligned}\|g\|_{\mathscr{O}_b^m(B^1(\eta_1))} &\leq \eta_1\|g_1\|_{\mathscr{O}_b^m(B_{\mathbb{C}}(\eta_1))} \leq \eta_1 \sup_{|z|=\eta_2} |g_1(z)| = \frac{\eta_1}{\eta_2} \sup_{|z|=\eta_2} |g(z)| \\ &= \frac{\eta_1}{\eta_2}\|g\|_{\mathscr{O}_b^m(B^1(\eta_2))},\end{aligned}$$

completing the proof of (5.3).

Let $h \in \mathscr{O}_b^m(B_{\mathbb{C}^n}(\eta_2))$ with $h(0) = 0$. We claim

$$\|h\|_{\mathscr{O}_b^m(B_{\mathbb{C}^n}(\eta_1))} \leq \frac{\eta_1}{\eta_2}\|h\|_{\mathscr{O}_b^m(B_{\mathbb{C}^n}(\eta_2))}. \tag{5.4}$$

Indeed, for $0 \neq w \in B_{\mathbb{C}^n}(\eta_1)$, apply (5.3) to $g(z) := h(zw/|w|)$, to see $|h(w)| \leq \frac{\eta_1}{\eta_2}\|h\|_{\mathscr{O}_b^m(B^n(\eta_2))}$. Taking the supremum over all such w yields (5.4).

(5.2) is an immediate consequence of (5.4). □

Lemma 5.8. *Fix $0 < \eta_1 < \eta_2$. Then $\mathscr{B}_{\eta_2}^{n,1} \subseteq \mathscr{A}^{n,\eta_1}$ and for $f \in \mathscr{B}_{\eta_2}^{n,1}$,*

$$\|f\|_{\mathscr{A}^{n,\eta_1}} \leq C\|f\|_{\mathscr{B}_{\eta_2}^{n,1}},$$

where C can be chosen to depend only on n, η_2, and η_1.

Similarly for $s \in (0,\infty)$, $\mathscr{B}_{\eta_2}^{n,1} \subseteq \mathscr{C}^s(B^n(\eta_1))$ and for $f \in \mathscr{C}^s(B^n(\eta_1))$,

$$\|f\|_{\mathscr{C}^s(B^n(\eta_1))} \leq C\|f\|_{\mathscr{B}_{\eta_2}^{n,1}},$$

where C can be chosen to depend only on s, n, η_2, and η_1.

Proof. It suffices to prove both results for $\eta_2 = 1$ and $\eta_1 \in (0,1)$, by rescaling. When $\eta_2 = 1$, we extend f to a holomorphic function $\mathscr{E}(f) \in \mathscr{O}_b^1(B_{\mathbb{C}^n}(1))$, and use the well-known representation:

$$\mathscr{E}f(z) = \frac{(n-1)!}{2\pi^n} \int\limits_{\partial B_{\mathbb{C}^n}(1)} \mathscr{E}f(\zeta)\frac{1-\overline{z}\cdot\zeta}{|\zeta - z|^{2n}}\, d\sigma(\zeta),$$

where σ denotes the surface area measure on $\partial B_{\mathbb{C}^n}(1)$. From here, the results follows easily. □

Lemma 5.9. *Fix $\eta_1 > 0$, $D > 0$. For $0 < \gamma \leq \frac{\eta_1}{D}$ and $f : B^n(\eta_1) \to \mathbb{C}$, define $f_\gamma : B^n(D) \to \mathbb{C}$ by $f_\gamma(t) = f(\gamma t)$.*

(i) *Let $m \in \mathbb{N}$ with $m \geq 1$ and $s \in (0,1]$. Then, for $0 < \gamma \leq \min\{\frac{\eta_1}{D}, 1\}$, we have for $f \in \mathscr{C}^{m+s}(B^n(\eta_1))$ with $f(0) = 0$,*

$$\|f_\gamma\|_{\mathscr{C}^{m+s}(B^n(D))} \leq \gamma(15(D+1)+1)\|f\|_{\mathscr{C}^{m+s}(B^n(\eta_1))}.$$

(ii) *For $0 < \gamma \leq \frac{\eta_1}{D}$, we have for $f \in \mathscr{A}^{n,\eta_1}$ with $f(0) = 0$,*

$$\|f_\gamma\|_{\mathscr{A}^{n,D}} \leq \frac{\gamma D}{\eta_1}\|f\|_{\mathscr{A}^{n,\eta_1}}.$$

Proof. We begin with (i). Using $0 < \gamma \leq \min\{\frac{\eta_1}{D}, 1\}$, it follows immediately from the definitions that

$$\begin{aligned} \sum_{1\leq|\alpha|\leq m} \|\partial_x^\alpha f_\gamma\|_{\mathscr{C}^s(B^n(D))} &= \sum_{1\leq|\alpha|\leq m} \gamma^{|\alpha|}\|(\partial_x^\alpha f)(\gamma\cdot)\|_{\mathscr{C}^s(B^n(D))} \\ &\leq \sum_{1\leq|\alpha|\leq m} \gamma^{|\alpha|}\|\partial_x^\alpha f\|_{\mathscr{C}^s(B^n(\eta_1))} \leq \gamma\|f\|_{\mathscr{C}^{m+s}(B^n(\eta_1))}. \end{aligned} \tag{5.5}$$

Since $f_\gamma(0) = f(0) = 0$, we have (using the Fundamental Theorem of Calculus)

$$\begin{aligned} \|f_\gamma\|_{C^1(B^n(D))} &= \|f_\gamma\|_{C^0(B^n(D))} + \sum_{|\alpha|=1} \|\partial_x^\alpha f_\gamma\|_{C^0(B^n(D))} \\ &\leq (D+1)\sum_{|\alpha|=1} \|\partial_x^\alpha f_\gamma\|_{C^0(B^n(D))} \leq (D+1)\gamma\|f\|_{C^1(B^n(\eta_1))}. \end{aligned} \tag{5.6}$$

It is easy to see, directly from the definitions, that (for any function g on any ball B),[10]

$$\|g\|_{\mathscr{C}^s(B)} \leq 5\|g\|_{C^{0,s}(B)} \leq 15\|g\|_{C^{0,1}(B)} \leq 15\|g\|_{C^1(B)} \leq 15\|g\|_{\mathscr{C}^{m+s}(B)}. \tag{5.7}$$

Thus, using (5.6), we have

$$\begin{aligned} \|f_\gamma\|_{\mathscr{C}^s(B^n(D))} &\leq 15\|f_\gamma\|_{C^1(B^n(D))} \leq 15(D+1)\gamma\|f\|_{C^1(B^n(\eta_1))} \\ &\leq 15(D+1)\gamma\|f\|_{\mathscr{C}^{m+s}(B^n(\eta_1))}. \end{aligned}$$

Combining this with (5.5) completes the proof of (i).

We turn to (ii). Let $f \in \mathscr{A}^{n,\eta_1}$ with $f(0) = 0$, so that $f(t) = \sum_{|\alpha|>0} c_\alpha t^\alpha$, and $\|f\|_{\mathscr{A}^{n,\eta_1}} = \sum_{|\alpha|>0} |c_\alpha|\eta_1^{|\alpha|}$. For $0 < \gamma \leq \frac{\eta_1}{D}$ we have $f_\gamma(t) = \sum_{|\alpha|>0} c_\alpha \gamma^{|\alpha|} t^\alpha$, and therefore $f_\gamma \in \mathscr{A}^{n,D}$ and we have

[10] See [10, Lemma 8.1] for a result like (5.7)–in the proof of that result, one can see how the constants 5 and 15 arise. However, these particular constants are not essential for what follows.

$$\|f_\gamma\|_{\mathscr{A}^{n,D}} = \sum_{|\alpha|>0} |c_\alpha|(\gamma D)^{|\alpha|} = \sum_{|\alpha|>0} |c_\alpha|(\eta_1)^{|\alpha|}\left(\frac{\gamma D}{\eta_1}\right)^{|\alpha|} \le \left(\frac{\gamma D}{\eta_1}\right)\|f\|_{\mathscr{A}^{n,\eta_1}},$$

completing the proof of (ii). □

Lemma 5.10. *Let $\eta_1, \eta_2 > 0$, $n_1, n_2 \in \mathbb{N}$, and let $\mathscr{V}$ be a Banach space. Suppose $f \in \mathscr{A}^{n_1,\eta_1}(\mathscr{V})$, $g \in \mathscr{A}^{n_2,\eta_2}(\mathbb{R}^{n_1})$ with $\|g\|_{\mathscr{A}^{n_2,\eta_2}(\mathbb{R}^{n_1})} \le \eta_1$. Then, $f \circ g \in \mathscr{A}^{n_2,\eta_2}(\mathscr{V})$ with $\|f \circ g\|_{\mathscr{A}^{n_2,\eta_2}} \le \|f\|_{\mathscr{A}^{n_1,\eta_1}}$.*

Proof. This is immediate from the definitions. □

Lemma 5.11. *Fix $0 < \eta_2 < \eta_1$, and suppose $f \in \mathscr{A}^{n,\eta_1}(\mathscr{V})$, where $\mathscr{V}$ is a Banach space. Then, for each $j = 1, \ldots, n$, $\frac{\partial}{\partial t_j} f(t) \in \mathscr{A}^{n,\eta_2}(\mathscr{V})$ and $\|\frac{\partial}{\partial t_j} f\|_{\mathscr{A}^{n,\eta_2}(\mathscr{V})} \le C\|f\|_{\mathscr{A}^{n,\eta_1}(\mathscr{V})}$, where C can be chosen to depend only on η_1 and η_2.*

Proof. Without loss of generality, we prove the result for $j = 1$. We let e_1 denote the first standard basis element: $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Suppose $f(t) = \sum c_\alpha \frac{t^\alpha}{\alpha!}$. Then, $\frac{\partial}{\partial t_j} f(t) = \sum_{\alpha_1>0} c_\alpha \frac{t^{\alpha-e_1}}{(\alpha-e_1)!}$. Hence,

$$\begin{aligned}\left\|\frac{\partial}{\partial t_1} f\right\|_{\mathscr{A}^{n,\eta_2}} &= \sum_{\alpha_1>0} \frac{\|c_\alpha\|_{\mathscr{V}}}{(\alpha-e_1)!}\eta_2^{|\alpha-e_1|} = \sum_{\alpha} \frac{\|c_\alpha\|_{\mathscr{V}}}{\alpha!}\eta_1^{|\alpha|}\left(\frac{\eta_2}{\eta_1}\right)^{|\alpha|}\frac{\alpha_1}{\eta_1} \\ &\le \left(\sup_\alpha \left(\frac{\eta_2}{\eta_1}\right)^{|\alpha|}\frac{\alpha_1}{\eta_1}\right)\|f\|_{\mathscr{A}^{n,\eta_1}},\end{aligned}$$

completing the proof. □

6. Some additional notation

If $f : M \to N$ is a C^1 map between C^1 manifolds, we write $df(x) : T_xM \to T_xN$ for the usual differential. We extend this to be a complex linear map $df(x) : \mathbb{C}T_xM \to \mathbb{C}T_xN$, where $\mathbb{C}T_xM = T_xM \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexified tangent space. Even if the manifold M has additional structure (e.g., in the case of a complex manifold), $df(x)$ is defined in terms of the underlying real manifold structure.

When working on $\mathbb{R}^r \times \mathbb{C}^n$ we will often use coordinates (t, z) where $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We write

$$\frac{\partial}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial t_2} \\ \vdots \\ \frac{\partial}{\partial t_r} \end{bmatrix}, \quad \frac{\partial}{\partial z} = \begin{bmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \\ \vdots \\ \frac{\partial}{\partial z_n} \end{bmatrix}, \quad \frac{\partial}{\partial \overline{z}} = \begin{bmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \frac{\partial}{\partial \overline{z}_2} \\ \vdots \\ \frac{\partial}{\partial \overline{z}_n} \end{bmatrix}.$$

At times we will instead use coordinates (u, w) where $u \in \mathbb{R}^r$ and $w \in \mathbb{C}^n$ and define $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial w}$, and $\frac{\partial}{\partial \overline{w}}$ similarly.

For a function $F(t,z) = (F_1(t,z), \ldots, F_m(t,z)) : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{C}^m$ we write

$$d_t F = \begin{bmatrix} \frac{\partial F_1}{\partial t_1} & \cdots & \frac{\partial F_1}{\partial t_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial t_1} & \cdots & \frac{\partial F_m}{\partial t_r} \end{bmatrix}, \; d_z F = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_n} \end{bmatrix}, \; d_{\overline{z}} F = \begin{bmatrix} \frac{\partial F_1}{\partial \overline{z}_1} & \cdots & \frac{\partial F_1}{\partial \overline{z}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial \overline{z}_1} & \cdots & \frac{\partial F_m}{\partial \overline{z}_n} \end{bmatrix}.$$

We identify $\mathbb{R}^r \times \mathbb{R}^{2n} \cong \mathbb{R}^r \times \mathbb{C}^n$ via the map $(t_1, \ldots, t_r, x_1, \ldots, x_{2n}) \mapsto (t_1, \ldots, t_r, x_1 + ix_{n+1}, \ldots, x_n + ix_{2n})$. Thus, given a function $G(t,z) : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{R}^s \times \mathbb{C}^m$, we may also think of F as function $G(t,x) = (G_1(t,x), \ldots, G_{s+2m}(t,x)) : \mathbb{R}^r \times \mathbb{R}^{2n} \to \mathbb{R}^s \times \mathbb{R}^{2m}$. For such a function, we write

$$d_{(t,x)} G = \begin{bmatrix} \frac{\partial G_1}{\partial t_1} & \cdots & \frac{\partial G_1}{\partial t_r} & \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_{2n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_{s+2m}}{\partial t_1} & \cdots & \frac{\partial G_{s+2m}}{\partial t_r} & \frac{\partial G_{s+2m}}{\partial x_1} & \cdots & \frac{\partial G_{s+2m}}{\partial x_{2n}} \end{bmatrix}.$$

We write $I_{N\times N} \in \mathbb{M}^{N\times N}$ to denote the $N \times N$ identity matrix, and $0_{a\times b} \in \mathbb{M}^{a\times b}$ to denote the $a \times b$ zero matrix.

7. The main technical result

In this section, we state and prove the main technical result needed to prove Theorem 4.18.

Fix $s_0 \in (0, \infty) \cup \{\omega\}$ and let $X_1, \ldots, X_r, L_1, \ldots, L_n$ be complex vector fields on $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ with:

- If $s_0 \in (0, \infty)$, $X_k, L_j \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1); \mathbb{C}^{r+2n})$.
- If $s_0 = \omega$, $X_k, L_j \in \mathscr{A}^{r+2n,1}(\mathbb{C}^{r+2n})$.

We suppose:

- $X_k(0) = \frac{\partial}{\partial t_k}$, $L_j(0) = \frac{\partial}{\partial \overline{z}_j}$.
- $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, $[X_{k_1}, X_{k_2}](\zeta), [X_k, L_j](\zeta), [L_{j_1}, L_{j_2}](\zeta) \in \mathrm{span}_{\mathbb{C}}\{X_1(\zeta), \ldots, X_r(\zeta), L_1(\zeta), \ldots, L_n(\zeta)\}$.

Under these hypotheses, Nirenberg's theorem on the integrability of elliptic structures[11] implies that there exists a map $\Phi_4 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, with $\Phi_4(0) = 0$,

[11] Originally, Nirenberg considered only the case of C^∞ vector fields and worked in the case when $X_1, \ldots, X_r$ were real.

Φ_4 is a diffeomorphism onto its image (which is an open neighborhood of $0 \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$), and such that $\Phi_4^* X_k(u,w), \Phi_4^* L_j(u,w) \in \mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_r},\frac{\partial}{\partial \overline{w}_1},\ldots,\frac{\partial}{\partial \overline{w}_r}\}$, $\forall(u,w)$ (here we are giving the domain space $\mathbb{R}^r\times\mathbb{C}^n$ coordinates (u,w)). Our goal in this section is to give a quantitative version of this result which gives Φ_4 the optimal regularity (namely, when $s_0\in(0,\infty)$, Φ_4 is in $\mathscr{C}^{s_0+2}$, and when $s_0=\omega$, Φ_4 is real analytic).

As discussed in Section 1.2, for future applications we need keep track of what the constants depend on in this section, and need to make the statement of the results more precise than would be required just for the main results of this paper. To ease notation, we introduce notions of "admissible" constants. These are constants which only depend on certain parameters. The use of these constants greatly simplifies notation in both the statements of the results and the proofs.

Definition 7.1. If $s_0\in(0,\infty)$, for $s\geq s_0$ if we say C is an $\{s\}$-admissible constant, it means that we assume $X_k, L_j \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{C}^{r+2n})$, $\forall j,k$. C can then be chosen to depend only on n, r, s, s_0, and upper bounds for $\|X_k\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$ and $\|L_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$, $1\leq k\leq r$, $1\leq j\leq n$. For $s\leq s_0$, we define $\{s\}$-admissible constants to be $\{s_0\}$-admissible constants.

Definition 7.2. If $s_0=\omega$, we say C is an $\{\omega\}$-admissible constant if C can be chosen to depend only on n, r, and upper bounds for $\|X_k\|_{\mathscr{A}^{2n+r,1}}$, $\|L_j\|_{\mathscr{A}^{2n+r,1}}$, $1\leq k\leq r$, $1\leq j\leq n$.

We write $A\lesssim_{\{s\}} B$ to mean $A\leq CB$, where C is a positive $\{s\}$-admissible constant. We write $A\approx_{\{s\}} B$ for $A\lesssim_{\{s\}} B$ and $B\lesssim_{\{s\}} A$.

Theorem 7.3. *There exists an $\{s_0\}$-admissible constant $K_2\geq 1$ and a map $\Phi_4 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1)\to B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ such that*

(i)
- *If $s_0\in(0,\infty)$, $\Phi_4\in\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{R}^r\times\mathbb{C}^n)$ and $\|\Phi_4\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}\lesssim_{\{s\}} 1$, $\forall s>0$.*
- *If $s_0=\omega$, $\Phi_4\in\mathscr{A}^{2n+r,2}(\mathbb{R}^r\times\mathbb{C}^n)$ and $\|\Phi_4\|_{\mathscr{A}^{2n+r,2}}\leq 1$. In particular, Φ_4 extends to a real analytic function on $B_{\mathbb{R}^r\times\mathbb{C}^n}(2)$.*

(ii) *$\Phi_4(0)=0$ and $d_{(t,x)}\Phi_4(0)=K_2^{-1}I_{(r+2n)\times(r+2n)}$. See Section 6 for the notation $d_{(t,x)}$.*

(iii) *$\forall\zeta\in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, $\det d_{(t,x)}\Phi_4(\zeta)\approx_{\{s_0\}} 1$.*

(iv) *$\Phi_4(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))\subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ is an open set and $\Phi_4 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1)\to\Phi_4(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$ is a diffeomorphism.*[12]

[12] Here, and in the rest of the paper, we say $F: U_1\to U_2$ is a diffeomorphism if F is a bijection and dF is everywhere nonsingular.

(v)

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = K_2^{-1}(I+\mathcal{A}) \begin{bmatrix} \Phi_4^* X \\ \Phi_4^* L \end{bmatrix},$$

where $\mathcal{A} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \mathbb{M}^{(n+r)\times(n+r)}(\mathbb{C})$, $\mathcal{A}(0)=0$ *and*

- *If* $s_0 \in (0,\infty)$, $\|\mathcal{A}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \lesssim_{\{s\}} 1$, $\forall s>0$ *and*

$$\|\mathcal{A}\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}.$$

- *If* $s_0=\omega$, $\|\mathcal{A}\|_{\mathscr{A}^{2n+r,1}(\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}$.

In either case, note that this implies $(I+\mathcal{A})$ *is an invertible matrix on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

(vi) *Suppose* Z *is another complex vector field on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$. *Then,*

- *If* $s_0 \in (0,\infty)$, $\|\Phi_4^* Z\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} \|Z\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$, $\forall s>0$.
- *If* $s_0=\omega$, $\|\Phi_4^* Z\|_{\mathscr{A}^{2n+r,1}} \lesssim_{\{\omega\}} \|Z\|_{\mathscr{A}^{2n+r,1}}$.

Remark 7.4. In Theorem 7.3 (and in the rest of this section), we have written $s>0$ to mean $s\in(0,\infty)$ and similarly for other such inequalities. For example if $s_0\in(0,\infty)$ and we write $s\geq s_0$, it means $s\in[s_0,\infty)$.

Remark 7.5. Proofs of results like Theorem 7.3 in the literature only prove that Φ_4 is $\mathscr{C}^{s_0+1}$ (instead of $\mathscr{C}^{s_0+2}$); and each of the estimates is similarly off by a derivative.[13]

Remark 7.6. When $s_0=\omega$, the hypothesis $X_k, L_j \in \mathscr{A}^{r+2n,1}(\mathbb{C}^{r+2n})$ can be replaced with the slightly weaker hypothesis $X_k, L_j \in \mathscr{B}_1^{r+2n,r+2n}$; one can achieve the same result with the same proof. However, our applications use $X_k, L_j \in \mathscr{A}^{r+2n,1}(\mathbb{C}^{r+2n})$, so we use this space instead. In any case, it is straightforward to see (using Lemmas 5.5 and 5.8) that either choice yields an equivalent theorem.

7.1. A reduction

To prove Theorem 7.3, we prove the following proposition. For it we use the same notation and setting as Theorem 7.3.

Proposition 7.7. *There exist* $\{s_0\}$*-admissible constants* $K_1\geq 1$ *and* $\eta_3\in(0,1]$ *and a map* $\Phi_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ *such that:*

(i) • *If* $s_0\in(0,\infty)$, $\Phi_3\in\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3);\mathbb{R}^r\times\mathbb{C}^n)$ *and* $\|\Phi_3\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s>0$.

[13] However, the results in this section concerning real analytic vector fields are standard.

- *If* $s_0 = \omega$, $\Phi_3 \in \mathscr{A}^{2n+r,2\eta_3}(\mathbb{R}^r \times \mathbb{C}^n)$ *and* $\|\Phi_3\|_{\mathscr{A}^{2n+r,2\eta_3}} \leq 1$. *In particular,* Φ_3 *extends to a real analytic function on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(2\eta_3)$.

(ii) $\Phi_3(0) = 0$ *and* $d_{(t,x)}\Phi_3(0) = K_1^{-1} I_{(r+2n)\times(r+2n)}$.

(iii) $\Phi_3(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ *is open and* $\Phi_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \Phi_3(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))$ *is a diffeomorphism.*

(iv)

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = K_1^{-1}(I + \mathcal{A}_3) \begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix},$$

where $\mathcal{A}_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \mathbb{M}^{(n+r)\times(n+r)}(\mathbb{C})$, $\mathcal{A}_3(0) = 0$ *and*

- *If* $s_0 \in (0,\infty)$, $\|\mathcal{A}_3\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
- *If* $s_0 = \omega$, $\|\mathcal{A}_3\|_{\mathscr{A}^{n,\eta_3}} \lesssim_{\{\omega\}} 1$.

First we see how Theorem 7.3 follows from Proposition 7.7.

Proof of Theorem 7.3. Let Φ_3, K_1, η_3, and $\mathcal{A}_3$ be as in Proposition 7.7. It follows from Proposition 7.7 (ii) that $\det d_{(t,x)}\Phi_3(0) \approx_{\{s_0\}} 1$. Next we claim that if $\hat{\eta} \approx_{\{s_0\}} 1$ is chosen sufficiently small (with $\hat{\eta} \leq \eta_3$), then $\det d_{(t,x)}\Phi_3(\zeta) \approx_{\{s_0\}} 1$, $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\hat{\eta})$. Indeed:

- Suppose $s_0 \in (0,\infty)$. Note $\|\Phi_3\|_{C^2(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \leq \|\Phi_3\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s_0\}} 1$. Using the fact that $\det d_{(t,x)}\Phi_3(0) \approx_{\{s_0\}} 1$, if $\hat{\eta} \approx_{\{s_0\}} 1$ is chosen sufficiently small (with $\hat{\eta} \leq \eta_3$), we have $\det d_{(t,x)}\Phi_3(\zeta) \approx_{\{s_0\}} 1$, $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\hat{\eta})$.
- Suppose $s_0 = \omega$. By Lemmas 5.5 and 5.8, $\|\Phi_3\|_{C^2(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s_0\}} \|\Phi_3\|_{\mathscr{B}^{2n+r,2n+r}_{2\eta_3}} \leq \|\Phi_3\|_{\mathscr{A}^{2n+r,\eta_3}} \lesssim_{\{s_0\}} 1$. Thus, using the fact that $\det d_{(t,x)}\Phi_3(0) \approx_{\{s_0\}} 1$, if $\hat{\eta} \approx_{\{s_0\}} 1$ is chosen sufficiently small (with $\hat{\eta} \leq \eta_3$), we have $\det d_{(t,x)}\Phi_3(\zeta) \approx_{\{s_0\}} 1$, $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\hat{\eta})$.

For $\gamma \leq \hat{\eta}$, define $\Psi_\gamma : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\hat{\eta})$ by $\Psi_\gamma(\zeta) = \gamma\zeta$. We will set $\Phi_4 := \Phi_3 \circ \Psi_\gamma$ for appropriately chosen γ. Consider,

$$\frac{1}{\gamma}\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = \Psi_\gamma^* \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = \Psi_\gamma^* K_1^{-1}(I + \mathcal{A}_3) \begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix} = K_1^{-1}(I + \mathcal{A}_3 \circ \Psi_\gamma) \begin{bmatrix} (\Phi_3 \circ \Psi_\gamma)^* X \\ (\Phi_3 \circ \Psi_\gamma)^* L \end{bmatrix}.$$

Since $\mathcal{A}_3(0) = 0$, using Proposition 7.7 (iv) and Lemma 5.9, we have:

- If $s_0 \in (0,\infty)$, $\|\mathcal{A}_3 \circ \Psi_\gamma\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s_0\}} \gamma\|\mathcal{A}_3\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s_0\}} \gamma$. Thus, by taking γ to be a sufficiently small $\{s_0\}$-admissible constant, we have $\|\mathcal{A}_3 \circ \Psi_\gamma\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \leq \frac{1}{4}$.
- If $s_0 = \omega$, we have $\|\mathcal{A}_3 \circ \Psi_\gamma\|_{\mathscr{A}^{2n+r,1}} \leq \frac{\gamma}{\eta_3}\|\mathcal{A}_3\|_{\mathscr{A}^{2n+r,\eta_3}} \lesssim_{\{\omega\}} \gamma$. Also, set $R(t,z) := d\Phi_3(t,z) - K_1^{-1} I_{(2n+r)\times(2n+r)}$, so that $R(0,0) = 0$, and by Lemma 5.11, $R \in \mathscr{A}^{2n+r,\eta_3}(\mathbb{M}^{(2n+r)\times(2n+r)})$ and $\|R\|_{\mathscr{A}^{2n+r,\eta_3}} \lesssim_{\{\omega\}} 1$. We have $\|R \circ \Psi_\gamma\|_{\mathscr{A}^{2n+r,1}} \leq$

$\frac{\gamma}{\eta_3}\|R\|_{\mathscr{A}^{2n+r,\eta_3}} \lesssim_{\{\omega\}} \gamma$. Thus, by taking γ to be a sufficiently small $\{\omega\}$-admissible constant, we have $\|\mathcal{A}_3 \circ \Psi_\gamma\|_{\mathscr{A}^{2n+r,1}} \leq \frac{1}{4}$ and $\|R \circ \Psi_\gamma\|_{\mathscr{A}^{2n+r,1}} \leq (2K_1)^{-1}$.

Taking γ as above and setting $\Phi_4 = \Phi_3 \circ \Psi_\gamma$, Theorem 7.3 (i), (ii), (iii), (iv), and (v) follow with $K_2 = \gamma^{-1}K_1$ and $\mathcal{A} = \mathcal{A}_3 \circ \Psi_\gamma$.

We turn to (vi). Recall,

$$\Phi_4^* Z(u,w) = d\Phi_4(u,w)^{-1} Z(\Phi_4(u,w)). \tag{7.1}$$

If $s_0 \in (0,\infty)$, we have from (i) and Lemma 5.3 that $\|Z \circ \Phi_4\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} \|Z\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$. Also, by (i), (iii), and Remark 5.2 we have $\|(d\Phi_4)^{-1}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$. Using these estimates, (7.1), and Proposition 5.1, (vi) follows in the case $s_0 \in (0,\infty)$.

If $s_0 = \omega$, (i) and Lemma 5.10 show $\|Z\circ\Phi_4\|_{\mathscr{A}^{2n+r,1}} \lesssim_{\{\omega\}} \|Z\|_{\mathscr{A}^{2n+r,1}}$. Letting R be as above, we have $d\Phi_4 = \gamma K_1^{-1}(I + K_1 R\circ\Psi_\gamma)$. Since $\|K_1 R\circ\Psi_\gamma\|_{\mathscr{A}^{2n+r,1}(\mathbb{M}^{(2n+r)\times(2n+r)})} \leq 1/2$, and since $\mathscr{A}^{2n+r,1}(\mathbb{M}^{(2n+r)\times(2n+r)})$ is a Banach Algebra (Lemma 5.6), it follows (by using the Neumann series for $(I + K_1 R\circ\Psi_\gamma)^{-1}$) that $\|(d\Phi_4)^{-1}\|_{\mathscr{A}^{2n+r,1}} \leq 2K_1\gamma^{-1} \lesssim_{\{\omega\}} 1$. Using these estimates, (7.1), and Proposition 5.1, (vi) follows in the case $s_0 = \omega$, completing the proof. □

We now turn to the proof of Proposition 7.7, which encompasses the rest of Section 7. We do this by presenting a series of increasingly general versions of the proposition, and reducing each to the previous; eventually culminating with the full Proposition 7.7. The outline of this proof is:

- In Section 7.2 we present a quantitative version of the holomorphic Frobenius theorem; this result is standard.
- In Section 7.3 we prove the special case of Proposition 7.7 when the vector fields are all assumed to be real analytic and commute. We do this by reducing it to the holomorphic case. This procedure is standard.
- In Section 7.4 we present an easily checkable special case of the real analytic setting using elliptic PDEs. This is a generalization of part of Malgrange's approach [5].
- In Section 7.5 we use elliptic PDEs to reduce the case of vector fields which are a small perturbation of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \bar{z}}$ to the previous case. This is a generalization of part of Malgrange's approach [5].
- In Section 7.6 we use a simple scaling argument to study vector fields which might be a large perturbation of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \bar{z}}$; we do this by reducing to the previous case.
- In Section 7.7 we complete the proof by using some simple linear algebra.

Remark 7.8. In each subsection which follows we use notions of admissible constants which are specific to that section; we are explicit about what we mean in each subsection. In each subsection, we prove progressively stronger results, eventually culminating in the proof of Proposition 7.7; we do this by reducing each result to the weaker results which proceed it. The admissible constants in each result are defined so that constants which are admissible in the result we are proving are admissible in the weaker results which we reduce it to. So that, for example, the main result in Section 7.5 is reduced to the main result in Section 7.4; and in this application of the main result in Section 7.4, each constant which is admissible in the sense of Section 7.4 is admissible in the sense of Section 7.5. Thus the various notions of admissible constants seamlessly glue together to yield Proposition 7.7.

7.2. The holomorphic Frobenius theorem

Fix $\eta_0 > 0$. In this section we work on $\mathbb{C}^r \times \mathbb{C}^{2n}$ with complex coordinates $(\sigma, \zeta) = (\sigma_1, \ldots, \sigma_r, \zeta_1, \ldots, \zeta_{2n})$. We are given holomorphic vector fields:

$$X_k = \frac{\partial}{\partial \sigma_k} + \sum_{l=1}^{2n} b_{1,k}^l(\sigma,\zeta)\frac{\partial}{\partial \zeta_l} + \sum_{l=1}^{r} b_{2,k}^l(\sigma,\zeta)\frac{\partial}{\partial \sigma_l}, \quad 1 \leq k \leq r,$$

$$L_j = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_j} + i\frac{\partial}{\partial \zeta_{j+n}}\right) + \sum_{l=1}^{2n} b_{3,j}^l(\sigma,\zeta)\frac{\partial}{\partial \zeta_l} + \sum_{l=1}^{r} b_{4,j}^l(\sigma,\zeta)\frac{\partial}{\partial \sigma_l}, \quad 1 \leq j \leq n,$$

where $b_{c,d}^e \in \mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_0))$, $\forall c, d, e$ (see Section 5.1 for the definition of the space $\mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_0))$). We also assume $b_{c,d}^e(0,0) = 0$, $\forall c, d, e$.

We assume $[L_{j_1}, L_{j_2}] = 0$, $[L_j, X_k] = 0$, $[X_{k_1}, X_{k_2}] = 0$, $\forall j_1, j_2, k_1, k_2, j, k$. Take C_1 so that $\|b_{c,d}^e\|_{\mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_0))} \leq C_1$, $\forall c, d, e$.

Definition 7.9. We say C is an admissible constant if C can be chosen to depend only on η_0, n, r, and C_1.

We write $A \lesssim B$ for $A \leq CB$, where C is an admissible constant. We write $A \approx B$ for $A \lesssim B$ and $B \lesssim A$.[14]

Proposition 7.10. *There exists an admissible constant $\eta_1 > 0$ and $w_1, \ldots, w_n \in \mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_1))$ such that:*

- $w_l(0) = 0$ *and* $dw_l(0) = d\zeta_l + i d\zeta_{l+n}$.
- $\|w_l\|_{\mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_1))} \lesssim 1$, $\forall l$.
- $L_j w_l = 0$, $X_k w_l = 0$, $\forall j, k, l$.

In what follows, we use the exponentiation of holomorphic vector fields. So that if V is a holomorphic vector field on an open set $\Omega \subseteq \mathbb{C}^N$, it makes sense to define

[14] We use similar notation in the following sections without explicitly defining it.

$(t,z)\mapsto e^{tV}z$, for $z\in\Omega$ and t in a neighborhood of $0\in\mathbb{C}$ (depending on z). If $\Omega'\Subset\Omega$ is a relatively compact open set, then the map $e^{tV}z$ exists for $z\in\Omega'$ and $t\in B_{\mathbb{C}}(\delta)$, where δ can be chosen to depend only on upper bounds for $\mathrm{dist}(\Omega',\partial\Omega)^{-1}$, $\|Z\|_{\mathscr{O}_b^N(\Omega)}$, and N. Furthermore, $\|e^{tZ}z-z\|_{\mathscr{O}_b^N(B_{\mathbb{C}}(\delta)\times\Omega')}$ can be bounded in terms of upper bounds for $\|Z\|_{\mathscr{O}_b^N(\Omega)}$, δ, and N. This is all proved using the standard Contraction Mapping Principle argument. See Chapter I, Section 1 of [4] for a proof of this standard fact.

Proof of Proposition 7.10. Let $Z_1,\ldots,Z_n$ be given by $Z_j=\frac{1}{2}\left(\frac{\partial}{\partial\zeta_j}-i\frac{\partial}{\partial\zeta_{j+n}}\right)$, and set

$$\begin{aligned}&\Psi(t_1,\ldots,t_r,u_1,\ldots,u_n,v_1,\ldots,v_n)\\&\quad:=e^{t_1X_1}e^{t_2X_2}\cdots e^{t_rX_r}e^{u_1L_1}e^{u_2L_2}\cdots e^{u_nL_n}e^{v_1Z_1}e^{v_2Z_2}\cdots e^{v_nZ_n}0.\end{aligned}$$

By the above discussion, there exists an admissible constant $\eta'>0$ with $\Psi\in\mathscr{O}_b^{r+2n}(B_{\mathbb{C}^{r+2n}}(\eta'))$ and $\|\Psi\|_{\mathscr{O}_b^{r+2n}(B_{\mathbb{C}^{r+2n}}(\eta'))}\lesssim 1$.

Since $\frac{\partial}{\partial t_k}\big|_{t=0,u=0,v=0}\Psi(t,u,v)=X_k(0)=\frac{\partial}{\partial\sigma_k}$, $\frac{\partial}{\partial u_j}\big|_{t=0,u=0,v=0}\Psi(t,u,v)=L_j(0)=\frac{1}{2}\left(\frac{\partial}{\partial\zeta_j}+i\frac{\partial}{\partial\zeta_{j+n}}\right)$, and $\frac{\partial}{\partial v_j}\big|_{t=0,u=0,v=0}\Psi(t,u,v)=Z_j(0)=\frac{1}{2}\left(\frac{\partial}{\partial\zeta_j}-i\frac{\partial}{\partial\zeta_{j+n}}\right)$, we have (where $I_{a\times a}$ denotes the $a\times a$ identity matrix and $0_{a\times b}$ denotes the $a\times b$ zero matrix):

$$d_{t,u,v}\Psi(0,0,0)=\left[\begin{array}{c|c|c}I_{r\times r}&0_{r\times n}&0_{r\times n}\\\hline 0_{n\times r}&\frac{1}{2}I_{n\times n}&\frac{1}{2}I_{n\times n}\\\hline 0_{n\times r}&\frac{i}{2}I_{n\times n}&-\frac{i}{2}I_{n\times n}\end{array}\right].$$

In particular, $d_{t,u,v}\Psi(0,0,0)$ is invertible and

$$(d_{t,u,v}\Psi(0,0,0))^{-1}=\left[\begin{array}{c|c|c}I_{r\times r}&0_{r\times n}&0_{r\times n}\\\hline 0_{n\times r}&I_{n\times n}&-iI_{n\times n}\\\hline 0_{n\times r}&I_{n\times n}&iI_{n\times n}\end{array}\right].\tag{7.2}$$

Since $\Psi(0,0,0)=(0,0,0)$, the holomorphic Inverse Function Theorem applies to show that there exist admissible constants $\eta'',\eta_1>0$ such that

$$\Psi:B_{\mathbb{C}^{r+2n}}(\eta'')\to\Psi(B_{\mathbb{C}^{r+2n}}(\eta''))$$

is a biholomorphism, $B_{\mathbb{C}^{r+2n}}(\eta_1)\subseteq\Psi(B_{\mathbb{C}^{r+2n}}(\eta''))$, and $\|\Psi^{-1}\|_{\mathscr{O}_b^{2n+r}(B_{\mathbb{C}^{r+2n}}(\eta_1))}\lesssim 1$.

We give $B_{\mathbb{C}^{r+2n}}(\eta'')$ holomorphic coordinates $(t_1,\ldots,t_r,u_1,\ldots,u_n,v_1\ldots,v_n)$. Set $V_j(t_1,\ldots,t_r,u_1,\ldots,u_n,v_1,\ldots,v_n)=v_j$. Define, for $1\le j\le n$, $w_j\in\mathscr{O}_b^1(B_{\mathbb{C}^{r+2n}}(\eta_1))$ by

$$w_j(\sigma,\zeta):=V_j\circ\Psi^{-1}(\sigma,\zeta).$$

Note $\|w_j\|_{\mathscr{O}^1_b(B_{\mathbb{C}^{r+2n}}(\eta_1))} \lesssim 1$ and $w_j(0) = 0$.

Because the X_ks and L_js commute, we have $\Psi_* \frac{\partial}{\partial t_k} = X_k$ and $\Psi_* \frac{\partial}{\partial u_j} = L_j$. Thus, since $\frac{\partial}{\partial t_k} V_l = 0$ and $\frac{\partial}{\partial u_j} V_l = 0$, we have $X_k w_l = 0$ and $L_j w_l = 0$.

Finally, we compute $dw_j(0) = dV_j(0) d\Psi^{-1}(0)$. $dV_j(0)$ is the row vector which has 1 in the $r+n+j$ component and 0 in all other components and $d\Psi^{-1}(0)$ is given in (7.2). Thus, $dw_j(0)$ is the vector which equals 1 in the $r+j$ component, i in the $r+n+j$ component, and 0 in all other components. I.e., $dw_j(0) = d\zeta_j + i d\zeta_{j+n}$. □

7.3. Real analytic vector fields

Fix $\eta_0 > 0$. Let $X_1, \ldots, X_r, L_1, \ldots, L_n$ be real analytic vector fields on $\mathbb{R}^r \times \mathbb{C}^n \cong \mathbb{R}^{r+2n}$ of the form

$$X_k = \frac{\partial}{\partial t_k} + A_k \frac{\partial}{\partial t} + B_k \frac{\partial}{\partial z} + E_k \frac{\partial}{\partial \overline{z}}, \quad 1 \leq k \leq r,$$
$$L_j = \frac{\partial}{\partial \overline{z}_j} + C_j \frac{\partial}{\partial t} + D_j \frac{\partial}{\partial z} + F_j \frac{\partial}{\partial \overline{z}}, \quad 1 \leq j \leq n.$$

Here we are thinking of A_k, B_k, C_j, D_j, E_k, and F_j as real analytic row vectors: $A_k, C_j \in \mathscr{B}^{r+2n,r}_{\eta_0}$, $B_k, D_j, E_k, F_j \in \mathscr{B}^{r+2n,n}_{\eta_0}$ (see Section 5.1 for the definition of $\mathscr{B}^{r+2n,\cdot}_{\eta_0}$). We assume $A_k(0) = 0$, $B_k(0) = 0$, $C_j(0) = 0$, $D_j(0) = 0$, $E_k(0) = 0$, and $F_j(0) = 0$, and we assume the Xs and Ls all commute: $[X_{k_1}, X_{k_2}] = 0$, $[X_k, L_j] = 0$, $[L_{j_1}, L_{j_2}] = 0$, $\forall j_1, j_2, k_1, k_2, j, k$.

Definition 7.11. We say K is an admissible constant if K can be chosen to depend only on η_0, n, r, and upper bounds for $\|A_k\|_{\mathscr{B}^{r+2n,r}_{\eta_0}}$, $\|B_k\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}$, $\|C_j\|_{\mathscr{B}^{r+2n,r}_{\eta_0}}$, $\|D_j\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}$, $\|E_k\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}$, and $\|F_j\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}$, $\forall j, k$.

Proposition 7.12. *There exists an admissible constant $\eta_2 > 0$ and a map*

$$\Phi_1 : B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2) \to B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0)$$

such that:

- $\Phi_1 \in \mathscr{B}^{r+2n,r+2n}_{\eta_2}$ *with* $\|\Phi_1\|_{\mathscr{B}^{r+2n,r+2n}_{\eta_2}} \lesssim 1$.
- $\Phi_1(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2)) \subseteq B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0)$ *is open and* $\Phi_1 : B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2) \to \Phi_1(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2))$ *is a real analytic diffeomorphism.*
- $\Phi_1(0) = 0$ *and* $d_{t,x}\Phi_1(0) = I_{(r+2n)\times(r+2n)}$.
- $$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = (I + \mathcal{A}_1) \begin{bmatrix} \Phi_1^* X \\ \Phi_1^* L \end{bmatrix},$$

  where $\mathcal{A}_1(0) = 0$, $\mathcal{A}_1 \in \mathscr{B}^{r+2n,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}$, *and* $\|\mathcal{A}_1\|_{\mathscr{B}^{r+2n,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \leq 1$.

To prove Proposition 7.12 we start with a conditional lemma.

Lemma 7.13. *We take the same setting as Proposition 7.12. Suppose there is an admissible constant $\eta_1 > 0$ and functions $w_1, \ldots, w_n \in \mathscr{B}^{r+2n,1}_{\eta_1}$ such that: $w_l(0) = 0$, $dw_l(0) = dz_l$, $\|w_l\|_{\mathscr{B}^{r+2n,1}_{\eta_1}} \lesssim 1$, and $L_j w_l = 0$, $X_k w_l = 0$, $\forall j, k, l$. Then, the conclusions of Proposition 7.12 hold.*

Proof. We define $\Psi : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_1) \to \mathbb{R}^r \times \mathbb{C}^n$ by

$$\Psi(t, z) = (t, w_1(t, z), \ldots, w_n(t, z)).$$

I.e., by identifying $\mathbb{R}^{2n} \cong \mathbb{C}^n$ via the map $(x_1, \ldots, x_{2n}) \mapsto (x_1 + ix_{n+1}, \ldots, x_n + ix_{2n})$, we have

$$\Psi(t, x) = (t, \mathrm{Re}(w_1)(t, x), \ldots, \mathrm{Re}(w_n)(t, x), \mathrm{Im}(w_1)(t, x), \ldots, \mathrm{Im}(w_n)(t, x)).$$

Note that $\Psi(0, 0) = 0$. Since $dw_j(0) = dz_j$ it follows that $d_{t,x}\Psi(0) = I_{(r+2n)\times(r+2n)}$. Thus, the Inverse Function Theorem applies to Ψ to show that there exists admissible constants $\eta', \eta'' > 0$ such that $\Psi : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta') \to \Psi(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta'))$ is a real analytic diffeomorphism, $B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta'') \subseteq \Psi(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta'))$, and $\Phi_1 := \Psi^{-1} : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta'') \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta')$ satisfies $\Phi_1 \in \mathscr{B}^{r+2n,r+2n}_{\eta''}$ with $\|\Phi_1\|_{\mathscr{B}^{r+2n,r+2n}_{\eta''}} \lesssim 1$.

Using coordinates (u, w) on $\mathbb{R}^r \times \mathbb{C}^n$, since $L_j w_l = 0$ and $X_k w_l = 0$, $\forall j, k, l$, we have

$$\begin{aligned}&\Phi_1^* X_k(u, w), \Phi_1^* L_j(u, w) \in \mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_r}, \frac{\partial}{\partial \overline{w}_1}, \ldots, \frac{\partial}{\partial \overline{w}_n}\right\},\\ &\quad\forall (u, w) \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta'').\end{aligned}$$

Since $d_{t,x}\Phi_1(0) = d_{t,x}\Psi(0)^{-1} = I_{(r+2n)\times(r+2n)}$ and $X_k(0) = \frac{\partial}{\partial t_k}$, $L_j(0) = \frac{\partial}{\partial \overline{w}_j}$ we have

$$\begin{bmatrix}\Phi_1^* X\\ \Phi_1^* L\end{bmatrix} = (I + M)\begin{bmatrix}\frac{\partial}{\partial u}\\ \frac{\partial}{\partial \overline{w}}\end{bmatrix},$$

where M is a real analytic matrix, $M(0) = 0$, and $\|M\|_{\mathscr{B}^{2n+r,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta''/2}} \lesssim 1$ (here $\eta''/2$ can be replaced with any fixed number in $(0, \eta'')$). Since $M(0) = 0$, for $\eta_2 \in (0, \eta''/2]$ we have, using Lemma 5.7,

$$\|M\|_{\mathscr{B}^{2n+r,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \lesssim \eta_2.$$

By taking $\eta_2 > 0$ to be a sufficiently small admissible constant, we have

$$\|M\|_{\mathscr{B}^{2n+r,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \leq \frac{1}{2}.$$

We define $I+\mathcal{A}_1 := (I+M)^{-1}$. Then we have $\mathcal{A}_1(0)=0$, $\|\mathcal{A}_1\|_{\mathscr{B}^{2n+r,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \leq 1$ (since $\mathscr{B}^{2n+r,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}$ is a Banach algebra and we have used the Neumann series for $(1+M)^{-1}$), and

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = (I+\mathcal{A}_1)\begin{bmatrix} \Phi_1^* X \\ \Phi_1^* L \end{bmatrix},$$

as desired, completing the proof. □

Proof of Proposition 7.12. We need to show that there exist functions $w_1,\ldots,w_n$ as in Lemma 7.13. By the definition of $\mathscr{B}^{r+2n,\cdot}_{\eta_0}$, the functions A_k, B_k, C_j, D_j, E_k, and F_j extend to holomorphic functions $\mathscr{E}(A_k),\mathscr{E}(C_j)\in\mathscr{O}^r_b(B_{\mathbb{C}^{r+2n}}(\eta_0))$, $\mathscr{E}(B_k),\mathscr{E}(D_j),\mathscr{E}(E_k)$, $\mathscr{E}(F_j)\in\mathscr{O}^n_b(B_{\mathbb{C}^{r+2n}}(\eta_0))$, with

$$\|\mathscr{E}(A_k)\|_{\mathscr{O}^r_b},\|\mathscr{E}(C_j)\|_{\mathscr{O}^r_b},\|\mathscr{E}(B_k)\|_{\mathscr{O}^n_b},\|\mathscr{E}(D_j)\|_{\mathscr{O}^n_b},\|\mathscr{E}(E_k)\|_{\mathscr{O}^n_b},\|\mathscr{E}(F_j)\|_{\mathscr{O}^n_b}\lesssim 1.$$

We give $\mathbb{C}^r\times\mathbb{C}^{2n}$ coordinates (σ,ζ). Let

$$\frac{\partial}{\partial\zeta_\cdot} := \begin{bmatrix} \frac{\partial}{\partial\zeta_1} \\ \vdots \\ \frac{\partial}{\partial\zeta_n} \end{bmatrix}, \quad \frac{\partial}{\partial\zeta_{\cdot+n}} := \begin{bmatrix} \frac{\partial}{\partial\zeta_{n+1}} \\ \vdots \\ \frac{\partial}{\partial\zeta_{2n}} \end{bmatrix}.$$

We extend X_k and L_j to holomorphic vector fields on $\mathbb{C}^r\times\mathbb{C}^{2n}$, by setting

$$\begin{aligned}
\mathscr{E}(X_k) &= \frac{\partial}{\partial\sigma_j} + \mathscr{E}(A_k)\frac{\partial}{\partial\sigma} + \mathscr{E}(B_k)\frac{1}{2}\left(\frac{\partial}{\partial\zeta_\cdot} - i\frac{\partial}{\partial\zeta_{\cdot+n}}\right) + \mathscr{E}(E_k)\frac{1}{2}\left(\frac{\partial}{\partial\zeta_\cdot} + i\frac{\partial}{\partial\zeta_{\cdot+n}}\right), \\
\mathscr{E}(L_j) &= \frac{1}{2}\left(\frac{\partial}{\partial\zeta_j} + i\frac{\partial}{\partial\zeta_{j+n}}\right) + \mathscr{E}(C_j)\frac{\partial}{\partial\sigma} + \mathscr{E}(D_j)\frac{1}{2}\left(\frac{\partial}{\partial\zeta_\cdot} - i\frac{\partial}{\partial\zeta_{\cdot+n}}\right) \\
&\quad + \mathscr{E}(F_j)\frac{1}{2}\left(\frac{\partial}{\partial\zeta_\cdot} + i\frac{\partial}{\partial\zeta_{\cdot+n}}\right).
\end{aligned}$$

I.e., we have extended each t_k to the complex variable σ_k and each x_j to the complex variable ζ_j. Since the Xs and Ls commute, the same is true of the $\mathscr{E}(X)$s and $\mathscr{E}(L)$s by analytic continuation: $[\mathscr{E}(X_{k_1}),\mathscr{E}(X_{k_2})]=0$, $[\mathscr{E}(L_{j_1}),\mathscr{E}(L_{j_2})]=0$, $[\mathscr{E}(X_k),\mathscr{E}(L_j)]=0$, $\forall k_1,k_2,j_1,j_2,j,k$.

Proposition 7.10 applies to $\mathscr{E}(X_1),\ldots,\mathscr{E}(X_r),\mathscr{E}(L_1),\ldots,\mathscr{E}(L_n)$ and each constant which is admissible in the sense of Proposition 7.10 is admissible in the sense of this section. This shows that there exists an admissible constant $\eta_1>0$ and functions $\hat{w}_1,\ldots,\hat{w}_n\in\mathscr{O}^1_b(B_{\mathbb{C}^{r+2n}}(\eta_1))$, with $\|\hat{w}_l\|_{\mathscr{O}^1_b}\lesssim 1$, $\hat{w}_l(0)=0$, $d\hat{w}_l(0)=d\zeta_l+id\zeta_{l+n}$, and $\mathscr{E}(L_j)\hat{w}_l=0$, $\mathscr{E}(X_k)\hat{w}_l=0$, $\forall j,k,l$.

Define, for $(t,x)\in B_{\mathbb{R}^r\times\mathbb{R}^{2n}}(\eta_1)$,

$$w_l(t_1,\ldots,t_r,x_1,\ldots,x_{2n}) := \hat{w}_l(t_1+i0,\ldots,t_r+i0,x_1+i0,\ldots,x_{2n}+i0).$$

Note that $\hat{w}_l$ is the analytic extension of w_l and therefore $\|w_l\|_{\mathscr{B}^{r+2n,1}_{\eta_1}} \lesssim 1$. Also, $dw_l(0) = dx_l + idx_{l+n} = dz_l$, $w_l(0) = \hat{w}_l(0) = 0$. Finally, since $\mathscr{E}(X_k)\hat{w}_l = 0$ and $\mathscr{E}(L_j)\hat{w}_l = 0$ we have $X_k w_l = 0$ and $L_j w_l = 0$, $\forall j, k, l$. Thus Lemma 7.13 applies, completing the proof. □

7.4. Vector fields satisfying an additional equation

Fix $s_0 \in (0, \infty)$. We let $X_1, \ldots, X_r, L_1, \ldots, L_n$ be $\mathscr{C}^{s_0+1}$ complex vector fields on $B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ of the following form:

$$X = \frac{\partial}{\partial t} + A\frac{\partial}{\partial t} + B\frac{\partial}{\partial z} + E\frac{\partial}{\partial \overline{z}}, \quad L = \frac{\partial}{\partial \overline{z}} + C\frac{\partial}{\partial t} + D\frac{\partial}{\partial z} + F\frac{\partial}{\partial \overline{z}}.$$

Here we are using matrix notation; so that X is the column vector $[X_1, \ldots, X_r]^\top$, $\frac{\partial}{\partial t} = \left[\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right]^\top$, similarly for L, $\frac{\partial}{\partial z}$, and $\frac{\partial}{\partial \overline{z}}$, and A, B, C, D, E, and F are matrices of the appropriate size. Thus, if we let A_k denote the kth row of A, and similarly for B, C, D, E, and F we have

$$X_k = \frac{\partial}{\partial t_k} + A_k\frac{\partial}{\partial t} + B_k\frac{\partial}{\partial z} + E_k\frac{\partial}{\partial \overline{z}}, \quad L_j = \frac{\partial}{\partial \overline{z}_j} + C_j\frac{\partial}{\partial t} + D_j\frac{\partial}{\partial z} + F_j\frac{\partial}{\partial \overline{z}}.$$

We assume:

- $A \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{M}^{r\times r}(\mathbb{C}))$, $B, E \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{M}^{r\times n}(\mathbb{C}))$, $C \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{M}^{n\times r}(\mathbb{C}))$, $D, F \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{M}^{n\times n}(\mathbb{C}))$.
- $A(0) = 0_{r\times r}$, $B(0) = 0_{r\times n}$, $C(0) = 0_{n\times r}$, $D(0) = 0_{n\times n}$, $E(0) = 0_{r\times n}$, and $F(0) = 0_{n\times n}$.
- The Xs and Ls commute: $[X_{k_1}, X_{k_2}] = 0$, $[L_{j_1}, L_{j_2}] = 0$, and $[X_k, L_j] = 0$, $\forall j_1, j_2, k_1, k_2, j, k$.
-
$$\sum_{k=1}^r \frac{\partial A_k}{\partial t_k} + \sum_{j=1}^n \frac{\partial C_j}{\partial z_j} = 0, \quad \sum_{k=1}^r \frac{\partial B_k}{\partial t_k} + \sum_{j=1}^n \frac{\partial D_j}{\partial z_j} = 0, \quad \sum_{k=1}^r \frac{\partial E_k}{\partial t_k} + \sum_{j=1}^n \frac{\partial F_j}{\partial z_j} = 0. \quad (7.3)$$

Definition 7.14. We say K is an admissible constant if K can be chosen to depend only on n, r, and s_0.

Proposition 7.15. *There exists an admissible constant $\gamma > 0$ such that if*

$$\|A\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))}, \|B\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))}, \|C\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))},$$
$$\|D\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))}, \|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \leq \gamma,$$

then there exists an admissible constant $\eta_2 > 0$ and a map $\Phi_1 : B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2) \to B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ such that:

- $\Phi_1 \in \mathscr{B}^{r+2n,r+2n}_{\eta_2}$ *with* $\|\Phi_1\|_{\mathscr{B}^{r+2n,r+2n}_{\eta_2}} \lesssim 1$.
- $\Phi_1(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_2)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ *is open and* $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_2) \to \Phi_1(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_2))$ *is a real analytic diffeomorphism.*
- $\Phi_1(0) = 0$ *and* $d_{t,x}\Phi_1(0) = I_{(r+2n)\times(r+2n)}$.
- $$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = (I + \mathcal{A}_1) \begin{bmatrix} \Phi_1^* X \\ \Phi_1^* L \end{bmatrix},$$

  where $\mathcal{A}_1(0) = 0$, $\mathcal{A}_1 \in \mathscr{B}^{r+2n,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}$, *and* $\|\mathcal{A}_1\|_{\mathscr{B}^{r+2n,\mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \leq 1$.

Proof. To prove the proposition, we will show that if $\gamma > 0$ is a sufficiently small admissible constant, then A, B, C, D, E, and F are real analytic, and there exists an admissible constant $\eta_0 > 0$ such that

$$\|A_k\|_{\mathscr{B}^{r+2n,r}_{\eta_0}}, \|B_k\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}, \|C_j\|_{\mathscr{B}^{r+2n,r}_{\eta_0}}, \|D_j\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}, \|E_k\|_{\mathscr{B}^{r+2n,n}_{\eta_0}}, \|F_j\|_{\mathscr{B}^{r+2n,n}_{\eta_0}} \lesssim 1, \quad \forall j,k. \tag{7.4}$$

The result will then follow immediately from Proposition 7.12.

The equation $[X_{k_1}, X_{k_2}] = 0$ can be equivalently rewritten as the following three equations:

$$\frac{\partial A_{k_2}}{\partial t_{k_1}} - \frac{\partial A_{k_1}}{\partial t_{k_2}} = A_{k_2}\frac{\partial}{\partial t}A_{k_1} - A_{k_1}\frac{\partial}{\partial t}A_{k_2} + B_{k_2}\frac{\partial}{\partial z}A_{k_1} - B_{k_1}\frac{\partial}{\partial z}A_{k_2} + E_{k_2}\frac{\partial}{\partial \overline{z}}A_{k_1} - E_{k_1}\frac{\partial}{\partial \overline{z}}A_{k_2}, \tag{7.5}$$

$$\frac{\partial B_{k_2}}{\partial t_{k_1}} - \frac{\partial B_{k_1}}{\partial t_{k_2}} = A_{k_2}\frac{\partial}{\partial t}B_{k_1} - A_{k_1}\frac{\partial}{\partial t}B_{k_2} + B_{k_2}\frac{\partial}{\partial z}B_{k_1} - B_{k_1}\frac{\partial}{\partial z}B_{k_2} + E_{k_2}\frac{\partial}{\partial \overline{z}}B_{k_1} - E_{k_1}\frac{\partial}{\partial \overline{z}}B_{k_2}, \tag{7.6}$$

$$\frac{\partial E_{k_2}}{\partial t_{k_1}} - \frac{\partial E_{k_1}}{\partial t_{k_2}} = A_{k_2}\frac{\partial}{\partial t}E_{k_1} - A_{k_1}\frac{\partial}{\partial t}E_{k_2} + B_{k_2}\frac{\partial}{\partial z}E_{k_1} - B_{k_1}\frac{\partial}{\partial z}E_{k_2} + E_{k_2}\frac{\partial}{\partial \overline{z}}E_{k_1} - E_{k_1}\frac{\partial}{\partial \overline{z}}E_{k_2}. \tag{7.7}$$

We write (7.5), (7.6), and (7.7) as the following equation:

$$\left(\left(\frac{\partial A_{k_2}}{\partial t_{k_1}} - \frac{\partial A_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}, \left(\frac{\partial B_{k_2}}{\partial t_{k_1}} - \frac{\partial B_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}, \left(\frac{\partial E_{k_2}}{\partial t_{k_1}} - \frac{\partial E_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}\right) = \Gamma_1((A,B,E), \nabla(A,B,E)), \tag{7.8}$$

where Γ_1 is an explicit constant coefficient bilinear form depending only on n and r. Similarly, $[L_{j_1}, L_{j_2}] = 0$ can be written as:

$$\left(\left(\frac{\partial C_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial C_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}, \left(\frac{\partial D_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial D_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}, \left(\frac{\partial F_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial F_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}\right)$$
$$= \Gamma_2((C, D, F), \nabla(C, D, F)). \tag{7.9}$$

Finally, $[X_k, L_j] = 0$ can be written as:

$$\left(\left(\frac{\partial C_j}{\partial t_k} - \frac{\partial A_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}, \left(\frac{\partial D_j}{\partial t_k} - \frac{\partial B_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}, \left(\frac{\partial F_j}{\partial t_k} - \frac{\partial E_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}\right)$$
$$= \Gamma_3((A, B, C, D, E, F), \nabla(A, B, C, D, E, F)). \tag{7.10}$$

Combining (7.8), (7.9), (7.10), and (7.3) we see that (A, B, C, D, E, F) satisfies the following equation:

$$\mathcal{E}(A, B, C, D, E, F) = \Gamma((A, B, C, D, E, F), \nabla(A, B, C, D, E, F)), \tag{7.11}$$

where Γ is an explicit constant coefficient, bilinear form, depending only on n and r, and $\mathcal{E}$ is the following explicit operator (which depends only on n and r):

$$\begin{aligned}\mathcal{E}(A, B, C, D, E, F) = \Bigg(&\left(\frac{\partial A_{k_2}}{\partial t_{k_1}} - \frac{\partial A_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}, \left(\frac{\partial C_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial C_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}, \\ &\left(\frac{\partial C_j}{\partial t_k} - \frac{\partial A_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}, \sum_{k=1}^{r} \frac{\partial A_k}{\partial t_k} + \sum_{j=1}^{n} \frac{\partial C_j}{\partial z_j}, \\ &\left(\frac{\partial B_{k_2}}{\partial t_{k_1}} - \frac{\partial B_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}, \left(\frac{\partial D_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial D_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}, \\ &\left(\frac{\partial D_j}{\partial t_k} - \frac{\partial B_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}, \sum_{k=1}^{r} \frac{\partial B_k}{\partial t_k} + \sum_{j=1}^{n} \frac{\partial D_j}{\partial z_j}, \\ &\left(\frac{\partial E_{k_2}}{\partial t_{k_1}} - \frac{\partial E_{k_1}}{\partial t_{k_2}}\right)_{1\leq k_1<k_2\leq r}, \left(\frac{\partial F_{j_2}}{\partial \overline{z}_{j_1}} - \frac{\partial F_{j_1}}{\partial \overline{z}_{j_2}}\right)_{1\leq j_1<j_2\leq n}, \\ &\left(\frac{\partial F_j}{\partial t_k} - \frac{\partial E_k}{\partial \overline{z}_j}\right)_{\substack{1\leq j\leq n\\ 1\leq k\leq r}}, \sum_{k=1}^{r} \frac{\partial E_k}{\partial t_k} + \sum_{j=1}^{n} \frac{\partial F_j}{\partial z_j}\Bigg).\end{aligned}$$

Lemma B.5 shows that $\mathcal{E}$ is elliptic.

Proposition B.1, applied to (7.11), shows that there is an admissible $\gamma > 0$ such that if

$$\|A\|_{\mathscr{C}^{s_0+1}}, \|B\|_{\mathscr{C}^{s_0+1}}, \|C\|_{\mathscr{C}^{s_0+1}}, \|D\|_{\mathscr{C}^{s_0+1}}, \|E\|_{\mathscr{C}^{s_0+1}}, \|F\|_{\mathscr{C}^{s_0+1}} \leq \gamma,$$

then there exists an admissible $\eta_0 > 0$ such that (7.4) holds. Now the result follows from Proposition 7.12. $\square$

Remark 7.16. We only use Proposition 7.15 in the special case $A \equiv 0$, $C \equiv 0$, $E \equiv 0$, and $F \equiv 0$; however the proof in this special case is no easier than the more general case covered in Proposition 7.15.

7.5. Vector fields with small error

Fix $s_0 \in (0, \infty)$. We consider $\mathscr{C}^{s_0+1}$ complex vector fields, $X_1, \ldots, X_r, L_1, \ldots, L_n$, on $B_{\mathbb{R}^r \times \mathbb{C}^n}(2)$ of the following form:

$$X = \frac{\partial}{\partial t} + E\frac{\partial}{\partial z}, \quad L = \frac{\partial}{\partial \overline{z}} + F\frac{\partial}{\partial z}.$$

Here we are again using the matrix notation from Section 7.4.

We assume:

(I) $E \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2); \mathbb{M}^{r\times n}(\mathbb{C}))$, $F \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2); \mathbb{M}^{n\times n}(\mathbb{C}))$.
(II) $E(0) = 0$, $F(0) = 0$.
(III) $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(2)$, $[X_{k_1}, X_{k_2}](\zeta), [L_{j_1}, L_{j_2}](\zeta), [X_k, L_j](\zeta) \in \operatorname{span}_{\mathbb{C}}\{X_1(\zeta), \ldots, X_r(\zeta), L_1(\zeta), \ldots, L_n(\zeta)\}$, $\forall j, k, l$.

Remark 7.17. Assumption (III) is equivalent to assuming $X_1, \ldots, X_r, L_1, \ldots, L_n$ commute. Indeed, under (III) and because of the form of X and L, we have $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(2)$,

$$\begin{aligned} &[X_{k_1}, X_{k_2}](\zeta), [L_{j_1}, L_{j_2}](\zeta), [X_k, L_j](\zeta) \\ &\in \operatorname{span}_{\mathbb{C}}\{X_1(\zeta), \ldots, X_r(\zeta), L_1(\zeta), \ldots, L_n(\zeta)\} \bigcap \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right\} = \{0\}. \end{aligned}$$

Definition 7.18. For $s > s_0$ if we say C is an $\{s\}$-admissible constant, it means that we assume $E \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2); \mathbb{M}^{r\times n}(\mathbb{C}))$ and $F \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2); \mathbb{M}^{n\times n}(\mathbb{C}))$. C can be chosen to depend only on n, r, s_0, s, and upper bounds for $\|E\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))}$ and $\|F\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))}$. For $0 < s \leq s_0$ we say C is an $\{s\}$-admissible constant if C can be chosen to depend only on n, r, and s_0.

Proposition 7.19. *There exists $\sigma = \sigma(n, r, s_0) > 0$ such that if $\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))}$, $\|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))} \leq \sigma$, then there exists an $\{s_0\}$-admissible constant $\eta_3 > 0$ and a map $\Phi_2 : B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r \times \mathbb{C}^n}(2)$ such that:*

- $\Phi_2 \in \mathscr{C}^{s_0+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3); \mathbb{R}^r \times \mathbb{C}^n)$ *and* $\|\Phi_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$.

- $\Phi_2(0) = 0$, $d_{t,x}\Phi_2(0) = I_{(r+2n)\times(r+2n)}$.
- $\Phi_2(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(2)$ *is open and* $\Phi_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \Phi_2(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))$ *is a* $\mathscr{C}^{s_0+2}$ *diffeomorphism.*
-

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = (I + \mathcal{A}_2) \begin{bmatrix} \Phi_2^* X \\ \Phi_2^* L \end{bmatrix},$$

where $\mathcal{A}_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C})$, $\mathcal{A}_2(0) = 0$, *and* $\|\mathcal{A}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$.

To prove Proposition 7.19, we prove the following lemma.

Lemma 7.20. *Fix* $\gamma > 0$. *There exists* $\sigma = \sigma(n, r, s_0, \gamma) > 0$ *such that if* $\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma$, *there exists* $H \in \mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2); \mathbb{R}^r \times \mathbb{C}^n)$ *such that*

(i) $H(t, z) = (t, z) + R(t, z)$, $R(0, 0) = 0$, $d_{t,x}R(0, 0) = 0_{(r+2n)\times(r+2n)}$.
(ii) $\|H\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
(iii) $H : B_{\mathbb{R}^r\times\mathbb{C}^n}(2) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(3)$ *is injective,* $H(B_{\mathbb{R}^r\times\mathbb{C}^n}(2)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(3)$ *is open, and* $H : B_{\mathbb{R}^r\times\mathbb{C}^n}(2) \to H(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))$ *is a diffeomorphism.*
(iv) $B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \subseteq H(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))$.
(v) $\|H^{-1}\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
(vi) *Let* $V_k := H_* X_k$ *and* $W_j = H_* L_j$. *Then there exists a matrix* $M \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1); \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C}))$ *with* $M(0) = 0$ *and such that:*
   - $\|M\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
   - *If*

$$\begin{bmatrix} \widetilde{X} \\ \widetilde{L} \end{bmatrix} := (I + M) \begin{bmatrix} V \\ W \end{bmatrix},$$

then

$$\widetilde{X} = \frac{\partial}{\partial u} + B\frac{\partial}{\partial w}, \quad \widetilde{L} = \frac{\partial}{\partial \overline{w}} + D\frac{\partial}{\partial w},$$

where we are using the matrix notation from Section 7.4. We have

$$\|B\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}, \|D\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \leq \gamma,$$
$$\|B\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}, \|D\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} 1, \quad \forall s > 0,$$

and $B(0) = 0$, $D(0) = 0$. *Finally, if we let* B_k *denote the kth row of* B, *and similarly for* D_j, *we have*

$$\sum_{k=1}^{r} \frac{\partial B_k}{\partial u_k} + \sum_{j=1}^{n} \frac{\partial D_j}{\partial w_j} = 0. \tag{7.12}$$

- $\widetilde{X}_1, \ldots, \widetilde{X}_r, \widetilde{L}_1, \ldots, \widetilde{L}_n$ *commute on* $B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$.

First we see why Lemma 7.20 gives Proposition 7.19

Proof of Proposition 7.19. Take $\gamma = \gamma(n, r, s_0) > 0$ as in Proposition 7.15. We take $\sigma = \sigma(n, r, s_0, \gamma) > 0$ as in Lemma 7.20. With this choice of σ and γ, Lemma 7.20 shows that Proposition 7.15 applies to the vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_r, \widetilde{L}_1, \ldots, \widetilde{L}_n$ from Lemma 7.20 (and constants which are admissible in the sense of Proposition 7.15 are $\{s_0\}$-admissible in the sense of this section)–here we are taking $A \equiv 0$, $C \equiv 0$, $E \equiv 0$, and $F \equiv 0$ in Proposition 7.15.

Thus, we obtain an $\{s_0\}$-admissible constant $\eta_2 > 0$ and a map $\Phi_1 : B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_2) \to B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ as in Proposition 7.15. Set $\eta_3 := \eta_2/2$. For each $s > 0$, we have, using Lemma 5.8,

$$\|\Phi_1\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \leq C_{s,\eta_2} \|\Phi_1\|_{\mathscr{B}^{r+2n, r+2n}_{\eta_2}} \lesssim_{\{s_0\}} C_{s,\eta_2},$$

where C_{s,η_2} can be chosen to depend only on s and η_2. We conclude,

$$\|\Phi_1\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1, \quad \forall s > 0.$$

Similarly, if $\mathcal{A}_1$ is as in Proposition 7.15, we have $\mathcal{A}_1(0) = 0$, and using Lemma 5.8, $\|\mathcal{A}_1\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} \|\mathcal{A}_1\|_{\mathscr{B}^{r+2n, \mathbb{M}^{(n+r)\times(n+r)}}_{\eta_2}} \leq 1$, $\forall s > 0$, and

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial w} \end{bmatrix} = (I + \mathcal{A}_1) \begin{bmatrix} \Phi_1^* \widetilde{X} \\ \Phi_1^* \widetilde{L} \end{bmatrix}.$$

We have, with M and H as in Lemma 7.20,

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial w} \end{bmatrix} &= (I + \mathcal{A}_1)(I + M \circ \Phi_1) \begin{bmatrix} \Phi_1^* V \\ \Phi_1^* W \end{bmatrix} = (I + \mathcal{A}_1)(I + M \circ \Phi_1) \begin{bmatrix} (H^{-1} \circ \Phi_1)^* X \\ (H^{-1} \circ \Phi_1)^* L \end{bmatrix} \\ &=: (I + \mathcal{A}_2) \begin{bmatrix} \Phi_2^* X \\ \Phi_2^* L \end{bmatrix}, \end{aligned}$$

where $\Phi_2 = H^{-1} \circ \Phi_1$ and $I + \mathcal{A}_2 = (I + \mathcal{A}_1)(I + M \circ \Phi_1)$. Since we have already noted $\|\Phi_1\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$ and $\|\mathcal{A}_1\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$, and Lemma 7.20 gives $\|M\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$, $\forall s > 0$, it follows from Proposition 5.1 and Lemma 5.3 that $\|\mathcal{A}_2\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$. Since $\mathcal{A}_1(0) = 0$, $M(0) = 0$, and $\Phi_1(0) = 0$, we have $\mathcal{A}_2(0) = 0$. Since $\|H^{-1}\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$ (by Lemma 7.20) and $\|\Phi_1\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, for all $s > 0$, it follows from Lemma 5.3

that $\|\Phi_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s>0$. $\Phi_2(0) = H^{-1}(\Phi_1(0)) = H^{-1}(0) = 0$, since $H(0)=0$. $d_{t,x}\Phi_2(0) = (d_{t,x}H(0))^{-1}d_{t,x}\Phi_1(0) = I\cdot I = I$. Finally, that Φ_2 is a diffeomorphism onto its image follows from the corresponding results for H and Φ_1 in Lemma 7.20 and Proposition 7.15. □

Proof of Lemma 7.20. Let $\sigma_0 = \sigma_0(n, r, s_0, \gamma) > 0$ be a small constant (depending only on n, r, s_0, and γ), to be chosen later. We will find H of the form $H(t,z) = (t,z)+R(t,z)$, where $R(t,z) = (0, R_2(t,z))$, $R_2 \in \mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2);\mathbb{C}^n)$, $R_2(0,0)=0$, $dR_2(0,0)=0$, and $\|R_2\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma_0$. Note that if $\sigma_0>0$ is sufficiently small (depending only on n and r), (iii) and (iv) follow immediately from the inverse function theorem. Moreover, we will also have

$$\inf_{(t,z)\in B_{\mathbb{R}^r\times\mathbb{C}^n}(2)} |\det dH(t,z)| \geq \frac{1}{2}. \tag{7.13}$$

Henceforth, we take $\sigma_0>0$ so small that these consequences hold.

We begin by studying an arbitrary $H(t,z)$ of the form $H(t,z) = (t,z) + (0, R_2(t,z))$ with $\|R_2\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma_0$, $R_2(0,0)=0$, $dR_2(0,0)=0$ (we will later specialize to a specific choice of R_2). In what follows, for $s>0$ if we write $A\lesssim_s B$, it means that we assume $R_2\in\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2);\mathbb{C}^n)$ and $A\leq CB$ where C is a positive $\{s\}$-admissible constant which is also allowed to depend on an upper bound for $\|R_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))}$. At the end of the proof, we will choose a particular R_2 with $\|R_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))} \lesssim_{\{s\}} 1$; once we do this, $\lesssim_s$ and $\lesssim_{\{s\}}$ will denote the same thing.

For such H, by the above remarks, it makes sense to consider $H^{-1}: B_{\mathbb{R}^r\times\mathbb{C}^n}(1)\to B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2)$. Moreover, it follows from Lemma 5.4 (using (7.13)) that

$$\|H^{-1}\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_s 1. \tag{7.14}$$

Set $H_1(t,z)=t$, $H_2(t,z)=z+R_2(t,z)$ so that $H(t,z)=(H_1(t,z),H_2(t,z))$. We have the following obvious equalities:

$$\begin{aligned} &d_tH_1 = I, d_zH_1 = 0, d_{\overline{z}}H_1 = 0, d_tH_2 = d_tR_2, d_zH_2 = I + d_zR_2,\\ &d_{\overline{z}}H_2 = d_{\overline{z}}R_2, d_t\overline{H_2} = d_t\overline{R_2}, d_z\overline{H_2} = d_z\overline{R_2}, d_{\overline{z}}\overline{H_2} = I + d_{\overline{z}}\overline{R_2}. \end{aligned} \tag{7.15}$$

Using the notation from Section 6, we have (thinking of H mapping the (t,z) variable to the (u,w) variable):

$$\begin{aligned} H_*\frac{\partial}{\partial t} &= (d_tH_1(t,z))^\top\frac{\partial}{\partial u} + (d_tH_2(t,z))^\top\frac{\partial}{\partial w} + (d_t\overline{H_2}(t,z))^\top\frac{\partial}{\partial \overline{w}}\Big|_{(t,z)=H^{-1}(u,w)},\\ H_*\frac{\partial}{\partial z} &= (d_zH_1(t,z))^\top\frac{\partial}{\partial u} + (d_zH_2(t,z))^\top\frac{\partial}{\partial w} + (d_z\overline{H_2}(t,z))^\top\frac{\partial}{\partial \overline{w}}\Big|_{(t,z)=H^{-1}(u,w)},\\ H_*\frac{\partial}{\partial \overline{z}} &= (d_{\overline{z}}H_1(t,z))^\top\frac{\partial}{\partial u} + (d_{\overline{z}}H_2(t,z))^\top\frac{\partial}{\partial w} + (d_{\overline{z}}\overline{H_2}(t,z))^\top\frac{\partial}{\partial \overline{w}}\Big|_{(t,z)=H^{-1}(u,w)}. \end{aligned}$$

Thus, if $V = H_*X$ and $W = H_*L$, using (7.15) we have

$$\begin{aligned}V(u,w) &= \frac{\partial}{\partial u} + \left[(d_t R_2(t,z))^\top + E(t,z)(I + d_z R_2(t,z))^\top\right]\frac{\partial}{\partial w}\\ &\quad + \left[(d_t \overline{R_2}(t,z))^\top + E(t,z)(d_z \overline{R_2}(t,z))^\top\right]\frac{\partial}{\partial \overline{w}}\bigg|_{(t,z)=H^{-1}(u,w)},\\ W(u,w) &= \frac{\partial}{\partial \overline{w}} + \left[(d_{\overline{z}} R_2(t,z))^\top + F(t,z)(I + d_z R_2(t,z))^\top\right]\frac{\partial}{\partial w}\\ &\quad + \left[(d_{\overline{z}} \overline{R_2}(t,z))^\top + F(t,z)(d_z \overline{R_2}(t,z))^\top\right]\frac{\partial}{\partial \overline{w}}\bigg|_{(t,z)=H^{-1}(u,w)}.\end{aligned}$$

Our goal is to pick $\sigma = \sigma(n, r, s_0, \gamma) > 0$ so that the conclusions of the lemma hold for

$$\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma.$$

We will choose σ at the end of the proof; but we will ensure $\sigma \leq 1$, so that we may henceforth assume $\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq 1$. Using this and the assumption $\|R_2\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma_0$, we have, by taking $\sigma_0 > 0$ sufficiently small (depending only on n and r),

$$\inf_{(t,z)\in B_{\mathbb{R}^r\times\mathbb{C}^n}(2)} \left|\det\left[I + (d_{\overline{z}}\overline{R_2}(t,z))^\top + F(t,z)(d_z\overline{R_2}(t,z))^\top\right]\right| \geq \frac{1}{2}.$$

Thus, $I + d_{\overline{z}}\overline{R_2}^\top + F d_z \overline{R_2}^\top$ is invertible on $B_{\mathbb{R}^r\times\mathbb{C}^n}(2)$ and Remark 5.2 implies

$$\left\|\left(I + d_{\overline{z}}\overline{R_2}^\top + F d_z\overline{R_2}^\top\right)^{-1}\right\|_{\mathscr{C}_{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2);\mathbb{M}^{n\times n})} \lesssim_s 1. \tag{7.16}$$

We define a matrix $M(u,w) : H(B_{\mathbb{R}^r\times\mathbb{C}^n}(2)) \to \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C})$ by

$$I + M\circ H := \left[\begin{array}{c|c} I_{r\times r} & -\left(d_t\overline{R_2}^\top + E d_z\overline{R_2}^\top\right)\left(I + d_{\overline{z}}\overline{R_2}^\top + F d_z\overline{R_2}^\top\right)^{-1} \\ \hline 0_{n\times r} & \left(I + d_{\overline{z}}\overline{R_2}^\top + F d_z\overline{R_2}^\top\right)^{-1}\end{array}\right],$$

where each part of the above equation is evaluated at (t,z) and we are using notation like $d_t\overline{R_2}^\top$ to mean $(d_t\overline{R_2}(t,z))^\top$. In particular, since $B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \subseteq H(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))$ (by (iv) which we have already verified), M is defined on $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

By (7.16) and Proposition 5.1, we have $\|I + M\circ H\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))} \lesssim_s 1$. Combining this with (7.14), Lemma 5.3 shows

$$\|M\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_s 1, \quad \forall s > 0.$$

Also, since $dR_2(0) = 0$ and $H(0) = 0$, we have $M(0) = 0$. Set

$$\begin{bmatrix} \widetilde{X} \\ \widetilde{L} \end{bmatrix} := (I + M) \begin{bmatrix} V \\ W \end{bmatrix}. \tag{7.17}$$

Note that

$$\widetilde{X} = \frac{\partial}{\partial u} + B\frac{\partial}{\partial w}, \quad \widetilde{L} = \frac{\partial}{\partial \overline{w}} + D\frac{\partial}{\partial w}, \tag{7.18}$$

where B and D depend on E, F, and R_2, and (in what follows each function is evaluated at (t, z) unless otherwise mentioned):

$$\begin{aligned} B(u,w) &= B[E,F,R_2](u,w) \\ &= \left(d_t R_2^\top + E(I + d_z R_2^\top)\right) \\ &- \left(d_t \overline{R_2}^\top + E d_z \overline{R_2}^\top\right)\left(I + d_{\overline{z}} \overline{R_2}^\top + F d_z \overline{R_2}^\top\right)^{-1}\left(d_{\overline{z}} R_2^\top + F(I + d_z R_2^\top)\right)\Big|_{(t,z)=H^{-1}(u,w)}, \end{aligned} \tag{7.19}$$

$$\begin{aligned} D(u,w) &= D[E,F,R_2](u,w) \\ &= \left(I + d_{\overline{z}} \overline{R_2}^\top + F d_z \overline{R_2}^\top\right)^{-1} \left(d_{\overline{z}} R_2^\top + F(I + d_z R_2^\top)\right)\Big|_{(t,z)=H^{-1}(u,w)}. \end{aligned} \tag{7.20}$$

Note that since $E(0) = 0$, $F(0) = 0$, $dR_2(0) = 0$, and $H^{-1}(0) = 0$, we have $B(0) = 0$ and $D(0) = 0$. Let $\sigma_1 = \sigma_1(n, r, s_0, \gamma) \in (0, 1]$ be a small constant to be chosen later. At the end of the proof, we will take $\sigma \le \sigma_1$ so we may assume $\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))} \le \sigma_1$. We have, using (7.14), (7.16), Proposition 5.1, and Lemma 5.3,

$$\|D\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s_0\}} \|d_{\overline{z}} R_2^\top + F(I + d_z R_2^\top)\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))} \lesssim_{\{s_0\}} \sigma_0 + \sigma_1,$$

and

$$\|D\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_s \|d_{\overline{z}} R_2^\top + F(I + d_z R_2^\top)\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(3/2))} \lesssim_s 1, \quad \forall s > 0.$$

Similarly, we have

$$\|B\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s_0\}} \sigma_0 + \sigma_1, \quad \|B\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_s 1, \quad \forall s > 0.$$

In particular, if we take σ_0 and σ_1 sufficiently small (depending only on n, r, s_0, and γ), we have

$$\|B\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))}, \|D\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \le \gamma.$$

Next we claim that $\widetilde{X}_1, \ldots, \widetilde{X}_r, \widetilde{L}_1, \ldots, \widetilde{L}_n$ commute. We are given that $X_1, \ldots, X_r$, $L_1, \ldots, L_n$ commute (see Remark 7.17), and it follows that $V_1, \ldots, V_r, W_1, \ldots, W_n$ commute. Since $I + M(u, w)$ is clearly an invertible matrix by its definition, (7.17) shows $\forall (u, w) \in H(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))$, $\forall j, k, j_1, j_2, k_1, k_2$,

$$\begin{aligned}&[\widetilde{X}_{k_1}, \widetilde{X}_{k_2}](u, w), [\widetilde{L}_{j_1}, \widetilde{L}_{j_2}](u, w), [\widetilde{X}_k, \widetilde{L}_j](u, w)\\ &\quad \in \operatorname{span}_{\mathbb{C}}\{V_1(u, w), \ldots, V_r(u, w), W_1(u, w), \ldots, W_n(u, w)\}\\ &\quad = \operatorname{span}_{\mathbb{C}}\{\widetilde{X}_1(u, w), \ldots, \widetilde{X}_r(u, w), \widetilde{L}_1(u, w), \ldots, \widetilde{L}_n(u, w)\}.\end{aligned}$$

Because of the form of $\widetilde{X}$ and $\widetilde{L}$ given in (7.18) this implies $\widetilde{X}_1, \ldots, \widetilde{X}_r, \widetilde{L}_1, \ldots, \widetilde{L}_n$ commute (just as in Remark 7.17).

So far we have shown that if we have R_2 as above with $R_2(0) = 0$, $dR_2(0) = 0$, $\|R_2\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(2))} \le \sigma_0$, and have $\|R_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(3/2))} \lesssim_{\{s\}} 1$, then all of the conclusions of the lemma hold, except possibly for (7.12). Thus all that remains to show is that we can pick such an R_2 so that (7.12) holds (provided σ is small enough). To do this we use Proposition B.4.

Given E, F, and R_2, we define $B = B[E, F, R_2]$ and $D = D[E, F, R_2]$ by (7.19) and (7.20). We let $B_{k,l}$ denote the (k, l) component of the matrix B, and similarly for D. For $(t, z) \in B_{\mathbb{R}^r \times \mathbb{C}^n}(2)$ and $1 \le m \le n$,

$$\Psi_m(E, F, R_2)(t, z) := \sum_{k=1}^{r} \frac{\partial B_{k,m}(u, w)}{\partial u_k} + \sum_{j=1}^{n} \frac{\partial D_{j,m}(u, w)}{\partial w_j}\bigg|_{(u,w)=H(t,z)}.$$

Set $\Psi(E, F, R_2) := (\Psi_1(E, F, R_2), \ldots, \Psi_n(E, F, R_2))$. Note that (7.12) follows from $\Psi(E, F, R_2) = 0$, so our goal is to solve for R_2 (in terms of E and F) so that $\Psi(E, F, R_2) = 0$.

Letting $R(t, z) = (0, R_2(t, z))$, for any function $K(t, x)$ we have

$$\frac{\partial}{\partial u_k} K(H^{-1}(u, w))\bigg|_{(u,w)=H(t,z)} = dK(t, z)(I + dR(t, z))^{-1} e_k,$$

where e_k is the kth standard basis element–what is important is that the right hand side is a function of $dK(t, z)$ and $dR_2(t, z)$. Similar comments hold for $\frac{\partial}{\partial y_l} K(H^{-1}(s, w))$ where $w_j = y_j + i y_{j+n}$. Thus, using the formulas for B and D in (7.19) and (7.20), using the notation of Proposition B.4, and writing $z_j = x_j + i x_{j+n}$ we see that there is a smooth function g, taking values in $\mathbb{C}^n$, which vanishes at the origin, such that

$$\Psi(E, F, R_2)(t, x) = g(\mathscr{D}^1 E(t, x), \mathscr{D}^1 F(t, x), \mathscr{D}^2 R_2(t, x)).$$

Furthermore, the function g depends only on n and r. Also it is easy to see that g is quasi-linear in R_2 in the sense of (B.3).[15]

[15] It is not necessary for what follows that g be quasi-linear; though the proof of Proposition B.4 is simpler in the quasi-linear case.

To apply Proposition B.4, we wish to show that g is elliptic in R_2 at $E = 0$, $F = 0$, $R_2 = 0$, in the sense of that proposition. I.e., define $\mathcal{E}_2$ as in Proposition B.4; we wish to show $\mathcal{E}_2$ is elliptic. Note the map

$$R_2 \mapsto \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi(0,0,\epsilon R_2)$$

is a second order, constant coefficient differential operator acting on R_2 whose principal symbol is $\mathcal{E}_2$. Thus we wish to show that this operator is elliptic.

To make the dependance of H on R_2 explicit, we write H_{R_2} in place of H. I.e., $H_{R_2}(t,z) = (t,z) + (0, R_2(t,z))$. It suffices to compute $\frac{d}{d\epsilon}\big|_{\epsilon=0}\Psi(0,0,\epsilon R_2)$ in the case $R_2 \in C^\infty$. In that case, we have $H_{\epsilon R_2}^{-1}(u,w) = (u,w) - \epsilon(0, R_2(u,w)) + O(\epsilon^2)$, and for example,

$$\begin{aligned}\epsilon(d_t R_2)(H_{\epsilon R_2}^{-1}(u,w)) &= \epsilon(d_t R_2)(u,w) + O(\epsilon^2) \text{ and}\\ \epsilon(d_t R_2)(H_{\epsilon R_2}(t,z)) &= \epsilon(d_t R_2)(t,z) + O(\epsilon^2),\end{aligned} \tag{7.21}$$

and similarly for d_t replaced by $d_{\overline{z}}$. Here, $O(\epsilon^2)$ it denotes a term which is C^∞ in the variables (t,z) or (u,w) and every derivative, of every order ≥ 0, in these variables is $O(\epsilon^2)$ as $\epsilon \to 0$.

Thus, using the formulas (7.19) and (7.20), we have

$$B[0,0,\epsilon R_2](s,w) = \epsilon d_t R_2(s,w)^\top + O(\epsilon^2), \quad D[0,0,\epsilon R_2](s,w) = \epsilon d_{\overline{z}} R_2(s,w)^\top + O(\epsilon^2). \tag{7.22}$$

We write $R_2(t,z) = (R_{2,1}(t,z), \ldots, R_{2,n}(t,z))$. We also write $d_t R_2(s,w)_{l,k}$ for the (l,k) component of the matrix $d_t R_2$, and similarly for $d_{\overline{z}} R_2$ (see the discussion of this notation in Section 6). Using this notation, plugging (7.22) into the definition of Ψ, and using (7.21), we have for $1 \leq m \leq n$,

$$\begin{aligned}&\Psi_m(0,0,\epsilon R_2)(t,z)\\ &\quad= \epsilon \sum_{k=1}^{r} \frac{\partial}{\partial u_k}(d_t R_2)(u,w)_{m,k} + \epsilon \sum_{j=1}^{n} \frac{\partial}{\partial w_j}(d_{\overline{z}} R_2)(u,w)_{m,j} + O(\epsilon^2)\bigg|_{(u,w)=H_{\epsilon R_2}(t,z)}\\ &\quad= \epsilon \sum_{k=1}^{r} \frac{\partial^2}{\partial t_k^2} R_{2,m}(t,z) + \epsilon \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} R_{2,m}(t,z) + O(\epsilon^2).\end{aligned}$$

We conclude

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi(0,0,\epsilon R_2) = \left(\sum_{k=1}^{r} \frac{\partial^2}{\partial t_k^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j}\right) R_2,$$

and is therefore elliptic, as desired.

We apply Proposition B.4 with $D = 2$, $\eta = 3/2$, and

$$N = \{R_2 \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2);\mathbb{C}^n) : \|R_2\|_{\mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2);\mathbb{C}^n)} < \sigma_0\}.$$

We conclude that there exists $\sigma_2 > 0$ (depending only on n, r, s_0, and σ_0–since g depends only on n and r) so that if $\|E\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))}, \|F\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma_2$, then we may find $R_2 = R_2(E,F) \in N$ so that $\Psi(E,F,R_2) = 0$. The conclusions of Proposition B.4 show that this R_2 satisfies $R_2(0) = 0$, $dR_2(0) = 0$, and $\|R_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(3/2))} \lesssim_{\{s\}} 1$, $\forall s > 0$. Setting $\sigma := \min\{\sigma_1, \sigma_2\}$ completes the proof. □

7.6. Commuting vector fields

Fix $\eta_0 > 0$, $s_0 \in (0,\infty)\cup\{\omega\}$, and let $X_1,\ldots,X_r, L_1,\ldots,L_n$ be complex $\mathscr{C}^{s_0+1}$ vector fields on $B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$ of the form

$$X = \frac{\partial}{\partial t} + E\frac{\partial}{\partial z}, \quad L = \frac{\partial}{\partial \overline{z}} + F\frac{\partial}{\partial z},$$

where $E(0) = 0$, $F(0) = 0$, we are using the matrix notation from Section 7.4, and:

- If $s_0 \in (0,\infty)$, $E \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0); \mathbb{M}^{r\times n}(\mathbb{C}))$ and $F \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0); \mathbb{M}^{n\times n}(\mathbb{C}))$.
- If $s_0 = \omega$, $E \in \mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{r\times n}(\mathbb{C}))$ and $F \in \mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{n\times n}(\mathbb{C}))$.

We suppose $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$,

$$[X_{k_1}, X_{k_2}](\zeta), [L_{j_1}, L_{j_2}](\zeta), [X_k, L_j](\zeta) \in \operatorname{span}_{\mathbb{C}}\{X_1(\zeta),\ldots,X_r(\zeta), L_1(\zeta),\ldots,L_n(\zeta)\},$$
$$\forall j, k, j_1, j_2, k_1, k_2.$$

As in Remark 7.17, this is the same as assuming the vector fields commute.

Definition 7.21. If $s_0 \in (0,\infty)$, for $s \in [s_0,\infty)$, if we say C is an $\{s\}$-admissible constant, it means that we assume $E, F \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))$. C can then be chosen to depend only on s, s_0, n, r, η_0, and upper bounds for $\|E\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))}$ and $\|F\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))}$. For $s \in (0, s_0)$, we define $\{s\}$-admissible constants to be $\{s_0\}$-admissible constants.

Definition 7.22. If $s_0 = \omega$, we say C is an $\{\omega\}$-admissible constant if C can be chosen to depend only on n, r, η_0, and upper bounds for $\|E\|_{\mathscr{A}^{r+2n,\eta_0}}$ and $\|F\|_{\mathscr{A}^{r+2n,\eta_0}}$.

Proposition 7.23. *There exist $\{s_0\}$-admissible constants $\eta_3 > 0$, $K_1 \geq 1$ and a map $\Phi_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$ such that:*

(i) • *If $s_0 \in (0,\infty)$, $\Phi_3 \in \mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3); \mathbb{R}^r \times \mathbb{C}^n)$ and $\|\Phi_3\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$.*

- *If* $s_0 = \omega$, $\Phi_3 \in \mathscr{A}^{r+2n,2\eta_3}(\mathbb{R}^r \times \mathbb{C}^n)$ *and* $\|\Phi_3\|_{\mathscr{A}^{r+2n,2\eta_3}} \leq \eta_0$.

(ii) $\Phi_3(0) = 0$ *and* $d_{t,x}\Phi_3(0) = K_1^{-1} I_{(r+2n)\times(r+2n)}$.

(iii) $\Phi_3(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$ *is open and* $\Phi_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \Phi_3(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))$ *is a diffeomorphism.*

(iv)

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = K_1^{-1}(I + \mathcal{A}_2) \begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix},$$

where $\mathcal{A}_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C})$, $\mathcal{A}_2(0) = 0$, *and:*

- *If* $s_0 \in (0, \infty)$, $\|\mathcal{A}_2\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
- *If* $s_0 = \omega$, $\|\mathcal{A}_2\|_{\mathscr{A}^{2n+r,\eta_3}} \lesssim 1$.

Remark 7.24. If $s_0 \in (0, \infty)$ we will show η_3 depends only on n, r, and s_0. For $s_0 = \omega$, we will take $K_1 = 1$. This is not important in the sequel, however.

Proof of Proposition 7.23 when $s_0 = \omega$**.** Since $E \in \mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{r\times n}) \subseteq \mathscr{B}_{\eta_0}^{r+2n,\mathbb{M}^{r\times n}}$, $F \in \mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{n\times n}) \subseteq \mathscr{B}_{\eta_0}^{r+2n,\mathbb{M}^{n\times n}}$, and

$$\|E\|_{\mathscr{B}_{\eta_0}^{r+2n,\mathbb{M}^{r\times n}}} \leq \|E\|_{\mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{r\times n})} \lesssim_{\{\omega\}} 1, \text{ and}$$

$$\|F\|_{\mathscr{B}_{\eta_0}^{r+2n,\mathbb{M}^{n\times n}}} \leq \|F\|_{\mathscr{A}^{r+2n,\eta_0}(\mathbb{M}^{n\times n})} \lesssim_{\{\omega\}} 1,$$

we see that Proposition 7.12 applies to the vector fields $X_1, \ldots, X_r, L_1, \ldots, L_n$ and every constant which is admissible in the sense of that proposition is $\{\omega\}$-admissible here.

Thus, we obtain an $\{\omega\}$-admissible constant $\eta_2 > 0$ and a map $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_2) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$ as in that proposition. Letting $\mathcal{A}_1$ be the matrix from that proposition, and setting $\eta_3 := \eta_2/4$, we have (using Lemma 5.8)

$$\|\Phi_1\|_{\mathscr{A}^{r+2n,2\eta_3}} \lesssim_{\{\omega\}} \|\Phi_1\|_{\mathscr{B}_{\eta_2}^{r+2n,r+2n}} \lesssim_{\{\omega\}} 1,$$

$$\|\mathcal{A}_1\|_{\mathscr{A}^{r+2n,\eta_3}} \lesssim_{\{\omega\}} \|\mathcal{A}_1\|_{\mathscr{B}_{\eta_2}^{r+2n,\mathbb{M}^{(n+r)\times(n+r)}}} \leq 1.$$

Taking $\Phi_3 := \Phi_1$, $\mathcal{A}_2 := \mathcal{A}_1$, and $K_1 := 1$, all of the conclusions of Proposition 7.23 now follow from the corresponding conclusions in Proposition 7.12. □

We now turn to the proof of Proposition 7.23 when $s_0 \in (0, \infty)$. Because of the definition of $\{s\}$-admissible constants, it suffices to prove the result just for $s \in [s_0, \infty)$, and that is how we will proceed. We begin with a lemma.

Lemma 7.25. *Define for* $\gamma > 0$, $\Psi_\gamma : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0/\gamma) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$ *by* $\Psi_\gamma(t, z) = (\gamma t, \gamma z)$. *Let* $X_k^\gamma := \gamma\Psi_\gamma^* X_k$ *and* $L_j^\gamma := \gamma\Psi_\gamma^* L_j$. *Then,*

$$X^\gamma = \frac{\partial}{\partial t} + E_\gamma \frac{\partial}{\partial z}, \quad L^\gamma = \frac{\partial}{\partial \overline{z}} + F_\gamma \frac{\partial}{\partial z}, \tag{7.23}$$

where $E_\gamma(0) = 0$, $F_\gamma(0) = 0$, *and for* $0 < \gamma \leq \min\{\eta_0/2, 1\}$, $s \in [s_0, \infty)$,

$$\|E_\gamma\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2);\mathbb{M}^{r\times n})}, \|F_\gamma\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2);\mathbb{M}^{n\times n})} \lesssim_{\{s\}} \gamma. \tag{7.24}$$

Finally, $X_1^\gamma, \ldots, X_r^\gamma, L_1^\gamma, \ldots, L_n^\gamma$ *commute.*

Proof. That $X_1^\gamma, \ldots, X_r^\gamma, L_1^\gamma, \ldots, L_n^\gamma$ commute follows immediately from the same property of $X_1, \ldots, X_r, L_1, \ldots, L_n$. Note that (7.23) holds with $E_\gamma(t, z) = E(\gamma t, \gamma z)$ and $F_\gamma(t, z) = F(\gamma t, \gamma z)$. Thus, since $E(0) = 0$ and $F(0) = 0$, the same is true for E_γ and F_γ, and we have for $0 < \gamma \leq \min\{\eta_0/2, 1\}$, using Lemma 5.9,

$$\|E_\gamma\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq 46\gamma \|E\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))} \lesssim_{\{s\}} \gamma,$$

and similarly for F_γ. This completes the proof. □

Proof of Proposition 7.23 when $s_0 \in (0, \infty)$. Let $\sigma = \sigma(n, r, s_0) > 0$ be the constant from Proposition 7.19. For $\gamma \leq \eta_0/2$, define Ψ_γ, X^γ, L^γ, E_γ, and F_γ as in Lemma 7.25. By (7.24), if $\gamma \in (0, \eta_0/2]$ is a sufficiently small $\{s_0\}$-admissible constant (without loss of generality, $\gamma \leq 1$), we have

$$\|E_\gamma\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))}, \|F_\gamma\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(2))} \leq \sigma.$$

With this choice of γ, Proposition 7.19 applies to the vector fields X^γ and L^γ to yield a constant $\eta_3 = \eta_3(n, r, s_0) > 0$ and a map $\Phi_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(2)$ as in that result, and any constant which is $\{s\}$-admissible in that proposition is $\{s\}$-admissible in the sense of this section. Set $\Phi_3 := \Psi_\gamma \circ \Phi_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$. We take $K_1 := \gamma^{-1} \geq 1$. Since γ is $\{s_0\}$-admissible and $\|\Phi_2\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s$ (by Proposition 7.19), we have $\|\Phi_3\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, $\forall s$. Also, $\Phi_3(0) = \Psi_\gamma(\Phi_2(0)) = \Psi_\gamma(0) = 0$, and $d_{t,x}\Phi_3(0) = \gamma d_{t,x}\Phi_2(0) = K_1^{-1} I_{(2n+r)\times(2n+r)}$. That Φ_3 is a diffeomorphism onto its image follows from the corresponding result about Φ_2 in Proposition 7.19. Finally, if $\mathcal{A}_2$ is as in Proposition 7.19, we have,

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = (I + \mathcal{A}_2) \begin{bmatrix} \Phi_2^* X^\gamma \\ \Phi_2^* L^\gamma \end{bmatrix} = (I + \mathcal{A}_2) K_1^{-1} \begin{bmatrix} \Phi_2^* \Psi_\gamma^* X \\ \Phi_2^* \Psi_\gamma^* L \end{bmatrix} = (I + \mathcal{A}_2) K_1^{-1} \begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix}.$$

All of the desired estimates for $\mathcal{A}_2$ are stated in Proposition 7.19 and this completes the proof. □

7.7. Proof of Proposition 7.7

Using the matrix notation of Section 7.4 we may write

$$X = \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial t} + B_2 \frac{\partial}{\partial z} + B_3 \frac{\partial}{\partial \overline{z}}, \quad L = \frac{\partial}{\partial \overline{z}} + B_4 \frac{\partial}{\partial t} + B_5 \frac{\partial}{\partial z} + B_6 \frac{\partial}{\partial \overline{z}},$$

where each B_l takes values in matrices of an appropriate size, $B_l(0) = 0$ for each l, and

- If $s_0 \in (0, \infty)$, $\|B_l\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$, $\forall s > 0$.
- If $s_0 = \omega$, $\|B_l\|_{\mathscr{A}^{r+2n,1}} \lesssim_{\{\omega\}} 1$.

Define M to be the $(r+n) \times (r+n)$ matrix:

$$M := \left[\begin{array}{c|c} B_1 & B_3 \\ \hline B_4 & B_6 \end{array}\right].$$

We have

$$\begin{bmatrix} X \\ L \end{bmatrix} = (I + M) \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \overline{z}} \end{bmatrix} + \begin{bmatrix} B_2 \\ B_5 \end{bmatrix} \frac{\partial}{\partial z},$$

and $M(0) = 0$.

- If $s_0 \in (0, \infty)$, we have $\|M\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s_0\}} 1$. Thus, by taking $\eta_0 > 0$ to be a sufficiently small $\{s_0\}$-admissible constant and using that $M(0) = 0$, we have

$$\inf_{\zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0)} |\det(I + M(\zeta))| \geq \frac{1}{2}.$$

  Remark 5.2 shows that $\|(I + M)^{-1}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0))} \lesssim_{\{s\}} 1$.
- If $s_0 = \omega$, we have $\|M\|_{\mathscr{A}^{2n+r,1}} \lesssim_{\{\omega\}} 1$. Since $M(0) = 0$, Lemma 5.7 implies $\|M\|_{\mathscr{A}^{2n+r,\eta_0}} \lesssim_{\{\omega\}} \eta_0$, for $\eta_0 \in (0, 1]$. Thus, by taking $\eta_0 > 0$ to be a sufficiently small $\{\omega\}$-admissible constant we have

$$\|M\|_{\mathscr{A}^{2n+r,\eta_0}(\mathbb{M}^{(r+n) \times (r+n)})} \leq \frac{1}{2}.$$

  Since $\mathscr{A}^{2n+r,\eta_0}(\mathbb{M}^{(r+n) \times (r+n})$ is a Banach algebra (Proposition 5.1) it follows that $\|(I + M)^{-1}\|_{\mathscr{A}^{2n+r,\eta_0}(\mathbb{M}^{(r+n) \times (r+n)})} \leq 2$; here we have used the Neumann series for $(1 + M)^{-1}$.

In either case we have an $\{s_0\}$-admissible constant $\eta_0 > 0$ so that $(I + M)^{-1}$ satisfies good estimates on $B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0)$.

Define vector fields $\widehat{X}_1, \ldots, \widehat{X}_r, \widehat{L}_1, \ldots, \widehat{L}_n$ on $B_{\mathbb{R}^r \times \mathbb{C}^n}(\eta_0)$ by

$$\begin{bmatrix} \widehat{X} \\ \widehat{L} \end{bmatrix} = (I+M)^{-1} \begin{bmatrix} X \\ L \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \overline{z}} \end{bmatrix} + (I+M)^{-1} \begin{bmatrix} B_2 \\ B_5 \end{bmatrix} \frac{\partial}{\partial z}.$$

Thus, we have

$$\widehat{X} = \frac{\partial}{\partial t} + \widehat{E}\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \overline{z}} + \widehat{F}\frac{\partial}{\partial z},$$

where $\widehat{E}(0)=0$, $\widehat{F}(0)=0$ and using Proposition 5.1 and the bounds for $(I+M)^{-1}$,

- If $s_0 \in (0,\infty)$, $\|\widehat{E}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))}, \|\widehat{F}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))} \lesssim_{\{s\}} 1$, $\forall s>0$.
- If $s_0=\omega$, $\|\widehat{E}\|_{\mathscr{A}^{2n+r,\eta_0}}, \|\widehat{F}\|_{\mathscr{A}^{2n+r,\eta_0}} \lesssim_{\{\omega\}} 1$.

Furthermore, we have $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$,

$$[\widehat{X}_{k_1}, \widehat{X}_{k_2}](\zeta), [\widehat{L}_{j_1}, \widehat{L}_{j_2}](\zeta), [\widehat{X}_k, \widehat{L}_j](\zeta) \in \operatorname{span}_{\mathbb{C}}\{\widehat{X}_1(\zeta), \ldots, \widehat{X}_r(\zeta), \widehat{L}_1(\zeta), \ldots, \widehat{L}_n(\zeta)\},$$

which follows from the corresponding assumption on the Xs and Ls (and the fact that $(I+M)^{-1}$ is an invertible matrix).

Proposition 7.23 applies to the vector fields $\widehat{X}, \widehat{L}$, and any constant which is $\{s\}$-admissible in the sense of that proposition is $\{s\}$-admissible in the sense of this section. We obtain $\{s_0\}$-admissible constants $\eta_3>0$, $K_1\geq 1$, a map $\Phi_3 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$, and a matrix $\mathcal{A}_2 : B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3) \to \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C})$ as in that proposition. (i), (ii), and (iii) follow immediately from the corresponding results in Proposition 7.23.

Next we establish (iv). We have, from Proposition 7.23,

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} &= K_1^{-1}(I+\mathcal{A}_2)\begin{bmatrix} \Phi_3^*\widehat{X} \\ \Phi_3^*\widehat{L} \end{bmatrix} = K_1^{-1}(I+\mathcal{A}_2)(I+M\circ\Phi_3)^{-1}\begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix} \\ &=: K_1^{-1}(I+\mathcal{A}_3)\begin{bmatrix} \Phi_3^* X \\ \Phi_3^* L \end{bmatrix}, \end{aligned}$$

where $I+\mathcal{A}_3 := (I+\mathcal{A}_2)(I+M\circ\Phi_3)^{-1}$. Since $M(0)=0$, $\Phi_3(0)=0$, and $\mathcal{A}_2(0)=0$, we have $\mathcal{A}_3(0)=0$. Also, we have

- If $s_0\in(0,\infty)$, since $\|(I+M)^{-1}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0))} \lesssim_{\{s\}} 1$, $\|\Phi_3\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$, and $\Phi_3(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_0)$, it follows from Lemma 5.3 that $\|(I+M\circ\Phi_3)^{-1}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$. Combining this with $\|\mathcal{A}_2\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$ (see Proposition 7.23), Proposition 5.1 implies $\|\mathcal{A}_3\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(\eta_3))} \lesssim_{\{s\}} 1$.
- If $s_0=\omega$, since $\|(I+M)^{-1}\|_{\mathscr{A}^{2n+r,\eta_0}} \leq 2$ and since $\|\Phi_3\|_{\mathscr{A}^{r+2n,\eta_3}} \leq \eta_0$, it follows from Lemma 5.10 that $\|(I+M\circ\Phi_3)^{-1}\|_{\mathscr{A}^{r+2n,\eta_3}} \leq 2$. Since $\|\mathcal{A}_2\|_{\mathscr{A}^{r+2n,\eta_3}} \lesssim_{\{\omega\}} 1$ (see Proposition 7.23), Proposition 5.1 implies $\|\mathcal{A}_3\|_{\mathscr{A}^{r+2n,\eta_3}} \lesssim_{\{\omega\}} 1$.

The above comments complete the proof.

8. Proof of the main result

In this section, we prove Theorem 4.18; Theorem 1.1 is an immediate consequence of Theorem 4.18. Throughout this section, fix $s \in (0, \infty] \cup \{\omega\}$ and let M be a $\mathscr{C}^{s+2}$ manifold. As in the rest of the paper, we give $\mathbb{R}^r \times \mathbb{C}^n$ coordinates $(t_1, \ldots, t_r, z_1, \ldots, z_n)$.

Lemma 8.1. *Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ elliptic structure on M of dimension (r, n). Then, $\forall \zeta_0 \in M$, there exists a neighborhood V_0 of ζ_0, $\mathscr{C}^{s+1}$ sections $L_1, \ldots, L_n, X_1, \ldots, X_r$ of $\mathscr{L}$ over V_0, and a $\mathscr{C}^{s+2}$ diffeomorphism $\Psi_0 : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V_0$ such that:*

(i) $\Psi_0(0) = \zeta_0$.
(ii) $\forall \zeta \in V_0$, $L_1(\zeta), \ldots, L_n(\zeta), X_1(\zeta), \ldots, X_r(\zeta)$ *is a basis for* $\mathscr{L}_\zeta$.
(iii) $\Psi_0^* L_j(0) = \frac{\partial}{\partial \bar{z}_j}$, $\Psi_0^* X_k(0) = \frac{\partial}{\partial t_k}$, $1 \leq j \leq n$, $1 \leq k \leq r$.
(iv) *For* $1 \leq j \leq n$, $1 \leq k \leq r$,
- *If* $s \in (0, \infty]$, $\Psi_0^* L_j, \Psi_0^* X_k \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{C}^{2n+r})$.
- *If* $s = \omega$, $\Psi_0^* L_j, \Psi_0^* X_k \in \mathscr{A}^{2n+r,1}(\mathbb{C}^{2n+r})$.

(v) $\forall j_1, j_2, k_1, k_2, j, k$, $\forall \xi \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,

$$\begin{aligned}&[\Psi_0^* L_{j_1}, \Psi_0^* L_{j_2}](\xi), [\Psi_0^* X_k, \Psi_0^* L_j](\xi), [\Psi_0^* X_{k_1}, \Psi_0^* X_{k_2}](\xi)\\ &\in \operatorname{span}_{\mathbb{C}}\{\Psi_0^* L_1(\xi), \ldots, \Psi_0^* L_n(\xi), \Psi_0^* X_1(\xi), \ldots, \Psi_0^* X_r(\xi)\}.\end{aligned}$$

Proof. Note that, by the definition of elliptic structures of dimension (r, n), we have $\dim \mathscr{L}_\zeta = n + r$, $\forall \zeta \in M$ and $\dim M = 2n + r$ (see Remark 3.8). By Lemma A.2 we may pick a basis $y_1, \ldots, y_r$ of $\mathscr{L}_{\zeta_0} \cap \overline{\mathscr{L}_{\zeta_0}}$ with $y_1, \ldots, y_r \in T_{\zeta_0} M$ (i.e., $y_1, \ldots, y_r$ are *real*). Extend $y_1, \ldots, y_r$ to a basis $l_1, \ldots, l_n, y_1, \ldots, y_r$ of $\mathscr{L}_{\zeta_0}$.

By the definition of a $\mathscr{C}^{s+1}$ bundle, we may find a neighborhood U_1 of ζ_0 and $\mathscr{C}^{s+1}$ sections $Z_1, \ldots, Z_K$ of $\mathscr{L}$ over U_0 such that $\forall \zeta \in U_0$, $\operatorname{span}_{\mathbb{C}}\{Z_1(\zeta), \ldots, Z_K(\zeta)\} = \mathscr{L}_\zeta$. Without loss of generality, reorder $Z_1, \ldots, Z_K$ so that $Z_1(\zeta_0), \ldots, Z_{n+r}(\zeta_0)$ form a basis of $\mathscr{L}_{\zeta_0}$. By continuity, there exists a neighborhood $U_2 \subseteq U_1$ of ζ_0 such that $\forall \zeta \in U_2$, $Z_1(\zeta), \ldots, Z_{n+r}(\zeta)$ are linearly independent. We conclude $\forall \zeta \in U_2$, $Z_1(\zeta), \ldots, Z_{n+r}(\zeta)$ forms a basis for $\mathscr{L}_\zeta$.

Let $M \in \mathbb{M}^{(n+r)\times(n+r)}(\mathbb{C})$ be the invertible matrix such that

$$M \begin{bmatrix} Z_1(\zeta_0) \\ \vdots \\ Z_{n+r}(\zeta_0) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_r \\ l_1 \\ \vdots \\ l_n \end{bmatrix}.$$

For $\zeta \in U_2$ set

$$\begin{bmatrix} \widehat{X}_1(\zeta) \\ \vdots \\ \widehat{X}_r(\zeta) \\ \widehat{L}_1(\zeta) \\ \vdots \\ \widehat{L}_n(\zeta) \end{bmatrix} := M \begin{bmatrix} Z_1(\zeta) \\ \vdots \\ Z_{n+r}(\zeta) \end{bmatrix}.$$

Since M is a (constant) invertible matrix, we have $\forall \zeta \in U_2$, $\widehat{L}_1(\zeta), \ldots, \widehat{L}_n(\zeta), \widehat{X}_1(\zeta), \ldots, \widehat{X}_r(\zeta)$ forms a basis for $\mathscr{L}_\zeta$, and $\widehat{L}_1, \ldots, \widehat{L}_n, \widehat{X}_1, \ldots, \widehat{X}_r$ are $\mathscr{C}^{s+1}$ sections of $\mathscr{L}$ over U_2.

By the definition of a $\mathscr{C}^{s+2}$ manifold (see also Remark 2.16) there exists a $\mathscr{C}^{s+2}$ diffeomorphism $\Psi_1 : B_{\mathbb{R}^{2n+r}}(\epsilon_1) \to V_1$, where $V_1 \subseteq U_2$ is a neighborhood of ζ_0, $\Psi_1(0) = \zeta_0$. Since $\widehat{X}_1(\zeta_0) = y_1, \ldots, \widehat{X}_r(\zeta_0) = y_r$ are real, since $\mathscr{L}_{\zeta_0} + \overline{\mathscr{L}_{\zeta_0}} = \mathbb{C}T_{\zeta_0}M$, and since $\widehat{X}_1(\zeta_0), \ldots, \widehat{X}_r(\zeta_0), \widehat{L}_1(\zeta_0), \ldots, \widehat{L}_n(\zeta_0)$ forms a basis for $\mathscr{L}_{\zeta_0}$, we have

$$\begin{aligned} \operatorname{span}_{\mathbb{R}}\{&2\mathrm{Re}(\widehat{L}_1)(\zeta_0), \ldots, 2\mathrm{Re}(\widehat{L}_n)(\zeta_0), 2\mathrm{Im}(\widehat{L}_1)(\zeta_0), \ldots, 2\mathrm{Im}(\widehat{L}_n)(\zeta_0), \\ &\widehat{X}_1(\zeta_0), \ldots, \widehat{X}_n(\zeta_0)\} = T_{\zeta_0}M. \end{aligned} \tag{8.1}$$

Pulling (8.1) back via Ψ_1 we have

$$\begin{aligned} &2\mathrm{Re}(\Psi_1^*\widehat{L}_1)(0), \ldots, 2\mathrm{Re}(\Psi_1^*\widehat{L}_n)(0), 2\mathrm{Im}(\Psi_1^*\widehat{L}_1)(0), \ldots, 2\mathrm{Im}(\Psi_1^*\widehat{L}_n)(0), \\ &\quad \Psi_1^*\widehat{X}_1(0), \ldots, \Psi_1^*\widehat{X}_n(0) \end{aligned}$$

forms a basis for $T_0\mathbb{R}^{2n+r}$.

We give $\mathbb{R}^{2n+r} \cong \mathbb{R}^r \times \mathbb{R}^{2n}$ coordinates $(t_1, \ldots, t_r, x_1, \ldots, x_{2n})$. Let $C \in \mathbb{M}^{(r+2n)\times(r+2n)}(\mathbb{R})$ denote the (constant) invertible matrix such that

$$C \begin{bmatrix} \frac{\partial}{\partial t_1} \\ \vdots \\ \frac{\partial}{\partial t_r} \\ \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{2n}} \end{bmatrix} = \begin{bmatrix} \Psi_1^*\widehat{X}_1(0) \\ \vdots \\ \Psi_1^*\widehat{X}_r(0) \\ 2\mathrm{Re}(\Psi_1^*\widehat{L}_1)(0) \\ \vdots \\ 2\mathrm{Re}(\Psi_1^*\widehat{L}_n)(0) \\ 2\mathrm{Im}(\Psi_1^*\widehat{L}_1)(0) \\ \vdots \\ 2\mathrm{Im}(\Psi_1^*\widehat{L}_n)(0) \end{bmatrix}.$$

Set $A = C^\top$ and we identify A with the corresponding invertible linear transformation $\mathbb{R}^{r+2n} \to \mathbb{R}^{r+2n}$. Then for $\epsilon_2 > 0$ sufficiently small, we set $\Psi_2 := \Psi_1 \circ A : B_{\mathbb{R}^{r+2n}}(\epsilon_2) \to V_1$. Then, $\Psi_2(0) = \Psi_1(0) = \zeta_0$, Ψ_2 is a $\mathscr{C}^{s+2}$ diffeomorphism onto its image (which is a neighborhood of ζ_0), and if we identify $\mathbb{R}^{r+2n} \cong \mathbb{R}^r \times \mathbb{C}^n$ via the map $(t_1, \ldots, t_r, x_1, \ldots, x_{2n}) \mapsto (t_1, \ldots, t_r, x_1 + ix_n, \ldots, x_n + ix_{2n})$, then,

$$\Psi_2^* \widehat{L}_j(0) = \frac{\partial}{\partial \overline{z}_j}, \quad \Psi_2^* \widehat{X}_k(0) = \frac{\partial}{\partial t_k}, \quad 1 \le j \le n, 1 \le k \le r,$$

$$\Psi_2^* \widehat{L}_j, \Psi_2^* \widehat{X}_k \in \mathscr{C}^{s+1}_{\mathrm{loc}}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\epsilon_2); \mathbb{C}^{2n+r}).$$

Take $\epsilon_3 \in (0, \epsilon_2)$ such that, $\forall 1 \le j \le n, 1 \le k \le r$,

- If $s_0 \in (0, \infty]$, $\Psi_2^* \widehat{L}_j, \Psi_2^* \widehat{X}_k \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r \times \mathbb{C}^n}(\epsilon_3); \mathbb{C}^{2n+r})$.
- If $s_0 = \omega$, $\Psi_2^* \widehat{L}_j, \Psi_2^* \widehat{X}_k \in \mathscr{A}^{2n+r, \epsilon_3}(\mathbb{C}^{2n+r})$.

Define $D_{\epsilon_3} : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{R}^r \times \mathbb{C}^n$ by $D_{\epsilon_3}(t, z) = (\epsilon_3 t, \epsilon_3 z)$, and define $\Psi_0 : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V_1$ by $\Psi_0 := \Psi_2 \circ D_{\epsilon_3}$. Letting $V_0 = \Psi_0(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)) \subseteq V_1$ we have $\Psi_0 : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V_0$ is a $\mathscr{C}^{s+2}$ diffeomorphism. Set $L_j := \epsilon_3 \widehat{L}_j$ and $X_k := \epsilon_3 \widehat{X}_k$. With these choices, all of the conclusions of the lemma follow from the above remarks.

We include a few additional comments regarding the proof of (v). Since $\forall \zeta \in V_0 \subseteq V_1 \subseteq U_2$, we have $L_1(\zeta), \ldots, L_n(\zeta), X_1(\zeta), \ldots, X_r(\zeta)$ form a basis for $\mathscr{L}_\zeta$, we have $\forall \zeta \in V_0$, $\forall j_1, j_2, k_1, k_2, j, k$,

$$\begin{aligned}
&[L_{j_1}, L_{j_2}](\zeta), [X_k, L_j](\zeta), [X_{k_1}, X_{k_2}](\zeta) \in \mathscr{L}_\zeta \\
&\quad = \mathrm{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_n(\zeta), X_1(\zeta), \ldots, X_r(\zeta)\}.
\end{aligned}$$

Pulling this back via Ψ_0 yields (v) and completes the proof. □

Lemma 8.2. *Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ elliptic structure on M of dimension (r, n). Then, $\forall \zeta_0 \in M$, there exists a neighborhood V of ζ_0, $\mathscr{C}^{s+1}$ sections $L_1, \ldots, L_n, X_1, \ldots, X_r$ of $\mathscr{L}$ over V, and a $\mathscr{C}^{s+2}$ diffeomorphism $\Psi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V$ such that*

- *$\Psi(0) = \zeta_0$.*
- *$\forall \zeta \in V$, $L_1(\zeta), \ldots, L_n(\zeta), X_1(\zeta), \ldots, X_r(\zeta)$ is a basis for $\mathscr{L}_\zeta$.*
- *$\forall \xi \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,*

$$\begin{aligned}
&\mathrm{span}_{\mathbb{C}}\{\Psi^* L_1(\xi), \ldots, \Psi^* L_n(\xi), \Psi^* X_1(\xi), \ldots, \Psi^* X_r(\xi)\} \\
&\quad = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n} \right\}.
\end{aligned}$$

Proof. Let $L_1, \ldots, L_n, X_1, \ldots, X_r$ and $\Psi_0 : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to V_0$ be as in Lemma 8.1. If $s \in (0, \infty) \cup \{\omega\}$, set $s_0 := s$. If $s = \infty$, set $s_0 := 1$. The conclusions of Lemma 8.1 show that Theorem 7.3 applies (with this choice of s_0) to the vector fields $\Psi_0^* L_1, \ldots, \Psi_0^* L_n, \Psi_0^* X_1, \ldots, \Psi_0^* X_r$ and yields $\Phi_4 \in \mathscr{C}^{s+2}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1); \mathbb{R}^r \times \mathbb{C}^n)$ as in that theorem. In particular, Φ_4 is a diffeomorphism onto its image, $\Phi_4(0) = 0$, and since $I + \mathcal{A}(\xi)$ from Theorem 7.3 (v) is invertible, $\forall \xi \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$, Theorem 7.3 (v) shows $\forall \xi \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,

$$\begin{aligned}&\operatorname{span}_{\mathbb{C}}\{\Phi_4^*\Psi_0^*L_1(\xi),\ldots,\Phi_4^*\Psi_0^*L_n(\xi),\Phi_4^*\Psi_0^*X_1(\xi),\ldots,\Phi_4^*\Psi_0^*X_n(\xi)\}\\&=\operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial\overline{z}_1},\ldots,\frac{\partial}{\partial\overline{z}_n},\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\right\}.\end{aligned}$$

Setting $\Psi := \Psi_0 \circ \Phi_4$, the result follows with $V := \Psi(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq V_0$, by using the above mentioned properties of Φ_4 combined with the conclusions of Lemma 8.1. □

Proof of Theorem 4.18. (i)⇒(ii): This is obvious.

(ii)⇒(i): Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ elliptic structure on M of dimension (r,n). We wish to construct a $\mathscr{C}^{s+2}$ E-atlas on M of dimension (r,n), compatible with its $\mathscr{C}^{s+2}$ structure, such that $\mathscr{L}$ is the $\mathscr{C}^{s+1}$ elliptic structure associated to this E-manifold structure. For each $\zeta_0 \in M$, let $\Psi_{\zeta_0} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to V_{\zeta_0}$ be the function Ψ from Lemma 8.2 with this choice of ζ_0; so that V_{ζ_0} is a neighborhood of ζ_0 and Ψ_{ζ_0} is a $\mathscr{C}^{s+2}$ diffeomorphism satisfying the conclusions of that lemma. In particular, it follows from that lemma that $\forall\zeta \in V_{\zeta_0}$,

$$\begin{aligned}\mathscr{L}_\zeta = \operatorname{span}_{\mathbb{C}}\Bigg\{&d\Psi_{\zeta_0}(\Psi_{\zeta_0}^{-1}(\zeta))\frac{\partial}{\partial t_1},\ldots,d\Psi_{\zeta_0}(\Psi_{\zeta_0}^{-1}(\zeta))\frac{\partial}{\partial t_r},d\Psi_{\zeta_0}(\Psi_{\zeta_0}^{-1}(\zeta))\frac{\partial}{\partial\overline{z}_1},\ldots,\\&d\Psi_{\zeta_0}(\Psi_{\zeta_0}^{-1}(\zeta))\frac{\partial}{\partial\overline{z}_n}\Bigg\}.\end{aligned}\tag{8.2}$$

We claim $\{(\Psi_{\zeta_0}^{-1}, V_{\zeta_0}) : \zeta_0 \in M\}$ is the desired atlas. Indeed, that $\Psi_{\zeta_1}^{-1}\circ\Psi_{\zeta_2}$ is a $\mathscr{C}^{s+2}_{\mathrm{loc}}$ map follows from Lemmas 2.13 and 2.14. To see that $\Psi_{\zeta_1}^{-1}\circ\Psi_{\zeta_2}$ is an E-map, note that, for $1 \le k \le r$,

$$d\left(\Psi_{\zeta_1}^{-1}\circ\Psi_{\zeta_2}\right)(\xi)\frac{\partial}{\partial t_k} = d\Psi_{\zeta_1}^{-1}(\Psi_{\zeta_2}(\xi))d\Psi_{\zeta_2}(\xi)\frac{\partial}{\partial t_k}.$$

(8.2) shows $d\Psi_{\zeta_2}(\xi)\frac{\partial}{\partial t_k} \in \mathscr{L}_{\Psi_{\zeta_2}(\xi)}$, and applying (8.2) again shows

$$d\Psi_{\zeta_1}^{-1}(\Psi_{\zeta_2}(\xi))d\Psi_{\zeta_2}(\xi)\frac{\partial}{\partial t_k} \in \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r},\frac{\partial}{\partial\overline{z}_1},\ldots,\frac{\partial}{\partial\overline{z}_n}\right\}.$$

Similarly, for $1 \le j \le n$,

$$d\left(\Psi_{\zeta_1}^{-1}\circ\Psi_{\zeta_2}\right)(\xi)\frac{\partial}{\partial\overline{z}_j} \in \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r},\frac{\partial}{\partial\overline{z}_1},\ldots,\frac{\partial}{\partial\overline{z}_n}\right\}.$$

It follows that $\Psi_{\zeta_1}^{-1}\circ\Psi_{\zeta_2}$ is an E-map. Thus, since $\{V_{\zeta_0} : \zeta_0 \in M\}$ is an open cover for M we have that $\{(\Psi_{\zeta_0}^{-1}, V_{\zeta_0}) : \zeta_0 \in M\}$ is a $\mathscr{C}^{s+2}$ E-atlas on M. Since each $\Psi_{\zeta_0} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to V_{\zeta_0}$ is a $\mathscr{C}^{s+2}$ diffeomorphism (by Lemma 8.2, where $V_{\zeta_0} \subseteq M$ is given the original $\mathscr{C}^{s+2}$ manifold structure), we see that the $\mathscr{C}^{s+2}$ E-manifold structure induced by the above atlas is compatible with the original $\mathscr{C}^{s+2}$ manifold structure on M. That $\mathscr{L}$ is the $\mathscr{C}^{s+1}$ elliptic structure associated to this E-manifold structure follows from (8.2).

Finally, we turn to the uniqueness of this E-manifold structure. Suppose M is given two $\mathscr{C}^{s+2}$ E-manifold structures, compatible with the $\mathscr{C}^{s+2}$ manifold structure, such that $\mathscr{L}$ is the $\mathscr{C}^{s+1}$ elliptic structure associated to both of these E-manifold structures. That the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ diffeomorphism follows immediately because both copies of M have the same underlying $\mathscr{C}^{s+2}$ manifold structure. That the identity map is an E-map follows from Lemma 4.17. This shows that the identity map is a $\mathscr{C}^{s+2}$ E-diffeomorphism, which completes the proof. □

Appendix A. Linear algebra

Let $\mathscr{V}$ be a real vector space and let $\mathscr{V}^{\mathbb{C}} = \mathscr{V} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. We consider $\mathscr{V} \hookrightarrow \mathscr{V}^{\mathbb{C}}$ as a real subspace by identifying v with $v \otimes 1$. There are natural maps:

$$\mathrm{Re} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}, \quad \mathrm{Im} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}, \quad \text{complex conjugation} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}^{\mathbb{C}},$$

defined as follows. Every $v \in \mathscr{V}^{\mathbb{C}}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$, with $v_1, v_2 \in V$. Then, $\mathrm{Re}(v) := v_1$, $\mathrm{Im}(v) := v_2$, and $\overline{v} := v_1 \otimes 1 - v_2 \otimes i$.

Lemma A.1. *Let $\mathscr{L} \subseteq \mathscr{V}^{\mathbb{C}}$ be a finite dimensional complex subspace. Then,* $\dim(\mathscr{L} + \overline{\mathscr{L}}) + \dim(\mathscr{L} \bigcap \overline{\mathscr{L}}) = 2\dim(\mathscr{L})$.

Proof. It is a standard fact that $\dim(\mathscr{L} + \overline{\mathscr{L}}) + \dim(\mathscr{L} \bigcap \overline{\mathscr{L}}) = \dim(\mathscr{L}) + \dim(\overline{\mathscr{L}})$. Using that $w \mapsto \overline{w}$, $\mathscr{L} \to \overline{\mathscr{L}}$ is an anti-linear isomorphism, the result follows. □

Lemma A.2. *Let $\mathscr{X} \subseteq \mathscr{V}^{\mathbb{C}}$ be a finite dimensional subspace of dimension r, and suppose $\overline{\mathscr{X}} = \mathscr{X}$. Then there exist $x_1, \ldots, x_r \in \mathscr{X} \cap V$ such that $x_1, \ldots, x_r$ is a basis for $\mathscr{X}$.*

Proof. Let $l_1, \ldots, l_r$ be a basis for $\mathscr{X}$. Since $\mathscr{X} = \overline{\mathscr{X}}$, $\mathrm{Re}(l_j), \mathrm{Im}(l_j) \in \mathscr{X}$, and clearly $\mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_r), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_r)$ form a spanning set for $\mathscr{X}$. Extracting a basis from this spanning set yields the result. □

Appendix B. Elliptic PDEs

In this section, we state quantitative versions of some standard results regarding nonlinear elliptic PDEs. All of the results in this section are well-known, and we make no effort to state these results in the greatest possible generality: we content ourselves with the simplest settings which are sufficient for our purposes.

B.1. Real analyticity for a nonlinear elliptic equation

It is a classical result that the solutions to real analytic, nonlinear elliptic PDEs are themselves real analytic; see, e.g., [6]. We require a quantitative version of (a special case of) this fact, which follows from standard proofs.

Let $\mathcal{E}$ be a constant coefficient, first order, linear partial differential operator

$$\mathcal{E} : C^\infty(\mathbb{R}^n; \mathbb{C}^{m_1}) \to C^\infty(\mathbb{R}^n; \mathbb{C}^{m_2}),$$

where $m_2 \geq m_1$. We may think of $\mathcal{E}$ as an $m_2 \times m_1$ matrix of constant coefficient partial differential operators of order ≤ 1.

Let $\Gamma : \mathbb{C}^{m_1} \times \mathbb{C}^{nm_1} \to \mathbb{C}^{m_2}$ be a bilinear map. Fix $R > 0$ and we consider the equation for $b : B_{\mathbb{R}^n}(R) \to \mathbb{C}^{m_1}$ given by

$$\mathcal{E}b = \Gamma(b, \nabla b). \tag{B.1}$$

Proposition B.1. *Fix $s_0 > 1$ and suppose $\mathcal{E}$ is elliptic. Then, $\exists \gamma = \gamma(\mathcal{E}, \Gamma, R, s_0) > 0$, $\eta_0 = \eta_0(\mathcal{E}, \Gamma, R, s_0) > 0$ such that the following holds. If $b \in \mathscr{C}^{s_0}(B_{\mathbb{R}^n}(R); \mathbb{C}^{m_1})$ is a solution to (B.1) and $\|b\|_{\mathscr{C}^{s_0}(B_{\mathbb{R}^n}(R))} \leq \gamma$, then $b \in \mathscr{B}^{n,m_1}_{\eta_0}$ and $\|b\|_{\mathscr{B}^{n,m_1}_{\eta_0}} \leq C$, where $C = C(\mathcal{E}, \Gamma, R, s_0)$. See Section 5.1 for the definition of $\mathscr{B}^{n,m_1}_{\eta_0}$.*

We outline a proof of Proposition B.1 by following the proof from [6], which becomes somewhat simpler in this special case and is therefore easier to extract the needed quantitative estimates. In what follows, we write $A \lesssim B$ to mean $A \leq CB$, where C can be chosen to depend only on $\mathcal{E}$, Γ, R, and s_0. Throughout the rest of this section, we take the setting of Proposition B.1; in particular, we are given a solution $b \in \mathscr{C}^{s_0}(B_{\mathbb{R}^n}(R); \mathbb{C}^{m_1})$ to (B.1) as in that proposition. Our goal is to pick γ and η_0 so that the conclusions of the proposition hold.

Without loss of generality, by possibly shrinking s_0, we may assume $s_0 = 1 + \mu$, where $\mu \in (0, 1)$. Thus, the space $\mathscr{C}^{s_0}(B)$ coincides with the Hölder space $C^{1,\mu}(B)$ for any ball B,[16] which allows us to use the results from [6] which deal with Hölder spaces. For the rest of the section, we continue to use the notation $\mathscr{C}^{j+\mu}$ for $j \in \mathbb{N}$, but (just in this section) the reader is free to interpret it either as $\mathscr{C}^{j+\mu}$ or $C^{j,\mu}$; indeed in this section we only deal with $\mu \in (0, 1)$ fixed and $\mathscr{C}^{j+\mu}(\Omega)$, $C^{j,\mu}(\Omega)$ for bounded Lipschitz domains Ω in which case these two spaces have equivalent norms.

First we need a quantitative version of the classical fact that the solution b is smooth. This is discussed in an appendix to [12]. There it is shown that $\exists \gamma_1 = \gamma_1(\mathcal{E}, \Gamma) > 0$ such that if $\|b\|_{\mathscr{C}^{1+\mu}(B_{\mathbb{R}^n}(R))} \leq \gamma_1$, then $b \in \mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2); \mathbb{C}^{m_1})$ with $\|b\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2))} \lesssim \|b\|_{\mathscr{C}^{1+\mu}(B_{\mathbb{R}^n}(R))}$. We will choose $\gamma \leq \gamma_1$, so we may henceforth assume $b \in \mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2); \mathbb{C}^{m_1})$ with $\|b\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2))} \lesssim \gamma$.

[16] That $\mathscr{C}^{1+\mu}(\Omega) = C^{1,\mu}(\Omega)$ for a bounded, Lipschitz domain Ω (and $\mu \neq 0, 1$) is classical and follows easily from [16, Theorem 1.118 (i)].

For $\eta, h > 0$, set $D^n(\eta; h) := \{x + iy : x, y \in \mathbb{R}^n, |x| < \eta, |y| < h(\eta - |x|)\}$ and set, for $s > 0$,

$$\mathscr{D}^{n,m_1}_{\eta,h,s} := \big\{ f : B_{\mathbb{R}^n}(\eta) \to \mathbb{C}^{m_1} \;\big|\; f \text{ is real analytic and extends to a holomorphic function } E(f) \in \mathscr{C}^s(D^n(\eta; h); \mathbb{C}^{m_1}) \big\}.$$

With the norm

$$\|f\|_{\mathscr{D}^{n,m_1}_{\eta,h,s}} := \|E(f)\|_{\mathscr{C}^s(D^n(\eta;h);\mathbb{C}^{m_1})},$$

$\mathscr{D}^{n,m_1}_{\eta,h,s}$ is a Banach space.

Lemma B.2. *There exists a bounded linear map*

$$\mathcal{P} : \mathscr{C}^\mu(B_{\mathbb{R}^n}(R/2); \mathbb{C}^{m_1}) \to \mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2); \mathbb{C}^{m_1})$$

such that $\mathcal{E}^*\mathcal{E}\mathcal{P} = I$ and $\exists h = h(\mathcal{E}, R) > 0$ such that $\mathcal{P}$ restricts to a bounded map

$$\mathcal{P} : \mathscr{D}^{n,m_1}_{R/2,h,\mu} \to \mathscr{D}^{n,m_1}_{R/2,h,2+\mu}$$

and such that if we set $V_0 := \mathcal{P}\mathcal{E}^*\Gamma(b, \nabla b)$ and $H := b - V_0$, then $\|V_0\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$, $\|H\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$, and $H \in \mathscr{D}^{n,m_1}_{R/2,h,2+\mu}$ with $\|H\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \leq C_1\gamma$. Here, $C_1 = C_1(\mathcal{E}, \Gamma, R, s_0) > 0$.

Comments on the proof. This is essentially a special case of Theorems A, B, and C of [6]; here we are applying these theorems to the elliptic operator $\mathcal{E}^*\mathcal{E}$ and using that $\mathcal{E}^*\mathcal{E}H = 0$ by the definitions. In [6], these theorems were stated on the subspace of functions which vanish at 0, though this is not an essential point. Moreover, in the special case we are interested in, $\mathcal{E}^*\mathcal{E}$ is essentially the Laplacian (see (B.4) for $\mathcal{E}^*\mathcal{E}$ in the case we are interested in). In this case, the above result follows from standard methods. □

Define $\mathcal{T}(V) := \mathcal{P}\mathcal{E}^*\Gamma(H + V, \nabla(H + V))$; by the definition of V_0, $\mathcal{T}(V_0) = V_0$.

Lemma B.3. *Let $C_1 > 0$ be as in Lemma B.2. If $\gamma = \gamma(\mathcal{E}, \Gamma, R, s_0) > 0$ is sufficiently small and $\|V_1\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}, \|V_2\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$ then,*

$$\|\mathcal{T}(V_1)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma,$$
$$\|\mathcal{T}(V_1) - \mathcal{T}(V_2)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq \frac{1}{2}\|V_1 - V_2\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}.$$

The same results hold for $\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2); \mathbb{C}^{m_1})$ replaced by $\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}$, throughout.

Proof. Since $\|V_1\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$ and $\|H\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$, it follows from Proposition 5.1 that $\|\Gamma(H+V_1, \nabla(H+V_1))\|_{\mathscr{C}^{1+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \lesssim (C_1\gamma)^2$. Lemma B.2 implies $\|\mathcal{T}(V_1)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \lesssim (C_1\gamma)^2$; and so if γ is sufficiently small it follows that $\|\mathcal{T}(V_1)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma$. Similarly, again using Proposition 5.1, we have

$$\begin{aligned}&\|\Gamma(V_1-V_2, \nabla(H+V_1))\|_{\mathscr{C}^{1+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}, \|\Gamma(H+V_2, \nabla(V_1-V_2))\|_{\mathscr{C}^{1+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}\\ &\quad\lesssim \gamma\|V_1-V_2\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}.\end{aligned}$$

Since $\mathcal{T}(V_1) - \mathcal{T}(V_2) = \mathcal{P}\mathcal{E}^*(\Gamma(V_1-V_2, \nabla(H+V_1)) - \Gamma(H+V_2, \nabla(V_1-V_2)))$ it follows from Lemma B.2 that

$$\|\mathcal{T}(V_1) - \mathcal{T}(V_2)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \lesssim \gamma\|V_1-V_2\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})}.$$

Taking γ sufficiently small, we have

$$\|\mathcal{T}(V_1) - \mathcal{T}(V_2)\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq \frac{1}{2}\|V_1-V_2\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})},$$

as desired. The same proof works with $\mathscr{C}^{j+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})$ replaced by $\mathscr{D}^{n,m_1}_{R/2,h,j+\mu}$, throughout. □

Proof of Proposition B.1. By taking $\gamma > 0$ sufficiently small, as in Lemma B.3, we see that V_0 is the unique fixed point of the strict contraction $\mathcal{T}$, acting on the complete metric space $\{V : \|V\|_{\mathscr{C}^{2+\mu}(B_{\mathbb{R}^n}(R/2);\mathbb{C}^{m_1})} \leq C_1\gamma\}$. This fixed point agrees with the fixed point of $\mathcal{T}$ when acting on the complete metric space $\{V : \|V\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \leq C_1\gamma\}$ (on which is it a strict contraction by Lemma B.3). We conclude $\|V_0\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \leq C_1\gamma \lesssim 1$. Since $\|H\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \leq C_1\gamma \lesssim 1$, by Lemma B.2, and since $b = H + V_0$, we have $\|b\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \leq 2C_1\gamma \lesssim 1$. Taking $\eta_0 = \eta_0(R/2, h) > 0$ sufficiently small we have $B_{\mathbb{R}^n}(\eta_0) \subseteq D^n(R/2; h)$ and therefore,

$$\|b\|_{\mathscr{B}^{n,m_1}_{\eta_0}} \leq \|b\|_{\mathscr{D}^{n,m_1}_{R/2,h,2+\mu}} \lesssim 1,$$

completing the proof. □

B.2. Existence for a nonlinear elliptic equation

Fix $D > 0$, $m_1, m_2 \in \mathbb{N}$. For functions $A : B_{\mathbb{R}^n}(D) \to \mathbb{C}^{m_1}$ and $B : B_{\mathbb{R}^n}(D) \to \mathbb{C}^{m_2}$ we write

$$\mathscr{D}^1 A = (\partial_x^\alpha A)_{|\alpha|\leq 1}, \quad \mathscr{D}^2 B = (\partial_x^\alpha B)_{|\alpha|\leq 2}, \quad \mathscr{D}_2 B = (\partial_x^\alpha B)_{|\alpha|=2},$$

so that, for example, $\mathscr{D}^2 B$ is the vector of all partial derivatives of B up to order 2, and $\mathscr{D}_2 B$ is the vector of all partial derivatives of order exactly 2.

Fix a C^∞ function g. We consider the equation

$$g(\mathscr{D}^1 A(x), \mathscr{D}^2 B(x)) = 0. \tag{B.2}$$

Here, g is a C^∞ function defined on a neighborhood of the origin, takes values in $\mathbb{C}^{m_2}$, and satisfies $g(0,0) = 0$. Our goal is to give conditions on g so that given A (sufficiently small), we can find $B = B(A)$ so that (B.2) holds; and we wish to further understand how the regularity of B depends on the regularity of A, in a quantitative way.

Though it is not necessary for the results which follow, we assume (B.2) is quasilinear in B, which is sufficient for our purposes and simplifies the proof. That is, we assume

$$g(\mathscr{D}^1 A(x), \mathscr{D}^2 B(x)) = g_1(A(x), \mathscr{D}^1 B(x))\mathscr{D}_2 B(x) + g_2(\mathscr{D}^1 A(x), \mathscr{D}^1 B(x)), \tag{B.3}$$

where g_1 and g_2 are smooth on a neighborhood of the origin, g_1 takes values in matrices of an appropriate size, and $g_2(0,0) = 0$.

Finally, let $\mathcal{E}_2$ denote the second order partial differential operator

$$\mathcal{E}_2 B := g_1(0,0)\mathscr{D}_2 B,$$

so that $\mathcal{E}_2$ is an $m_2 \times m_2$ matrix of constant coefficient partial differential operators of order ≤ 2.

Proposition B.4. *Suppose $\mathcal{E}_2$ is elliptic. Fix $s_0 > 0$ and a neighborhood $N \subseteq \mathscr{C}^{2+s_0}(B_{\mathbb{R}^n}(D); \mathbb{C}^{m_2})$ of 0. Then, there exists a neighborhood $W \subseteq \mathscr{C}^{1+s_0}(B_{\mathbb{R}^n}(D); \mathbb{C}^{m_2})$ of 0 and a map $\mathscr{B} : W \to N$ such that $g(\mathscr{D}^1 A(x), \mathscr{D}^2 \mathscr{B}(A)(x)) = 0$ for $x \in B^n(D)$, $A \in W$. This map satisfies $\mathscr{D}^1 \mathscr{B}(A)(0) = 0$, $\forall A \in W$, and $\|\mathscr{B}(A)\|_{\mathscr{C}^{2+s_0}(B_{\mathbb{R}^n}(D);\mathbb{C}^{m_2})} \leq C\|A\|_{\mathscr{C}^{1+s_0}(B_{\mathbb{R}^n}(D);\mathbb{C}^{m_1})}$, where C does not depend on $A \in W$. Finally, for $\eta \in (0, D)$, let R_η denote the restriction map $R_\eta : f \mapsto f\big|_{B_{\mathbb{R}^n}(\eta)}$. Then, for $s \geq s_0$, $\eta \in (0, D)$, $R_\eta \circ \mathscr{B} : \mathscr{C}^{1+s}(B_{\mathbb{R}^n}(D); \mathbb{C}^{m_1}) \cap W \to \mathscr{C}^{2+s}(B_{\mathbb{R}^n}(\eta); \mathbb{C}^{m_2})$, and $\|R_\eta \circ \mathscr{B}(A)\|_{\mathscr{C}^{2+s}(B_{\mathbb{R}^n}(\eta);\mathbb{C}^{m_2})} \leq C_{s,\eta}$, where $C_{s,\eta}$ can be chosen to depend on an upper bound for $\|A\|_{\mathscr{C}^{1+s}(B_{\mathbb{R}^n}(D);\mathbb{C}^{m_1})}$ and does not depend on $A \in W$ in any other way. It can depend on any of the other ingredients in the problem.*

See [12] for a discussion of this proposition.

B.3. An elliptic operator

In this section, we discuss a particular first order, overdetermined, constant coefficient, linear, elliptic operator which is needed in this paper. For $t \in \mathbb{R}^r$ and $z \in \mathbb{C}^n$, we consider functions $A(t,z) : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{C}^r$ and $B(t,z) : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{C}^n$. We define

$$\mathcal{E}(A,B) = \left(\left(\frac{\partial A_{k_1}}{\partial t_{k_2}} - \frac{\partial A_{k_2}}{\partial t_{k_1}} \right)_{1\le k_1<k_2\le r}, \left(\frac{\partial A_k}{\partial \overline{z}_j} - \frac{\partial B_j}{\partial t_k} \right)_{\substack{1\le k\le r\\ 1\le j\le n}}, \right.$$
$$\left. \left(\frac{\partial B_{j_1}}{\partial \overline{z}_{j_2}} - \frac{\partial B_{j_2}}{\partial \overline{z}_{j_1}} \right)_{1\le j_1<j_2\le n}, \sum_{k=1}^{r} \frac{\partial A_k}{\partial t_k} + \sum_{j=1}^{n} \frac{\partial B_j}{\partial z_j} \right).$$

Lemma B.5. $\mathcal{E}$ *is elliptic.*

Proof. It is straightforward to directly compute $\mathcal{E}^*\mathcal{E}$ to see

$$\mathcal{E}^*\mathcal{E}(A,B) = -\left(\sum_{k=1}^{r} \frac{\partial^2}{\partial t_k^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \right)(A,B), \tag{B.4}$$

and the result follows. □

There is another way to interpret (B.4). Indeed, we identify (A, B) with the one form $\Psi := A_1 dt_1 + \cdots + A_r dt_r + B_1 d\overline{z}_1 + \cdots + B_n d\overline{z}_n$. We let d denote the usual de Rham complex acting in the t variable and $\overline{\partial}$ denote the usual $\overline{\partial}$-complex acting in the z variable. We let d^* and $\overline{\partial}^*$ denote the adjoints of these two complexes. Then $\mathcal{E}(A,B)$ can be identified with $(d+\overline{\partial}, -(d+\overline{\partial})^*)\Psi$. And so $\mathcal{E}^*\mathcal{E}$ can be identified with $(d+\overline{\partial})^*(d+\overline{\partial}) + (d+\overline{\partial})(d+\overline{\partial})^* = d^*d + dd^* + \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* + (\overline{\partial}^* d + d\overline{\partial}^*) + (d^*\overline{\partial} + \overline{\partial} d^*) = d^*d + dd^* + \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$.

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