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CADENCE TRACKING FOR SWITCHED FES-CYCLING WITH UNKNOWN TIME-VARYING INPUT DELAY

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ABSTRACT

Functional electrical stimulation (FES) induced exercise, such as motorized FES-cycling, is commonly used in rehabilitation for lower limb movement disorders. A challenge in closed-loop FES control is the presence of an input delay between the application (and removal) of the electrical stimulus and the production of muscle force. Moreover, switching between motor control and FES control of various muscle groups can be destabilizing. This paper examines the development of a control method and state-dependent trigger condition to account for the time-varying input delayed response. Uniformly ultimately bounded tracking for a switched uncertain nonlinear dynamic system with input delays is achieved.

INTRODUCTION[1]

A common rehabilitative technique for people with lower limb movement disorders is to induce exercise via functional electrical stimulation (FES) [1–4]. Closed-loop FES control has several challenges. For instance, fatigue causes muscle force to decay under a constant stimulation intensity [5], uncertainty exists in disturbances to the system and parameters in the dynamic model [6], and the complex nonlinear mapping from electrical input to generated muscle force is unknown [7]. In addition, more complicated functional tasks (e.g., FES cycling) require switching control between different muscle groups and potentially a motor [8]. Moreover, the complex electrophysiological process involved in human muscle contraction due to an external electric field results in an input delay.

Results for continuous exercises (e.g., leg extensions), focus on the delay resulting from the application of an electrical stimulus [9–13]. However, for exercises like cycling that require limb coordination by switching between multiple muscle groups, residual forces resulting from the delayed response after removing the electrical stimulus is also an important consideration. Such residual forces may come from antagonistic muscles, leading to unfavorable biomechanics and an increased rate of fatigue. Fatigue is undesired because it reduces the number of effective repetitive limb movements and may result in even greater destabilizing delay effects [9]. To date, no closed-loop FES controller has been developed for switched system dynamics to include the effects of input delay. FES controllers have been developed recently to account for the delayed response of muscle. In [12], an unknown constant delay and exact model knowledge of the lower limb dynamics are assumed and a global asymptotic tracking controller is developed. A uniformly ultimately bounded result was developed in [13] for uncertain dynamics with a known delay. In [10] and [11] an input delay that is both unknown and time-varying is assumed for a continuous leg extension exercise.

Over the last decade, the stability of general input delayed systems has been studied extensively [14–20]. The result in [16] considers uncertain dynamics and a known input delay. However, it is often difficult to measure the input de-

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lay [21], so more recent papers have assumed the delay is unknown [17]. Switching in the presence of unknown delays has been explored recently [18–20]. The result in [18] considers time-varying state delays, while time-varying input delays and non-time-varying input delays are considered in [19] and [20], respectively. Lyapunov–Krasovskii functionals are developed in [19] for a certain class of nonlinear systems to yield input-to-state stability, whereas a linear system was examined in [20]. The aforementioned results do not account for the unique circumstances associated with switched FES-cycling, such as the need for complex state-dependent switching and the need to compensate for residual antagonistic forces resulting from delays after the stimulation has been withdrawn. For example, the result in this paper combines a controller that is robust to the time-varying delay and a delay-dependent trigger method to appropriately schedule the activation of muscle groups and the motor. Specifically, this paper examines the development of switching conditions and the associated stability analysis to account for the time-varying input delayed response of closed-loop FES control. It is the first paper to develop a controller for a switched uncertain nonlinear dynamic FES system with unknown time-varying input delays.

DYNAMICS

Throughout the paper, delayed functions are defined as

$$h_{\tau} \triangleq \begin{cases} h(t-\tau(t)) & t-\tau(t) \geq t_0 \\ 0 & t-\tau(t) < t_0 \end{cases},$$

where $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time and time is denoted by $t \in \mathbb{R}_{\geq 0}$. The time-varying electromechanical delay, i.e., the delay between the application/removal of the current and the onset/elimination of muscle force production is denoted by $\tau : \mathbb{R}_{\geq 0} \to \mathbb{S}$, where $\mathbb{S} \subset \mathbb{R}$ denotes the set of delay values [21]. The motorized cycle-rider system can be modeled as [8][2]

$$\begin{aligned} \tau_M(q,\dot{q},\tau,t) + \tau_e(q,t) = M(q)\ddot{q} + V(q,\dot{q})\dot{q} \\ + G(q) + P(q,\dot{q}) + b_c\dot{q} + d(t), \end{aligned} \tag{1}$$

where $q : \mathbb{R}_{\geq 0} \to \mathcal{Q}$, $\dot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $\ddot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ denote the measurable crank angle and velocity, and unmeasured acceleration, respectively. The set $\mathcal{Q} \subseteq \mathbb{R}$ denotes all possible crank angles. The inertial effects, gravitational effects, centripetal-Coriolis effects, and passive viscoelastic tissue forces are denoted as $M : \mathcal{Q} \to \mathbb{R}_{>0}$, $G : \mathcal{Q} \to \mathbb{R}$, $V : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$, and $P : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$, respectively. The viscous damping effects and disturbances applied about the crank axis are denoted by $b_c \in \mathbb{R}_{>0}$ and $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively. The torque contributions due to the motor and FES induced muscle contractions are denoted as $\tau_e : \mathcal{Q} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\tau_M : \mathcal{Q} \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively defined as

$$\tau_e(q,t) \triangleq B_e u_E(q,t), \tag{2}$$

$$\tau_M(q,\dot{q},\tau,t) \triangleq \sum_{m \in \mathcal{M}} B_m(q,\dot{q})\, u_m(q_\tau,\dot{q}_\tau,\tau,t), \tag{3}$$

where the unknown motor control effectiveness is denoted by $B_e \in \mathbb{R}_{>0}$. The control effectiveness for the electrically stimulated muscle groups in (3) are denoted by $B_m : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$, where $m \in \mathcal{M} \triangleq \{RH, RQ, RG, LH, LQ, LG\}$ indicates the right (R) and left (L) hamstrings (H), quadriceps femoris (Q), and gluteal (G) muscle groups. The delayed electrical stimulation input (i.e., pulse width) delivered to the rider's muscles, denoted by $u_m : \mathcal{Q} \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\forall m \in \mathcal{M}$, and the control current to the electric motor denoted by $u_E : \mathcal{Q} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, are defined as

$$u_m(q_\tau,\dot{q}_\tau,\tau,t) \triangleq k_m \sigma_{m,\tau}(q_\tau,\dot{q}_\tau)\, u_\tau, \tag{4}$$

$$u_E(q,t) \triangleq k_e \sigma_e(q)\, u_e(t), \tag{5}$$

where $k_m, k_e \in \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$ are selectable constants. The delayed switching signals denoted by $\sigma_{m,\tau}(q_\tau,\dot{q}_\tau)$, $\forall m \in \mathcal{M}$ indicate which muscle groups received the delayed FES input u_τ at the time $t-\tau(t)$. The subsequently designed non-delayed FES control and motor inputs are denoted by $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $u_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively. The state-dependent trigger condition and switching function, denoted by $\sigma_m(q,\dot{q})$ is designed to activate the muscles at the appropriate location at time t. The designed piecewise left-continuous switching signal for each muscle group is denoted as $\sigma_m : \mathcal{Q} \times \mathbb{R} \to \{0, 1\}$ and is defined as

$$\sigma_m(q,\dot{q}) \triangleq \begin{cases} 1, & q_\alpha \in \mathcal{Q}_m, q \in \mathcal{Q}_e \\ 1, & q_\beta \in \mathcal{Q}_m \\ 0, & \text{otherwise} \end{cases}, \tag{6}$$

$\forall m \in \mathcal{M}$, where trigger conditions $q_\alpha, q_\beta : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ are defined as $q_\alpha \triangleq f_1(q,\dot{q})$ and $q_\beta \triangleq f_2(q,\dot{q})$, where f_1 and f_2 are designed to stimulate the rider's muscles sufficiently prior to the crank entering the FES region and for stimulation to cease sufficiently prior to the crank leaving the FES region. The functions f_1 and f_2 utilize the fact that the delay can be lower and upper bounded as determined from experimental results such as [21] and do not require explicit knowledge of the actual delay. This allows q_α and q_β to act as trigger conditions that adjust the activation/deactivation of the FES input based on the delay bounds.

[2] For notational brevity, all explicit dependence on time, t, within the terms $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ is suppressed.

Because the motor delay is negligible, the switching signal to activate the motor is implementable at time t. Hence, in (5), $\sigma_e : \mathcal{Q} \to \{0,1\}$ denotes a piecewise left-continuous switching signal for the motor and is defined as

$$\sigma_e(q) \triangleq \begin{cases} 1, & q \in \mathcal{Q}_e \\ 0, & q \notin \mathcal{Q}_e \end{cases}. \tag{7}$$

Definitions for the subsequent FES regions, denoted by $\mathcal{Q}_m \subset \mathcal{Q}$, and switching laws are based on [8], where each muscle group is stimulated in specific regions of the crank cycle (i.e., when kinematically efficient). In this manner, $\mathcal{Q}_m$ is defined for each muscle group as

$$\mathcal{Q}_m \triangleq \{q \in \mathcal{Q} \mid T_m(q) > \varepsilon_m\}, \tag{8}$$

$\forall m \in \mathcal{M}$, where $\varepsilon_m \in (0, \max(T_m)]$ is the lower threshold for each torque transfer ratio denoted by $T_m : \mathcal{Q} \to \mathbb{R}$, which limits the FES regions such that each muscle group can only contribute to forward pedaling (i.e., positive crank motion). The union of all muscle regions defined in (8) represents the entire FES region, denoted by $\mathcal{Q}_{FES}$, and defined as $\mathcal{Q}_{FES} \triangleq \bigcup_{m\in\mathcal{M}} \{\mathcal{Q}_m\}$. The motor regions (i.e., kinematic deadzones) are defined as $\mathcal{Q}_e \triangleq \mathcal{Q} \setminus \mathcal{Q}_{FES}$. Substituting (3)-(5) into (1) yields[3]

$$B_M^\tau u_\tau + B_E u_e = M\ddot{q} + V\dot{q} + G + P + b_c\dot{q} + d, \tag{9}$$

where $B_M^\tau \triangleq \sum_{m\in\mathcal{M}} B_m(q,\dot{q}) k_m \sigma_{m,\tau}(q_\tau,\dot{q}_\tau)$ and $B_E(q) \triangleq B_e k_e \sigma_e(q)$.

The parameters in (9) capture the torques that affect the dynamics of the combined cycle-rider system, but the exact value of these parameters are unknown for each rider and the cycle. However, the subsequently designed FES and motor controllers only require known bounds on the aforementioned parameters. The switched system in (9) has the following properties [8].

Property 1. $c_m \le M \le c_M$, where $c_m, c_M \in \mathbb{R}_{>0}$ are known constants.

Property 2. $|V| \le c_V|\dot{q}|$, where $c_V \in \mathbb{R}_{>0}$ is a known constant and $|\cdot|$ denotes the absolute value.

Property 3. $|G| \le c_G$, where $c_G \in \mathbb{R}_{>0}$ is a known constant.

Property 4. $|P| \le c_{P1} + c_{P2}|\dot{q}|$, where $c_{P1}, c_{P2} \in \mathbb{R}_{>0}$ are known constants.

Property 5. $b_c\dot{q} \le c_c|\dot{q}|$, where $c_c \in \mathbb{R}_{>0}$ is a known constant.

Property 6. $|d| \le c_d$, where $c_d \in \mathbb{R}_{>0}$ is a known constant.

Property 7. $\frac{1}{2}\dot{M} = V$, by skew-symmetry.

[3]For notational brevity, all functional dependencies are hereafter suppressed unless required for clarity of exposition.

Property 8. The muscle control effectiveness B_m is lower and upper bounded $\forall m \in \mathcal{M}$, and thus, when $\sum_{m\in\mathcal{M}} \sigma_{m,\tau} > 0$, $c_b \le B_M^\tau \le c_B$, where $c_b, c_B \in \mathbb{R}_{>0}$ are known constants.

Property 9. $c_e \le B_e \le c_E$, where $c_e, c_E \in \mathbb{R}_{>0}$ are known constants.

Property 10. The mismatch between the actual input delay $\tau(t)$ and the constant estimated input delay, denoted by $\hat{\tau} \in \mathbb{R}_{>0}$, is bounded by a known constant $\bar{\tilde{\tau}} \in \mathbb{R}_{>0}$ such that $\sup_{t\in\mathbb{R}_{\ge 0}} |\tau - \hat{\tau}| \le \bar{\tilde{\tau}}$.

CONTROL DEVELOPMENT

In this paper, the control objective is for the pedal crank to track a desired cadence $\dot{q}_d : \mathbb{R}_{\ge 0} \to \mathbb{R}$ despite the uncertainties in the dynamic model and an unknown time-varying input delay. To facilitate the subsequent analysis, measurable auxiliary tracking errors, denoted by $e : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and $r : \mathbb{R}_{\ge 0} \to \mathbb{R}$ are defined as[4]

$$e \triangleq q_d - q, \tag{10}$$

$$r \triangleq \dot{e} + \alpha e + \eta e_u, \tag{11}$$

where $\alpha, \eta \in \mathbb{R}_{\ge 0}$ are selectable constants. To incorporate a delay-free input term in the closed-loop error system, an auxiliary error signal, denoted by $e_u : \mathbb{R}_{\ge 0} \to \mathbb{R}$, is defined as

$$e_u \triangleq -\int_{t-\hat{\tau}}^{t} u(\theta)\, d\theta. \tag{12}$$

The open-loop error system can be obtained by taking the time derivative of (11), multiplying by M, adding and subtracting $B_M^\tau u_{\hat{\tau}} + e$, and using (9), (10), and (12) to obtain

$$\begin{aligned} M\dot{r} = &-Vr - e + \chi - B_E u_e - M\eta u \\ &+ B_M^\tau (u_{\hat{\tau}} - u_\tau) + (M\eta - B_M^\tau) u_{\hat{\tau}}, \end{aligned} \tag{13}$$

where the auxiliary term $\chi : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ is defined as

$$\begin{aligned} \chi \triangleq &M\ddot{q}_d + V(\dot{q}_d + \alpha e + \eta e_u) + G \\ &+ P + b_c\dot{q} + d + M\alpha\dot{e} + e. \end{aligned}$$

From Properties 1-6, χ can be bounded as

$$|\chi| \le \Phi + \rho(\|z\|)\|z\|, \tag{14}$$

[4]The control objective can be quantified in terms of the first time derivative of e, (i.e., $\dot{e}$).

where $\Phi \in \mathbb{R}_{>0}$ is a known constant, $\rho(\cdot)$ is a positive, radially unbounded, and strictly increasing function, and $z \in \mathbb{R}^3$ is a vector of error signals defined as

$$z \triangleq \left[\, e \; r \; e_u \,\right]^T. \tag{15}$$

Based on (13) and (14), and the subsequent stability analysis, the FES control input is designed as

$$u = k_s r, \tag{16}$$

where $k_s \in \mathbb{R}_{>0}$ is a selectable constant. The motor control input is designed as

$$u_e = k_1 \operatorname{sgn}(r) + (k_2 + k_3)\, r, \tag{17}$$

where $\operatorname{sgn}(\cdot)$ denotes the signum function, and $k_1, k_2, k_3 \in \mathbb{R}_{>0}$ are selectable constants. Substituting (16) and (17) into (13) yields the closed-loop error system

$$\begin{aligned} M\dot{r} = & -Vr - B_E\left(k_1 \operatorname{sgn}(r) + (k_2+k_3)\, r\right) \\ & + k_s B_M^{\tau}\left(r_{\hat{\tau}} - r_{\tau}\right) + \left(M\eta - B_M^{\tau}\right) k_s r_{\hat{\tau}} \\ & - M\eta k_s r - e + \chi. \end{aligned} \tag{18}$$

Based on the subsequent stability analysis and the closed-loop error system in (18), let the Lyapunov-Krasovskii functionals $Q_1, Q_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be defined as

$$Q_1 \triangleq \varepsilon_1 \omega_1 k_s \int_{t-\hat{\tau}}^{t} r(\theta)^2\, d\theta, \tag{19}$$

$$Q_2 \triangleq \frac{\omega_2 k_s}{\hat{\tau}} \int_{t-\hat{\tau}}^{t} \int_{s}^{t} r(\theta)^2\, d\theta ds, \tag{20}$$

where $\varepsilon_1, \varepsilon_2, \omega_1, \omega_2 \in \mathbb{R}_{>0}$ are selectable constants. To facilitate the subsequent stability analysis, auxiliary bounding constants $\beta_1, \beta_2, \delta_1, \delta_2 \in \mathbb{R}_{>0}$ are defined as

$$\beta_1 \triangleq \min\left(\alpha - \frac{\varepsilon_2 \eta^2}{2},\ \frac{1}{4} c_m \eta k_s,\ \frac{\omega_2}{3 k_s \hat{\tau}^2} - \frac{1}{2\varepsilon_2} - \frac{\omega_1 k_s}{\varepsilon_1}\right), \tag{21}$$

$$\beta_2 \triangleq \min\left(\alpha - \frac{\varepsilon_2 \eta^2}{2},\ c_e k_2 - 2\varepsilon_1 \omega_1 k_s - \omega_2 k_s,\right.$$

$$\left. \frac{\omega_2}{3 k_s \hat{\tau}^2} - \frac{1}{2\varepsilon_2} - \frac{\omega_1 k_s}{\varepsilon_1}\right), \tag{22}$$

$$\delta_1 \triangleq \min\left(\frac{\beta_1}{2},\ \frac{\omega_2}{3\hat{\tau}\varepsilon_1\omega_1},\ \frac{1}{3\hat{\tau}}\right), \tag{23}$$

$$\delta_2 \triangleq \min\left(\frac{\beta_2}{2},\ \frac{\omega_2}{3\hat{\tau}\varepsilon_1\omega_1},\ \frac{1}{3\hat{\tau}}\right). \tag{24}$$

STABILITY ANALYSIS

To facilitate the analysis, switching times are denoted by $\left\{t_n^i\right\}, i \in \{u, e\}, n \in \{0, 1, 2, ...\}$, which represent the time instances when B_M^{τ} becomes nonzero $(i = u)$, or the time instances when B_M^{τ} becomes zero $(i = e)$. Let $V_L : \mathbb{R}^5 \to \mathbb{R}_{>0}$ denote a positive definite, continuously differentiable, common Lyapunov function candidate defined as

$$V_L \triangleq \frac{1}{2} e^2 + \frac{1}{2} M r^2 + \frac{1}{2} \omega_1 e_u^2 + Q_1 + Q_2. \tag{25}$$

The common Lyapunov function candidate V_L satisfies the following inequalities:

$$\lambda_1 \left\|y\right\|^2 \leq V_L \leq \lambda_2 \left\|y\right\|^2, \tag{26}$$

where $y \in \mathbb{R}^5$ is defined as

$$y \triangleq \left[\, z \; \sqrt{Q_1} \; \sqrt{Q_2} \,\right]^T, \tag{27}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ are known constants defined as

$$\lambda_1 \triangleq \frac{1}{2} \min\left(1, c_m, \omega_1\right), \quad \lambda_2 \triangleq \max\left(1, \frac{c_M}{2}, \frac{\omega_1}{2}\right).$$

For use in the following stability analysis, let $\mathcal{D} \triangleq B_{\gamma} \cap \mathbb{R}^5$ where B_{γ} denotes a closed ball of radius γ centered at the origin, where $\gamma \in \mathbb{R}_{>0}$ is a known constant, and let[5]

$$S_{\mathcal{D}} \triangleq \left\{ y \in \mathcal{D} \mid \|y\| < \inf\left\{\rho^{-1}\left(\left(\sqrt{\kappa}, \infty\right)\right)\right\}\right\}, \tag{28}$$

where $\kappa \triangleq c_m \eta k_s \min\left(\frac{1}{2}\beta_1, 2\beta_2\right)$.

[5]For a set A, the inverse image is defined as $\rho^{-1}(A) \triangleq \{a \mid \rho(a) \in A\}$.

Theorem 1. *The closed-loop error system in (18) is uniformly ultimately bounded in the sense that*

$$\|y(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}\|y(t_0)\| - \frac{v}{\lambda_1\lambda_3}} \cdot \exp\left(-\frac{\lambda_3}{2}(t-t_0)\right) + \sqrt{\frac{v}{\lambda_1\lambda_3}}, \tag{29}$$

where $v \triangleq \frac{2\Phi^2}{c_m \eta k_s} + \frac{\bar{\bar{\tau}}\Upsilon^2}{k_s}$, $\Upsilon \in \mathbb{R}_{>0}$ *is a known constant,* $\lambda_3 \triangleq \lambda_2^{-1}\min(\delta_1, \delta_2)$ $\forall t \in [t_0, \infty)$, *provided* $\|y(t_n^u)\|,\ \|y(t_n^e)\| \in S_{\mathcal{D}}$, *and the following gain conditions are satisfied.*

$$\alpha \geq \frac{\varepsilon_2 \eta^2}{2}, \quad \omega_2 \geq 3k_s\hat{\tau}^2\left(\frac{1}{2\varepsilon_2} + \frac{\omega_1 k_s}{\varepsilon_1}\right), \tag{30}$$

$$\max(|c_M\eta - c_b|, |c_m\eta - c_B|) \leq \varepsilon_1\omega_1, \quad k_3 \geq \frac{c_b k_s}{c_e}, \tag{31}$$

$$\bar{\bar{\tau}} \leq \frac{1}{c_B^2 k_s^2}(2c_m\eta - 8\varepsilon_1\omega_1 - 4\omega_2), \tag{32}$$

$$k_1 \geq \frac{1}{c_e}(c_b k_s \hat{\tau}\Upsilon + \Phi), \quad k_2 \geq \frac{k_s}{c_e}(2\varepsilon_1\omega_1 + \omega_2). \tag{33}$$

Proof. When $B_M^\tau > 0$, the delay effect is present in the system because the rider's muscles are receiving stimulation (i.e., $t \in [t_n^u, t_{n+1}^e)$). Furthermore, since B_M^τ and B_E are discontinuous, the time derivative of (25) exists almost everywhere (a.e.) within $t \in [t_0, \infty)$. After using (10)-(12), (18), and applying the Leibniz Rule for (19)-(20), the time derivative of (25) can be expressed as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{=}\ & e(r - \alpha e - \eta e_u) + \tfrac{1}{2}\dot{M}r^2 + \omega_1 e_u k_s(r_{\hat{\tau}} - r) \\ & + r\left(-Vr - e + \chi + k_s B_M^\tau (r_{\hat{\tau}} - r_\tau) - M\eta k_s r\right. \\ & \left. + (M\eta - B_M^\tau)k_s r_{\hat{\tau}}\right) - B_E\left(k_1 \operatorname{sgn}(r) + (k_2 + k_3)r\right) \\ & + \varepsilon_1\omega_1 k_s\left(r^2 - r_{\hat{\tau}}^2\right) + \tfrac{\omega_2 k_s}{\hat{\tau}}\left(\hat{\tau}r^2 - \int_{t-\hat{\tau}}^t r(\theta)^2 d\theta\right).\end{aligned} \tag{34}$$

According to the switching laws in (6) and (7), when $B_M^\tau > 0$, $B_E = 0$ or $B_E > 0$. The more restrictive case is when $B_E = 0$ (i.e., the system is being controlled only by the delayed FES input). Hence, the subsequent proof does not include details for the case when $B_M^\tau > 0$ and $B_E > 0$ since (34) with $B_E > 0$ can be upper bounded by (34) with $B_E = 0$.

Using Properties 1, 7, and 8, canceling common terms, selecting ε_1 and ω_1 such that $\max(|c_M\eta - c_b|, |c_m\eta - c_B|) \leq \varepsilon_1\omega_1$, and setting $B_E = 0$ in (34) yields

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq}\ & -\alpha e^2 + \eta|ee_u| + k_s c_B|r(r_{\hat{\tau}} - r_\tau)| \\ & + \varepsilon_1\omega_1 k_s|rr_{\hat{\tau}}| - c_m\eta k_s r^2 + \varepsilon_1\omega_1 k_s\left(r^2 - r_{\hat{\tau}}^2\right) \\ & + \omega_1 k_s(|e_u r_{\hat{\tau}}| + |e_u r|) + |r||\chi| \\ & + \tfrac{\omega_2 k_s}{\hat{\tau}}\left(\hat{\tau}r^2 - \int_{t-\hat{\tau}}^t r(\theta)^2 d\theta\right).\end{aligned} \tag{35}$$

To facilitate the analysis, Young's Inequality is used to obtain the following inequalities:

$$|ee_u| \leq \frac{1}{2\varepsilon_2\eta}e_u^2 + \frac{\varepsilon_2\eta}{2}e^2, \tag{36}$$

$$|rr_{\hat{\tau}}| \leq \frac{1}{2}r^2 + \frac{1}{2}r_{\hat{\tau}}^2, \tag{37}$$

$$|e_u r_{\hat{\tau}}| \leq \frac{1}{2\varepsilon_1}e_u^2 + \frac{\varepsilon_1}{2}r_{\hat{\tau}}^2, \tag{38}$$

$$|e_u r| \leq \frac{1}{2\varepsilon_1}e_u^2 + \frac{\varepsilon_1}{2}r^2. \tag{39}$$

Substituting (14) and (36)-(39) into (35), and completing the squares on $|r||\chi|$, yields

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq}\ & -\left(\alpha - \tfrac{\varepsilon_2\eta^2}{2}\right)e^2 + \left(\tfrac{1}{2\varepsilon_2} + \tfrac{\omega_1 k_s}{\varepsilon_1}\right)e_u^2 - \tfrac{1}{4}c_m\eta k_s r^2 \\ & + k_s c_B|r(r_{\hat{\tau}} - r_\tau)| + \tfrac{2}{c_m\eta k_s}\left(\rho^2(\|z\|)\|z\|^2 + \Phi^2\right) \\ & - k_s\left(\tfrac{1}{2}c_m\eta - 2\varepsilon_1\omega_1 - \omega_2\right)r^2 - \tfrac{\omega_2 k_s}{\hat{\tau}}\int_{t-\hat{\tau}}^t r(\theta)^2 d\theta.\end{aligned} \tag{40}$$

Using (16) and the Cauchy-Schwarz inequality, an upper bound for e_u^2 is obtained as

$$e_u^2 \leq \hat{\tau}k_s^2\int_{t-\hat{\tau}}^t r(\theta)^2 d\theta, \tag{41}$$

and an upper bound for Q_2 can be obtained as

$$Q_2 \leq \omega_2 k_s\int_{t-\hat{\tau}}^t r(\theta)^2 d\theta. \tag{42}$$

Using (19), (41), and (42) the following upper bound can be developed

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq}\ & -\left(\alpha - \tfrac{\varepsilon_2\eta^2}{2}\right)e^2 - \left(\tfrac{\omega_2}{3k_s\hat{\tau}^2} - \tfrac{1}{2\varepsilon_2} - \tfrac{\omega_1 k_s}{\varepsilon_1}\right)e_u^2 \\ & - \tfrac{1}{4}c_m\eta k_s r^2 - \tfrac{\omega_2}{3\hat{\tau}\varepsilon_1\omega_1}Q_1 - \tfrac{1}{3\hat{\tau}}Q_2 \\ & + k_s c_B|r(r_{\hat{\tau}} - r_\tau)| + \tfrac{2}{c_m\eta k_s}\left[\rho^2(\|z\|)\|z\|^2 + \Phi^2\right] \\ & - k_s\left(\tfrac{1}{2}c_m\eta - 2\varepsilon_1\omega_1 - \omega_2\right)r^2.\end{aligned} \tag{43}$$

Provided that $\|y(t)\| \in S_{\mathcal{D}},\ \forall t \in [t_0,\infty)$ it can be proven that $\dot{r} \leq \Upsilon$, which will allow the Mean Value Theorem to be used to further upper bound (43). Specifically, from Properties 1-6, 8, and 9, (14) and (18), and the fact that $\|y\| \geq \|z\|$,

$$\dot{r} \leq c_1 + c_2\|y\| + c_3\|y\|^2 + c_4\|y_\tau\| + c_5\|y_{\hat{\tau}}\| \leq \Upsilon,$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}_{>0}$ are known constants. Subsequently, by using the Mean Value Theorem, the definition of β_1 in (21), the fact that $\|y\| \geq \|z\|$, completing the squares, and imposing the aforementioned gain conditions in (30)-(33), the following upper bound for (43) is obtained as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\left(\tfrac{\beta_1}{2} - \tfrac{2}{c_m\eta k_s}\rho^2(\|y\|)\right)\|z\|^2 \\ & -\tfrac{\beta_1}{2}\|z\|^2 - \tfrac{\omega_2}{3\hat{\tau}\varepsilon_1\omega_1}Q_1 - \tfrac{1}{3\hat{\tau}}Q_2 \\ & +\tfrac{2\Phi^2}{c_m\eta k_s} + \tfrac{\bar{\tilde{\tau}}\Upsilon^2}{k_s}.\end{aligned} \tag{44}$$

Provided $\|y(t_n^u)\| \in S_{\mathcal{D}}$ and using (23) and (29), (44), can be upper bounded as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\delta_1\|y\|^2 + \nu. \tag{45}$$

From (26), the bound in (45) can be further bounded as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\frac{\delta_1}{\lambda_2}V_L + \nu, \tag{46}$$

$\forall t \in \left[t_n^u, t_{n+1}^e\right)$.

When $B_M^\tau = 0$, the delay effect is absent from the system (i.e., $t \in \left[t_n^e, t_{n+1}^u\right)$). According to the switching laws in (6) and (7), when $B_M^\tau = 0$, $B_E > 0$ (i.e., the system is controlled by the motor only). Using Properties 1 and 7-9, canceling common terms, choosing ε_1 and ω_1 such that $c_M\eta - c_b \leq \varepsilon_1\omega_1$, and setting $B_M^\tau = 0$ in (34), an upper bound for (34) can be obtained as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\alpha e^2 + \eta|ee_u| + |r||\chi| - c_e k_1|r| + c_b k_s r r_{\hat{\tau}} \\ & -(c_e(k_2+k_3) + c_m\eta k_s)r^2 + \omega_1 k_s(|e_u r_{\hat{\tau}}| + |e_u r|) \\ & +\varepsilon_1\omega_1 k_s\left(r^2 - r_{\hat{\tau}}^2\right) + \varepsilon_1\omega_1 k_s|r r_{\hat{\tau}}| \\ & +\tfrac{\omega_2 k_s}{\hat{\tau}}\left(\hat{\tau}r^2 - \int_{t-\hat{\tau}}^{t} r(\theta)^2 d\theta\right).\end{aligned} \tag{47}$$

After substituting (14) and (36)-(39) into (47), using the Mean Value Theorem, selecting the gain conditions according to (30)-(33), and completing the squares on $|r||\chi|$, (47) can be upper bounded as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\left(\alpha - \tfrac{\varepsilon_2\eta^2}{2}\right)e^2 + \left(\tfrac{1}{2\varepsilon_2} + \tfrac{\omega_1 k_s}{\varepsilon_1}\right)e_u^2 \\ & -(c_e k_2 - 2\varepsilon_1\omega_1 k_s - \omega_2 k_s)r^2 \\ & +\tfrac{1}{4c_m\eta k_s}\rho^2(\|z\|)\|z\|^2 - \tfrac{\omega_2 k_s}{\hat{\tau}}\int_{t-\hat{\tau}}^{t} r(\theta)^2 d\theta.\end{aligned} \tag{48}$$

After following a similar development as the case when $B_M^\tau > 0$, (48) can be upper bounded as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\frac{\delta_2}{\lambda_2}V_L, \tag{49}$$

$\forall t \in \left[t_n^e, t_{n+1}^u\right)$ provided that $\|y(t_n^e)\| \in S_{\mathcal{D}}$. The result in (49) can be further upper bounded by adding the constant ν and substituting the decay rate $\lambda_3 \triangleq \lambda_2^{-1}\min(\delta_1, \delta_2)$ to yield

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\lambda_3 V_L + \nu. \tag{50}$$

Hence, (46) can be used with (50) to verify (25) is a common Lyapunov function across all regions of the crank cycle. Furthermore, the decay rate in (50) represents the most conservative decay rate across all regions (i.e., $\forall t \in [t_0,\infty)$). Solving the differential inequality in (50) yields the following bound

$$\begin{aligned}V_L(t) \leq & \left(V_L(t_0) - \lambda_3^{-1}\nu\right) \\ & \cdot\exp(-\lambda_3(t-t_0)) + \lambda_3^{-1}\nu.\end{aligned} \tag{51}$$

Sufficient conditions for $\|y(t)\| \in S_{\mathcal{D}},\ \forall t \in [t_0,\infty)$, are $\|y(t_n^u)\|, \|y(t_n^e)\| \in S_{\mathcal{D}}$. Therefore, provided that $\|y(t_n^u)\|$, $\|y(t_n^e)\| \in S_{\mathcal{D}}$ and the aforementioned gain conditions are met, (25) can be used with (51) to yield the exponential bound in (29). From (25) and (50), $e, r, e_u \in \mathcal{L}_\infty$. By (16) and (17), $u, u_e \in \mathcal{L}_\infty$ and the remaining signals are bounded. □

CONCLUSION

This paper examines the development of switching conditions and the associated Lyapunov stability analysis to account for the time-varying input delayed response of closed-loop FES-cycling. A controller is developed to provide cadence tracking that is robust to a time-varying input delay, uncertain nonlinear lower limb dynamics, and bounded unknown additive disturbances. Switching conditions are developed to activate and deactivate the FES such that the muscles will generate torque when entering kinematically efficient regions of the cycle and stop generating torque when entering inefficient regions. Because it is currently unknown how the delay varies with time, a constant delay estimate is used to provide robust cadence tracking for a time-varying input delay. A focus of future efforts will be to quantify the FES delay and how it varies with time. Additional efforts will then focus on implementing a time-varying estimate of the time-varying input delay in the control design to yield a more precise estimate and hence improved cadence tracking.

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