Kronecker Powers of Tensors and Strassen's Laser Method

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Abstract

We answer a question, posed implicitly in [18, §11], [11, Rem. 15.44] and explicitly in [9, Problem 9.8], showing the border rank of the Kronecker square of the little Coppersmith-Winograd tensor is the square of the border rank of the tensor for all $q > 2$, a negative result for complexity theory. We further show that when $q > 4$, the analogous result holds for the Kronecker cube. In the positive direction, we enlarge the list of explicit tensors potentially useful for the laser method. We observe that a well-known tensor, the 3×3 determinant polynomial regarded as a tensor, $\det_3 \in \mathbb{C}^9 \otimes \mathbb{C}^9 \otimes \mathbb{C}^9$, could potentially be used in the laser method to prove the exponent of matrix multiplication is two. Because of this, we prove new upper bounds on its Waring rank and rank (both 18), border rank and Waring border rank (both 17), which, in addition to being promising for the laser method, are of interest in their own right. We discuss "skew" cousins of the little Coppersmith-Winograd tensor and indicate why they may be useful for the laser method. We establish general results regarding border ranks of Kronecker powers of tensors, and make a detailed study of Kronecker squares of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

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1 Introduction

The exponent ω of matrix multiplication is defined as

$$\omega := \inf\{\tau \mid \mathbf{n} \times \mathbf{n} \text{ matrices may be multiplied using } O(\mathbf{n}^\tau) \text{ arithmetic operations}\}.$$

The exponent is a fundamental constant governing the complexity of the basic operations in linear algebra. It is conjectured that $\omega = 2$. There was steady progress in the research for upper bounds from 1968 to 1988: after Strassen's famous $\omega < 2.81$ [39], Bini et al. [8], using border rank (see below), showed $\omega < 2.78$, then a major breakthrough by Schönhage [36] (the asymptotic sum inequality) was used to show $\omega < 2.55$, then Strassen's *laser method* was introduced and used by Strassen to show $\omega < 2.48$, and refined by Coppersmith and Winograd to show $\omega < 2.3755$ [18]. Then there was no progress until 2011 when a series of improvements by Stothers, Williams, and Le Gall [38, 45, 33] lowered the upper bound to the current state of the art $\omega < 2.373$.

Strassen's 1968 result is obtained by an explicit algorithm for multiplying matrices. This algorithm is more efficient than the standard one in practical implementation as soon as the size of the matrices is around 1000×1000, see [6]. Bini et al. exhibited a matrix multiplication algorithm that is in principle implementable exactly (at a cost of a constant size blow-up which does not effect the exponent) but as presented is only a sequence of algorithms that limits to an exact one. This gave rise to the notion of *border rank* to describe this phenomenon. To explain border rank, it is best to adopt the language of *tensors*.

A bilinear map $b : \mathbb{C}^{\mathbf{a}} \times \mathbb{C}^{\mathbf{b}} \to \mathbb{C}^{\mathbf{c}}$ may be regarded as a trilinear form $\widehat{b} : \mathbb{C}^{\mathbf{a}} \times \mathbb{C}^{\mathbf{b}} \times \mathbb{C}^{\mathbf{c}} \to \mathbb{C}$ defined by $\widehat{b}(X, Y, \alpha) = \alpha \cdot b(X, Y)$ where $b(X, Y)$ is regarded as a column vector of $\mathbb{C}^{\mathbf{c}}$, α is regarded as a row vector and $\cdot$ is the row-column multiplication. In this language, matrix multiplication, as a trilinear map, becomes $M_{\mathbf{l},\mathbf{m},\mathbf{n}}(X, Y, Z) = \mathrm{trace}(XYZ)$, where X, Y, Z are matrices of size $\mathbf{l} \times \mathbf{m}$, $\mathbf{m} \times \mathbf{n}$ and $\mathbf{n} \times \mathbf{l}$, respectively. It is known [11, §14.1] that the complexity of performing a bilinear map is captured, up to a factor of four, by the *tensor rank* of the corresponding tensor. Thus, this geometric quantity may be used to determine ω.

Let A, B, C be fixed vector spaces. A tensor $T \in A \otimes B \otimes C$ has *rank one* if $T = a \otimes b \otimes c$ for some $a \in A$, $b \in B$, $c \in C$. The *rank* of T, denoted $\mathbf{R}(T)$, is the smallest r such that T is sum of r rank one tensors. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T is the limit of a sequence of rank r tensors. One has $\underline{\mathbf{R}}(T) \leq \mathbf{R}(\mathbf{T})$ and the inequality can be strict: Let $T = a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$, then $\mathbf{R}(\mathbf{T}) = 3$ and $\underline{\mathbf{R}}(T) = 2$ as

$$T = \lim_{t \to 0} \frac{1}{t}[(a_1 + ta_2) \otimes (b_1 + tb_2) \otimes (c_1 + tc_2) - a_1 \otimes b_1 \otimes c_1].$$

Bini [7] proved that the border rank of matrix multiplication also captures its complexity. More precisely,

$$\omega = \inf\{\tau : \underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \in O(\mathbf{n}^{\tau})\}.$$

Schönhage's advance comes from his discovery that it can be more efficent to perform two matrix multiplications together than one at a time. For tensors $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, define a new tensor $T \oplus T' \in (A \oplus A') \otimes (B \oplus B') \otimes (C \otimes C')$ whose computation is equivalent to computing T and T'. He gave explicit examples of matrix multiplication tensors where $\underline{\mathbf{R}}(T \oplus T') \ll \underline{\mathbf{R}}(T) + \underline{\mathbf{R}}(T')$. To explain how he exploited this we need some more definitions:

Given $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, the *Kronecker product* of T and T' is the tensor $T \boxtimes T' := T \otimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$, regarded as 3-way tensor. Given $T \in A \otimes B \otimes C$, the *Kronecker powers* of T are $T^{\boxtimes N} \in A^{\otimes N} \otimes B^{\otimes N} \otimes C^{\otimes N}$, defined iteratively. We have $\mathbf{R}(T \boxtimes T') \leq \mathbf{R}(T)\mathbf{R}(T')$, and similarly for border rank. The matrix multiplication tensor has the following important self-reproducing property: $M_{\langle \mathbf{l},\mathbf{m},\mathbf{n} \rangle} \boxtimes M_{\langle \mathbf{l}',\mathbf{m}'\mathbf{n}' \rangle} = M_{\langle \mathbf{l}\mathbf{l}',\mathbf{m}\mathbf{m}',\mathbf{n}\mathbf{n}' \rangle}$.

Given $T, T' \in A \otimes B \otimes C$, we say that T *degenerates* to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$, the closure of the orbit of T under the natural action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$. Here $GL(A)$ denote the general linear group of invertible linear maps $A \to A$. Border rank is upper semi-continuous under degeneration: if T' is a degeneration of T, then $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T)$.

Schönhage observed that if one takes a high Kronecker power of $(M_{\langle \mathbf{l},\mathbf{m},\mathbf{n}\rangle} \oplus M_{\langle \mathbf{l}',\mathbf{m}',\mathbf{n}'\rangle})$, that because of the reproducing property, it will be a sum of matrix multiplication tensors, some of them quite large. One can then perform a degeneration to obtain a single very large matrix multiplication tensor and exploit the strict sub-additivity to get an upper bound on this large matrix multiplication tensor. This is his celebrated *asymptotic sum inequality*.

After Schönhage, Strassen realized that the starting tensor need not be a sum of matrix multiplication tensors, as long as some high power of it degenerates to a large matrix multiplication tensor. This gave rise to his *laser method*, where the starting tensor "resembles" the sum of disjoint matrix multiplication tensors. All upper bounds since 1984 are obtained via Strassen's laser method. The best starting tensor for Strassen's method (so far) was discovered by Coppersmith and Winograd, the *big Coppersmith-Winograd tensor*.

In 2014 [4] gave an explanation for the limited progress since 1988, followed by further explanations in [3, 2, 13, 1]: there are limitations to the laser method applied to the big Coppersmith-Winograd tensor and other auxiliary tensors. These limitations are referred to as *barriers*. Our main motivation is to eventually overcome these barriers via auxilary tensors that avoid them, or, failing that, to prove structural results explaining the failure. We deal with the *little Coppersmith-Winograd tensor*, which was known to potentially avoid the barriers and a new series of tensors that are skew versions of the little Coppersmith-Winograd tensor that we show also potentially avoid the barriers. We are interested in two kinds of barriers: to proving the exponent is two, and barriers to proving the exponent is less than 2.3.

▶ Remark 1. A different approach to upper bounds was introduced by Cohn and Umans [15] using the Fourier-transform on finite groups. One can show $\omega < 2.41$ by this method [13, 14].

Definitions and notation

Let A, B, C be complex vector spaces. We will work with tensors in $A \otimes B \otimes C$. Let $GL(A)$ denote the general linear group of invertible linear maps $A \to A$. Unless stated otherwise, we write $\{a_i\}$ for a basis of A, and similarly for bases of B and C. Often we assume that all tensors involved in the discussion belong to the same space $A \otimes B \otimes C$; this is not restrictive, since we may re-embed the spaces A, B, C into larger spaces whenever it is needed. We say that two tensors are *isomorphic* if they are the same up to a change of bases in A,B and C.

One may define border rank in terms of degeneration: $\underline{\mathbf{R}}(T) \leq r$ if and only if $M_{\langle 1\rangle}^{\oplus r}$ degenerates to T. The *border subrank* of T, denoted $\underline{\mathbf{Q}}(T)$, is the largest q such that T degenerates to $M_{\langle 1\rangle}^{\oplus q}$.

The *asymptotic rank* of T is $\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty} \underline{\mathbf{R}}(T^{\boxtimes N})^{1/N}$. Thus $\omega = \log_{\mathbf{m}} \underset{\sim}{\mathbf{R}}(M_{\langle \mathbf{m}\rangle})$ for any $\mathbf{m} > 2$. The *asymptotic subrank* of T is $\underset{\sim}{\mathbf{Q}}(T) = \lim_{N\to\infty} \underline{\mathbf{Q}}(T^{\boxtimes N})^{1/N}$. These limits exist and are finite, see [41]. Moreover $\underset{\sim}{\mathbf{R}}(T) \leq \underline{\mathbf{R}}(T)$ and $\underset{\sim}{\mathbf{Q}}(T) \geq \underline{\mathbf{Q}}(T)$.

A tensor $T \in A \otimes B \otimes C$ is *concise* if the induced linear maps $T_A : A^* \to B \otimes C$, $T_B : B^* \to A \otimes C$, $T_C : C^* \to A \otimes B$ are injective. We say that a concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ has *minimal rank* (resp. *minimal border rank*) if $\mathbf{R}(T) = m$ (resp. $\underline{\mathbf{R}}(T) = m$).

The laser method and the Coppersmith-Winograd tensors

So far, the best upper bounds for ω have been obtained using the laser method applied to the big Coppersmith-Winograd tensor, which is

$$T_{CW,q} := \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 + \\ + a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0 \in (\mathbb{C}^{q+2})^{\otimes 3}.$$

It was used to obtain the current world record $\omega < 2.373$ and all bounds below $\omega < 2.41$. The barrier identified in [4] said that $T_{CW,q}$ cannot be used to prove $\omega < 2.3$ using the standard laser method, and a geometric identification of this barrier in terms of asymptotic subrank was given in [13]: $\underset{\sim}{\mathbf{Q}}(M_{\langle \mathbf{n} \rangle}) = \mathbf{n}^2$ which is maximal, which is used to show any tensor with non-maximal asymptotic subrank cannot be used to prove $\omega = 2$ by the laser method, and Strassen [43] had shown $\underset{\sim}{\mathbf{Q}}(T_{CW,q})$ is non-maximal.

The second best tensor for the laser method so far has been the little Coppersmith-Winograd tensor, which is

$$T_{cw,q} := \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 \in (\mathbb{C}^{q+1})^{\otimes 3}. \tag{1}$$

The laser method was used to prove the following inequality:

▶ **Theorem 2.** *[18] For all k and q,*

$$\omega \leq \log_q\left(\frac{4}{27}(\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}))^{\frac{3}{k}}\right). \tag{2}$$

More precisely, the ingredients needed for the proof but not the statement appears in [18]. It was pointed out in [11, Ex. 15.24] that the statement holds with $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k})^{\frac{3}{k}}$ replaced by $\underset{\sim}{\mathbf{R}}(T_{cw,q})^3$ and the proof implicitly uses (2). The equation does appear in [29, Thm. 5.1.5.1].

An easy calculation shows $\underline{\mathbf{R}}(T_{cw,q}) = q+2$ (one more than minimal). Applying Theorem 2 to $T_{cw,8}$ with $k = 1$ gives $\omega \leq 2.41$ [18]. Theorem 2 shows that, unlike $T_{CW,q}$, $T_{cw,2}$ is not subject to the barriers of [4, 3, 2, 13] for proving $\omega = 2$, and $T_{cw,q}$, for $2 \leq q \leq 10$ are not subject to the barriers for proving $\omega < 2.3$. Thus, if any Kronecker power of $T_{cw,q}$ for $2 \leq q \leq 10$ is strictly sub-multiplicative, one can get new upper bounds on ω, and if it were the case that $\underset{\sim}{\mathbf{R}}(T_{cw,2}) = 3$, one would obtain that ω is two. Hence the questions:

▶ **Question 3.** *For given q, k, what is $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k})$? Does there exist $q \in \{2, \ldots, 10\}$ and $k \in \mathbb{N}$ such that $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}) < [\underline{\mathbf{R}}(T_{cw,q})]^k]$?*

▶ Remark 4. Although we know little about asymptotic rank of explicit tensors beyond matrix multiplication, most tensors have asymptotic rank less than their border rank: For all tensors $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, with $m > 3$, outside a set of measure zero (more precisely, for all tensors outside a proper subvariety), Lickteig showed that $\underline{\mathbf{R}}(T) = \lceil \frac{m^3}{3m-2} \rceil$ [35]. Strassen [42, Lemma 3.5] implicitly showed that for any $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, if $\underline{\mathbf{R}}(T) > m^{\frac{2\omega}{3}} > m^{1.6}$, then $\underset{\sim}{\mathbf{R}}(T) < \underline{\mathbf{R}}(T)$. It is worth recalling Strassen's proof: any $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is a degeneration of $M_{\langle 1,m,m \rangle} \in \mathbb{C}^{m^2} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, so $T^{\boxtimes 3}$ is a degeneration of $M_{\langle m^2,m^2,m^2 \rangle} = M_{\langle 1,m,m \rangle} \boxtimes M_{\langle m,1,m \rangle} \boxtimes M_{\langle m,m,1 \rangle}$. In particular $\underline{\mathbf{R}}(T^{\boxtimes 3}) \leq \underline{\mathbf{R}}(M_{\langle m^2,m^2,m^2 \rangle})$ and $\underset{\sim}{\mathbf{R}}(T)^3 = \underset{\sim}{\mathbf{R}}(T^{\boxtimes 3}) \leq \underset{\sim}{\mathbf{R}}(M_{\langle m^2,m^2,m^2 \rangle}) = m^{2\omega}$, so $\underset{\sim}{\mathbf{R}}(T) \leq m^{\frac{2\omega}{3}}$. Since $\omega < 2.4$ we conclude. In particular, note that $\underset{\sim}{\mathbf{R}}(T) \leq m^{1.6}$ for all $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$.

2 Results

2.1 Lower bounds for Kronecker powers of $T_{cw,q}$

We address Problem 9.8 in [9], which was motivated by Theorem 2: Is $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 2}) < (q+2)^2$? We give an almost complete answer:

▶ **Theorem 5.** *For all $q > 2$, $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 2}) = (q+2)^2$, and $15 \leq \underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) \leq 16$.*

We also examine the Kronecker cube:

▶ **Theorem 6.** *For all $q > 4$, $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 3}) = (q+2)^3$.*

Proofs are given in §4.

Proposition 25 below, combined with the proofs of Theorems 6 and 5, implies

▶ **Corollary 7.** *For all $q > 4$ and all N,*

$$\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes N}) \geq (q+1)^{N-3}(q+2)^3,$$

and $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes N}) \geq 36 \times 5^{N-2}$.

Previously, in [10] it had been shown that $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes N}) \geq (q+1)^N + 2^N - 1$ for all q, N, whereas the bound in Corollary 7 is $(q+1)^N + 3(q+1)^{N-1} + 3(q+1)^{N-2} + (q+1)^{N-3}$.

Previous to this work one might have hoped to prove $\omega < 2.3$ simply by using the Kronecker square of, e.g., $T_{cw,7}$. Now, the smallest possible calculation to give a new upper bound on ω from a tensor that has been used in the laser method would be e.g., to prove the fourth Kronecker power of a small Coppersmith-Winograd tensor achieves the lower bound of Corollary 7 (which we do not expect to happen). Of course, one could work directly with the matrix multiplication tensor, in which case the cheapest possible upper bound would come from proving the border rank of the 6×6 matrix multiplication tensor equaled its known lower bound of 69 from [30].

The following corollary of Theorems 5 and 6 is immediate by the semi-continuity property of border rank, as most tensors of border rank $m+1$ in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ may be degenerated to $T_{cw,m-1}$, in fact the set of tensors of border rank $m+1$ is an orbit closure and $T_{cw,m-1}$ lives on the boundary of the orbit.

▶ **Corollary 8.** *Most tensors $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank $m+1$, satisfy $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2 = (m+1)^2$ for $m \geq 4$ and $\underline{\mathbf{R}}(T^{\boxtimes 3}) = \underline{\mathbf{R}}(T)^3 = (m+1)^3$ for $m \geq 6$. More precisely all tensors outside of a Zariski closed subset of the set of tensors of border rank $m+1$. In particular the set of such is of full measure.*

2.2 A skew cousin of $T_{cw,q}$

In light of the negative results for complexity theory above, one might try to find a better tensor than $T_{cw,q}$ that is also not subject to the barriers. In [16], when q is even, we introduced a skew cousin of the big Coppersmith-Winograd tensor, which has the largest symmetry group of any tensor in its space satisfying a natural genericity condition. However this tensor turns out not to be useful for the laser method. Inspired by it, we introduce a skew cousin of the small Coppersmith-Winograd tensor when q is even:

$$T_{skewcw,q} := \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + \sum_{\xi=1}^{\frac{q}{2}} (a_\xi \otimes b_{\xi+\frac{q}{2}} - a_{\xi+\frac{q}{2}} \otimes b_\xi) \otimes c_0 \in (\mathbb{C}^{q+1})^{\otimes 3}. \quad (3)$$

In the language of [11], $T_{skewcw,q}$ has the same "block structure" as $T_{cw,q}$, which immediately implies Theorem 2 also holds for $T_{skewcw,q}$:

▶ **Theorem 9.** *For all k,*

$$\omega \leq \log_q\left(\frac{4}{27}(\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes k}))^{\frac{3}{k}}\right). \quad (4)$$

In particular, the known barriers do not apply to $T_{skewcw,2}$ for proving $\omega = 2$ and to any $T_{skewcw,q}$ for $q \leq 10$ for proving $\omega < 2.3$. Unfortunately, we have

▶ **Proposition 10.** $\underline{\mathbf{R}}(T_{skewcw,q}) \geq q + 3$.

Proposition 10 is proved in §4.

Thus $\underline{\mathbf{R}}(T_{skewcw,q}) > \underline{\mathbf{R}}(T_{cw,q})$ for all q, in particular $\underline{\mathbf{R}}(T_{skewcw,2}) = 5$, as for all $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, $\underline{\mathbf{R}}(T) \leq 5$.

However, unlike $T_{cw,2}$, substantial strict sub-multiplicativity holds for the Kronecker square of $T_{skewcw,2}$:

▶ **Theorem 11.** $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) \leq 17$.

▶ Remark 12. Regarding border rank strict submultiplicativity of Kronecker powers for other explicit tensors, little is known. For matrix multiplication, the only explicit drop under a Kronecker power that is known to our knowledge is [37]: $\underline{\mathbf{R}}(M_{\langle 2\rangle}^{\boxtimes 2}) \leq 46 < 49$.

Previous to this work, we are only aware of one class of tensors other than $M_{\langle 2\rangle}$ for which any bound on the Kronecker squares other than the trivial $\underline{\mathbf{R}}(T^{\boxtimes 2}) \leq \underline{\mathbf{R}}(T)^2$ is known. In [12], they show that

$$\begin{aligned} T_{CGJ,m} :=& a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 + (\textstyle\sum_{i=1}^3 a_i) \otimes (\sum_{j=1}^3 b_j) \otimes (\sum_{j=1}^3 c_k) \\ &+ 2(a_1 + a_2) \otimes (b_1 + b_3) \otimes (c_2 + c_3) + a_3 \otimes (\textstyle\sum_{s=4}^m b_s \otimes c_s) \in \mathbb{C}^3 \otimes \mathbb{C}^m \otimes \mathbb{C}^m \end{aligned}$$

satisfies $\underline{\mathbf{R}}(T_{CGJ,m}) = m + 2$ and $\underline{\mathbf{R}}(T_{CGJ,m}^{\otimes 2}) \leq (m+2)^2 - 1$. Of course, for any tensor T, $\underline{\mathbf{R}}(T^{\boxtimes 2}) \leq \underline{\mathbf{R}}(T^{\otimes 2})$, and strict inequality, e.g., with $M_{\langle 2\rangle}$ is possible. This is part of a general theory in [12] for constructing examples with a drop of one when the last non-trivial secant variety is a hypersurface.

We also show

▶ **Theorem 13.** $\mathbf{R}(T_{skewcw,2}^{\boxtimes 2}) \leq 18$.

Theorems 11 and 13 are proved in §5.

2.3 Two familiar tensors with no known laser method barriers

Recall from above that either $\underset{\sim}{\mathbf{R}}(T_{cw,2}) = 3$ or $\underset{\sim}{\mathbf{R}}(T_{skewcw,2}) = 3$ would imply $\omega = 2$.

Let $\det_3 \in (\mathbb{C}^9)^{\otimes 3}$ and $\mathrm{perm}_3 \in (\mathbb{C}^9)^{\otimes 3}$ be the 3×3 determinant and permanent polynomials considered as tensors. We observe that if either of these has minimal asymptotic rank, then $\omega = 2$: either $\underset{\sim}{\mathbf{R}}(\det_3) = 9$ or $\underset{\sim}{\mathbf{R}}(\mathrm{perm}_3) = 9$ would imply $\omega = 2$. This observation is an immediate consequence of the following lemma:

▶ **Lemma 14.** *We have the following isomorphisms of tensors:*

$$T_{cw,2}^{\boxtimes 2} \cong \mathrm{perm}_3$$
$$T_{skewcw,2}^{\boxtimes 2} \cong \det_3.$$

Lemma 14 is proved in §3.

Lemma 14 thus implies Theorems 11 and 13 may be restated as saying $\underline{\mathbf{R}}(\det_3) \leq 17$ and $\mathbf{R}(\det_3) \leq 18$. Although it is not necessarily relevant for complexity theory, we actually prove stronger statements, which are important for geometry:

A symmetric tensor $T \in S^3\mathbb{C}^m \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ has *Waring rank one* if $T = a \otimes a \otimes a$ for some $a \in \mathbb{C}^3$. The *Waring rank* of T, denoted $\mathbf{R}_S(T)$, is the smallest r such that T is sum of r tensors of Waring rank one. The *Waring border rank* of T, denoted $\underline{\mathbf{R}}_S(T)$, is the smallest r such that T is limit of a sequence of tensors of Waring rank r.

We actually show:

▶ **Theorem 15.** $\mathbf{R}_S(\det_3) \leq 18$.

and

▶ **Theorem 16.** $\underline{\mathbf{R}}_S(\det_3) \leq 17$.

Proofs are respectively given in §5.1 and §5.2.

2.4 Generic tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

▶ Remark 17. A generic tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ has border rank five. Our numerical experiments suggest that for all $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\underline{\mathbf{R}}(T^{\boxtimes 2}) \leq 22 < 25. \tag{5}$$

This is obtained by starting with a tensor whose entries are obtained from making draws according to a uniform distribution on $[-1, 1]$, and proving the result for that tensor. The data to perform an example of this computation is available in Appendix A at `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`.

▶ **Problem 18.** *Write a proof of* (5)*. Even better, give a geometric proof.*

The inequality (5) is not too surprising because $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is *secant defective*, in the sense that by a dimension count, one would expect the maximum border rank of a tensor to be 4, but the actual maximum is 5. This means that for a generic tensor, there is a 8 parameter family of rank 5 decompositions, and it is not surprising that the naïve 64-parameter family of decompositions of the square might have decompositions of lower border rank on the boundary.

3 Symmetries of tensors and the proof of Lemma 14

3.1 Symmetry groups of tensors and polynomials

The group $GL(A) \times GL(B) \times GL(C)$ acts naturally on $A \otimes B \otimes C$. The map $\Phi : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ has a two dimensional kernel $\ker \Phi = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda\mu\nu = 1\} \simeq (\mathbb{C}^*)^2$.

In particular, the group $(GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^{\times 2}$ is naturally identified with a subgroup of $GL(A \otimes B \otimes C)$. Given $T \in A \otimes B \otimes C$, the *symmetry group* of a tensor T is the stabilizer of T in $(GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^{\times 2}$, that is

$$G_T := \{g \in (GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^{\times 2} \mid \ g \cdot T = T\}. \tag{6}$$

Let $\mathfrak{S}_k$ be the permutation group on k elements. We record the following observation:

▶ **Proposition 19.** *For any tensor $T \in A \otimes B \otimes C$, $G_{T^{\boxtimes N}} \supset \mathfrak{S}_N$.*

Proof. Write $T^{\boxtimes N} = \sum_{I,J,K} T^{I,J,K} a_I \otimes b_J \otimes c_K$ where $I = (i_1, \dots, i_N)$, $a_I = a_{i_1} \otimes \cdots \otimes a_{i_N}$, etc.. For $\sigma \in \mathfrak{S}_N$, define $\sigma \cdot T = \sum_{I,J,K} T^{IJK} a_{\sigma(I)} \otimes b_{\sigma(J)} \otimes c_{\sigma(K)}$. Since $T^{IJK} = T^{i_1 j_1 k_1} \cdots T^{i_N j_N k_N}$ we have $T^{IJK} = T^{\sigma(I),\sigma(J),\sigma(K)}$ and we conclude. ◀

▶ Remark 20. For a symmetric tensor (equivalently, a homogeneous polynomial), $T \in S^d A$, one may also consider the symmetry group $G_T^s := \{g \in GL(A) \mid g \cdot T = T\}$ where the action is the induced action on polynomials.

3.2 Proof of Lemma 14

Write $(-1)^\sigma$ for the sign of a permutation σ. Let

$$\det{}_3 = \sum_{\sigma,\tau\in\mathfrak{S}_3} (-1)^\tau a_{\sigma(1)\tau(1)} \otimes b_{\sigma(2)\tau(2)} \otimes c_{\sigma(3)\tau(3)},$$
$$\mathrm{perm}_3 = \sum_{\sigma,\tau\in\mathfrak{S}_3} a_{\sigma(1)\tau(1)} \otimes b_{\sigma(2)\tau(2)} \otimes c_{\sigma(3)\tau(3)}$$

be the 3×3 determinant and permanent polynomials regarded as tensors in $\mathbb{C}^9 \otimes \mathbb{C}^9 \otimes \mathbb{C}^9$.

Proof of Lemma 14. After the change of basis $\tilde{b}_0 := -b_0$ and $\tilde{c}_1 := c_2$, $\tilde{c}_2 := -c_1$, we obtain

$$\begin{aligned} T_{skewcw,2} = {} & a_0 \otimes b_1 \otimes \tilde{c}_2 - a_0 \otimes b_2 \otimes \tilde{c}_1 + a_2 \otimes \tilde{b}_0 \otimes c_1 \\ & - a_1 \otimes \tilde{b}_0 \otimes \tilde{c}_2 + a_1 \otimes b_2 \otimes c_0 - a_2 \otimes b_1 \otimes c_0. \end{aligned}$$

This shows that, after identifying the three spaces, $T_{skewcw,2} = a_0 \wedge a_1 \wedge a_2$ is the unique (up to scale) skew-symmetric tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. In particular, $T_{skewcw,2}$ is invariant under the action of SL_3 on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Consequently, the stabilizer of $T_{skewcw,2}^{\boxtimes 2}$ in $GL(\mathbb{C}^9)$ contains (and in fact equals) $SL_3^{\times 2} \rtimes \mathbb{Z}_2$. This is the stabilizer of the determinant polynomial $\det_3$. Since the determinant is characterized by its stabilizer, we conclude.

The tensor $T_{cw,2}$ is symmetric and, after identifying the three spaces, it coincides with $a_0(a_1^2 + a_2^2) \in S^3\mathbb{C}^3$. After the change of basis $\tilde{a}_1 := a_1 + a_2$, $\tilde{a}_2 := a_1 - a_2$, we obtain $T_{cw,2} = a_0\tilde{a}_1\tilde{a}_2 \in S^3\mathbb{C}^3$ is the square-free monomial of degree 3. The stabilizer of $T_{cw,2}$ under the action of GL_3 on $S^3\mathbb{C}^3$ is $\mathbb{T}_3^{SL} \rtimes \mathfrak{S}_3$, where $\mathbb{T}_3^{SL}$ denotes the torus of diagonal matrices with determinant one, and $\mathfrak{S}_3$ acts permuting the three basis elements.

Consequently, the stabilizer of $T_{cw,2}^{\boxtimes 2}$ in $GL(\mathbb{C}^9)$ contains (and in fact equals) $(\mathbb{T}_3^{SL} \rtimes \mathfrak{S}_3)^{\times 2} \rtimes \mathbb{Z}_2$. This is the stabilizer of the permanent polynomial perm_3. Since the permanent is characterized by its stabilizer, we conclude. ◀

▶ Remark 21. For the reader's convenience, here are short proofs that $\det_m, \mathrm{perm}_m$ are characterized by their stabilizers: To see $\det_m$ is characterized by its stabilizer, note that $SL_m \times SL_m = SL(E) \times SL(F)$ acting on $S^m(E \otimes F)$ decomposes it to

$$\bigoplus_{|\pi|=m} S_\pi E \otimes S_\pi F$$

which is multiplicity free, with the only trivial module $S_{1^m}E \otimes S_{1^m}F = \Lambda^m E \otimes \Lambda^m F$. To see that perm_m is characterized by its stabilizer, take the above decomposition and consider the $\mathbb{T}^{SL(E)} \times \mathbb{T}^{SL(F)}$-invariants, these are the weight zero spaces $(S_\pi E)_0 \otimes (S_\pi F)_0$. By [24], one has the decomposition of the weight zero spaces as $\mathfrak{S}_m^E \times \mathfrak{S}_m^F$-modules to $(S_\pi E)_0 \otimes (S_\pi F)_0 = [\pi]_E \otimes [\pi]_F$. The only such that is trivial is the case $\pi = (d)$.

▶ Remark 22. Even Kronecker powers of $T_{skewcw,2}$ are invariant under $SL_3^{\times 2k}$, and coincide, up to a change of basis, with the *Pascal determinants* (see, e.g., [27, §8.3]), $T_{skewcw,2}^{\boxtimes 2k} = PasDet_{k,3}$, the unique, up to scale, tensor spanning $(\Lambda^3\mathbb{C}^3)^{\otimes 2k} \subset S^3((\mathbb{C}^3)^{\otimes 2k})$.

▶ Remark 23. One can regard the 3×3 determinant and permanent as trilinear maps $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$, where the three copies of $\mathbb{C}^3$ are the first, second and third column of a 3×3 matrix. From this point of view, the trilinear map given by the determinant is $T_{skewcw,2}$ as a tensor and the one given by the permanent is $T_{cw,2}$ as a tensor. This perspective, combined with the notion of product rank, immediately provides the upper bounds $\underline{\mathbf{R}}(\mathrm{perm}_3) \leq 16$ (which is also a consequence of Lemma 14) and $\underline{\mathbf{R}}(\det_3) \leq 20$, see [19, 26].

▶ Remark 24. A similar change of basis as the one performed in the second part of proof of Lemma 14 shows that, up to a change of basis, $T_{skewcw,q} \in \Lambda^3\mathbb{C}^{q+1}$. In particular, its even Kronecker powers are symmetric tensors.

4 Koszul flattenings and lower bounds for Kronecker powers

In this section we review Koszul flattenings, prove a result on propagation of Koszul flattening lower bounds under Kronecker products, and prove Theorems 5 and 6. We give two proofs of Theorem 5 because the first is elementary and method of the second generalizes to give the proof of Theorem 6.

4.1 Definition

Respectively fix bases $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ of the vector spaces A, B, C. Given $T = \sum_{ijk} T^{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$, define the linear map

$$\begin{aligned} T_A^{\wedge p} : \Lambda^p A \otimes B^* &\to \Lambda^{p+1} A \otimes C \\ X \otimes \beta &\mapsto \textstyle\sum_{ijk} T^{ijk} \beta(b_j)(a_i \wedge X) \otimes c_k. \end{aligned}$$

Then [31, Proposition 4.1.1] states

$$\underline{\mathbf{R}}(T) \geq \frac{\operatorname{rank}(T_A^{\wedge p})}{\binom{\dim(A)-1}{p}}. \tag{7}$$

This type of lower bound has a long history: in general, one takes the space $A \otimes B \otimes C$ and linearly embeds it into a large space of matrices. Then if a rank one tensor maps to a rank q matrix, a rank r tensor maps to a rank at most rq matrix, so the size $rq + 1$ minors give equations testing for border rank r. In this case the size of the matrices is $\binom{\mathbf{a}}{p}\mathbf{b} \times \binom{\mathbf{a}}{p+1}\mathbf{c}$ and a rank one tensor maps to a matrix of rank $\binom{\mathbf{a}-1}{p}$. Here $\mathbf{a} = \dim A$, $\mathbf{b} = \dim B$ and $\mathbf{c} = \dim C$.

In practice, one takes a subspace $A'^* \subseteq A^*$ of dimension $2p+1$ and restricts T (considered as a trilinear form) to $A'^* \times B^* \times C^*$ to get an optimal bound, so the denominator $\binom{\dim(A)-1}{p}$ is replaced by $\binom{2p}{p}$ in (7). Write $\phi : A \to A/(A'^*)^{\perp} =: A'$ for the projection onto the quotient: the corresponding Koszul flattening map gives a lower bound for $\underline{\mathbf{R}}(\phi(T))$, which, by linearity, is a lower bound for $\underline{\mathbf{R}}(T)$. The case $p = 1$ is equivalent to Strassen's equations [40]. There are numerous expositions of Koszul flattenings and their generalizations, see, e.g., [27, §7.3], [5, §7.2], [20], [28, §2.4], or [21].

Proof of Proposition 10. Write $q = 2u$. Fix a space $A' = \langle e_0, e_1, e_2 \rangle$. Define $\phi : A \to A'$ by

$$\begin{aligned} \phi(a_0) &= e_0, \\ \phi(a_i) &= e_1 \quad \text{for } i = 1, \ldots, u, \\ \phi(a_s) &= e_2 \quad \text{for } s = u+1, \ldots, q. \end{aligned}$$

As an element of $\Lambda^3 A$, we have $T_{skewcw,q} = a_0 \wedge \sum_{i=1}^{u} a_i \wedge a_{u+i}$.

We prove that if $T = T_{skewcw,q}$ then $\operatorname{rank}(T_{A'}^{\wedge 1}) = 2(q+2)+1$. This provides the lower bound $\underline{\mathbf{R}}(T) \geq \left\lceil \frac{2(q+2)+1}{2} \right\rceil = q + 3$.

We record the images via $T_{A'}^{\wedge 1}$ of a basis of $A' \otimes B^*$. Fix the range of $i = 1, \dots, u$:

$$\begin{aligned}
T_{A'}^{\wedge 1}(e_0 \otimes \beta_0) &= (e_0 \wedge e_1) \otimes \textstyle\sum_{i=1}^{u} c_{u+i} - (e_0 \wedge e_2) \otimes \sum_{i=1}^{u} c_i,\\
T_{A'}^{\wedge 1}(e_0 \otimes \beta_i) &= (e_0 \wedge e_2) \otimes c_0,\\
T_{A'}^{\wedge 1}(e_0 \otimes \beta_{u+i}) &= (e_0 \wedge e_1) \otimes c_0,\\
T_{A'}^{\wedge 1}(e_1 \otimes \beta_0) &= (e_1 \wedge e_2) \otimes \textstyle\sum_{i=1}^{u} c_{u+i},\\
T_{A'}^{\wedge 1}(e_1 \otimes \beta_i) &= (e_0 \wedge e_1) \otimes c_{u+i} + e_1 \wedge e_2 \otimes c_0,\\
T_{A'}^{\wedge 1}(e_1 \otimes \beta_{u+i}) &= e_0 \wedge e_1 \otimes c_i,\\
T_{A'}^{\wedge 1}(e_2 \otimes \beta_0) &= (e_1 \wedge e_2) \otimes \textstyle\sum_{i=1}^{u} c_i,\\
T_{A'}^{\wedge 1}(e_2 \otimes \beta_i) &= e_0 \wedge e_2 \otimes c_{u+i},\\
T_{A'}^{\wedge 1}(e_2 \otimes \beta_{u+i}) &= (e_0 \wedge e_2) \otimes c_i - e_1 \wedge e_2 \otimes c_0.
\end{aligned}$$

Notice that the image of $\sum_{i=1}^{u}(e_1 \otimes \beta_i) - \sum_{i=1}^{u}(e_2 \otimes \beta_{u+i}) - e_0 \otimes \beta_0$ is (up to scale) $e_1 \wedge e_2 \otimes c_0$.

This shows that the image of $T_{A'}^{\wedge 1}$ contains

$$\Lambda^2 A' \otimes c_0 + e_1 \wedge e_2 \otimes \langle \textstyle\sum_{i=1}^{u} c_i, \sum_{i=1}^{u} c_{u+i} \rangle + \langle e_0 \wedge e_1, e_0 \wedge e_2 \rangle \otimes \langle c_1, \dots, c_q \rangle.$$

These summands are in disjoint subspaces, so we conclude

$$\operatorname{rank}(T_{A'}^{\wedge 1}) \geq 3 + 2 + 2q = 2q + 5.$$ ◀

4.2 Propagation of lower bounds under Kronecker products

A tensor $T \in A \otimes B \otimes C$, with $\dim B = \dim C$ is 1_A*-generic* if $T(A^*) \subseteq B \otimes C$ contains a full rank element. Here is a partial multiplicativity result for Koszul flattening lower bounds under Kronecker products:

▶ **Proposition 25.** *Let $T_1 \in A_1 \otimes B_1 \otimes C_1$ with $\dim B_1 = \dim C_1$ be a tensor with a Koszul flattening lower bound for border rank $\underline{\mathbf{R}}(T) \geq r$ given by ${T_1}_{A_1}^{\wedge p}$ (possibly after a restriction ϕ). Let $T_2 \in A_2 \otimes B_2 \otimes C_2$, with $\dim B_2 = \dim C_2 = \mathbf{b}_2$ be 1_{A_2}-generic. Then*

$$\underline{\mathbf{R}}(T_1 \boxtimes T_2) \geq \left\lceil \frac{\operatorname{rank}({T_1}_{A_1}^{\wedge p}) \cdot \mathbf{b}_2}{\binom{2p}{p}} \right\rceil. \tag{8}$$

In particular, if $\frac{\operatorname{rank}({T_1}_{A_1}^{\wedge p})}{\binom{2p}{p}} \in \mathbb{Z}$, *then* $\underline{\mathbf{R}}(T_1 \boxtimes T_2) \geq r\mathbf{b}_2$.

Proof. After applying a restriction ϕ as described above, we may assume $\dim A_1 = 2p + 1$ so that the lower bound for T_1 is

$$\underline{\mathbf{R}}(T_1) \geq \left\lceil \frac{\operatorname{rank}({T_1}_{A_1}^{\wedge p})}{\binom{2p}{p}} \right\rceil.$$

Let $\alpha \in A_2^*$ be such that $T(\alpha) \in B_2 \otimes C_2$ has full rank $\mathbf{b}_2$, which exists by 1_{A_2}-genericity. Define $\psi : A_1 \otimes A_2 \to A_1$ by $\psi = \mathrm{Id}_{A_1} \otimes \alpha$ and set $\Psi := \psi \otimes \mathrm{Id}_{B_1 \otimes C_1 \otimes B_2 \otimes C_2}$. Then $(\Psi(T_1 \boxtimes T_2)_{A_1}^{\wedge p})$ provides the desired lower bound.

Indeed, the linear map $(\Psi(T_1 \boxtimes T_2)_{A_1}^{\wedge p})$ coincides with ${T_1}_{A_1}^{\wedge p} \boxtimes T_1(\alpha)$. Since matrix rank is multiplicative under Kronecker product, we conclude. ◀

4.3 First proof of Theorem 5

When $q = 3$, the result is true by a direct calculation using the $p = 2$ Koszul flattening with a sufficiently generic $\mathbb{C}^5 \subset A^*$, which is left to the reader. In what follows we treat the case $q > 3$.

Write $a_{ij} = a_i \otimes a_j \in A^{\otimes 2}$ and similarly for $B^{\otimes 2}$ and $C^{\otimes 2}$. Let $A' = \langle e_0, e_1, e_2 \rangle$ and define the linear map $\phi_2 : A^{\otimes 2} \to A'$ via

$$\begin{aligned}
&\phi_2(a_{00}) = \phi_2(a_{01}) = \phi_2(a_{10}) = e_0 + e_1,\\
&\phi_2(a_{11}) = e_0,\\
&\phi_2(a_{02}) = \phi_2(a_{20}) = e_1 + e_2\\
&\phi_2(a_{33}) = \phi_2(a_{21}) = e_2\\
&\phi_2(a_{0i}) = \phi_2(a_{i0}) = e_1 \quad \text{for } i = 3, \dots, q\\
&\phi_2(a_{ij}) = 0 \quad \text{for all other pairs } (i, j).
\end{aligned}$$

Write $T_q := T_{cw,q}^{\boxtimes 2}|_{A^{*\prime} \otimes B^{*\otimes 2} \otimes C^{*\otimes 2}}$. Consider the $p = 1$ Koszul flattening $(T_q)^{\wedge 1}_{A'} : A' \otimes B^{\otimes 2*} \to \Lambda^2 A' \otimes C^{\otimes 2}$.

We are going to prove that $\operatorname{rank}((T_q)^{\wedge 1}_{A'}) = 2(q+2)^2$. This provides the lower bound $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 2}) \geq (q+2)^2$ and equality follows because of the submultiplicativity properties of border rank under Kronecker product.

We proceed by induction on q. When $q = 4$ one does a direct computation with the $p = 1$ Koszul flattening, which is left to the reader, and which provides the base of the induction.

Write $W_j = a_0 \otimes b_j \otimes c_j + a_i \otimes b_0 \otimes c_j + a_i \otimes b_i \otimes c_0$. Then $T_{cw,q} = \sum_{j=1}^q W_j$, so that $T_{cw,q}^{\boxtimes 2} = \sum_{ij} W_i \boxtimes W_j$.

If $q \geq 4$, write $T_{cw,q} = T_{cw,q-1} + W_q$, so $T_{cw,q}^{\boxtimes 2} = T_{cw,q-1}^{\boxtimes 2} + T_{cw,q-1} \boxtimes W_q + W_q \boxtimes T_{cw,q-1} + W_q \boxtimes W_q$. Let $S_q = (T_{cw,q-1} \boxtimes W_q + W_q \boxtimes T_{cw,q-1} + W_q \boxtimes W_q)|_{A' \otimes B^{*\otimes 2} \otimes C^{*\otimes 2}}$.

Write $U_1 = A' \otimes \langle \beta_{ij} : i, j = 0, \dots, q-1 \rangle$ and $U_2 = A' \otimes \langle \beta_{qi}, \beta_{iq} : i = 0, \dots, q \rangle$ so that $U_1 \oplus U_2 = A' \otimes B^{\otimes 2*}$. Similarly, define $V_1 = \Lambda^2 A' \otimes \langle c_{ij} : i, j = 0, \dots, q-1 \rangle$ and $V_2 = \Lambda^2 A' \otimes \langle c_{qi}, c_{iq} : i = 0, \dots, q \rangle$, so that $V_1 \oplus V_2 = \Lambda^2 A' \otimes C^{\otimes 2}$. Observe that $(T_{q-1})^{\wedge 1}_{A'}$ is identically 0 on U_2 and its image is contained in V_1. Moreover, the image of U_1 under $(S_q)^{\wedge 1}_{A'}$ is contained in V_1. Representing the Koszul flattening in blocks, we have

$$(T_{q-1})^{\wedge 1}_{A'} = \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad (S_q)^{\wedge 1}_{A'} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}$$

therefore $\operatorname{rank}((T_q)^{\wedge 1}_{A'}) \geq \operatorname{rank}(M_{11} + N_{11}) + \operatorname{rank}(N_{22})$.

First, we prove that $\operatorname{rank}(M_{11} + N_{11}) \geq \operatorname{rank}(M_{11}) = 2(q+1)^2$. This follows by a degeneration argument. Consider the linear map given by pre-composing the Koszul flattening with the projection onto U_1. Its rank is semicontinuous under degeneration. Since $T_{cw,q}^{\boxtimes 2}$ degenerates to $T_{cw,q-1}^{\boxtimes 2}$, we deduce $\operatorname{rank}(M_{11} + N_{11}) \geq \operatorname{rank}(M_{11})$. The equality $\operatorname{rank}(M_{11}) = 2(q+1)^2$ follows by the induction hypothesis.

We show that $\operatorname{rank}(N_{22}) = 2(2q+3)$. The following equalities are modulo V_1. Moreover, each equality is modulo the tensors resulting from the previous ones. They are all straightforward applications of the Koszul flattening map, which in these cases, can always be performed on some copy of $W_i \boxtimes W_j$.

$$
\begin{aligned}
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{qj}) \equiv e_1 \wedge e_0 \otimes c_{qj} \quad \text{for } j = 3, \dots, q\\
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{jq}) \equiv e_1 \wedge e_0 \otimes c_{jq} \quad \text{for } j = 3, \dots, q\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{3q}) \equiv e_0 \wedge e_1 \otimes c_{0q}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{q3}) \equiv e_0 \wedge e_1 \otimes c_{q0}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{q1}) \equiv e_0 \wedge e_1 \otimes c_{q1}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{1q}) \equiv e_0 \wedge e_1 . \otimes c_{1q}
\end{aligned}
$$

Further passing modulo $\langle e_0 \wedge e_1 \rangle \otimes C$, we obtain

$$
\begin{aligned}
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{0q}) \equiv e_0 \wedge e_2 \otimes c_{2q}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{q0}) \equiv e_0 \wedge e_2 \otimes c_{q2}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{q2}) \equiv e_0 \wedge e_2 \otimes c_{0q}\\
&(S_q)^{\wedge 1}_{A'}(e_0 \otimes \beta_{2q}) \equiv e_0 \wedge e_2 \otimes c_{q0}\\
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{20}) \equiv e_1 \wedge e_2 \otimes c_{0q}\\
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{02}) \equiv e_1 \wedge e_2 \otimes c_{q0}\\
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{q0}) \equiv e_1 \wedge e_2 \otimes c_{2q}\\
&(S_q)^{\wedge 1}_{A'}(e_1 \otimes \beta_{0q}) \equiv e_1 \wedge e_2 \otimes c_{q2},
\end{aligned}
$$

and modulo the above,

$$
\begin{aligned}
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{qj}) \equiv e_2 \wedge (e_0 + e_1) \otimes c_{qj} \quad \text{for } j = 3, \dots, q\\
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{jq}) \equiv e_2 \wedge (e_0 + e_1) \otimes c_{jq} \quad \text{for } j = 3, \dots, q\\
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{q1}) \equiv e_2 \wedge (e_0 + e_1) \otimes c_{q1}\\
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{1q}) \equiv e_2 \wedge (e_0 + e_1) \otimes c_{1q}.
\end{aligned}
$$

Finally passing modulo $\langle e_1 \wedge e_2 \rangle$, we have

$$
\begin{aligned}
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{q0}) \equiv e_2 \wedge e_0 \otimes c_{q1}\\
&(S_q)^{\wedge 1}_{A'}(e_2 \otimes \beta_{0q}) \equiv e_2 \wedge e_0 \otimes c_{1q}.
\end{aligned}
$$

All the tensors listed above are linearly independent. Adding all the contributions together, we obtain

$$\operatorname{rank}((S_q)^{\wedge 1}_{A'}) = [2(q-3)+1] + 4 + 8 + 2 + [2(q-3)+1] + 4 = 2(2q+3)$$

as desired, and since $2(q+3)^2 = 2(q+1)^2 + 2(2q+3)$, this concludes the proof. ◀

4.4 A short detour on computing ranks of equivariant maps

We briefly explain how to exploit Schur's Lemma (see, e.g., [23, §1.2]) to compute the rank of an equivariant linear map. This is a standard technique, used extensively e.g., in [32, 25] and will reduce the proof of Theorems 5 and 6 to the computation of the ranks of specific linear maps in small dimension.

Let G be a reductive group. In the proof of Theorems 5 and 6, G will be the product of symmetric groups. Let Λ_G be the set of irreducible representations of G. For $\lambda \in \Lambda_G$, let W_λ denote the corresponding irreducible module.

Suppose U, V are two representations of G. Write $U = \bigoplus_{\lambda \in \Lambda_G} W_\lambda^{\oplus m_\lambda}$, $V = \bigoplus_{\lambda \in \Lambda_G} W_\lambda^{\oplus \ell_\lambda}$, where m_λ is the multiplicity of W_λ in U and ℓ_λ is the multiplicity of W_λ in V. The direct summand corresponding to λ is called the *isotypic component* of type λ.

Let $f : U \to V$ be a G-equivariant map. By Schur's Lemma [23, §1.2], f decomposes as $f = \oplus f_\lambda$, where $f_\lambda : W_\lambda^{\oplus m_\lambda} \to W_\lambda^{\oplus \ell_\lambda}$. Consider multiplicity spaces M_λ, L_λ with $\dim M_\lambda = m_\lambda$ and $\dim L_\lambda = \ell_\lambda$ so that $W_\lambda^{\oplus m_\lambda} \simeq M_\lambda \otimes W_\lambda$ as a G-module, where G acts trivially on M_λ and similarly $W_\lambda^{\oplus \ell_\lambda} \simeq L_\lambda \otimes W_\lambda$.

By Schur's Lemma, the map $f_\lambda : M_\lambda \otimes W_\lambda \to L_\lambda \otimes W_\lambda$ decomposes as $f_\lambda = \phi_\lambda \otimes \mathrm{Id}_{[\lambda]}$, where $\phi_\lambda : M_\lambda \to L_\lambda$. Thus $\mathrm{rank}(f)$ is uniquely determined by $\mathrm{rank}(\phi_\lambda)$ for $\lambda \in \Lambda_G$.

The ranks $\mathrm{rank}(\phi_\lambda)$ can be computed via restrictions of f. For every λ, fix a vector $w_\lambda \in W_\lambda$, so that $M_\lambda \otimes \langle w_\lambda \rangle$ is a subspace of U. Here and in what follows, for a subset $X \subset V$, $\langle X \rangle$ denotes the span of X. Then the rank of the restriction of f to $M_\lambda \otimes \langle w_\lambda \rangle$ coincides with the rank of ϕ_λ.

We conclude

$$\mathrm{rank}(f) = \textstyle\sum_\lambda \mathrm{rank}(\phi_\lambda) \cdot \dim W_\lambda.$$

The second proof of Theorem 5 and proof of Theorem 6 will follow the algorithm described above, exploiting the symmetries of $T_{cw,q}$. Consider the action of the symmetry group $\mathfrak{S}_q$ on $A \otimes B \otimes C$ defined by permuting the basis elements with indices $\{1, \dots, q\}$. More precisely, a permutation $\sigma \in \mathfrak{S}_q$ induces the linear map defined by $\sigma(a_i) = a_{\sigma(i)}$ for $i = 1, \dots, q$ and $\sigma(a_0) = a_0$. The group $\mathfrak{S}_q$ acts on B, C similarly, and the simultaneous action on the three factors defines an $\mathfrak{S}_q$-action on $A \otimes B \otimes C$. The tensor $T_{cw,q}$ is invariant under this action.

4.5 Second Proof of Theorem 5

When $q = 3$, as before, one uses the $p = 2$ Koszul flattening with a sufficiently generic $\mathbb{C}^5 \subset A^*$.

For $q \geq 4$, we apply the $p = 1$ Koszul flattening map to the same restriction of $T_{cw,q}^{\boxtimes 2}$ as the first proof, although to be consistent with the code at the website, we use the less appealing swap of the roles of a_2 and a_3 in the projection ϕ.

Since $T_{cw,q}$ is invariant under the action of $\mathfrak{S}_q$, $T_{cw,q}^{\boxtimes 2}$ is invariant under the action of $\mathfrak{S}_q \times \mathfrak{S}_q$, acting on $A^{\otimes 2} \otimes B^{\otimes 2} \otimes C^{\otimes 2}$. Let $\Gamma := \mathfrak{S}_{q-3} \times \mathfrak{S}_{q-3}$ where $\mathfrak{S}_{q-3}$ is the permutation group on $\{4, \dots, q\}$, so $T_{cw,q}^{\boxtimes 2}$ is invariant under the action of Γ. Note that Γ acts trivially on A', so $(T_q)_{A'}^{\wedge 1}$ is Γ-equivariant, because in general, Koszul flattenings are equivariant under the product of the three general linear groups, which is $GL(A') \times GL(B^{\otimes 2}) \times GL(C^{\otimes 2})$ in our case. (We remind the reader that $T_q := T_{cw,q}^{\boxtimes 2}|_{A^{*\prime} \otimes B^{*\otimes 2} \otimes C^{*\otimes 2}}$.) We now apply the method described in §4.4 to compute $\mathrm{rank}((T_q)_{A'}^{\wedge 1})$.

Let [triv] denote the trivial $\mathfrak{S}_{q-3}$-representation and let V denote the standard representation, that is the Specht module associated to the partition $(q-4, 1)$ of $q-3$. We have $\dim[\mathrm{triv}] = 1$ and $\dim V = q - 4$. (When $q = 4$ only the trivial representation appears.)

The spaces B, C are isomorphic as $\mathfrak{S}_{q-3}$-modules and they decompose as $B = C = [\mathrm{triv}]^{\oplus 5} \oplus V$. After fixing a 5-dimensional multiplicity space $\mathbb{C}^5$ for the trivial isotypic component, we write $B^* = C = \mathbb{C}^5 \otimes [\mathrm{triv}] \oplus V$. To distinguish the two $\mathfrak{S}_{q-3}$-actions, we write $B \otimes B = ([\mathrm{triv}]_L^{\oplus 5} \oplus V_L) \otimes ([\mathrm{triv}]_R^{\oplus 5} \oplus V_R)$.

Thus,

$$\begin{aligned} B^{*\otimes 2} = C^{\otimes 2} =& (\mathbb{C}^5 \otimes [\mathrm{triv}]_L \oplus V_L) \otimes (\mathbb{C}^5 \otimes [\mathrm{triv}]_R \oplus V_R) \\ =& (\mathbb{C}^5 \otimes \mathbb{C}^5) \otimes ([\mathrm{triv}]_L \otimes [\mathrm{triv}]_R) \quad \oplus \\ & \mathbb{C}^5 \otimes ([\mathrm{triv}]_L \otimes V_R) \quad \oplus \\ & \mathbb{C}^5 \otimes (V_L \otimes [\mathrm{triv}]_R) \quad \oplus \\ & (V_L \otimes V_R). \end{aligned}$$

Write $W_1, \ldots, W_4$ for the four irreducible representations in the decomposition above and let $M_1, \ldots, M_4$ be the four corresponding multiplicity spaces.

Recall from [22] that a basis of V is given by standard Young tableaux of shape $(q-4,1)$ (with entries in $4, \ldots, q$ for consistency with the action of $\mathfrak{S}_{q-3}$); let w_{std} be the vector corresponding to the standard tableau having $4, 6, \ldots, q$ in the first row and 5 in the second row. We refer to [22, §7] for the straightening laws of the tableaux. Let w_{triv} be a generator of the trivial representation [triv].

For each of the four isotypic components in the decomposition above, we fix a vector $w_i \in W_i$ and explicitly realize the subspaces $M_i \otimes \langle w_i \rangle$ of $B^{*\otimes 2}$ as follows:

W_i	w_i	$\dim M_i$	$M_i \otimes \langle w_i \rangle$
$[\mathrm{triv}]_L \otimes [\mathrm{triv}]_R$	$w_{\mathrm{triv}} \otimes w_{\mathrm{triv}}$	25	$\langle \beta_{ij} : i,j = 0,\ldots,3 \rangle \oplus$ $\langle \sum_{j=4}^{q} \beta_{ij} : i = 0,\ldots,3 \rangle \oplus$ $\langle \sum_{i=4}^{q} \beta_{ij} : j = 0,\ldots,3 \rangle \oplus$ $\langle \sum_{i,j=4}^{q} \beta_{ij} \rangle$
$[\mathrm{triv}]_L \otimes V_R$	$w_{\mathrm{triv}} \otimes w_{std}$	5	$\langle \beta_{i5} - \beta_{i4} : i = 0,\ldots,3 \rangle \oplus$ $\langle \sum_{i=4}^{q} (\beta_{i5} - \beta_{i4}) \rangle$
$V_L \otimes [\mathrm{triv}]_R$	$w_{std} \otimes w_{\mathrm{triv}}$	5	$\langle \beta_{5j} - \beta_{4j} : j = 0,\ldots,3 \rangle \oplus$ $\langle \sum_{j=4}^{q} (\beta_{5j} - \beta_{4j}) \rangle$
$V_L \otimes V_R$	$w_{std} \otimes w_{std}$	1	$\langle \beta_{55} - \beta_{45} - \beta_{54} + \beta_{44} \rangle$.

The subspaces in $C^{\otimes 2}$ are realized similarly.

Since $(T_{cw,q}^{\boxtimes 2})^{\wedge 1}_{A'}$ is Γ-equivariant, by Schur's Lemma, it has the isotypic decomposition $(T_{cw,q}^{\boxtimes 2})^{\wedge 1}_{A'} = f_1 \oplus f_2 \oplus f_3 \oplus f_4$, where

$$f_i : A' \otimes (M_i \otimes W_i) \to \Lambda^2 A' \otimes W_i.$$

As explained in §4.4, it suffices to compute the ranks of the four restrictions $\Phi_i : A' \otimes M_i \otimes \langle w_i \rangle \to \Lambda^2 A' \otimes M_i \otimes \langle w_i \rangle$.

Using the bases presented in the fourth column of the table above, we write down the four matrices representing the maps $\Phi_1, \ldots, \Phi_4$.

The map Φ_4 is represented by the 3×3 matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so $\mathrm{rank}(\Phi_4) = 2$.

The map Φ_2 is represented by the 15×15 matrix (here $q' = q - 3$)

$$\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & q' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q' & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

We prove the matrix above and those that follow are as asserted for all q in §7. The proof goes by showing each entry must be a low degree polynomial in q, and then one simply tests enough small cases to fix the polynomials. Thus $\mathrm{rank}(\Phi_2) = 12$, and similarly for Φ_3.

The map Φ_1 is represented by a 75×75 matrix that can be presented in block form

$$\begin{pmatrix}
-X & Y & 0 \\
-Z & 0 & Y \\
0 & -Z & X
\end{pmatrix}$$

with X the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

Y the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & q' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & q' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & q' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & q' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & q' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q' & 0 & 0 & 0 & 1
\end{pmatrix},$$

and Z the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We compute $\operatorname{rank}(\Phi_1) = 72$.

Although these matrices are of fixed size, they are obtained via intermediate tensors whose dimensions depend on q, which created a computational challenge. Two ways of addressing the challenge (including the one utilized in the code) are explained in §7.

The relevant matrices and the implementation of the method of §7 to justify them for all q, together with the code for the computation of their ranks are available at the website `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix D. The ranks are bounded below by taking a matrix M (which has some entries depending linearly on q), multiplying it on the left by a rectangular matrix P whose entries are rational functions of q, and on the right by a rectangular matrix Q whose entries are constant, to obtain a square matrix PMQ that is upper triangular with ± 1 on the diagonal, and thus its rank is its dimension. Finally one checks that the least common multiple of the denominators of the entries of P has no integral solution when $q > 4$.

Adding all the contributions gives

$$\begin{aligned}
\operatorname{rank}(T_{A'}^{\wedge 1}) &= 2 \cdot \dim(V \otimes V) + 12 \cdot \dim([\mathrm{triv}] \otimes V) + \\
&\quad + 12 \cdot \dim(V \otimes [\mathrm{triv}]) + 72 \cdot \dim([\mathrm{triv}] \otimes [\mathrm{triv}]) = \\
&\quad = 2 \cdot (q-4)^2 + 12 \cdot (q-4) + 12 \cdot (q-4) + 72 \cdot 1 = 2(q+2)^2.
\end{aligned}$$

This concludes the proof of Theorem 5.

▶ Remark 26. One might have hoped to exploit the full symmetry group $\mathfrak{S}_q \times \mathfrak{S}_q$ to simplify the argument further. However there is no choice of a restriction map ψ which is $\mathfrak{S}_{q-s} \times \mathfrak{S}_{q-s}$-invariant for $s < 3$ that gives rise to a Koszul flattening map of sufficiently high rank to prove the result.

4.6 Proof of Theorem 6

We will use a Koszul flattening with $p = 2$, so we need a 5 dimensional subspace of $(A^*)^{\otimes 3}$. Let

$$A'^* := \left\langle \begin{array}{c}
\alpha_{000}, \\
\sum_{i=1}^{q} (\alpha_{i00} + \alpha_{0i0} + \alpha_{00i}), \\
\alpha_{001} + \alpha_{010} + \alpha_{012} + \alpha_{102} + \alpha_{110} + \alpha_{121} + \alpha_{200} + \alpha_{211}, \\
\alpha_{022} + \alpha_{030} + \alpha_{031} + \alpha_{100} + \alpha_{103} - \alpha_{120} + \alpha_{210} + \alpha_{212} + \alpha_{300}, \\
\alpha_{002} + \alpha_{004} + \alpha_{011} + \alpha_{014} + \alpha_{020} + \alpha_{023} + \alpha_{032} + \alpha_{040} + \alpha_{100} + \alpha_{122} + \alpha_{220} + \alpha_{303}
\end{array} \right\rangle .$$

Write $\phi_3 : A^{\otimes 3} \to A'$ for the resulting projection map and, abusing notation, for the induced map $A^{\otimes 3} \otimes B^{\otimes 3} \otimes C^{\otimes 3} \to A' \otimes B^{\otimes 3} \otimes C^{\otimes 3}$. Write $T = \phi_3(T_{cw,q}^{\boxtimes 3})$, suppressing the q from the notation. Consider the Koszul flattening:

$$(T)^{\wedge 2}_{A'} : \Lambda^2 A' \otimes B^{*\otimes 2} \to \Lambda^3 A' \otimes C^{\otimes 2}.$$

We will show $\operatorname{rank}((T)^{\wedge 2}_{A'}) = 6(q+2)^3$, which implies $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 3}) \geq (q+2)^3$.

In order to compute $\operatorname{rank}((T)^{\wedge 2}_{A'})$, we follow the same strategy as before. The code to generate these matrices is available at `www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix D. The explanation of how we proved they are as asserted is outlined in §7.

The map $(T)^{\wedge 2}_{A'}$ is invariant under the action of $\Gamma = \mathfrak{S}_{q-4} \times \mathfrak{S}_{q-4} \times \mathfrak{S}_{q-4}$ where the first copy of $\mathfrak{S}_{q-4}$ permutes the basis elements with indices $5, \dots, q$ of the first factors, and similarly for the other copies of $\mathfrak{S}_{q-4}$. Let [triv] be the trivial $\mathfrak{S}_{q-4}$-representation and let V be the standard representation, namely the Specht module associated to the partition $(q-5, 1)$. Here $\dim V = q-5$, so if $q = 5$, only the trivial representation appears.

The $\mathfrak{S}_{q-4}$-isotypic decomposition of B (and C) is $\mathbb{C}^6 \otimes [\mathrm{triv}] \oplus V$ and this induces the decomposition of $B^{*\otimes 3} \simeq C^{\otimes 3}$ given by

$$\begin{aligned} B^{*\otimes 3} \simeq C^{\otimes 3} = & (\mathbb{C}^6)^{\otimes 3} \otimes ([\mathrm{triv}]_1 \otimes [\mathrm{triv}]_2 \otimes [\mathrm{triv}]_3) \oplus \\ & (\mathbb{C}^6)^{\otimes 2} \otimes [([\mathrm{triv}]_1 \otimes [\mathrm{triv}]_2 \otimes V_3) \oplus \\ & ([\mathrm{triv}]_1 \otimes V_2 \otimes [\mathrm{triv}]_3) \oplus \\ & (V_1 \otimes [\mathrm{triv}]_2 \otimes [\mathrm{triv}]_3)] \oplus \\ & (\mathbb{C}^6) \otimes [([\mathrm{triv}]_1 \otimes V_2 \otimes V_3) \oplus \\ & (V_1 \otimes V_2 \otimes [\mathrm{triv}]_3) \oplus \\ & (V_1 \otimes [\mathrm{triv}]_2 \otimes V_3)] \oplus \\ & V_1 \otimes V_2 \otimes V_3 \end{aligned}$$

consisting of eight isotypic components. As in the previous proof, for each of the eight irreducible components W_i, we consider $w_i \in W_i$ and we compute the rank of the restriction to $\Lambda^2 A' \otimes M_i \otimes \langle w_i \rangle$ of the Koszul flattening; call this restriction Φ_i.

The ranks of the restrictions are recorded in the following table:

W_i	$\dim(\Lambda^2 A' \otimes M_i)$	$\operatorname{rank}(\Phi_i)$
$[\mathrm{triv}]_1 \otimes [\mathrm{triv}]_2 \otimes [\mathrm{triv}]_3$	$6^3 \cdot \binom{5}{2} = 2160$	2058
$[\mathrm{triv}]_1 \otimes [\mathrm{triv}]_2 \otimes V_3$ (and permutations)	$6^2 \cdot \binom{5}{2} = 360$	294
$[\mathrm{triv}]_1 \otimes V_2 \otimes V_3$ (and permutations)	$6 \cdot \binom{5}{2} = 60$	42
$V_1 \otimes V_2 \otimes V_3$	$\binom{5}{2} = 10$	6

The relevant matrices and the implementation of §7 to justify them for all q, with the code computing their ranks are available at `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix D. As before, the ranks are bounded below by taking a matrix M (which has some entries depending linearly on q), multiplying it on the left by a rectangular matrix P whose entries are rational functions of q, and on the right by a rectangular matrix Q whose entries are constant, to obtain a square matrix PMQ that is upper triangular with ± 1 on the diagonal, and thus its rank is its dimension. Finally one checks that the least common multiple of the denominators of the entries of P has no integral solution when $q > 4$.

Adding all the contributions together, we obtain

$$\begin{aligned}\operatorname{rank}(T_{A'}^{\wedge 2}) = &6 \cdot \dim(V \otimes V \otimes V)+\\ &42 \cdot 3 \cdot \dim([\mathrm{triv}] \otimes V \otimes V)+\\ &294 \cdot 3 \cdot \dim([\mathrm{triv}] \otimes [\mathrm{triv}] \otimes V)+\\ &2058 \cdot \dim([\mathrm{triv}] \otimes [\mathrm{triv}] \otimes [\mathrm{triv}]) = 6 \cdot (q+2)^3.\end{aligned}$$

This concludes the proof.

5 Upper bounds for Waring rank and border rank of $\det_3$

5.1 Proof of Theorem 15

We give the rank 18 decomposition for $\det_3$ explicitly, as a collection of 18 linear forms on $\mathbb{C}^9 = \mathbb{C}^3 \otimes \mathbb{C}^3$ whose cubes add up to $\det_3$. The linear forms are given in coordinates recorded in the matrices below: the 3×3 matrix (ζ_{ij}) represents the linear forms $\sum_{ij} \zeta_{ij} x_{ij}$. This presentation highlights some of the symmetries of the decomposition.

Let $\vartheta = \exp(2\pi i/6)$ and let $\overline{\vartheta}$ be its inverse. The tensor $\det_3 = T_{skewcw,2}^{\boxtimes 2} = \det(x_{ij}) \in S^3(\mathbb{C}^3 \otimes \mathbb{C}^3)$ satisfies

$$\det_3 = \sum_1^{18} L_i^3$$

where $L_1, \ldots, L_{18}$ are the 18 linear forms given by the following coordinates:

$$\begin{array}{lll}
L_1 = \begin{pmatrix} -\vartheta & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \overline{\vartheta} \end{pmatrix} & L_2 = \begin{pmatrix} -\overline{\vartheta} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \vartheta \end{pmatrix} & L_3 = \begin{pmatrix} -\overline{\vartheta} & 0 & 0 \\ 0 & \frac{1}{3}\overline{\vartheta} & 0 \\ 0 & 0 & \overline{\vartheta} \end{pmatrix} \\
L_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\overline{\vartheta} \\ 0 & -\frac{1}{3}\vartheta & 0 \end{pmatrix} & L_5 = \begin{pmatrix} \overline{\vartheta} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{3}\vartheta & 0 \end{pmatrix} & L_6 = \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & 0 & -\vartheta \\ 0 & -\frac{1}{3}\vartheta & 0 \end{pmatrix} \\
L_7 = \begin{pmatrix} 0 & \frac{1}{3}\overline{\vartheta} & 0 \\ -\vartheta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & L_8 = \begin{pmatrix} 0 & \frac{1}{3}\overline{\vartheta} & 0 \\ -\overline{\vartheta} & 0 & 0 \\ 0 & 0 & -\overline{\vartheta} \end{pmatrix} & L_9 = \begin{pmatrix} 0 & \frac{1}{3}\vartheta & 0 \\ -\overline{\vartheta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
L_{10} = \begin{pmatrix} 0 & -\frac{1}{3}\vartheta & 0 \\ 0 & 0 & \overline{\vartheta} \\ -1 & 0 & 0 \end{pmatrix} & L_{11} = \begin{pmatrix} 0 & -\frac{1}{3}\overline{\vartheta} & 0 \\ 0 & 0 & \vartheta \\ -1 & 0 & 0 \end{pmatrix} & L_{12} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \\
L_{13} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} & L_{14} = \begin{pmatrix} 0 & 0 & 1 \\ \overline{\vartheta} & 0 & 0 \\ 0 & \frac{1}{3}\vartheta & 0 \end{pmatrix} & L_{15} = \begin{pmatrix} 0 & 0 & 1 \\ \vartheta & 0 & 0 \\ 0 & \frac{1}{3}\overline{\vartheta} & 0 \end{pmatrix} \\
L_{16} = \begin{pmatrix} 0 & 0 & \overline{\vartheta} \\ 0 & -\frac{1}{3}\vartheta & 0 \\ 1 & 0 & 0 \end{pmatrix} & L_{17} = \begin{pmatrix} 0 & 0 & \overline{\vartheta} \\ 0 & -\frac{1}{3}\overline{\vartheta} & 0 \\ -\overline{\vartheta} & 0 & 0 \end{pmatrix} & L_{18} = \begin{pmatrix} 0 & 0 & \vartheta \\ 0 & -\frac{1}{3}\overline{\vartheta} & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{array}.$$

The equality can be verified by hand. A Macaulay2 file performing the calculation is available at `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix B. ◀

5.2 Proof of Theorem 16

As in the case of Theorem 15, we prove Theorem 16 by explicitly giving 17 linear forms, depending on a parameter t, whose cubes provide a border rank 17 expression for $\det_3$. The algebraic numbers involved are more complicated than in the previous case.

The result was achieved by numerical methods, which allowed us to sparsify the decomposition and ultimately determine the value of the coefficients. The linear forms in the decomposition are described below.

Consider

$$L_1(t) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 t & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad L_2(t) = \begin{pmatrix} z_3 & 0 & 0 \\ z_4 & 0 & z_5 t \\ z_6 & 0 & 0 \end{pmatrix} \qquad L_3(t) = \begin{pmatrix} -z_{36} & z_7 t & 0 \\ -z_{38} & 0 & -z_{39} t \\ 0 & 0 & t \end{pmatrix}$$

$$L_4(t) = \begin{pmatrix} 0 & 0 & t \\ -z_{34} & 0 & 0 \\ 0 & z_8 t & -z_{35} t \end{pmatrix} \quad L_5(t) = \begin{pmatrix} 0 & -z_{19} t & -z_{20} t \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_6(t) = \begin{pmatrix} -z_{22} & z_9 t & 0 \\ -z_{23} & 0 & -z_{24} t \\ -z_{25} & 0 & 0 \end{pmatrix}$$

$$L_7(t) = \begin{pmatrix} z_{10} & z_{11} t & 0 \\ z_{12} & 0 & z_{13} t \\ z_{14} & 0 & 0 \end{pmatrix} \qquad L_8(t) = \begin{pmatrix} z_{15} & -t & 0 \\ z_{16} & 0 & z_{17} t \\ z_{18} & 0 & 0 \end{pmatrix} \qquad L_9(t) = \begin{pmatrix} 0 & z_{19} t & z_{20} t \\ 0 & z_{21} t & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L_{10}(t) = \begin{pmatrix} -z_{41} & 0 & 0 \\ 0 & 0 & 0 \\ -z_{44} & 0 & 0 \end{pmatrix} \qquad L_{11}(t) = \begin{pmatrix} z_{22} & 0 & 0 \\ z_{23} & 0 & z_{24} t \\ z_{25} & 0 & 0 \end{pmatrix} \qquad L_{12}(t) = \begin{pmatrix} -z_{31} & z_{26} t & 0 \\ 0 & z_{27} t & 0 \\ 0 & 0 & t \end{pmatrix}$$

$$L_{13}(t) = \begin{pmatrix} z_{28} & z_{29} t & 0 \\ z_{30} & 0 & -t \\ 0 & t & 0 \end{pmatrix} \qquad L_{14}(t) = \begin{pmatrix} z_{31} & z_{32} t & 0 \\ 0 & 0 & 0 \\ 0 & z_{33} t & -t \end{pmatrix} \qquad L_{15}(t) = \begin{pmatrix} 0 & 0 & -t \\ z_{34} & 0 & 0 \\ 0 & 0 & z_{35} t \end{pmatrix}$$

$$L_{16}(t) = \begin{pmatrix} z_{36} & z_{37} t & 0 \\ z_{38} & 0 & z_{39} t \\ 0 & z_{40} t & -t \end{pmatrix} \qquad L_{17}(t) = \begin{pmatrix} z_{41} & z_{42} t & 0 \\ 0 & z_{43} t & 0 \\ z_{44} & 0 & 0 \end{pmatrix}$$

The coefficients $z_1, \dots, z_{44}$ are algebraic numbers described as follows. Let y_* be a real root of the polynomial

$$\begin{aligned}
&x^{27} - 2x^{26} + 17x^{25} - 29x^{24} + 81x^{23} + 52x^{22} - 726x^{21} + 3451x^{20} - 10901x^{19} + 25738x^{18} - \\
&50663x^{17} + 72133x^{16} - 72973x^{15} + 10444x^{14} + 138860x^{13} - 308611x^{12} + 427344x^{11} \\
&- 267416x^{10} - 196096x^{9} + 762736x^{8} - 1236736x^{7} + 1092352x^{6} - 537600x^{5} - 42240x^{4} + \\
&684032x^{3} - 1136640x^{2} + 1146880x - 520192.
\end{aligned}$$

For $i = 1, \dots, 44$, we consider algebraic numbers y_j in the field extension $\mathbb{Q}[y_*]$, described as a polynomial of degree (at most) 26 in y_* with rational coefficients. Notice that all the y_j's are real. The expressions of the $y_1, \dots, y_{44}$ in terms of y_* are provided in the file `yy_exps` at `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix C. Let z_j be the unique real cubic root of y_j.

We are going to prove that, with this choice of coefficients z_j,

$$t^2 \det_3 + O(t^3) = \sum_{i=1}^{17} L_i(t)^3. \tag{9}$$

The condition $t^2 \det_3 + O(t^3) = \sum_{i=1}^{17} L_i(t)^3$ is equivalent to the fact that the degree 0 and the degree 1 components of $\sum_{i=1}^{17} L_i(t)^3$ vanish and that the degree 2 component equals $\det_3$. Given the sparse structure of the $L_i(t)$, this reduces to a system of 54 cubic equations in the 44 unknowns $z_1, \dots, z_{44}$. Our goal is to show that the algebraic numbers described above are a solution of this system.

We show that the z_i's satisfy each equation as follows. After evaluating the equations at the z_i's, there are two possible cases

1. all monomials appearing in the equation are elements of $\mathbb{Q}[y_*]$; we say that this is an equation of type 1; there are 14 such equations;
2. at least one monomial appearing in the equation is not an element of $\mathbb{Q}[y_*]$; we say that this is an equation of type 2; there are 40 such equations.

For equations of type 1, we provide expressions of each monomial in terms of y_*. To verify that each expression is indeed equal to the corresponding monomial, it suffices to compare the cube of the given expression and the expression obtained by evaluating the monomial at the y_j's. Finally, the equation can be verified in $\mathbb{Q}[y_*]$. This is performed by the file `checkingType1eqns.m2`.

For equations of type 2, let u be one of the monomials which do not belong to $\mathbb{Q}[y_*]$. We claim that it is possible to choose the monomial in such a way that $\mathbb{Q}[u^3] = \mathbb{Q}[y_*]$. For each equation, we choose one of the monomials and we verify the claim as follows. The element u^3 has an expression in terms of y_* which equals the chosen monomial evaluated at the y_i's. Let M_u be the 27×27 matrix with rational entries such that

$$(1, u^3, \cdots, u^{3 \cdot 26}) = \left(1, y_*, \ldots, y_*^{26}\right) \cdot M_u;$$

M_u can be computed directly by considering the expressions of the powers of u^3 in terms of y_*. Then $\mathbb{Q}[u^3] = \mathbb{Q}[y_*]$ if and only if M_u is full rank.

In particular y_* has an expression in terms of u^3, which can be computed inverting the matrix M_u. A consequence of this is that $\mathbb{Q}[u] = \mathbb{Q}[y_*, u]$.

At this point, we observe that $\mathbb{Q}[u]$ contains the other monomials occurring in the equation as well. To see this, we proceed as in the case of equations of type 1. For each monomial occurring in the equation, we provide an expression in terms of u (in fact, to speed up the calculation, we provide an expression in terms of u and y_*, which is equivalent to an expression in u because $\mathbb{Q}[u^3] = \mathbb{Q}[y_*]$ and y_* has a unique expression in terms of u^3); we compare the cube of this expression (appropriately reduced modulo the minimal polynomial of y_* and the relation between u^3 and y_*) with the expression obtained by evaluating the monomial at the y_i's (expressed in terms of y_*). This shows that all monomials occurring in the expression belong to $\mathbb{Q}[u]$, and verifies that the given expressions are indeed equal to the corresponding monomials. Finally, the equation is verified in $\mathbb{Q}[u]$ as in the case of type 1. This is performed by the file `checkingType2eqns.m2`. ◀

5.2.1 Discussion of how the decomposition was obtained

Many steps were accomplished by finding solutions of polynomial equations by nonlinear optimization. In each case, this was accomplished using a variant of Newton's method applied to the mapping of variable values to corresponding polynomial values. The result of this procedure in each case is limited precision machine floating point numbers.

First, we attempted to solve the equations describing a Waring rank 17 decomposition of $\det_3$ with nonlinear optimization, namely, $\det_3 = \sum_{i=1}^{17} (w_i')^{\otimes 3}$, where $w_i' \in \mathbb{C}^{3\times 3}$. Instead of finding a solution to working precision, we obtained a sequence of local refinements to an approximate solution where the norm of the defect is slowly converging to to zero, and some of the parameter values are exploding to infinity. Numerically, these are Waring decompositions of polynomials very close to $\det_3$.

Next, this approximate solution needed to be upgraded to a solution to equation (9).

We found a choice of parameters in the neighborhood of a solution, and then applied local optimization to solve to working precision. We used the following method: Consider the linear mapping $M : \mathbb{C}^{17} \to S^3(\mathbb{C}^{3\times 3})$, $M(e_i) = (w_i')^{\otimes 3}$, and let $M = U\Sigma V^*$ be its singular value decomposition (with respect to the standard inner products for the natural coordinate systems). We observed that the singular values seemed to be naturally partitioned by order of magnitude. We estimated this magnitude factor as $t_0 \approx 10^{-3}$, and wrote Σ' as Σ where we multiplied each singular value by $(t/t_0)^k$, with k chosen to agree with this observed partitioning, so that the constants remaining were reasonably sized. Finally, we let $M' = U\Sigma' V^*$, which has entries in $\mathbb{C}[[t]]$. M' is thus a representation of the map M with a parameter t.

Next, for each i, we optimized to find a best fit to the equation $(a_i + tb_i + t^2 c_i)^{\otimes 3} = M'(e_i)$, which is defined by polynomial equations in the entries of a_i, b_i and c_i. The a_i, b_i and c_i we constructed in this way proved to be a good initial guess to optimize equation (9), and we immediately saw quadratic convergence to a solution to machine precision. At this point, we greedily sparsified the solution by speculatively zero-ing values and re-optimizing, rolling back one step in case of failure. After sparsification, it turned out the c_i were not needed. The resulting matrices are those given in the proof.

To compute the minimal polynomials and other integer relationships between quantities, we used Lenstra-Lenstra-Lovász integer lattice basis reduction [34]. As an example, let $\zeta \in \mathbb{R}$ be approximately an algebraic number of degree k. Let N be a large number inversely proportional to the error of ζ. Consider the integer lattice with basis $\{e_i + \lfloor N\zeta^i \rceil e_{k+1}\} \subset \mathbb{Z}^{k+2}$, for $0 \le i \le k$. Then elements of this lattice are of the form $v_0 e_0 + \cdots + v_k e_k + E e_{k+1}$, where $E \approx Np(\zeta)$, $p = v_0 + v_1 x + \cdots x_k x^k$. Polynomials p for which ζ is an approximate root are distinguished by the property of having relatively small Euclidean norm in this lattice. Computing a small norm vector in an integer lattice is accomplished by LLL reduction of a known basis.

For example, the fact that the number field of degree 27 obtained by adjoining any z_α^3 to $\mathbb{Q}$ contains all the rest was determined via LLL reduction, looking for expressions of z_α^3 as a polynomial in z_β^3 for some fixed β. These expressions of z_α^3 in a common number field can be checked to have the correct minimal polynomial, and thus agree with our initial description of the z_α. LLL reduction was also used to find the expressions of values as polynomials in the primitive root of the various number fields.

After refining the known value of the parameters to $10,000$ bits of precision using Newton's method, LLL reduction was successful in identifying the minimal polynomials. The degrees were simply guessed, and the results checked by evaluating the computed polynomials in the parameters to higher precision.

▶ Remark 27. With the minimal polynomial information, it is possible to check that equation (9) is satisfied to any desired precision by the parameters.

6 Tight Tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

Following an analysis started in [17], we consider Kronecker squares of *tight* tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. We compute their symmetry groups and numerically provide bounds to their tensor rank and border rank, highlighting the submultiplicativity properties.

We refer to [44, 11, 17] for an exposition of the role of tightness in Strassen's work and in the laser method. We point out that $T_{CW,q}$ and $T_{cw,q}$ are tight. (If one uses the combinatorial definition of tightness, which depends on a choice of basis, they are not tight in their standard presentations.)

6.1 Tight tensors

Recall the map $\Phi : GL(A)\times GL(B)\times GL(C) \to GL(A\otimes B\otimes C)$ from Section 3.1 defining the action of $GL(A)\times GL(B)\times GL(C)$ on $A\otimes B\otimes C$. Its differential $d\Phi$ defines a map at the level of Lie algebras, mapping $\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)$ to a subalgebra of $\mathfrak{gl}(A\otimes B\otimes C)$, isomorphic to $(\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C))/\mathbb{C}^2$. Write $\mathfrak{g}_T \subseteq \mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)$ for the annihilator of T under this action.

A tensor $T \in A\otimes B\otimes C$ is *tight* if $\mathfrak{g}_T/\mathbb{C}^2$ contains a regular semisimple element. Tightness can be defined combinatorially with respect to a basis, see e.g. [17, Def. 1.3]. In particular, the combinatorial definition makes it clear that tightness depends on the support of a tensor in a given basis; we say that a support $\mathcal{S}$ is tight if every tensor having support $\mathcal{S}$ is tight.

Given concise tensors $T_1 \in A_1\otimes B_1\otimes C_1$ and $T_2 \in A_2\otimes B_2\otimes C_2$, [17, Theorem 4.1] shows that

$$\mathfrak{g}_{T_1\boxtimes T_2} \supseteq \mathfrak{g}_{T_1}\otimes \mathrm{Id}_{A_2\otimes B_2\otimes C_2} + \mathrm{Id}_{A_1\otimes B_1\otimes C_1}\otimes \mathfrak{g}_{T_2}; \tag{10}$$

moreover if $\mathfrak{g}_{T_1}=0$ and $\mathfrak{g}_{T_2}=0$ then equality holds $\mathfrak{g}_{T_1\boxtimes T_2}=0$.

The strict containment in (10) occurs, for instance, in the case of the matrix multiplication tensor. In [17], we posed the problem of characterizing tensors $T \in A\otimes B\otimes C$ such that $\mathfrak{g}_T\otimes \mathrm{Id}_{A\otimes B\otimes C} + \mathrm{Id}_{A\otimes B\otimes C}\otimes\mathfrak{g}_T$ is strictly contained in $\mathfrak{g}_{T^{\boxtimes 2}} \subset \mathfrak{gl}(A^{\otimes 2})+\mathfrak{gl}(B^{\otimes 2})+\mathfrak{gl}(C^{\otimes 2})$.

Proposition 29 provides several additional examples of tensors in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ for which this containment is strict.

6.2 Tight supports in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

From [17, Proposition 2.14], one obtains an exhaustive list of unextendable tight supports for tensors in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$, up to the action of $\mathbb{Z}_2\times\mathfrak{S}_3$, where $\mathfrak{S}_3$ acts permuting the factors and $\mathbb{Z}_2$ acts by reversing the order of the basis elements. In fact, tightness is invariant under the action of the full $\mathfrak{S}_3$ acting by permutation on the basis vectors. This additional simplification, pointed out by J. Hauenstein, provides the following list of 9 unextendable tight supports up to the action of $((\mathfrak{S}_3)^{\times 3})\rtimes\mathfrak{S}_3$.

$$\begin{aligned}
\mathcal{T}_1 &= \{(1,1,3),(1,2,2),(2,1,2),(3,3,1)\};\\
\mathcal{T}_2 &= \{(1,1,3),(1,3,2),(2,3,1),(3,2,2)\};\\
\mathcal{T}_3 &= \{(1,1,3),(1,2,2),(1,3,1),(2,1,2),(3,2,1)\};\\
\mathcal{T}_4 &= \{(1,1,3),(1,2,2),(2,1,2),(2,3,1),(3,2,1)\};\\
\mathcal{T}_5 &= \{(1,1,3),(1,2,2),(2,3,1),(3,1,2),(3,2,1)\};\\
\mathcal{T}_6 &= \{(1,1,3),(1,3,2),(2,2,2),(3,1,2),(3,3,1)\};\\
\mathcal{T}_7 &= \{(1,1,3),(1,2,2),(1,3,1),(2,1,2),(2,2,1),(3,1,1)\};\\
\mathcal{T}_8 &= \{(1,1,3),(1,3,2),(2,2,2),(2,3,1),(3,1,2),(3,2,1)\};\\
\mathcal{T}_9 &= \{(1,2,3),(1,3,2),(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1)\};
\end{aligned}$$

Supports $\mathcal{S}_2$ and $\mathcal{S}_3$ of [17] are equivalent to support $\mathcal{S}_1=\mathcal{T}_1$; supports $\mathcal{S}_8$ and $\mathcal{S}_{10}$ are equivalent to support $\mathcal{S}_6=\mathcal{T}_4$.

The following result characterizes tight tensors in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ up to isomorphism.

▶ **Proposition 28.** *Let $T\in\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ be a tight tensor with unextendable tight support in some basis. Then, up to permuting the three factors, T is isomorphic to exactly one of the following.*

$$
\begin{aligned}
T_1 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_3 \otimes c_1 \\
T_2 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_2 \otimes c_2 \\
T_3 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_1 \\
T_4 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_2 \otimes c_1 \\
T_5 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_1 \\
T_6 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_3 \otimes c_1 \\
T_7 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_1 + a_3 \otimes b_1 \otimes c_1 \\
T_8 :=& a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_1 \\
T_{9,\mu} :=& a_1 \otimes b_2 \otimes c_3 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_1 \otimes c_3 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_2 \\
&+ \mu \cdot a_3 \otimes b_2 \otimes c_1.
\end{aligned}
$$

Proof. The result of [17, Proposition 2.14] and the discussion above shows that T is, up to permutation of the factors, equivalent to a tensor with support $\mathcal{T}_i$ for some $i = 1, \dots, 9$.

For $i = 1, \dots, 8$, it is straightforward to verify that all tensors with support $\mathcal{T}_i$ are isomorphic, via the change of bases given by three diagonal matrices.

The case of $\mathcal{T}_9$ is slightly more involved but essentially the same argument shows that a tensor T with support $\mathcal{T}_9$ is isomorphic to $T_{9,\mu}$, for some μ.

Finally, we have to show that any two of the tensors in the statement are not isomorphic. For tensors having distinct supports, this is a consequence of Proposition 29 below: indeed, if T, T' are two of the tensors above, Proposition 29 shows that either $\dim \mathfrak{g}_T \neq \dim \mathfrak{g}_{T'}$ or $\dim \mathfrak{g}_{T^{\boxtimes 2}} \neq \dim \mathfrak{g}_{T'^{\boxtimes 2}}$.

As for the tensors with support $\mathcal{T}_9$, we proceed as follows. Let $T = T_{9,\mu}$ and $T' = T_{9,\mu'}$ with $\mu \neq \mu'$. We show that T is not isomorphic to T'. Suppose by contradiction that there is a triple of 3×3 matrices $g = (g_A, g_B, g_C) \in GL_3 \times GL_3 \times GL_3$ with $g(T) = T'$. One sees that in each case, g_A, g_B, g_C have to be diagonal matrices, and an explicit calculation shows that there is no triple of diagonal matrices such that $g(T) = T'$. ◀

We point out that T_7 is isomorphic to the Coppersmith-Winograd tensor $T_{CW,1}$, as well as to the structure tensor of the algebra $\mathbb{C}[x]/(x^3)$.

The tensors $T_{cw,2}$ and $T_{skewcw,2}$ are degenerations of $T_{9,\mu}$, respectively for $\mu = 1$ and $\mu = -1$. In particular, they do not have an unextendable tight support in some basis.

▶ **Proposition 29.** *For $i = 1, \dots, 9$, the following table records $\dim \mathfrak{g}_{T_i}$ and $\dim \mathfrak{g}_{T_i^{\boxtimes 2}}$.*

T	$\dim \mathfrak{g}_T$	$\dim \mathfrak{g}_{T^{\boxtimes 2}}$
T_1	5	22
T_2	3	9
T_3	5	13
T_4	4	9
T_5	3	7
T_6	2	5
T_7	6	28
T_8	1	2
$T_{9,-1}$	5	10
$T_{9,\mu}$ *(for $\mu \neq 0, -1$)*	1	2

In summary

$$\dim \mathfrak{g}_{T^{\boxtimes 2}} > 2 \dim \mathfrak{g}_T$$

for tight tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ with unextendable tight supports $\mathcal{T}_1, \dots, \mathcal{T}_7$.

Proof. For $T_1, \dots, T_8$ and for the $T_{9,-1}$, the proof follows by a direct calculation. The first part of the file `symmetryTightSupports.m2` at `www.math.tamu.edu/~jml/CGLVkronsupp.html`, Appendix E computes the dimension of the symmetry algebras of interest in these cases.

The second part of the file deals with the case $T_{9,\mu}$ when $\mu \neq -1$. By tightness, $\dim \mathfrak{g}_{T_{9,\mu}} \geq 1$.

Consider the linear map $\omega_{T_{9,\mu}} : \mathfrak{gl}(A) + \mathfrak{gl}(B) + \mathfrak{gl}(C) \to A \otimes B \otimes C$ defined by $(X, Y, Z) \mapsto (X, Y, Z).T_{9,\mu}$. Then $\mathfrak{g}_{T_{9,\mu}} = [\ker(\omega_{T_{9,\mu}})]/\mathbb{C}^2$, where $\mathbb{C}^2$ corresponds to $\ker d\Phi$.

The second part of the file `symmetryTightSupports.m2` computes a matrix representation of $\omega_{T_{9,\mu}}$, depending on a parameter μ (`t` in the file). Let F_μ be this 27×27 matrix representation. Then, it suffices to select a 24×24 submatrix whose determinant is a nonzero univariate polynomial in μ. If μ is a value for which $\dim \mathfrak{g}_{T_{9,\mu}} > 1$, then μ has to be a root of this univariate polynomial.

In the example computed in the file, we select a 24×24 submatrix whose determinant is $(\mu + 1)^6 \mu$, showing that the only possible values of μ for which $\dim \mathfrak{g}_{T_{9,\mu}} > 1$ are $\mu = 0$ or $\mu = -1$. The case $\mu = -1$ was considered separately. The case $\mu = 0$ does not correspond to a unextendable support, so it is not of interest. We point out that however, $\omega_{T_{9,0}} = 24$, namely $\dim \mathfrak{g}_{T_{9,0}} = 1$.

For $T_{9,\mu}^{\boxtimes 2}$, we follow essentially the same argument. By tightness, and (10), we obtain $\dim \mathfrak{g}_{T_{9,\mu}^{\boxtimes 2}} \geq 2$. The third part of `symmetryTightSupports.m2` computes a matrix representation of the map $\omega_{T_{9,\mu}^{\boxtimes 2}}$, depending on a parameter μ: this is a 729×243 matrix of rank at most 239.

In the example computed in the file, we select a 239×239 submatrix whose determinant is the univariate polynomial $\mu^8(\mu + 1)^{12}$. As before, we conclude. ◀

We also provide the values of the border rank of the tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ having unextendable tight support and numerical evidence for the values of border rank of their Kronecker square. They are recorded in the following table. The values of the border rank for the T_i's are straightforward to verify. The lower bounds for the Kronecker squares are obtained via Koszul flattenings. In the cases labeled by N/A the upper bounds coincide with the multiplicative upper bound; in the other cases, the upper bound is obtained via numerical method, and the last column of the table records the ℓ_2 distance (in the given basis) between the tensor obtained via the numerical approximation and the Kronecker square.

T	$\underline{\mathbf{R}}(T)$	$\underline{\mathbf{R}}(T^{\boxtimes 2})$	ℓ_2 error for upper bound in $T^{\boxtimes 2}$ decomposition
T_1	3	9	N/A
T_2	4	$[11, 14]$	0.000155951
T_3	4	$[11, 14]$	0.00517612
T_4	4	14	0.0144842
T_5	4	$[11, 15]$	0.0237172
T_6	4	$[11, 15]$	0.00951205
T_7	3	9	N/A
T_8	4	$[14, 16]$	N/A
$T_{9,-1}$	5	$[16, 19]$	0.0231353
$T_{9,\mu}$ (for $\mu \neq 0, -1$)	4	$[15, 16]$	N/A

7 Justification of the matrices

In this section, describe two ways of proving that the matrices appearing in the second proof of Theorem 5 and the proof of Theorem 6 are as asserted, one of which is carried out explicitly in the code at `http://www.math.tamu.edu/~jml/CGLVkronsupp.html`.

The computational issue is that, although the sizes of the matrices are fixed, they are obtained via intermediate matrices whose dimensions depend on q so one needs a way of encoding such matrices and tensors efficiently. The first method of proof critically relies on the definition of a class of tensors, which we call *box parameterized*, whose entries and dimensions depend on a parameter q in a very structured way. In this proof one shows the entries of the output matrices are low degree, say δ, polynomials in q, and then by computing the first $\delta+1$ cases directly, one has proven they are as asserted for all q. The second method, which is implemented in the code, does not rely on the structure to prove anything, but the structure allows an efficient coding of the tensors that significantly facilitates the computation.

A k-way sequence of tensors $T_q \in A_1^q \otimes \cdots \otimes A_k^q$ parametrized by $q \in \mathbb{N}$ is *basic box parameterized* if it is of the form

$$T_q = p(q) \textstyle\sum_{(i_1,\ldots,i_k)\in\Phi} t_{i_1,\ldots,i_k},$$

where $\{a_{\alpha,s}\}$ is a basis of A_α^q, $t_{i_1,\ldots,i_k} = a_{1,i_1} \otimes \cdots \otimes a_{k,i_k}$, p is a polynomial, and the index set Φ is defined by conditions $f_j q + h_j \le i_j \le g_j q + d_j$, $f_j, g_j \in \{0,1\}$, $h_j, d_j \in \mathbb{Z}_{\ge 0}$, for each j, and any number of equalities $i_j = i_k$ between indices.

We sometimes abuse notation and consider Φ to be its set of indices or the set of equations and inequalities defining the set of indices; no confusion should arise.

Tensor products of basic box parameterized tensors are basic box parameterized:

$$\begin{aligned}&(p_1(q) \textstyle\sum_{(i_1,\ldots,i_k)\in\Phi_1} t_{i_1,\ldots,i_k}) \otimes (p_2(q) \textstyle\sum_{(j_1,\ldots,j_l)\in\Phi_2} j_{i_1,\ldots,j_l}) \\ &= p_1(q)p_2(q) \textstyle\sum_{(i_1,\ldots,i_k,j_1,\ldots,j_l)\in\Phi_1\times\Phi_2} t_{i_1,\ldots,i_k,j_1,\ldots,j_l}.\end{aligned}$$

We next show that contraction of a basic box parameterized tensor is basic box parameterized when $q \ge \max_{i,j}\{|h_i - h_j|, |d_i - d_j|\}$, where i and j range over those indices related by equality to the ones being contracted. To do this, we first show they are closed under summing along a coordinate (with the same restriction on q), which we may take to be i_1 without loss of generality. (This corresponds to contracting with the vector $\sum_{i_1} a^*_{1,i_i} \in (A_1^q)^*$.) That is, we wish to show

$$p(q) \textstyle\sum_{(i_1,\ldots,i_k)\in\Phi} t_{i_2,\ldots,i_k}$$

is basic box parameterized with the above restriction on q. For this consider two cases. First, suppose there is a coordinate $j \ne 1$ so that $i_1 = i_j \in \Phi$. To construct the summed tensor, adjoin to Φ equalities $i_j = i_k$ for all k for which $i_1 = i_k \in \Phi$. Then, deleting i_1 from the indices and replacing the bounds on i_j with

$$\max(f_j q + h_j, f_1 q + h_1) \le i_j \le \min(g_j q + d_j, g_1 q + d_1)$$

yields the summed tensor. The max and the min can be replaced with one of their arguments provided $q \ge \max(|h_1 - h_j|, |d_1 - d_j|)$, so the sum is basic box parameterized with our restriction on q. Otherwise, suppose there is no coordinate so that $i_1 = i_j \in \Phi$. Then the summed tensor is $(g_1 q + d_1 - f_1 q - h_1 + 1)p(q) \sum_{(i_2,\ldots,i_k)\in\Phi} t_{i_2,\ldots,i_k}$, which is basic box parameterized.

Finally, to compute the contraction, say between indices i_j and i_k, adjoin $i_j = i_k$ as a condition to Φ and then sum over i_j and then over i_k using the previous technique.

Call a tensor *box parameterized* if it is a finite sum of basic box parameterized tensors. Clearly box parameterized tensors are closed under tensor products and contraction, possibly with an easily computed restriction on q.

Now, $T_{cw,q} \in (\mathbb{C}^{q+1})^{\otimes 3} = A \otimes B \otimes C$ is clearly box parameterized as a 3-way tensor. The tensors $\phi_2 \in A' \otimes (A^{\otimes 2})^*$ (where $\dim A' = 3$) and $\phi_3 \in A' \otimes (A^{\otimes 3})^*$ (where $\dim A' = 5$) defining the projection maps are box parameterized as 3-way and 4-way tensors, respectively. The tensors $KF_1 \in (A' \otimes B^{\otimes 2} \otimes C^{\otimes 2})^* \otimes ((A')^* \otimes B^{\otimes 2}) \otimes (\Lambda^2 A' \otimes C^{\otimes 2}))$ and $KF_2 \in (A' \otimes B^{\otimes 3} \otimes C^{\otimes 3})^* \otimes ((\Lambda^2 A')^* \otimes B^{\otimes 3}) \otimes (\Lambda^3 A' \otimes C^{\otimes 3}))$ defining the Koszul flattenings are also box parameterized, as they are the tensor product of tensors of fixed size with identity tensors, which are basic box parameterized. From this, we see that the corresponding Koszul flattenings are box parameterized, viewed in $A'^* \otimes B^{\otimes 2} \otimes \Lambda^2 A' \otimes C^{\otimes 2}$ as a 6-way tensor for the square and $\Lambda^2 A'^* \otimes B^{\otimes 3} \otimes \Lambda^3 A' \otimes C^{\otimes 3}$ as an 8-way tensor for the cube.

Finally, consider the change of basis map which block diagonalizes the flattening according to Schur's lemma. We explain the square case, the cube case is available in the Appendix. This change of basis is the Kronecker product of the 3×3 identity with the Kronecker square of the map represented by the following $q+1 \times q+1$ matrix

$$\begin{pmatrix}
1 &&&&&&&&& \\
& 1 &&&&&&&& \\
&& 1 &&&&&&& \\
&&& 1 &&&&&& \\
&&&& 1 & 1 & 1 & 1 & \cdots & 1 \\
&&&& -1 & 1 &&&& \\
&&&&& -1 & 1 &&& \\
&&&&&&& \ddots & 1 & \\
&&&&&&&& -1 & 1
\end{pmatrix}$$

Let E_0 denote the projection operator to the isotypic component of the trivial representation. In bases, this corresponds to the first five rows of the matrix above. Let E_1 denote the projection onto the standard representation, which corresponds to the sixth row. It is easy to see that the first 6 columns of the inverse is the matrix

$$\frac{1}{q-3}\begin{pmatrix}
q-3 &&&&& \\
& q-3 &&&& \\
&& q-3 &&& \\
&&& q-3 && \\
&&&& 1 & -(q-4) \\
&&&& 1 & 1 \\
&&&& \vdots & \\
&&&& 1 & 1
\end{pmatrix}$$

Write F_0 for the inclusion of the trivial representation into the space in the original basis, which is represented by the first five columns of this matrix, and F_1 for the inclusion of the standard representation which is represented by the sixth column. Write V_0 for the trivial representation of $\mathfrak{S}_{q-3}$ and V_1 for the standard representation. Then,

$$f_{V_i \boxtimes V_j} = (\mathrm{Id}_{A'} \boxtimes E_i \boxtimes E_j) \circ (T_{cw,q}^{\otimes 2})^{\wedge 2}_{A'} \circ (\mathrm{Id}_{\Lambda^2 A'} \boxtimes F_i \boxtimes F_j).$$

(These four maps were labeled $f_1, \ldots, f_4$ in §4.5.) Since E_i and $(q-3)F_i$ are clearly box parametrized, it follows that $(q-3)^2 f_{V_i \boxtimes V_j}$ is box parametrized. A similar argument shows that the cube $(q-3)^3 f_{V_i \boxtimes V_j \boxtimes V_k}$ is box parameterized.

At this point the first method shows the entries of the matrices are low degree polynomials in q so one can conclude by checking the first few cases.

The fact that all tensors involved are basic box parameterized guided us how to encode these maps efficiently so that they could be computed by direct calculation, which provides the second method and is described in Appendix D.

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