Deformed covariance in spherically symmetric vacuum models of loop quantum gravity: Consistency in Euclidean and self-dual gravity

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Different versions of consistent canonical realizations of hypersurface deformations of spherically symmetric spacetimes have been derived in models of loop quantum gravity, modifying the classical dynamics and sometimes also the structure of spacetime. Based on a canonical version of effective field theory, this paper provides a unified treatment, showing that modified spacetime structures are generic in this setting. The special case of Euclidean gravity demonstrates agreement also with existing operator calculations.

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I. INTRODUCTION

Several independent studies have shown that holonomy and inverse-triad corrections from loop quantum gravity (LQG) modify hypersurface-deformation brackets for spherically symmetric gravity and related midisuperspace models [1–10], thereby realizing a deformation of general covariance [11–14]. These modifications are closely related [15] to anomaly-free models of perturbative cosmological inhomogeneity constructed within the same framework [16–20], suggesting that modified spacetime structures may be a generic consequence of quantum-geometry effects in loop quantum gravity. In [21] (see also [22]), however, it has been shown that such modifications may be avoided if one uses self-dual connections and a densitized lapse function, as in [23–25], instead of real variables [26]. These models, valid for self-dual Lorentzian gravity with Barbero-Immirzi parameter $\gamma = \pm i$ or Euclidean gravity with Barbero-Immirzi parameter $\gamma = \pm 1$, are rather special because the Hamiltonian constraint simplifies considerably compared with general γ. It is therefore of interest to compare the structures encountered in various models in order to determine whether undeformed spacetime structures could be realized more broadly.

Such a comparison is not obvious, for instance because the modifications considered in [21] are different from those found in anomaly-free models using real variables. In particular, those modifications cannot be implemented in an anomaly-free manner for arbitrary choices of the Barbero-Immirzi parameter: We will show that the classical form of the constraint brackets can be retained only with a specific class of holonomy modifications for $\gamma = \pm i$ (self-dual Lorentzian gravity) or $\gamma = \pm 1$ (a special version of Euclidean gravity). More general treatments of the self-dual or Euclidean case, implemented in close analogy with the real connection formulation, lead to either anomalies or deformations of the spacetime structure. This result then allows us to draw conclusions about properties of the Hamiltonian constraint required for certain types of modifications to be consistent.

At a technical level, an analysis of the Hamiltonian constraint and its Poisson brackets indicates a formal relationship between modifications of spacetime structures and the appearance of spatial derivatives of the densitized triads (canonically conjugate to the connection). Spatial derivatives of the triad generically appear in the Hamiltonian constraints of gravitational theories because they are required for curvature components. But for $\gamma^2 = \pm 1$, and *only in this case*, they are completely absorbed in the connection components through the spin connection which, in combination with extrinsic-curvature components, forms the Ashtekar connection in the self-dual case [23], or the Ashtekar-Barbero connection in the real case [26].

This structural statement allows us to draw a first conclusion about the genericness of modified spacetime structures. Using standard arguments from effective field theory (generalized here to a canonical setting), modified brackets should be considered generic, unless one can show that the full quantum theory has a symmetry that protects the derivative structure of terms in the Hamiltonian constraint as encountered for self-dual variables, or more generally for $\gamma^2 = \pm 1$. No such symmetry is known. Although it has been shown that the real Ashtekar-Barbero connection, unlike the self-dual one, cannot be identified with the pullback of a spacetime connection, this result is of an "aesthetic nature" [27] and does not characterize the case of $\gamma^2 = \pm 1$ via a physical symmetry that could restrict possible quantum corrections. Moreover, applying this result in the present context would amount to presupposing the classical spacetime structure in a model of quantum gravity. In canonical quantum gravity, the structure of spacetime is determined intrinsically, based on the observation that spacetime symmetries of a gravitational theory are gauge transformations, generated in Hamiltonian form by the constraints that are to be quantized in order to define canonical quantum gravity. Poisson brackets of these constraints, or commutators of their operator versions, then encode the structure of spacetime. An analysis of possible consistent modifications of these brackets, such that they remain closed but possibly with nonclassical structure functions, shows whether the symmetries remain unviolated after quantization. As we will see, such modifications with intact (but possibly deformed) symmetry exist for any value of γ. Therefore, no value of γ is distinguished by the presence of a symmetry.

In this work, we will mainly focus on an interpretation of the constraints as representing Euclidean gravity. We will then be exempt from having to consider a possible role of reality conditions, the implementation of which remains poorly understood in a quantum theory of self-dual variables. However, as the constraints are formally identical in Euclidean gravity with $\gamma = \pm 1$ and self-dual Lorentzian gravity, our results can formally be used also in the latter case.

II. UNSOLVED GAUSS CONSTRAINT

The model considered in [21], following [24], consists of three canonical pairs of fields—$A_i(x)$ and $E^i(x)$ for $i = 1$, 2, 3 depending on the radial coordinate x of a spherically symmetric manifold—subject to three constraints. Two of the constraints function as generators of hypersurface deformations in spacetime and therefore encode the structure of spacetime. The third one, a Gauss constraint, implements an internal symmetry of $SO(2)$ rotations of two of the canonical pairs.

While the form of the Gauss constraint and the spatial generator of hypersurface deformations (the diffeomorphism constraint) is strictly determined by the canonical structure together with the corresponding Lie algebras of infinitesimal rotations and one-dimensional diffeomorphisms, respectively, there is much freedom in specifying the normal generator of hypersurface deformations, or the Hamiltonian constraint, even if the physical dynamics is fixed. The version used in [21,24] is rather special in that it is quadratic in the canonical fields and does not contain spatial derivatives of E^i (while first-order spatial derivatives of A_i do appear). In the first part of this section we will strengthen the result of [21] by showing that the consistent deformation found in this paper is unique within a family of models that preserve the quadratic nature and derivative structure of the Hamiltonian constraint. In the second part of this section, however, we will show that this rigidity is not stable within a larger class of models that determine the same classical dynamics but do not respect the restricted derivative structure (parametrized by the so-called Barbero-Immirzi parameter γ [26,28]). The following sections will then place our discussion in a setting of effective field theory and highlight the role played by the Gauss constraint.

A. Regaining the quadratic Hamiltonian constraint

In order to derive our rigidity result, we start from the condition that the Poisson brackets of constraints are closed and see what kind of restrictions it imposes on the form of constraints. The specific procedure follows the classical (and classic) result [29] that the full Hamiltonian constraint, up to second order in derivatives, can be regained uniquely from the classical hypersurface-deformation brackets, as specified in [30]. This procedure has already been applied to spherically symmetric models in [11], but only for modifications of the dependence of the Hamiltonian constraint on the triad variables E^i. Our calculations here differ from [11] in that we use connection variables A_i and take into account potential modifications of the dependence on these variables.

As already indicated, we assume for now that the Hamiltonian constraint is quadratic in the canonical fields without spatial derivatives of the triad E^i. This version of the constraint is realized in spherically symmetric gravity if one uses self-dual connection variables [23] in Lorentzian signature, or real Barbero-type variables [26] in Euclidean signature such that the Barbero-Immirzi parameter is equal to $\gamma = \pm 1$. (One should also smear the Hamiltonian constraint with a lapse function of density weight minus one to guarantee the quadratic nature.) This parameter is therefore fixed and does not appear in the remainder of this subsection. Working with

$$\{A_1(x), E^1(y)\} = 2G\delta(x,y) \tag{1}$$

and

$$\begin{aligned}\{A_2(x), E^2(y)\} &= G\delta(x,y),\\ \{A_3(x), E^3(y)\} &= G\delta(x,y)\end{aligned} \tag{2}$$

while all other brackets of basic variables vanish. [Note the missing factor of 2 in the last two brackets, compared with (1), which is a consequence of the fact that (A_2, E^2) and (A_3, E^3) encode the same degree of freedom after the Gauss constraint is implemented.]

$$\{A_1(x), E^1(y)\} = 2\{A_{2/3}(x), E^{2/3}(y)\} = 2\delta(x,y). \tag{3}$$

This canonical structure completely determines the Gauss constraint

$$G[\Lambda] = \frac{1}{2G}\int \mathrm{d}x\Lambda((E^1)' - 2E^2A_3 + 2E^3A_2) \tag{4}$$

and the diffeomorphism constraint

$$D[M] = \frac{1}{2G}\int \mathrm{d}xM(2A_3'E^3 + 2A_2'E^2 - A_1(E^1)') \tag{5}$$

but not the Hamiltonian constraint. Sometimes, it is convenient to combine the diffeomorphism constraint $D[M]$ and the Gauss constraint $G[\Lambda]$ to form the vector constraint

$$\begin{aligned}V[M] &= D[M] + G[A_1M]\\ &= \frac{1}{G}\int \mathrm{d}xM((A_3' + A_1A_2)E^3 + (A_2' - A_1A_3)E^2).\end{aligned} \tag{6}$$

We will now use these constraints and attempt to derive the most general form of the Hamiltonian constraint, purely quadratic in the canonical fields and with up to first derivatives of A_i but no derivatives of E^i, such that all constraints have closed Poisson brackets. With this assumption, we can write the local (unsmeared) constraint as

$$\begin{aligned}\mathcal{H} &= H^{110}E^1E^2 + H^{101}E^1E^3 + H^{011}E^2E^3\\ &\quad + H^{200}(E^1)^2 + H^{020}(E^2)^2 + H^{002}(E^3)^2,\end{aligned} \tag{7}$$

where we use the convention that $H[N] = (2G)^{-1}\int \mathrm{d}xN(x)\mathcal{H}$, H^{ijk} may be functions of A_1, A_2, A_3 and their spatial derivatives up to first order.

1. Diffeomorphism constraint

We first consider the bracket of the Hamiltonian and diffeomorphism constraints, writing it in local form as

$$\begin{aligned}\{\mathcal{H}(x), \mathcal{D}(y)\} = G\int \mathrm{d}z\bigg(&2\frac{\delta\mathcal{H}(x)}{\delta A_1(z)}\frac{\delta\mathcal{D}(y)}{\delta E^1(z)} - 2\frac{\delta\mathcal{H}(x)}{\delta E^1(z)}\frac{\delta\mathcal{D}(y)}{\delta A_1(z)}\\ &+ \frac{\delta\mathcal{H}(x)}{\delta A_2(z)}\frac{\delta\mathcal{D}(y)}{\delta E^2(z)} - \frac{\delta\mathcal{H}(x)}{\delta E^2(z)}\frac{\delta\mathcal{D}(y)}{\delta A_2(z)}\\ &+ \frac{\delta\mathcal{H}(x)}{\delta A_3(z)}\frac{\delta\mathcal{D}(y)}{\delta E^3(z)} - \frac{\delta\mathcal{H}(x)}{\delta E^3(z)}\frac{\delta\mathcal{D}(y)}{\delta A_3(z)}\bigg),\end{aligned} \tag{8}$$

where $D[M] = (2G)^{-1}\int \mathrm{d}xM(x)\mathcal{D}(x)$. If this bracket is to correspond to classical hypersurface deformations, it should be equal to

$$\{\mathcal{H}(x), \mathcal{D}(y)\} = 2G(\mathcal{H}'(x)\delta(x,y) + 2\mathcal{H}(x)\delta'(x,y)), \tag{9}$$

using the convention that a prime on a delta function always indicates a derivative with respect to the first argument. Therefore,

$$\delta'(x,y) = -\delta'(y,x). \tag{10}$$

If the bracket is of the given form, the smeared constraints have the bracket

$$\begin{aligned}\{H[\underset{\sim}{N}], D[M]\} &= \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y\underset{\sim}{N}(x)M(y)\{\mathcal{H}(x), \mathcal{D}(y)\}\\ &= \frac{1}{2G}\int \mathrm{d}x\mathrm{d}y\underset{\sim}{N}(x)M(y)((\partial_x\mathcal{H}(x))\delta(x,y)\\ &\quad - 2\mathcal{H}(x)\partial_y\delta(x,y))\\ &= -H[(\underset{\sim}{N}M)'] + 2H[\underset{\sim}{N}M']\\ &= -H[M\underset{\sim}{N}' - M'\underset{\sim}{N}]\end{aligned} \tag{11}$$

as required if $\underset{\sim}{N}$ has density weight minus one for the purpose of having a quadratic Hamiltonian constraint.

We proceed by evaluating the Poisson bracket. Considering the assumed dependence (7) of $\mathcal{H}$ on the canonical variables, we have

$$
\begin{aligned}
\{\mathcal{H}(x),\mathcal{D}(y)\} = 2G\int \mathrm{d}z\Bigg(&\left(\frac{\partial\mathcal{H}(x)}{\partial A_1(z)}\delta(x,z)+\frac{\partial\mathcal{H}(x)}{A_1'(z)}\delta'(x,z)\right)(-A_1(y)\delta'(y,z))-\frac{\partial\mathcal{H}(x)}{\partial E^1(z)}\delta(x,z)(-(E^1)'(y)\delta(y,z))\\
&+\left(\frac{\partial\mathcal{H}(x)}{\partial A_2(z)}\delta(x,z)+\frac{\partial\mathcal{H}(x)}{A_2'(z)}\delta'(x,z)\right)A_2'(y)\delta(y,z)-\frac{\partial\mathcal{H}(x)}{\partial E^2(z)}\delta(x,z)E^2(y)\delta'(y,z)\\
&+\left(\frac{\partial\mathcal{H}(x)}{\partial A_3(z)}\delta(x,z)+\frac{\partial\mathcal{H}(x)}{A_3'(z)}\delta'(x,z)\right)A_3'(y)\delta(y,z)-\frac{\partial\mathcal{H}(x)}{\partial E^3(z)}\delta(x,z)E^3(y)\delta'(y,z)\Bigg)\\
= 2G&\left(\frac{\partial\mathcal{H}(x)}{\partial A_2(x)}A_2'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3(x)}A_3'(x)+\frac{\partial\mathcal{H}(x)}{\partial E^1(x)}(E^1)'(x)\right)\delta(x,y)\\
&-\left(\frac{\partial\mathcal{H}(x)}{\partial A_1(x)}A_1(y)+\frac{\partial\mathcal{H}(x)}{\partial E^2(x)}E^2(y)+\frac{\partial\mathcal{H}(x)}{\partial E^3(x)}E^3(y)+\frac{\partial\mathcal{H}(x)}{\partial A_2'(x)}A_2'(y)+\frac{\partial\mathcal{H}(x)}{\partial A_3'(x)}A_3'(y)\right)\delta'(y,x)\\
&-\int \mathrm{d}z\frac{\partial\mathcal{H}(x)}{\partial A_1'(z)}A_1(y)\delta'(x,z)\delta'(y,z),
\end{aligned}
\tag{12}
$$

where we used (10).

The last term has a product of two derivatives of delta functions, which does not occur in (9). Integrating by parts can remove one of the derivatives, but it also gives a second-order derivative of a delta function which does not appear either in (9). The term, therefore, must be zero, so that we already know that $\mathcal{H}$ cannot depend on A_1'. In order to bring the remaining terms to a form close to (9), we use the identity

$$
\begin{aligned}
A(x)B(y)\delta'(y,x) &= A(x)\partial_y(B(y)\delta(y,x))-A(x)B'(y)\delta(x,y)\\
&= A(x)\partial_y(B(x)\delta(y,x))-A(x)B'(x)\delta(x,y)\\
&= A(x)B(x)\delta'(y,x)-A(x)B'(x)\delta(x,y)
\end{aligned}
\tag{13}
$$

and write

$$
\begin{aligned}
\{\mathcal{H}(x),\mathcal{D}(y)\} = 2G&\left(\frac{\partial\mathcal{H}(x)}{\partial A_1(x)}A_1'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_2(x)}A_2'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3(x)}A_3'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_2'(x)}A_2''(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3'(x)}A_3''(x)\right.\\
&\left.+\frac{\partial\mathcal{H}(x)}{\partial E^1(x)}(E^1)'(x)+\frac{\partial\mathcal{H}(x)}{\partial E^2(x)}(E^2)'(x)+\frac{\partial\mathcal{H}(x)}{\partial E^3(x)}(E^3)'(x)\right)\delta(x,y)\\
&+2G\left(\frac{\partial\mathcal{H}(x)}{\partial A_1(x)}A_1(x)+\frac{\partial\mathcal{H}(x)}{\partial A_2'(x)}A_2'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3'(x)}A_3'(x)+\frac{\partial\mathcal{H}(x)}{\partial E^2(x)}E^2(x)+\frac{\partial\mathcal{H}(x)}{\partial E^3(x)}E^3(x)\right)\delta'(x,y).
\end{aligned}
\tag{14}
$$

Since $\mathcal{H}$ does not depend on A_1', the first parentheses (multiplied by a delta function) are equal to $\mathcal{H}'$ without any further restriction on the dependence on other canonical variables. In order to evaluate the second parentheses, which according to (9) should equal $4G\mathcal{H}$, we use the quadratic form (7) and obtain the condition

$$
\begin{aligned}
&\frac{\partial\mathcal{H}(x)}{\partial A_1(x)}A_1(x)+\frac{\partial\mathcal{H}(x)}{\partial A_2'(x)}A_2'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3'(x)}A_3'(x)+H^{110}E^1E^2+H^{101}E^1E^3+2H^{011}E^2E^3+2H^{020}(E^2)^2+2H^{002}(E^3)^2\\
&\quad=2(H^{110}E^1E^2+H^{101}E^1E^3+H^{011}E^2E^3+H^{020}(E^2)^2+H^{002}(E^3)^2)
\end{aligned}
\tag{15}
$$

or

$$
\frac{\partial\mathcal{H}(x)}{\partial A_1(x)}A_1(x)+\frac{\partial\mathcal{H}(x)}{\partial A_2'(x)}A_2'(x)+\frac{\partial\mathcal{H}(x)}{\partial A_3'(x)}A_3'(x)=H^{110}E^1E^2+H^{101}E^1E^3+2H^{200}(E^1)^2
$$

after some cancellations. Comparing coefficients of E^iE^j in this equation, we obtain

$$\frac{\partial H^{110}}{\partial A_1}A_1+\frac{\partial H^{110}}{\partial A_2'}A_2'+\frac{\partial H^{110}}{\partial A_3'}A_3'=H^{110}, \tag{16}$$

$$\frac{\partial H^{101}}{\partial A_1}A_1+\frac{\partial H^{101}}{\partial A_2'}A_2'+\frac{\partial H^{101}}{\partial A_3'}A_3'=H^{101}, \tag{17}$$

$$\frac{\partial H^{011}}{\partial A_1}A_1+\frac{\partial H^{011}}{\partial A_2'}A_2'+\frac{\partial H^{011}}{\partial A_3'}A_3'=0, \tag{18}$$

$$\frac{\partial H^{200}}{\partial A_1}A_1+\frac{\partial H^{200}}{\partial A_2'}A_2'+\frac{\partial H^{200}}{\partial A_3'}A_3'=2H^{200}, \tag{19}$$

$$\frac{\partial H^{020}}{\partial A_1}A_1+\frac{\partial H^{020}}{\partial A_2'}A_2'+\frac{\partial H^{020}}{\partial A_3'}A_3'=0, \tag{20}$$

$$\frac{\partial H^{002}}{\partial A_1}A_1+\frac{\partial H^{002}}{\partial A_2'}A_2'+\frac{\partial H^{002}}{\partial A_3'}A_3'=0. \tag{21}$$

If we assume polynomial dependence of $\mathcal{H}$ on the connection variables, we can conclude that the coefficients H^{110} and H^{101} must be linear in A_1, A_2', and A_3', while H^{200} must be quadratic in these variables. The coefficients H^{011}, H^{020}, and H^{002} cannot depend on A_1, A_2', or A_3'.

2. Bracket of Hamiltonian constraints

The Poisson bracket of two Hamiltonian constraints can be computed in a similar way. Classically, we expect

$$\begin{aligned}\{\mathcal{H}(x),\mathcal{H}(y)\}=2G(&E^1(x)^2\mathcal{V}(x)\delta'(y,x)\\&-E^1(y)^2\mathcal{V}(y)\delta'(x,y))\end{aligned} \tag{22}$$

with the local vector constraint $\mathcal{V}(x)$ such that $V[M]=(2G)^{-1}\int \mathrm{d}xM(x)\mathcal{V}(x)$. If the spacetime structure is deformed, the bracket is multiplied by a nonconstant function β which, for a comparison with [21], we assume to depend only on the A_i. (This function should approach $\beta=1$ in some classical limit, usually for small A_i.) After using (7) and comparing coefficients of E^iE^j, we obtain the equations

$$\begin{aligned}&2\left(-2\frac{\partial H^{110}}{\partial A_1'}H^{200}-\frac{\partial H^{200}}{\partial A_1'}H^{110}\right)-\frac{\partial H^{110}}{\partial A_2'}H^{110}\\&\quad-2\frac{\partial H^{200}}{\partial A_2'}H^{020}-\frac{\partial H^{110}}{\partial A_3'}H^{101}-\frac{\partial H^{200}}{\partial A_3'}H^{011}\\&=4\beta(A_2'-A_1A_3),\end{aligned} \tag{23}$$

$$\begin{aligned}&2\left(-2\frac{\partial H^{101}}{\partial A_1'}H^{200}-\frac{\partial H^{200}}{\partial A_1'}H^{101}\right)-\frac{\partial H^{101}}{\partial A_2'}H^{110}\\&\quad-2\frac{\partial H^{200}}{\partial A_2'}H^{011}-\frac{\partial H^{101}}{\partial A_3'}H^{101}-\frac{\partial H^{200}}{\partial A_3'}H^{002}\end{aligned} \tag{24}$$

$$=4\beta(A_3'+A_1A_2), \tag{25}$$

which are sensitive to the modification function β, as well as several β-independent equations:

$$4\frac{\partial H^{200}}{\partial A_1'}H^{200}+\frac{\partial H^{200}}{\partial A_2'}H^{110}+\frac{\partial H^{200}}{\partial A_3'}H^{101}=0, \tag{26}$$

$$2\left(\frac{\partial H^{110}}{\partial A_1'}H^{110}+2\frac{\partial H^{020}}{\partial A_1'}H^{200}\right)+2\frac{\partial H^{110}}{\partial A_2'}H^{020}+\frac{\partial H^{020}}{\partial A_2'}H^{110}+\frac{\partial H^{110}}{\partial A_3'}H^{011}+\frac{\partial H^{020}}{\partial A_3'}H^{101}=0, \tag{27}$$

$$2\left(\frac{\partial H^{101}}{\partial A_1'}H^{101}+2\frac{\partial H^{002}}{\partial A_1'}H^{200}\right)+\frac{\partial H^{101}}{\partial A_2'}H^{011}+\frac{\partial H^{002}}{\partial A_2'}H^{110}+2\frac{\partial H^{101}}{\partial A_3'}H^{002}+\frac{\partial H^{002}}{\partial A_3'}H^{101}=0, \tag{28}$$

$$\begin{aligned}&2\left(2\frac{\partial H^{011}}{\partial A_1'}H^{200}+\frac{\partial H^{101}}{\partial A_1'}H^{110}+\frac{\partial H^{110}}{\partial A_1'}H^{101}\right)+\frac{\partial H^{011}}{\partial A_2'}H^{110}+2\frac{\partial H^{101}}{\partial A_2'}H^{020}+\frac{\partial H^{110}}{\partial A_2'}H^{011}\\&\quad+\frac{\partial H^{011}}{\partial A_3'}H^{101}+\frac{\partial H^{101}}{\partial A_3'}H^{011}+2\frac{\partial H^{110}}{\partial A_3'}H^{002}=0.\end{aligned} \tag{29}$$

Four additional equations,

$$2\frac{\partial H^{020}}{\partial A_1'}H^{110}+2\frac{\partial H^{020}}{\partial A_2'}H^{020}+\frac{\partial H^{020}}{\partial A_3'}H^{011}=0, \tag{30}$$

$$2\frac{\partial H^{002}}{\partial A_1'}H^{101}+\frac{\partial H^{002}}{\partial A_2'}H^{011}+2\frac{\partial H^{002}}{\partial A_3'}H^{002}=0, \tag{31}$$

$$2\left(\frac{\partial H^{011}}{\partial A_1'}H^{110}+\frac{\partial H^{020}}{\partial A_1'}H^{101}\right)+2\frac{\partial H^{011}}{\partial A_2'}H^{020}+\frac{\partial H^{020}}{\partial A_2'}H^{011}+\frac{\partial H^{011}}{\partial A_3'}H^{011}+2\frac{\partial H^{020}}{\partial A_3'}H^{002}=0, \tag{32}$$

$$2\left(\frac{\partial H^{011}}{\partial A_1'}H^{101}+\frac{\partial H^{002}}{\partial A_1'}H^{110}\right)+\frac{\partial H^{011}}{\partial A_2'}H^{011}+2\frac{\partial H^{002}}{\partial A_2'}H^{020}+2\frac{\partial H^{011}}{\partial A_3'}H^{002}+\frac{\partial H^{002}}{\partial A_3'}H^{011}=0, \tag{33}$$

are identically satisfied, given that H^{011}, H^{020}, and H^{002} cannot depend on A_i'. Because $\mathcal{H}$ cannot depend on A_1', we may simplify the set of equations to

$$-\frac{\partial H^{110}}{\partial A_2'}H^{110}-2\frac{\partial H^{200}}{\partial A_2'}H^{020}-\frac{\partial H^{110}}{\partial A_3'}H^{101}-\frac{\partial H^{200}}{\partial A_3'}H^{011} = 4\beta(A_2'-A_1A_3), \tag{34}$$

$$-\frac{\partial H^{101}}{\partial A_2'}H^{110}-2\frac{\partial H^{200}}{\partial A_2'}H^{011}-\frac{\partial H^{101}}{\partial A_3'}H^{101}-\frac{\partial H^{200}}{\partial A_3'}H^{002} = 4\beta(A_3'+A_1A_2), \tag{35}$$

$$\frac{\partial H^{200}}{\partial A_2'}H^{110}+\frac{\partial H^{200}}{\partial A_3'}H^{101}=0, \tag{36}$$

$$2\frac{\partial H^{110}}{\partial A_2'}H^{020}+\frac{\partial H^{110}}{\partial A_3'}H^{011}=0, \tag{37}$$

$$\frac{\partial H^{101}}{\partial A_2'}H^{011}+2\frac{\partial H^{101}}{\partial A_3'}H^{002}=0, \tag{38}$$

$$2\frac{\partial H^{101}}{\partial A_2'}H^{020}+\frac{\partial H^{110}}{\partial A_2'}H^{011}+\frac{\partial H^{101}}{\partial A_3'}H^{011}+2\frac{\partial H^{110}}{\partial A_3'}H^{002}=0. \tag{39}$$

3. Gauss constraint

The Gauss constraint further restricts the combinations of basic variables which can appear in the Hamiltonian constraint. The gauge-invariant combinations that contribute to the classical constraint are $E^1, (E^2)^2+(E^3)^2, A_2E^2+A_3E^3$, $A_2^2+A_3^2$, and $A_1(A_2E^2+A_3E^3)-(A_2'E^3-A_2'E^2)$. [The identity (13) is useful for seeing that the last combination has a vanishing Poisson bracket with the unsmeared Gauss constraint.] These expressions show that A_1, A_2', and A_3' can appear in gauge-invariant form only in combination with E^2 and E^3. It is therefore impossible to fulfill the condition that H^{200} be quadratic in A_1, A_2', and A_3' because H^{200} is defined as the E-independent coefficient of $(E^1)^2$ in the Hamiltonian constraint. For Hamiltonian constraints quadratic in E^i, we have $H^{200}=0$.

Equations (34) and (35) then simplify to

$$-\frac{\partial H^{110}}{\partial A_2'}H^{110}-\frac{\partial H^{110}}{\partial A_3'}H^{101}=4\beta(A_2'-A_1A_3), \tag{40}$$

$$-\frac{\partial H^{101}}{\partial A_2'}H^{110}-\frac{\partial H^{101}}{\partial A_3'}H^{101}=4\beta(A_3'+A_1A_2). \tag{41}$$

For $\beta=1$, these equations are obeyed by the classical $H^{110}_{\rm cl}=2(A_1A_2+A_3')$ and $H^{101}_{\rm cl}=2(A_1A_3-A_2')$, as they should. For $\beta\neq 1$, we can solve these two equations by $H^{110}=\beta_1H^{110}_{\rm cl}$ and $H^{101}=\beta_2H^{101}_{\rm cl}$, provided that β_1 and β_2 do not depend on spatial derivatives of A_i and are such that $\beta_1\beta_2=\beta$. Invariance under transformations generated by the Gauss constraint, which mix the terms of $H^{110}_{\rm cl}$ and $H^{101}_{\rm cl}$, implies that $\beta_1=\beta_2$, and therefore $\beta>0$ and $\beta_1=\beta_2=\sqrt{\beta}$. This modification function can be eliminated from the contributions of H^{110} and H^{101} to the constraint by absorbing it in the lapse function, thus moving the modification to the remaining contributions from $H^{020}=\beta^{-1/2}H^{020}_{\rm cl}$ and $H^{002}=\beta^{-1/2}H^{002}_{\rm cl}$. Therefore, the only nontrivial modification of the dynamics is in the contributions from H^{020} and H^{002} which, as already shown, can only depend on A_2 and A_3. Again invoking transformations generated by the Gauss constraint, the modified term $\beta^{-1/2}(H^{020}_{\rm cl}+H^{002}_{\rm cl})$ is an arbitrary (positive) function of $A_2^2+A_3^2$, which is equivalent to the modification found in [21] and therefore strengthens their result.

If we relax the condition that the Hamiltonian constraint not depend on spatial derivatives of the densitized triad, additional gauge invariant combinations are possible. For instance, the extrinsic-curvature component

$$K_1=A_1-\frac{(E^2)'E^3-E^2(E^3)'}{(E^2)^2+(E^3)^2} \tag{42}$$

is gauge invariant. Moreover, if spatial derivatives of the densitized triad are allowed, the Gauss constraint can be used to rewrite the Hamiltonian constraint without changing the on-shell behavior. For instance, the identity

$$\begin{aligned}A_1(A_2E^2+A_3E^3)+2E^2A_3'-2E^3A_2' &= (E^1)''+A_2(A_1E^2+2(E^3)') \\ &\quad +A_3(A_1E^3-2(E^2)')-\mathcal{G}'\end{aligned} \tag{43}$$

eliminates spatial derivatives of A_2 and A_3 from the Hamiltonian constraint, in favor of a second-order spatial derivative of E^1. This new form is much closer to the expression of the Hamiltonian constraint in extrinsic-curvature variables [31] and may allow different modified brackets than the quadratic version (7) even if one works with the reduced Ashtekar connections A_i.

The possibility of rewriting the Hamiltonian constraint by using the Gauss constraint explains why different formulations of the same classical theory may give rise to different modified brackets: The Gauss constraint depends on A_2 and A_3, and therefore, depending on how it is used in writing the Hamiltonian constraint, it restricts possible modifications. In extrinsic-curvature variables, this ambiguity does not appear because the Gauss constraint is solved explicitly.

From the perspective of effective field theory, applied here to the classical structure of up to second-order derivatives, restricting the dependence of the Hamiltonian constraint on spatial derivatives of E^i leads to nongeneric models. The classical constraint is quadratic in A_i, which, according to the field equations implied by the theory, amounts to terms with up to two derivatives. Any term that is consistent with the symmetries of the theory (generated by the constraints) and has up to two derivatives (temporal or spatial) should then be allowed for a generic model. Such theories should include terms with up to second-order spatial derivatives of E^i, in addition to the quadratic terms in A_i which contribute two time derivatives. (A higher-derivative theory beyond second order would be obtained by including quantum backreaction effects, which is not the purpose of this paper.)

B. Arbitrary Barbero-Immirzi parameter

We will now show that the preceding rigidity result is not stable within a class of models in which spatial derivatives of the densitized triad are allowed to appear. A suitable set of constraints that describes the same classical physics as, depending on the signature, Euclidean or self-dual gravity is obtained by letting the Barbero–Immirzi parameter vary, instead of fixing it to a specific value such that $\gamma^2 = \pm 1$. The modification found in [21] is therefore not generic. To this end, we will now switch to a general setting of spherically symmetric gravity in which the Barbero-Immirzi parameter and other numerical factors (as well as the gravitational constant G) are included.

Spherically symmetric gravity can be formulated as a Hamiltonian theory with phase space given by the canonical pairs, subject to three constraints. This setting has been formulated in [24] for self-dual variables and in [31] for real variables. In order to avoid having to impose reality conditions, we follow the latter notation, in which the canonical pairs (A_1, E^1), (A_2, E^2), and (A_3, E^3) are such that

$$\{A_1(x), E^1(y)\} = 2\gamma G\delta(x, y) \tag{44}$$

and

$$\{A_2(x), E^2(y)\} = \gamma G\delta(x, y), \tag{45}$$

$$\{A_3(x), E^3(y)\} = \gamma G\delta(x, y) \tag{46}$$

[a version of (1) and (2) for arbitrary real γ]. They are subject to the Gauss constraint

$$G[\Lambda] = \frac{1}{2\gamma G}\int \mathrm{d}x\Lambda((E^1)' + 2A_2E^3 - 2A_3E^2) \tag{47}$$

smeared with a multiplier Λ, the diffeomorphism constraint

$$D[N^x] = \frac{1}{2\gamma G}\int \mathrm{d}xN^x(-A_1(E^1)' + 2A_3'E^3 + 2A_2'E^2) \tag{48}$$

smeared with the shift vector N^x, and the Hamiltonian constraint

$$\begin{aligned} H[\underset{\sim}{N}] = &\frac{1}{2G}\int \mathrm{d}x\underset{\sim}{N}(2A_1E^1(A_2E^2 + A_3E^3) \\ &+ (A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2) + 2E^1(E^2A_3' - E^3A_2') \\ &+ (\epsilon - \gamma^2)(2K_1E^1(K_2E^2 + K_3E^3) \\ &+ ((K_2)^2 + (K_3)^2)((E^2)^2 + (E^3)^2))) \\ = &H^{\mathrm{E}}[\underset{\sim}{N}] + H^{\mathrm{L}}[\underset{\sim}{N}] \end{aligned} \tag{49}$$

smeared with the lapse function $\underset{\sim}{N}$ of density weight -1. The nonpolynomial relationship between the extrinsic-curvature components K_1, K_2, and K_3 with the basic variables is given below.

In all three constraints, the prime represents a derivative with respect to the radial coordinate x. Moreover, γ in (49) is the Barbero-Immirzi parameter [26,28] and $\epsilon = \pm 1$ the spacetime signature, such that $\epsilon = 1$ in the Euclidean case and $\epsilon = -1$ in the Lorentzian case. As usual, it is convenient to split the Hamiltonian constraint into the Euclidean part

$$\begin{aligned} H^{\mathrm{E}}[\underset{\sim}{N}] = &\frac{1}{2G}\int \mathrm{d}x\underset{\sim}{N}(2A_1E^1(A_2E^2 + A_3E^3) \\ &+ (A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2) \\ &+ 2E^1(E^2A_3' - E^3A_2')) \end{aligned} \tag{50}$$

and the "Lorentzian" contribution

$$\begin{aligned} H^{\mathrm{L}}[\underset{\sim}{N}] = &-\frac{\gamma^2 - \epsilon}{2G}\int \mathrm{d}x\underset{\sim}{N}(2K_1E^1(K_2E^2 + K_3E^3) \\ &+ ((K_2)^2 + (K_3)^2)((E^2)^2 + (E^3)^2)). \end{aligned} \tag{51}$$

Thus, $H[\underset{\sim}{N}] = H^{\mathrm{E}}[\underset{\sim}{N}]$ for $\gamma = \pm 1$ in the Euclidean signature ($\epsilon = 1$), while the Lorentzian contribution (a slight misnomer) also contributes in the Euclidean signature if $\gamma \neq \pm 1$. (The Lorentzian contribution is always required in the Lorentzian signature if one works with real γ such that the Poisson brackets are real.) The canonical variables A_1, E^2, and E^3 have density weight one.

The geometrical meaning of the phase-space variables is determined as follows: The fields E^1, E^2, and E^3, as the components of a spherically symmetric densitized triad, describe a spatial metric q_{ab} according to the line element

$$\begin{aligned} \mathrm{d}s^2 &= q_{ab}\mathrm{d}x^a\mathrm{d}x^b \\ &= \frac{(E^2)^2+(E^3)^2}{|E^1|}\mathrm{d}x^2 + |E^1|(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2). \end{aligned} \quad (52)$$

The densitized triad also determines a spin connection such that it is constant with respect to the resulting covariant derivative. The components of this spin connection are functions of the densitized triad and its first spatial derivatives:

$$\Gamma_1 = \frac{E^3(E^2)' - E^2(E^3)'}{(E^2)^2+(E^3)^2}, \quad (53)$$

$$\Gamma_2 = -\frac{1}{2}\frac{(E^1)'E^3}{(E^2)^2+(E^3)^2}, \quad (54)$$

$$\Gamma_3 = \frac{1}{2}\frac{(E^1)'E^2}{(E^2)^2+(E^3)^2}. \quad (55)$$

The densitized triad is canonically conjugate to components of extrinsic curvature, K_i, $i = 1, 2, 3$. Since the Γ_i depend only on E^i, one can add them to K_i without changing the latter's canonical relationships with E^i. In this way, the canonical connection components $A_i = \Gamma_i + \gamma K_i$ are obtained, using the Barbero-Immirzi parameter γ.

The constrained system is first class, with brackets of the constraints $D[N^x]$ and $H[N]$ according to Dirac's hypersurface deformations [30] (taking into account the density weight of $\underset{\sim}{N}$ in the Hamiltonian constraint used here). In particular, the bracketed $\{H[\underset{\sim}{N}], H[\underset{\sim}{M}]\}$ should be proportional to the diffeomorphism constraint, up to possible contributions from the Gauss constraint. We display the relevant derivations in a more general setting, following the observation [21] that, for $\gamma^2 = \epsilon$, the constraint brackets remain closed in the presence of a "magnetic-field" modification, replacing $B_1 := A_2^2 + A_3^2 - 1$ in the Euclidean part of the Hamiltonian constraint with an arbitrary function $f(A_2^2 + A_3^2 - 1)$. Our aim is to determine whether this modification can be carried over to the Lorentzian contribution.

We begin with the bracket of two modified Euclidean parts, $\{H^{\mathrm{E}}[\underset{\sim}{N}], H^{\mathrm{E}}[\underset{\sim}{M}]\}$. Thanks to antisymmetry of the brackets in $\underset{\sim}{N}$ and $\underset{\sim}{M}$, we need consider only those brackets of terms that lead to derivatives of delta functions. There are two such contributions,

$$\begin{aligned} &\{2A_1(x)E^2(x)(A_2(x)E^2(x)+A_3(x)E^3(x)), \\ &\quad 2E^1(y)(E^2(y)A_3(y)' - E^3(y)A_2(y)')\} \\ &\quad = (\cdots)\delta(x,y) - 4\gamma G A_1(x)E^1(x)E^1(y)(A_3(x)E^2(y) \\ &\quad - A_2(x)E^3(y))\partial_y\delta(x,y) \end{aligned} \quad (56)$$

and

$$\begin{aligned} &\{2E^1(x)(E^2(x)A_3(x)' - E^3(x)A_2(x)'), \\ &\quad 2E^1(y)(E^2(y)A_3(y)' - E^3(y)A_2(y)')\} \\ &\quad = (\cdots)\delta(x,y) - 4\gamma G E^1(x)E^1(y)((E^2(x)A_2(y)' \\ &\quad + E^3(x)A_3(y)')\partial_x\delta(x,y) \\ &\quad - (E^2(y)A_2(x)' + E^3(y)A_3(x)')\partial_y\delta(x,y)). \end{aligned} \quad (57)$$

With these two ingredients, we obtain

$$\begin{aligned} \{H^{\mathrm{E}}[\underset{\sim}{N}], H^{\mathrm{E}}[\underset{\sim}{M}]\} &= \frac{\gamma}{G}\int \mathrm{d}x(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')(E^1)^2 \\ &\quad \times (A_1(A_2E^3 - A_3E^2) + E^2A_2' + E^3A_3') \\ &= \gamma^2 V[(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{M}'\underset{\sim}{N})], \end{aligned} \quad (58)$$

where

$$V[\Lambda] = \frac{1}{\gamma G}\int \mathrm{d}x\Lambda(A_1(E^2A_3 - E^3A_2) + A_3'E^3 + A_2'E^2) \quad (59)$$

is the vector constraint (6), $V[\Lambda] = D[\Lambda] + G[A_1\Lambda]$, related to the diffeomorphism constraint D through a contribution from the Gauss constraint (47).

Using $\sqrt{\det q} = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}$ from (52), we can write the smearing function in (58) as

$$(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{M}'\underset{\sim}{N}) = \frac{|E^1|}{(E^2)^2+(E^3)^2}(N'M - M'N), \quad (60)$$

where $N = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}\underset{\sim}{N}$ and $M = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}\underset{\sim}{M}$ are lapse functions without density weight. The coefficient $|E^1|/((E^2)^2 + (E^3)^2)$ in (60) is, according to (52), the radial component of the inverse spatial metric, in agreement with the classical form of hypersurface-deformation brackets. The system is therefore anomaly-free for any modification f in (49) without any modification of the constraint brackets and the spacetime structure—provided the Lorentzian part does not contribute to the Hamiltonian constraint, that is in Euclidean gravity with $\gamma = \pm 1$ or in Lorentzian gravity with $\gamma = \pm i$. This is consistent with the results reported in [21].

It is easy to see that any function $f(A_2^2+A_3^2-1)$ can be used in the modified Euclidean part because this term does not produce derivatives of delta functions in the Poisson bracket of two Euclidean constraints. Moreover, because A_2 and A_3 are scalars without density weight, any such term has the correct Poisson bracket with the diffeomorphism constraint. However, if $\gamma^2\neq\epsilon$, the cross-term $\{H^{\rm E}[\underset{\sim}{N}],H^{\rm L}[\underset{\sim}{M}]\}$ in the Poisson bracket of two Hamiltonian constraints does receive a contribution from $f(A_2^2+A_3^2-1)$ in $H^{\rm E}[\underset{\sim}{N}]$ because $H^{\rm L}[\underset{\sim}{M}]$, written in the canonical variables A_i and E^i, contains spatial derivatives of E^i through Γ_i. An explicit calculation is therefore required to check whether the bracket can still be closed for $f(A_2^2+A_3^2-1)\neq A_2^2+A_3^2-1$.

We first compute the Poisson brackets of each individual term in $H^{\rm E}[\underset{\sim}{N}]$ with the full $H^{\rm L}[\underset{\sim}{M}]$: We obtain

$$
\begin{aligned}
&\frac{1}{G}\left\{\int {\rm d}xN(x)A_1(x)E^1(x)(A_2(x)E^2(x)+A_3(x)E^3(x)),H^{\rm L}[\underset{\sim}{M}]\right\}\\
&\quad=\frac{\gamma^2-\epsilon}{2\gamma^2G^2}\int {\rm d}x{\rm d}yN(x)M(y)((\cdots)\delta(x,y)\\
&\qquad-2A_1(x)E^1(x)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\{A_2(x)E^2(x)+A_3(x)E^3(x),\Gamma_1(y)\}\\
&\qquad+E^1(x)(A_2(x)E^2(x)+A_3(x)E^3(x))(E^2(y)^2+E^3(y)^2)\{A_1(x),-2(A_2(y)\Gamma_2(y)+A_3(y)\Gamma_3(y))+\Gamma_2(y)^2+\Gamma_3(y)^2\})\\
&\quad=\frac{\gamma^2-\epsilon}{2\gamma G}\int {\rm d}x{\rm d}yN(x)M(y)\bigg(-2A_1(x)E^1(x)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\frac{E^2(x)E^3(y)-E^2(y)E^3(x)}{E^2(y)^2+E^3(y)^2}\\
&\qquad+2E^1(x)(E^2(y)^2+E^3(y)^2)(A_2(x)E^2(x)+A_3(x)E^3(x))\\
&\qquad\times\frac{A_2(y)E^3(y)-A_3(y)E^2(y)-E^3(y)\Gamma_2(y)+E^2(y)\Gamma_3(y)}{E^2(y)^2+E^3(y)^2}\bigg)\partial_y\delta(x,y)\\
&\quad=-\frac{\gamma^2-\epsilon}{2\gamma G}\int {\rm d}xN(x)M'(x)E^1(A_2E^2+A_3E^3)((E^1)'+2A_2E^3-2A_3E^2)\\
&\quad=-(\gamma^2-\epsilon)G[NM'E^1(A_2E^2+A_3E^3)]
\end{aligned}\tag{61}
$$

up to terms that cancel out when inserted in the antisymmetric $\{H^{\rm E}[\underset{\sim}{N}],H^{\rm L}[\underset{\sim}{M}]\}+\{H^{\rm L}[\underset{\sim}{N}],H^{\rm E}[\underset{\sim}{M}]\}$. In the detailed calculations, we have used the explicit expressions for the Γ_i, from which we also obtain the useful identity

$$
\gamma(K_2E^2+K_3E^3)=A_2E^2+A_3E^3\tag{62}
$$

because $\Gamma_2E^2+\Gamma_3E^3$ is identically zero.

The second term,

$$
\begin{aligned}
&\frac{1}{2G}\left\{\int dxN(x)f(A_2(x)^2+A_3(x)^2-1)(E^2(x)^2+E^3(x)^2),H^{\rm L}[\underset{\sim}{M}]\right\}\\
&\quad=\frac{\gamma^2-\epsilon}{2\gamma^2G^2}\int {\rm d}x{\rm d}yN(x)M(y)\bigg((\cdots)\delta(x,y)\\
&\qquad-2\dot f(x)(E^2(x)^2+E^3(x)^2)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\{A_2(x)^2+A_3(x)^2,\Gamma_1(y)\}\bigg)\\
&\quad=\frac{\gamma^2-\epsilon}{2\gamma G}\int {\rm d}x{\rm d}yN(x)M(y)\bigg((\cdots)\delta(x,y)-2\dot f(x)(E^2(x)^2+E^3(x)^2)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\\
&\qquad\times\frac{2A_2(x)E^3(y)-A_3(x)E^2(y)}{E^2(y)^2+E^3(y)^2}\partial_y\delta(x,y)\bigg)\\
&\quad=2(\gamma^2-\epsilon)G[NM'\dot fE^1(A_2E^2+A_3E^3)]-\frac{\gamma^2-\epsilon}{2\gamma G}\int {\rm d}xNM'\dot fE^1(E^1)'(A_2E^2+A_3E^3),
\end{aligned}\tag{63}
$$

does not vanish on the constraint surface. Therefore, the function f, whose derivative by its argument we have denoted by $\dot{f}$, is now relevant for closed brackets. In particular, the last contribution containing $(E^1)'$ must be canceled by a corresponding term in the remaining bracket.

In this last bracket,

$$\begin{aligned}
B &:= \frac{1}{G}\left\{\int \mathrm{d}x N(x)E^1(x)(E^2(x)A_3(x)' - E^3(x)A_2(x)'), H^{\mathrm{L}}[\underset{\sim}{M}]\right\} \\
&= \frac{\gamma^2-\epsilon}{2\gamma^2 G^2}\int \mathrm{d}x\mathrm{d}y N(x)M(y)((\cdots)\delta(x,y) \\
&\quad + 2E^1(x)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\{E^2(x)A_3(x)' - E^3(x)A_2(x)', -\Gamma_1(y)\} \\
&\quad + 2E^1(x)E^1(y)(A_1(y)-\Gamma_1(y))\{E^2(x)A_3(x)' - E^3(x)A_2(x)', A_2(y)E^2(y)+A_3(y)E^3(y)\} \\
&\quad - 2E^1(x)(E^2(y)^2+E^3(y)^2)((A_2(y)-\Gamma_2(y))\{E^2(x)A_3(x)' - E^3(x)A_2(x)', \Gamma_2(y)\} \\
&\quad + (A_3(y)-\Gamma_3(y))\{E^2(x)A_3(x)' - E^3(x)A_2(x)', \Gamma_3(y)\})) \\
&= \frac{\gamma^2-\epsilon}{2\gamma G}\int \mathrm{d}x\mathrm{d}y N(x)M(y)((\cdots)\delta(x,y) \\
&\quad - 2E^1(x)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\frac{E^2(x)E^2(y)' + E^3(x)E^3(y)'}{E^2(y)^2+E^3(y)^2}\partial_x\delta(x,y) \\
&\quad + 2E^1(x)E^1(y)(A_2(y)E^2(y)+A_3(y)E^3(y))\frac{E^2(x)E^3(y) + E^3(x)E^2(y)}{E^2(y)^2+E^3(y)^2}\partial_x\partial_y\delta(x,y) \\
&\quad + 2(A_1(y)-\Gamma_1(y))E^1(x)E^1(y)(E^2(x)A_3(y) - E^3(x)A_2(y))\partial_x\delta(x,y) \\
&\quad + E^1(x)E^1(y)((A_2(y)-\Gamma_2(y))E^2(x) + (A_3(y)-\Gamma_3(y))E^3(y))\partial_x\delta(x,y)),
\end{aligned} \tag{64}$$

we have a contribution from a second-order derivative of the delta function. Integrating by parts once in this term and taking into account its contributions to NM' and $N'M$, respectively (noting that terms with $N'M'$ cancel out in the final antisymmetric bracket), we write

$$\begin{aligned}
B &= \frac{\gamma^2-\epsilon}{2\gamma G}\int \mathrm{d}x\mathrm{d}y N(x)M(y)\Bigg((\cdots)\delta(x,y) \\
&\quad - 2\frac{E^1(x)E^1(y)}{E^2(y)^2+E^3(y)^2}\Big((E^2(x)E^2(y)' + E^3(x)E^3(y)')(A_2(y)E^2(y)+A_3(y)E^3(y)) \\
&\quad + (E^3(y)E^2(y)' - E^2(y)E^3(y)')(E^2(x)A_3(y) - E^3(x)A_2(y))\Big)\partial_x\delta(x,y) \\
&\quad + E^1(x)E^1(y)(2A_1(y)(E^2(x)A_3(y) - E^3(x)A_2(y)) + E^1(x)E^1(y)'(A_2(y)E^2(x) + A_3(y)E^3(x)))\partial_x\delta(x,y) \\
&\quad - 2E^1(x)E^1(y)(A_2(y)E^2(y)' + A_3(y)E^3(y)' + A_2(y)'E^2(y) + A_3(y)'E^3(y) \\
&\quad - 2(A_2(y)E^2(y)+A_3(y)E^3(y))\frac{E^2(x)E^2(y)' + E^3(x)E^3(y)'}{E^2(y)^2+E^3(y)^2}\Big)\partial_x\delta(x,y)\Bigg) \\
&= \frac{\gamma^2-\epsilon}{2\gamma G}\int \mathrm{d}x\mathrm{d}y N(x)M(y)((\cdots)\delta(x,y) + 2E^1(x)E^1(y) \\
&\quad \times (A_1(y)(E^2(x)A_3(y) - E^3(x)A_2(y)) - (A_2(y)'E^2(y) + A_3(y)'E^3(y)))\partial_x\delta(x,y) \\
&= (\gamma^2-\epsilon)(D[(E^1)^2N'M] + G[A_1(E^1)^2N'M]) - \frac{\gamma^2-\epsilon}{2\gamma G}\int \mathrm{d}x N'ME^1(E^1)'(A_2E^2 + A_3E^3).
\end{aligned} \tag{65}$$

This result provides the diffeomorphism constraint as well as a term which cancels the previous nonconstraint contribution in (63), but only if $\dot{f} = 1$. Therefore, if the Lorentzian contribution is included, no modification of the classical $A_2^2 + A_3^2 - 1$ is allowed. The final bracket now equals

$$
\begin{aligned}
\{H[\underset{\sim}{N}],H[\underset{\sim}{M}]\} &= \{H^{\mathrm{E}}[\underset{\sim}{N}],H^{\mathrm{E}}[\underset{\sim}{M}]\} + \{H^{\mathrm{E}}[\underset{\sim}{N}],H^{\mathrm{L}}[\underset{\sim}{M}]\} - \{H^{\mathrm{E}}[\underset{\sim}{M}],H^{\mathrm{L}}[\underset{\sim}{N}]\} \\
&= \gamma^2 D[(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] + \gamma^2 G[A_1(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] \\
&\quad - (\gamma^2-\epsilon)G[E^1(A_2E^2+A_3E^3)(1-2\dot{f})(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] \\
&\quad - (\gamma^2-\epsilon)(D[(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] + G[A_1(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')]) \\
&= \epsilon(D[(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] + G[A_1(E^1)^2(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')]) \\
&\quad + (\gamma^2-\epsilon)G[E^1(A_2E^2+A_3E^3)(\underset{\sim}{N}'\underset{\sim}{M} - \underset{\sim}{N}\underset{\sim}{M}')] \\
&\approx -\epsilon D[(E^1)^2(\underset{\sim}{N}\underset{\sim}{M}' - \underset{\sim}{N}'\underset{\sim}{M})],
\end{aligned} \tag{66}
$$

using $\dot{f}=1$ in the last step because the bracket would not be closed otherwise. (Note that $\{H^{\mathrm{L}}[\underset{\sim}{N}],H^{\mathrm{L}}[\underset{\sim}{M}]\}=0$, which can most easily be seen if one uses the canonical variables K_i and E^i, of which no spatial derivatives appear in the Lorentzian contribution.)

III. CONNECTION VARIABLES IN A CANONICAL EFFECTIVE FIELD THEORY

We have seen a crucial difference between gravitational theories governed by the Euclidean Hamiltonian constraint H^{E} and the full $H^{\mathrm{E}}+H^{\mathrm{L}}$, respectively. Formally, the reason is the difference in derivative structures implied by the spin-connection terms in H^{L}: While H^{E} contains derivatives only of the spatial connection, H^{L} also contributes spatial derivatives of the triad. As a consequence, the two versions allow different modifications while maintaining closed brackets.

Derivative structures are best dealt with in a setting of effective field theory, in which one formulates generic theories by selecting the basic fields and the maximum order of derivatives to which they contribute, as well as relevant symmetries. For our purposes, we need an adaptation of the usual arguments to a canonical formulation, in which some derivatives may not be explicit because they appear only if some of the canonical equations are used, mainly in the relationship between momenta and "velocities."

In order to determine the correct derivative orders in a canonical theory, we must first choose which of the basic fields should play the role of configuration variables and therefore are considered free of time derivatives. We are looking for a canonical theory of triads, which will correspond to a spacetime metric or triad theory, and therefore choose as our basic fields a densitized spatial triad with momenta. The latter may be given in terms of a connection or extrinsic curvature. The derivative order depends on the quantum effects we wish to include. For now, we will analyze the classical setting and therefore consider up to second-order derivatives of the fields. Symmetries are implemented by the requirement that the constraint brackets be closed, and in the classical case amount to hypersurface-deformation brackets.

A. Basic strategy

In our explicit calculations of generic terms, we again follow the conventions of Sec. II. 2 and set $\gamma=1$ for simplicity. For our effective Hamiltonian, we choose to allow up to second order in derivatives of densitized triads. Since the conjugate momenta are of the form $A\sim\partial E$, using the equations of motion for $\dot{E}$, we have the following general form of the Hamiltonian constraint $H[\underset{\sim}{N}]=(2G)^{-1}\int\mathrm{d}x\underset{\sim}{N}(x)\mathcal{H}(x)$ with

$$
\begin{aligned}
\mathcal{H} = {}& \alpha^i(E^j,\partial E^j)A_i + \beta^{ij}(E^k)A_{ij} + \gamma^i(E)\partial A_i \\
& + Q(E,\partial E,\partial^2 E),
\end{aligned} \tag{67}
$$

where we have introduced the notation $\partial\equiv\partial/\partial x$, $A_{ij\cdots k}=A_iA_j\cdots A_k$, and $E^{ij\cdots k}=E^iE^j\cdots E^k$. We can already observe some preliminary restrictions on the coefficients $\alpha^i(E,\partial E)$ and $Q(E,\partial E,\partial E\partial E,\partial^2E)$. Both coefficients are initially allowed to depend on ∂E^i and ∂^2E^i. But since we only allow up to second-order derivatives in the Hamiltonian constraint, the dependence cannot be arbitrary. Specifically, we have

$$
\begin{cases}
\alpha^i = \bar{\alpha}^i(E) + \alpha^i_j(E)\partial E^j, \\
Q = \bar{Q}(E) + a_i(E)\partial E^i + b_{ij}(E)\partial E^i\partial E^j + c_i(E)\partial^2E^i.
\end{cases}
$$

We want the Hamiltonian density $\mathcal{H}$ to respect the classical symmetries,

$$
\begin{cases}
\{\mathcal{H}(x),\mathcal{G}(y)\}=0, \\
\{\mathcal{H}(x),\mathcal{D}(y)\}=2G(\partial\mathcal{H}(x)\delta_{xy}+2\mathcal{H}(x)\delta'_{xy}), \\
\{\mathcal{H}(x),\mathcal{H}(y)\}\approx -2G(\partial(E^{11}\mathcal{D}(x))\delta_{xy}+2E^{11}\mathcal{D}(x)\delta'_{xy}),
\end{cases} \tag{68}
$$

where $G[\Lambda]=(2G)^{-1}\int \mathrm{d}x\Lambda(x)\mathcal{G}(x)$ and $D[N]=(2G)^{-1}\int \mathrm{d}xN(x)\mathcal{D}(x)$ are the diffeomorphism and Gauss constraints, respectively. We have introduced the shorthand notation $\delta'_{xy} \coloneqq \partial_x\delta(x-y)$, and $\approx$ means "equal" when setting $\mathcal{G}=0$ in the final step of the calculation. These symmetries will impose restrictions on the coefficients $\alpha_i, \beta^{ij}, \gamma^i, Q$ in (67), telling us what a generic Hamiltonian constraint looks like.

B. Brackets

The first bracket, $\{\mathcal{H},\mathcal{G}\}$, represents the restriction to gauge-invariant terms for any allowed $\mathcal{H}$. Inserting (67), we have

$$\begin{aligned}\{\mathcal{H}(x),\mathcal{G}(y)\} = 2G\int \mathrm{d}z[&(\alpha^1+2\beta^{1j}A_j)\delta_{xz}+\gamma^1\delta'_{xz}](x)\delta'_{yz}+[(\alpha^2+2\beta^{2j}A_j)\delta_{xz}+\gamma^2\delta'_{xz}](x)(-A_3(y)\delta_{yz})\\ &-[(\delta_{xz}\partial_2+\delta'_{xz}\partial_{2'})(\alpha^i)A_i+(\delta_{xz}\partial_2+\delta'_{xz}\partial_{2'}+\delta''_{xz}\partial_{2''})Q+\delta_{xz}\partial_2\beta^{ij}A_{ij}+\delta_{xz}\partial_2\gamma^i\partial A_i](x)E^3(y)\delta_{yz}\\ &+[(\alpha^3+2\beta^{3j}A_j)\delta_{xz}+\gamma^3\delta'_{xz}](x)(A_2(y)\delta_{yz})-[(\delta_{xz}\partial_3+\delta'_{xz}\partial_{3'})(\alpha^i)A_i+(\delta_{xz}\partial_3+\delta'_{xz}\partial_{3'}+\delta''_{xz}\partial_{3''})Q\\ &+\delta_{xz}\partial_3\beta^{ij}A_{ij}+\delta_{xz}\partial_3\gamma^i\partial A_i](x)(-E^2(y)\delta_{yz})=0,\end{aligned}$$

where we have introduced further shorthand notation $\partial_i \coloneqq \partial/\partial E^i$ and $\partial_{i'} \coloneqq \partial/\partial(\partial_x E^i)$. To make the right-hand side of the equation vanish, we need several cancellations. We can do this by first making all functions depend on x using delta functions and integrating over z. Then we group terms with the same dependence on A_i and derivatives of δ_{xy} together and demand that each grouping vanish by itself. [Different orders of derivatives on δ may be dependent, for instance in $\delta'_{yx}A(x)=A(y)\delta'_{yx}+\partial_y A(y)\delta_{yx}$. Therefore, some δ' can produce terms that group with a δ.] This procedure produces several dozens of partial differential equations which we will list later along with those from the $\{\mathcal{H},\mathcal{D}\}$ bracket.

Inserting our form of $\mathcal{H}$ into the $\mathcal{H}-\mathcal{D}$ bracket, we obtain

$$\begin{aligned}\{\mathcal{H}(x),\mathcal{D}(y)\} = 2G\int \mathrm{d}z[&\delta_{xz}(\alpha^1+2\beta^{1j}A_j)+\gamma^1\delta'_{xz}](x)(-A_1(y)\delta'_{yz})\\ &-[(\delta_{xz}\partial_1+\delta'_{xz}\partial_{1'})(\alpha^i)A_i+\delta_{xz}\partial_1\beta^{ij}A_{ij}+\delta_{xz}\partial_1\gamma^i\partial A_i+(\delta_{xz}\partial_1+\delta'_{xz}\partial_{1'}+\delta''_{xz}\partial_{1''})(Q)](x)(-\partial E^1(y)\delta_{yz})\\ &+[\delta_{xz}(\alpha^2+2\beta^{2j}A_j)+\gamma^2\delta'_{xz}](x)(\partial A_2(y)\delta_{yz})\\ &-[(\delta_{xz}\partial_2+\delta'_{xz}\partial_{2'})(\alpha^i)A_i+\delta_{xz}\partial_2\beta^{ij}A_{ij}+\delta_{xz}\partial_2\gamma^i\partial A_i+(\delta_{xz}\partial_2+\delta'_{xz}\partial_{2'}+\delta''_{xz}\partial_{2''})(Q)](x)(E^2(y)\delta'_{yz})\\ &+[\delta_{xz}(\alpha^3+2\beta^{3j}A_j)+\gamma^3\delta'_{xz}](x)(\partial A_3(y)\delta_{yz})\\ &-[(\delta_{xz}\partial_3+\delta'_{xz}\partial_{3'})(\alpha^i)A_i+\delta_{xz}\partial_3\beta^{ij}A_{ij}+\delta_{xz}\partial_3\gamma^i\partial A_i+(\delta_{xz}\partial_3+\delta'_{xz}\partial_{3'}+\delta''_{xz}\partial_{3''})(Q)](x)(E^3(y)\delta'_{yz})\\ =2G(&\partial_x\mathcal{H}(x)\delta_{xy}+2\mathcal{H}(x)\delta'_{xy}).\end{aligned}$$

Similar to how we dealt with the condition of gauge invariance, we first integrate over z to make all functions depend on x, and then match term by term with the right-hand side, expanded in A_i and derivatives of δ_{xy}. Again, we obtain a few dozen partial differential equations.

We next list the partial differential equations that the coefficients of terms in $\mathcal{H}$ have to obey. These equations will completely determine the dependence on E^2 and E^3, leaving free functions of E^1 which the $\mathcal{H}-\mathcal{H}$ bracket will further restrict. These conditions then determine possible modifications of the classical $\mathcal{H}_{\mathrm{cl}}$. In the following equations, we use the differential operators $\hat{D} \coloneqq E^2\partial_2+E^3\partial_3$ and $\hat{C} \coloneqq E^2\partial_3-E^3\partial_2$.

1. The $\mathcal{H}-\mathcal{G}$ bracket

For β^{ij} and γ^i we have

$$\begin{cases}\hat{C}\beta^{11}=0\\ \hat{C}\beta^{12}=-\beta^{13}\\ \hat{C}\beta^{13}=\beta^{12}\end{cases} \begin{cases}\hat{C}\beta^{22}=-2\beta^{23}\\ \hat{C}\beta^{33}=2\beta^{23}\\ \hat{C}\beta^{23}=\beta^{22}-\beta^{33}\end{cases} \begin{cases}\hat{C}\gamma^1=0\\ \hat{C}\gamma^2=-\gamma^3\\ \hat{C}\gamma^3=\gamma^2.\end{cases} \tag{69}$$

For α^i we have

$$\begin{cases}\hat{C}\bar{\alpha}^1=0\\ \hat{C}\bar{\alpha}^2=-\bar{\alpha}^3\\ \hat{C}\bar{\alpha}^3=\bar{\alpha}^2\end{cases}\begin{cases}\hat{C}\alpha_1^1=0\\ \hat{C}\alpha_1^2=-\alpha_1^3\\ \hat{C}\alpha_1^3=\alpha_1^2\end{cases}\begin{cases}\hat{C}\alpha_2^1=-\alpha_3^1\\ \hat{C}\alpha_2^2=-\alpha_2^3-\alpha_3^2\\ \hat{C}\alpha_2^3=\alpha_2^2-\alpha_3^3\end{cases}\begin{cases}\hat{C}\alpha_3^1=\alpha_2^1\\ \hat{C}\alpha_3^2=\alpha_2^2-\alpha_3^3\\ \hat{C}\alpha_3^3=\alpha_2^3+\alpha_3^2.\end{cases} \tag{70}$$

For Q we have

$$\hat{C}\,\bar{Q}=0, \tag{71}$$

$$\begin{cases}\hat{C}a_1=0\\ \hat{C}a_2=-a_3\\ \hat{C}a_3=a_2\end{cases}\begin{cases}\hat{C}b_{11}=0\\ \hat{C}b_{12}=-b_{13}\\ \hat{C}b_{13}=b_{12}\end{cases}\begin{cases}\hat{C}b_{22}=-2b_{32}\\ \hat{C}b_{33}=2b_{32}\\ \hat{C}b_{23}=b_{22}-b_{33}\end{cases}\begin{cases}\hat{C}c_1=0\\ \hat{C}c_2=-c_3\\ \hat{C}c_3=c_2.\end{cases} \tag{72}$$

The remaining equations mix different coefficients:

$$\begin{cases}E^2a_3-E^3a_2=\bar{\alpha}^1\\ (-\alpha_j^1+2E^2b_{3j}-2E^3b_{2j})\partial E^j=-2(\partial E^2c_3-\partial E^3c_2)\\ E^2c_3-E^3c_2=\gamma^1\end{cases}\begin{cases}E^2\alpha_3^1-E^3\alpha_2^1=2\beta^{11}\\ E^2\alpha_3^2-E^3\alpha_2^2=2\beta^{12}-\gamma^3\\ E^2\alpha_3^3-E^3\alpha_2^3=2\beta^{13}+\gamma^2.\end{cases} \tag{73}$$

2. The $\mathcal{H}-\mathcal{D}$ bracket

For β^{ij} and γ^i we have

$$\begin{cases}\hat{D}\beta^{11}=0\\ \hat{D}\beta^{12}=\beta^{12}\\ \hat{D}\beta^{13}=\beta^{13}\end{cases}\begin{cases}\hat{D}\beta^{22}=2\beta^{22}\\ \hat{D}\beta^{33}=2\beta^{33}\\ \hat{D}\beta^{23}=2\beta^{23}\end{cases}\begin{cases}\hat{D}\gamma^1=0\\ \hat{D}\gamma^2=\gamma^2\\ \hat{D}\gamma^3=\gamma^3.\end{cases} \tag{74}$$

For α^i we have

$$\begin{cases}\hat{D}\bar{\alpha}^1=\bar{\alpha}^1\\ \hat{D}\bar{\alpha}^2=2\bar{\alpha}^2\\ \hat{D}\bar{\alpha}^3=2\bar{\alpha}^3\end{cases}\begin{cases}\hat{D}\alpha_1^1=0\\ \hat{D}\alpha_1^2=\alpha_1^2\\ \hat{D}\alpha_1^3=\alpha_1^3\end{cases}\begin{cases}\hat{D}\alpha_2^1=-\alpha_2^1\\ \hat{D}\alpha_2^2=0\\ \hat{D}\alpha_2^3=0\end{cases}\begin{cases}\hat{D}\alpha_3^1=-\alpha_3^1\\ \hat{D}\alpha_3^2=0\\ \hat{D}\alpha_3^3=0\end{cases}$$
$$\begin{cases}E^2\alpha_2^2+E^2\alpha_3^2=0\\ E^2\alpha_2^3+E^3\alpha_3^3=0.\end{cases} \tag{75}$$

For Q we have

$$\begin{cases}\hat{D}\,\bar{Q}=2\bar{Q}\\ E^2c_2+E^3c_3=0\\ E^2a_2+E^3a_3=0\end{cases}\begin{cases}c_1+2(b_{12}E^2+b_{13}E^3)=0\\ 3c_2+2(b_{22}E^2+b_{23}E^3)=0\\ 3c_3+2(b_{32}E^2+b_{33}E^3)=0,\end{cases} \tag{76}$$

$$\begin{cases}\hat{D}c_1=0\\ \hat{D}c_2=-c_2\\ \hat{D}c_3=-c_3\end{cases}\begin{cases}\hat{D}a_1=a_1\\ \hat{D}a_2=0\\ \hat{D}a_3=0\end{cases}\begin{cases}\hat{D}b_{11}=0\\ \hat{D}b_{12}=-b_{12}\\ \hat{D}b_{13}=-b_{13}\end{cases}\begin{cases}\hat{D}b_{22}=-2b_{22}\\ \hat{D}b_{33}=-2b_{33}\\ \hat{D}b_{23}=-2b_{23}.\end{cases} \tag{77}$$

One equation mixes different coefficients:

$$E^2\alpha_2^1+E^3\alpha_3^1=-\gamma^1. \tag{78}$$

3. The $\mathcal{H}-\mathcal{H}$ bracket

Matching term by term for $\mathcal{H}-\mathcal{H}$ is quite tedious, mainly because the classical bracket $\{\mathcal{H},\mathcal{H}\}$ is fully determined only after setting $\mathcal{G}=0$. For example, if there is a term $f(\alpha,\beta,\gamma,Q)\partial E^1$ on the left-hand side of $\{\mathcal{H}(x),\mathcal{H}(y)\}\approx-2G(E^{11}\partial_x\mathcal{D}(x)\delta_{xy}+2E^{11}\mathcal{D}(x)\delta'_{xy})$ which is not on the right-hand side, do we demand $f(\alpha,\beta,\gamma,Q)=0$ or do we demand $f(\alpha,\beta,\gamma,Q)\propto\mathcal{G}$ or $\partial\mathcal{G}$, or does $f(\alpha,\beta,\gamma,Q)\partial E^1$ combine with possible $f(\alpha,\beta,\alpha,Q)\times(-E^2A_3+E^3E_2)$ terms to become something proportional to $\mathcal{G}$? There are about 10^2 terms on the left-hand side of the $\mathcal{H}$-$\mathcal{H}$ bracket, each of which has several possibilities of respecting the symmetry (in the form of second-order polynomial equations of α, β, γ, Q). It is therefore necessary to check whether these $(10^2)^n, n\sim10^0$ possibilities are consistent with one another, rendering our current strategy impractical. Luckily, we can use an alternative strategy to find a subset of the most generic Hamiltonian by adding "semisymmetric Gaussian" terms to the classical Hamiltonian constraint.

C. Real vs self-dual variables

We define a *semisymmetric* term to be any term in a generic Hamiltonian constraint that is allowed by the $\{H,D\}$ and $\{H,G\}$ brackets. These terms are solutions to our previous partial differential equations (69)–(78). We define a *Gaussian* term to be any term that is a polynomial of $\mathcal{G}$ and $\partial^n\mathcal{G}$, with coefficients denoted collectively as $C(E)$, which may depend on densitized triads and its derivatives. Namely, for a semisymmetric Gaussian term $g(x) \coloneqq g[\mathcal{G}(x), \partial^n\mathcal{G}(x), C(E(x))]$ we demand

$$\begin{cases} g(x), \mathcal{G}(y)\} = 0, \\ \{g(x), \mathcal{D}(y)\} = 2G(\partial g(x)\delta_{xy} + 2g(x)\delta'_{xy}). \end{cases} \tag{79}$$

Any semisymmetric Gaussian term, $g[\mathcal{G}, \partial^n\mathcal{G}, C(E)]$, that we add to the classical Hamiltonian constraint $\mathcal{H}_{cl}$ is guaranteed to respect all our symmetries as shown below.

Suppose we add one semisymmetric Gaussian term $g[\mathcal{G}, \partial^n\mathcal{G}, C(E)]$ to the classical Hamiltonian constraint $\mathcal{H}_{\rm cl}$

$$H[\underset{\sim}{N}] = \frac{1}{2G}\int \mathrm{d}x \underset{\sim}{N}(x)(\mathcal{H}_{\rm cl} + g). \tag{80}$$

Since $\mathcal{H}_{\rm cl}$ respects all symmetries by definition and g is built out of semisymmetric Gaussian terms,

$$\{H[\underset{\sim}{N}], G[M]\} = 0 \tag{81}$$

is trivial. Similarly, the $H - D$ bracket is satisfied:

$$\begin{aligned} \{H[\underset{\sim}{N}], D[M]\} &= \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y \underset{\sim}{N}(x)M(y)(\{\mathcal{H}_{\rm cl}, \mathcal{D}\} + \{g, \mathcal{D}\}) \\ &= \frac{1}{2G}\int \mathrm{d}x\mathrm{d}y \underset{\sim}{N}(x)M(y)(\partial_x\mathcal{H}_{\rm cl}(x)\delta_{xy} + 2\mathcal{H}_{\rm cl}(x)\delta'_{xy} + \partial_x g(x)\delta_{xy} + 2g(x)\delta'_{xy}) \\ &= \frac{1}{2G}\int \mathrm{d}x\mathrm{d}y \underset{\sim}{N}(x)M(y)(\partial_x\mathcal{H}(x)\delta_{xy} + 2\mathcal{H}(x)\delta'_{xy}) = -H[M\underset{\sim}{N}' - M'\underset{\sim}{N}] \end{aligned} \tag{82}$$

because g is built out of semisymmetric Gaussian terms. The $H[\underset{\sim}{N}] - H[\underset{\sim}{M}]$ bracket then has additional terms compared with the classical case, given by $\{\mathcal{H}_{\rm cl}, g\}$ and $\{g, g\}$. Both terms are of the form $\{f, g\}$ with some semisymmetric f and share the property that $\int \mathrm{d}x\mathrm{d}y N(x)M(y)\{f(x), g(y)\}$ vanishes when $\mathcal{G} = 0$: In

$$\begin{aligned} &\int \mathrm{d}x\mathrm{d}y N(x)M(y)\{f(x), g[\mathcal{G}(y), \partial^n\mathcal{G}(y), C(E)]\} \\ &= \int \mathrm{d}x\mathrm{d}y N(x)M(y)\left(\{f(x), \mathcal{G}(y)\}\frac{\partial g}{\partial\mathcal{G}}(y) + \{f(x), \partial_y^n\mathcal{G}(y)\}\frac{\partial g}{\partial(\partial_y^n\mathcal{G})}(y) + \{f(x), C(E)\}\frac{\partial g}{\partial C(E)}\right) \\ &= \int \mathrm{d}x\mathrm{d}y N(x)M(y)\left(\{f(x), \mathcal{G}(y)\}\frac{\partial g}{\partial\mathcal{G}}(y) + \{f(x), C(E)\}\frac{\partial g}{\partial C(E)}\right) \\ &\quad + \int \mathrm{d}x\mathrm{d}y N(x)(-\partial_y)^n\left(M(y)\frac{\partial g}{\partial(\partial_y^n\mathcal{G})}(y)\right)\{f(x), \mathcal{G}(y)\}, \end{aligned} \tag{83}$$

the first and last terms vanish because f is semisymmetric, while $\partial g/\partial C(E) \approx 0$ because $C(E)$, by definition, represents coefficients in g of the Gauss constraint or its spatial derivatives.

With this result, we confirm that

$$\begin{aligned} \{H[\underset{\sim}{N}], H[\underset{\sim}{M}]\} = \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y \underset{\sim}{N}(x)\underset{\sim}{M}(y)(&\{\mathcal{H}_{\rm cl}(x), \mathcal{H}_{\rm cl}(y)\} + \{g[\mathcal{G}(x), \partial^n\mathcal{G}(x), C(E)], g[\mathcal{G}(y), \partial^n\mathcal{G}(y), C(E)]\} \\ &+ \{\mathcal{H}_{\rm cl}(x), g[\mathcal{G}(y), \partial^n\mathcal{G}(y), C(E)]\} + \{g[\mathcal{G}(x), \partial^n\mathcal{G}(x), C(E)], \mathcal{H}_{\rm cl}(y)\}) \\ \approx \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y \underset{\sim}{N}(x)\underset{\sim}{M}(y)&\{\mathcal{H}_{\rm cl}(x), \mathcal{H}_{\rm cl}(y)\} \end{aligned} \tag{84}$$

obeys the classical brackets for any semisymmetric g. Thus, semisymmetric Gaussian terms indeed preserve all symmetries.

When written in real variables, the classical Hamiltonian constraint contains a term with the second-order derivative of $E^1 \sim E^x$, given by $2\partial\Gamma_\phi E^x = -\partial(\partial E^x/(E^\varphi))E^x$. But when using self-dual variables, there are no second-order derivatives of triads. As already mentioned, this discrepancy is caused by the fact that $\mathcal{G} \approx 0$ is already solved in the real variable case. Indeed, using semisymmetric terms (see Appendix A) for constructing modifications we have the following allowed terms when using self-dual variables:

$$\mathcal{H}_2(A,E) = \mathcal{H}_{\rm cl}(A,E) + c_1(E^1)\left(\partial\mathcal{G} - \frac{1}{2}\frac{\partial((E^\varphi)^2)}{(E^\varphi)^2}\mathcal{G}\right) + \partial E^1[b_{11}(E^1)\partial E^1 + \tilde{C}_{\alpha_1^2}(E^1)(E^3A_2 - E^2A_3)], \tag{85}$$

where $\partial\mathcal{G} \sim \partial^2 E^1$ provides the second-order derivative. Note that the second semisymmetric term (proportional to ∂E^1) becomes a semisymmetric Gaussian term if we pick $b_{11} = \frac{1}{2}\tilde{C}_{\alpha_1^2}$.

Substituting $A_i = \gamma K_i + \Gamma_i$, $c_1 = E^1$, $b_{11} = \frac{1}{2}\tilde{C}_{\alpha_1^2} = 1/2$ in the classical Hamiltonian constraint and de-densitizing, we obtain

$$\mathcal{H}_2(K,E) = |E^x|^{-1/2}\left(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x - \left(1 - \left(\frac{\partial E^x}{2E^\varphi}\right)^2\right)E^\varphi + \frac{E^x\partial^2 E^x}{E^\varphi} - \frac{E^x\partial E^x\partial E^\varphi}{(E^\varphi)^2}\right), \tag{86}$$

where we used the Gauss constraint in real variables. This result matches the standard classical Hamiltonian constraint in real variables. Thus, including semisymmetric Gaussian terms in the quadratic constraint, it is equivalent to the classical one written in real variables.

Revisiting the setting of the previous section, it follows that a further restriction of our $\mathcal{H}$ to be only quadratic in densitized triads implies that all allowed modifications to the classical $\mathcal{H}_{\rm cl}$ are in the form of semisymmetric Gaussian terms:

$$\mathcal{H}_{\rm quad} = C_1(\partial A_3 E^{21} - \partial A_2 E^{31} + A_{12}E^{12} + A_{13}E^{13}) + C_2\left(A_{22} + A_{33} + \frac{C_3}{C_2}\right)(E^{22} + E^{33}) + C_4\partial E^1\mathcal{G} + C_5(A_2E^2 + A_3E^3)\mathcal{G}. \tag{87}$$

The first two terms are present in $\mathcal{H}_{\rm cl}$ while the last two are new semisymmetric Gaussian terms and all C_i are constants. However, the complexity of the general equations makes it difficult to show that all possible modifications to the Hamiltonian constraint up to second order in derivatives can be constructed from semisymmetric Gaussian terms.

IV. ELIMINATING THE GAUSS CONSTRAINT

Our analysis of gravitational theories in a setting of effective field theory has highlighted the role of the Gauss constraint, which implies that the hypersurface-deformation generators are not uniquely defined. Since the Gauss constraint contains a spatial derivative, and spatial derivatives of this constraint can also be added to the hypersurface-deformation generators, the derivative structure and therefore the possibility of modifications is ambiguous as long as the Gauss constraint remains unsolved. We will therefore now solve the Gauss constraint explicitly and analyze the resulting hypersurface-deformation generators and their brackets.

A. Gauge-invariant variables

We begin with the classical constraint

$$H[N] = \frac{1}{2G}\int \mathrm{d}x \frac{N}{\sqrt{E^1((E^2)^2 + (E^3)^2)}}(2E^1(E^2A_3' - E^3A_2') + 2A_1E^1(A_2E^2 + A_3E^3) + (A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2) + (\epsilon - \gamma^2)(2K_1E^1(K_2E^2 + K_3E^3) + (K_2^2 + K_3^2)((E^2)^2 + (E^3)^2)) \tag{88}$$

in which the lapse function no longer has a density weight. The next few transformations closely follow the derivations given in [31], but are presented here in a different form using vector notation.

The pairs (E^2, E^3) and (A_2, A_3) [as well as (K_2, K_3)] transform under the defining representation of SO(2) with respect to the Gauss constraint. It will be convenient to arrange them in 3-vectors, such that

$$\vec{E} = E^2\vec{e}_2 + E^3\vec{e}_3, \tag{89}$$

$$\vec{A} = A_2\vec{e}_2 + A_3\vec{e}_3, \tag{90}$$

$$\vec{K} = K_2\vec{e}_2 + K_3\vec{e}_3 \tag{91}$$

with standard basis vectors $\vec{e}_i$. Obvious invariant variables are therefore

$$E^\varphi = |\vec{E}| = \sqrt{(E^2)^2 + (E^3)^2}, \tag{92}$$

$$A_\varphi = |\vec{A}| = \sqrt{A_2^2 + A_3^2}, \tag{93}$$

$$K_\varphi = |\vec{K}| = \sqrt{K_2^2 + K_3^2}. \tag{94}$$

Moreover, we obtain another invariant α from the angle between $\vec{E}$ and $\vec{A}$,

$$\cos\alpha = \frac{\vec{E}\cdot\vec{A}}{E^\varphi A_\varphi}. \tag{95}$$

While E^1 and K_1 are also invariant, A_1 has a nontrivial transformation. A final gauge-invariant expression can be written as $A_1 + \beta'$, where

$$\cos\beta = \frac{\vec{e}_2\cdot\vec{A}}{A_\varphi}. \tag{96}$$

Using our definitions of α and β, we can write the unit vectors

$$\vec{e}_A = \frac{\vec{A}}{A_\varphi} = \vec{e}_2\cos(\beta) + \vec{e}_3\sin(\beta), \tag{97}$$

$$\vec{e}_E = \frac{\vec{E}}{E^\varphi} = \vec{e}_2\cos(\alpha+\beta) + \vec{e}_3\sin(\alpha+\beta). \tag{98}$$

From the last relation one can derive the spin-connection component $\Gamma_1 = -(\alpha+\beta)'$ [31]. Therefore, $\gamma^{-1}(A_1+\alpha'+\beta') = K_1$ is nothing but an extrinsic-curvature component. Since α and K_1 are gauge invariant, $A_1+\beta'$ must be gauge invariant, as claimed above.

Moreover, computing the extrinsic curvature and spin connection for a spherically symmetric triad [31] shows that the angular part $\vec{K}$ points in the same internal direction as the triad,

$$\vec{e}_K = \vec{e}_E, \tag{99}$$

while the angular part of the spin connection, $\vec{\Gamma}$, is orthogonal,

$$\vec{e}_\Gamma = -\vec{e}_1\times\vec{e}_E, \tag{100}$$

with coefficient

$$\Gamma_\varphi = -\frac{(E^1)'}{2E^\varphi}; \tag{101}$$

see (53). Therefore,

$$A_\varphi^2 = |\vec{A}|^2 = |\Gamma_\varphi\vec{e}_\Gamma + \gamma K_\varphi\vec{e}_K|^2 = \Gamma_\varphi^2 + \gamma^2K_\varphi^2. \tag{102}$$

The term in (88) containing spatial derivatives of the connection can now be written as

$$\begin{aligned} E^2A_3' - E^3A_2' &= \vec{e}_1\cdot(\vec{E}\times\vec{A}') = E^\varphi\vec{e}_1(\vec{e}_E\times(A_\varphi\vec{e}_A)' \\ &= E^\varphi(-A_\varphi'\sin(\alpha) + A_\varphi\beta'\cos(\alpha)). \end{aligned}$$

We then express connection terms through spin connection and extrinsic curvature, using

$$A_\varphi\sin(\alpha) = A_\varphi\vec{e}_A\cdot\vec{e}_\Gamma = \Gamma_\varphi \tag{103}$$

and

$$A_\varphi\cos(\alpha) = A_\varphi\vec{e}_A\cdot\vec{e}_K = \gamma K_\varphi. \tag{104}$$

Therefore,

$$\begin{aligned} E^2A_3' - E^3A_2' &= E^\varphi(-(A_\varphi\sin(\alpha)) + A_\varphi(\alpha'+\beta')\cos(\alpha)) \\ &= E^\varphi(-\Gamma_\varphi' + \gamma K_\varphi(\alpha'+\beta')). \end{aligned} \tag{105}$$

The angles in the last term can be combined with a similar contribution from the second term in (88), which adds A_1 to $\alpha'+\beta'$. [In (88), A_1 is multiplied with $A_2E^2 + A_3E^3 = \vec{A}\cdot\vec{E} = \gamma K_\varphi E^\varphi$, which does not depend on Γ_φ because $\vec{e}_\Gamma\cdot\vec{e}_E = 0$.] Since $\alpha'+\beta' = -\Gamma_1$ [31] and $A_1 - \Gamma_1 = \gamma K_1$, we have

$$E^2A_3' - E^3A_2' + A_1(A_2E^2 + A_3E^3) = E^\varphi(-\Gamma_\varphi' + \gamma^2K_\varphi K_1). \tag{106}$$

Thus, by using variables invariant under transformations generated by the Gauss constraint, we have been led to an expression in which all spatial derivatives of the connection have been replaced by spatial derivatives of the triad (through Γ_φ).

Again in [31], the Poisson brackets

$$\begin{aligned} \{K_\varphi(x), E^\varphi(y)\} &= G\delta(x,y), \\ \{K_1(x), E^1(y)\} &= 2G\delta(x,y) \end{aligned} \tag{107}$$

for the new gauge-invariant variables have been derived. If we express the diffeomorphism and Hamiltonian constraints in these variables, we restrict the previous theory to the solution space of the Gauss constraint. We obtain

$$D[N^x] = \frac{1}{2G}\int \mathrm{d}xN^x(2E^\varphi K_\varphi' - K_1(E^1)') \tag{108}$$

and

$$\begin{aligned} H[N] = \frac{1}{2G}\int \mathrm{d}x\frac{N}{\sqrt{E^1}}(&K_\varphi^2E^\varphi(\epsilon-\gamma^2) + 2\epsilon K_\varphi K_1E^1 \\ &+(\Gamma_\varphi^2 + \gamma^2K_\varphi^2 - 1)E^\varphi - 2E^1\Gamma_\varphi'). \end{aligned} \tag{109}$$

B. Modified constraint with classical brackets

In the Hamiltonian constraint, the two terms with $\gamma^2K_\varphi^2$ cancel out, showing that, for $\epsilon = -1$, we obtain the Hamiltonian constraint as considered in [31]. Our calculation here extends this result to the Euclidean signature, $\epsilon = 1$. Since all γ-dependent terms drop out of the final expression, it is no longer clear why $\gamma^2 = \epsilon$ should lead to different options for modified constraints. Nevertheless, the previous distinction between $\gamma^2 = \epsilon$ and $\gamma^2 \neq \epsilon$ can still be realized if we do not cancel the γ-dependent terms in (109) *before* we try to modify the constraint. In particular, the previous modification, using an arbitrary function of $f(A_2^2 + A_3^2 - 1)$, can still be implemented in the invariant version if we recognize the combination $\Gamma_\varphi^2 + \gamma^2K_\varphi^2 - 1$ as the correct substitute of $A_2^2 + A_3^2 - 1 = A_\varphi^2 - 1$. We therefore consider the modified constraint

$$H[N] = \frac{1}{2G}\int \mathrm{d}x \frac{N}{\sqrt{E^1}}(K_\varphi^2 E^\varphi(\epsilon - \gamma^2) + 2\epsilon K_\varphi K_1 E^1 + f(\Gamma_\varphi^2 + \gamma^2 K_\varphi^2 - 1)E^\varphi - 2E^1\Gamma_\varphi'). \quad (110)$$

Given the form of this new constraint, it is not obvious that it can lead to closed brackets for f not the identity because, compared with our previous derivation, we now have up to second-order spatial derivatives of the triad (through Γ_φ) instead of first-order derivatives of its momenta.

Thanks to antisymmetry of the Poisson bracket, the only terms that give nonzero contributions to $B_{NM} \coloneqq \{H[N], H[M]\}$ are combinations of a term from $H[N]$ depending on one of the K_i and a term from $H[M]$ depending on a (first or second order) spatial derivative of one of the E_i, or vice versa. Therefore,

$$\begin{aligned} B_{NM} = \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y \frac{N(x)M(y)}{\sqrt{E^1(x)E^1(y)}}\bigg(&-(\epsilon-\gamma^2)\{K_\varphi^2(x),(E^\varphi)'\}\frac{E^1(y)E^1(y)'E^\varphi(x)}{(E^\varphi(y))^2} \\ &- 2\epsilon\{K_\varphi(x),E^\varphi(y)'\}K_1(x)\frac{E^1(x)E^1(y)E^1(y)'}{(E^\varphi(y))^2} - 2\epsilon K_\varphi(x)\{K_1(x),E^1(y)'\}\frac{E^1(x)E^1(y)E^\varphi(y)'}{(E^\varphi(y))^2} \\ &- \{f,E^\varphi(y)'\}\frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi(y))^2} + 2\epsilon K_\varphi(x)\{K_1(x),f\}E^1(x)E^\varphi(y) \\ &+ 2\epsilon K_\varphi(x)\frac{E^1(y)}{E^\varphi(y)}\{K_1(x),E^1(y)''\}E^1(x)\bigg) - (N\leftrightarrow M). \end{aligned} \quad (111)$$

Integrating by parts, we obtain

$$\begin{aligned} B_{NM} &= \frac{1}{4G}\int \mathrm{d}x NM'\bigg((2(\epsilon-\gamma^2)K_\varphi\frac{(E^1)'}{E^\varphi} + 2\epsilon\frac{E^1}{(E^\varphi)^2}K_1(E^1)' + 4\epsilon K_\varphi(E^\varphi)'\frac{E^1}{(E^\varphi)^2} \\ &\quad - 4\epsilon\frac{E^1}{(E^\varphi)^2}E^\varphi K_\varphi' - 4\epsilon K_\varphi\frac{E^1(E^\varphi)'}{(E^\varphi)^2} + \frac{\partial f}{\partial K_\varphi}\frac{(E^1)'}{E^\varphi} - 4\epsilon K_\varphi E^\varphi\frac{\partial f}{\partial(E^1)'}) - (N\leftrightarrow M) \\ &= \frac{-\epsilon}{2G}\int \mathrm{d}x\frac{E^1}{(E^\varphi)^2}(NM'-N'M)(2E^\varphi K_\varphi' - K_1(E^1)') \\ &\quad + \frac{1}{4G}\int \mathrm{d}x(NM'-N'M)\bigg(2(\epsilon-\gamma^2)K_\varphi\frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi}\frac{(E^1)'}{E^\varphi} - 4\epsilon K_\varphi E^\varphi\frac{\partial f}{\partial(E^1)'}\bigg) \\ &= -\epsilon D\Big[\frac{E^1}{(E^\varphi)^2}(NM'-N'M)\Big] + \frac{1}{4G}\int \mathrm{d}x(NM'-N'M)\bigg(2(\epsilon-\gamma^2)K_\varphi\frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi}\frac{(E^1)'}{E^\varphi} - 4\epsilon K_\varphi E^\varphi\frac{\partial f}{\partial(E^1)'}\bigg). \end{aligned} \quad (112)$$

For a closed bracket, therefore,

$$2(\epsilon-\gamma^2)K_\varphi\frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi}\frac{(E^1)'}{E^\varphi} - 4\epsilon K_\varphi E^\varphi\frac{\partial f}{\partial(E^1)'} = 0. \quad (113)$$

Since f depends on K_φ and $(E^1)'$ only through $\frac{1}{4}(E^{1\prime})^2/(E^\varphi)^2 + \gamma^2 K_\varphi^2 - 1$, the chain rule implies that

$$\frac{\partial f}{\partial K_\varphi} = 2\gamma^2 K_\varphi \dot{f} \quad \text{and} \quad \frac{\partial f}{\partial (E^1)'} = \frac{1}{2(E^\varphi)^2}(E^1)'\dot{f}, \quad (114)$$

and (113) is equivalent to

$$2(\epsilon-\gamma^2)K_\varphi\frac{(E^1)'}{E^\varphi}(1-\dot{f}) = 0. \quad (115)$$

If $\gamma^2 = \epsilon$, the equation holds identically for any f. If $\gamma^2 \neq \epsilon$, however, $\dot{f} = 1$, and only the classical case is allowed. The modification found in [21] can therefore be found also in gauge-invariant variables, in which case the Hamiltonian constraint contains second-order derivatives of the triad, with the same restriction that it is allowed only for a specific value of γ.

C. Modified brackets

A generic modification which does not require a specific value of γ can be obtained for the theories considered here, as has been known for some time for real variables [1,5]. Since the Hamiltonian constraint in real variables has the same form as the general spherically symmetric constraint in gauge-invariant variables, the same modification can be transferred also to self-dual type variables ($\gamma^2 = \epsilon$) provided we implement it at the gauge-invariant level. At the level of variables that are not gauge invariant, this new modification (compared with [21]) is possible provided we use the Gauss constraint to reintroduce second-order derivatives of triads in the Hamiltonian constraint.

Starting with (109), the new modification is derived in a way very similar to the case of real variables, found in [1]. Nevertheless, we reproduce the calculation of brackets here for the sake of completeness. We modify (109) to

$$H[N] = \frac{1}{2G}\int \mathrm{d}x N(x)(E^1)^{-1/2}\Bigg(\epsilon f_1(K_\varphi)E^\varphi + 2\epsilon f_2(K_\varphi)E^1K_1 + \left(\frac{(E^{1\prime})^2}{4(E^\varphi)^2} - 1\right)E^\varphi + \frac{E^1(E^1)''}{E^\varphi} - \frac{E^1(E^1)'(E^\varphi)'}{(E^\varphi)^2}\Bigg) \tag{116}$$

with two functions, f_1 and f_2, that will be restricted further by the condition of having closed brackets. We first interpret this modification based on arguments within canonical effective field theory. We are now allowing for a nonquadratic dependence of the Hamiltonian constraint on K_φ. If K_φ is still a first-order time derivative, a nonquadratic dependence would be nongeneric unless we also allow for higher-order spatial derivatives of the densitized triad, which we do not do in (116).

However, modifying the Hamiltonian constraint in this form also modifies the equations of motion that classically imply the first-order nature of K_φ. An analysis of these modified equations of motion should then be performed in order to reveal the derivative order of the Hamiltonian constraint. Schematically, we obtain the modified derivative dependence of K_φ from the equation of motion

$$\dot{E}^1 = 2N\sqrt{E^1}f_2(K_\varphi) + N^1(E^1)', \tag{117}$$

$$\dot{E}^\varphi = N\sqrt{E^1}K_1\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi} + \frac{NE^\varphi}{2\sqrt{E^1}}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi} + (N^1E^\varphi)', \tag{118}$$

provided we can invert the function f_2. This can explicitly be done only in examples, which we restrict here to the common case of $f_1(K_\varphi)=\sin^2(K_\varphi)$, which implies $f_2(K_\varphi)=\sin(K_\varphi)\cos(K_\varphi)$ or $f_2(K_\varphi)^2=f_1(K_\varphi)(1-f_1(K_\varphi))$. The latter equation can be solved for

$$f_1(K_\varphi) = \frac{1}{2}\left(1 \pm \sqrt{1-4f_2(K_\varphi)^2}\right) = f_2(K_\varphi)^2 + f_2(K_\varphi)^4 + \cdots. \tag{119}$$

According to (117), $f_2(K_\varphi)$ is strictly of first order in derivatives, but $f_1(K_\varphi)$ is not polynomial in $f_2(K_\varphi)$, and therefore a derivative expansion of $f_1(K_\varphi)$ does not terminate. Similarly,

$$\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi} = \cos(2K_\varphi) = 1 - 2f_1(K_\varphi) = \sqrt{1-f_2(K_\varphi)^2} \tag{120}$$

has a derivative expansion that does not terminate. Therefore, K_1 has a nonterminating derivative expansion because $K_1\sqrt{1-f_2(K_\varphi)^2}$ must be of first order according to (117).

We conclude that the constraint (116) contains a derivative expansion in both space and time derivatives, which can consistently be truncated at any finite derivative order. The resulting effective theory is therefore meaningful, but it may not be the most general one because the derivative expansion results only from the K-dependent terms in (116) while we have not included higher-derivative corrections of the E-dependent terms. (Higher spatial derivatives may be expected from an expansion of nonlocal holonomies used in the Hamiltonian constraints for models of loop quantum gravity; see for instance [32,33]. However, it is difficult to find consistent constraint brackets with such modifications [5].) The mismatch does not violate (deformed) covariance because the constraint brackets still close. However, unless the symmetries implied by the closed constraints select only this specific derivative structure, the modified theory is not generic. (It resembles Born-Infeld type theories.) Since no other consistent modifications are known as of now, it remains unclear whether the apparently nongeneric model is selected by symmetries.

In order to confirm that the constraint brackets can be closed, we compute

$$\begin{aligned}\{H[N],H[M]\} = \frac{1}{4G^2}\int \mathrm{d}x\mathrm{d}y\frac{N(x)M(y)}{\sqrt{E^1(x)E^1(y)}}\Bigg(&-\epsilon\frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)}\{f_1(K_\varphi(x)),E^\varphi(y)'\}\\ &-2\epsilon\frac{E^1(x)E^1(y)E^1(y)'K_1(x)}{(E^\varphi)^2(y)}\{f_2(K_\varphi(x),E^\varphi(y)'\} + \epsilon\frac{f_2(K_\varphi(x))E^1(x)}{2E^\varphi(y)}\{K_1(x),(E^1(y)')^2\}\\ &+2\epsilon f_2(K_\varphi(x))E^1(x)\frac{E^1(y)}{E^\varphi(y)}\{K_1(x),E^1(y)''\} - 2\epsilon f_2(K_\varphi(x))E^1(x)\frac{E^1(y)E^\varphi(y)'}{E^\varphi(y)^2}\{K_1(x),E^1(y)'\}\Bigg)\\ &-(N\leftrightarrow M),\end{aligned} \tag{121}$$

writing only terms that produce nonzero contributions. All terms are multiplied with ϵ, and therefore the possibility of modifications does not depend on the spacetime signature.

The first two lines contain Poisson brackets of $f_1(K_\varphi)$ and $f_2(K_\varphi)$ and therefore lead to derivatives of the modification functions:

$$\frac{1}{G}\frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)}\{f_1(K_\varphi(x)),E^\varphi(y)'\} = \frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi}\partial_y\delta(x,y) \tag{122}$$

and

$$\frac{2}{G}\frac{E^1(x)E^1(y)E^1(y)'K_1(x)}{(E^\varphi)^2(y)}\{f_2(K_\varphi(x)),E^\varphi(y)'\} = 2\frac{E^1(x)E^1(y)E^1(y)'K_1(x)}{(E^\varphi)^2(y)}\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi}\partial_y\delta(x,y). \tag{123}$$

Another derivative of $f_2(K_\varphi)$ results from the second-order derivative of the delta function obtained after evaluating $\{K_1,(E^1)''\}$ in the fourth line of (121). This contribution follows from

$$2f_2(K_\varphi(x))\frac{E^1(x)E^1(y)}{E^\varphi(y)}\{K_1(x),E^1(y)''\} = 4f_2(K_\varphi(x))\frac{E^1(x)E^1(y)}{E^\varphi(y)}\partial_y^2\delta(x,y). \tag{124}$$

Upon integrating by parts twice in the resulting expression in (121), we initially produce a term with $N(x)M(y)''$ times a delta function without derivatives. Integrating over y, the delta function is eliminated, and we can integrate by parts once again to obtain a term with $N'M'$ (which cancels out in the antisymmetric bracket) and a term with NM' times the derivative of the entire coefficient in (124):

$$-4\left(f_2(K_\varphi)\frac{(E^1)^2}{E^\varphi}\right)' = -4\left(\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi}K_\varphi'\frac{(E^1)^2}{E^\varphi}+f_2(K_\varphi)\left(2\frac{E^1(E^1)'}{E^\varphi}-\frac{(E^1)^2(E^\varphi)'}{(E^\varphi)^2}\right)\right). \tag{125}$$

The last term [containing $(E^\varphi)'$] cancels out with the fifth line of (121), while only half the second term cancels out with the third line of (121), for any f_2. In order for the remaining terms to be proportional to the diffeomorphism constraint, only expressions proportional to K_1 or K_φ' can remain. Therefore, the other half of the second term in (125) must cancel out with (122), which requires

$$f_2(K_\varphi) = \frac{1}{2}\frac{\mathrm{d}f_1(K_\varphi)}{\mathrm{d}K_\varphi}. \tag{126}$$

Only two terms are then left, (123) and the first contribution in (125). They are both proportional to $\mathrm{d}f_2(K_\varphi)/\mathrm{d}K_\varphi$ and combine to form the diffeomorphism constraint

$$\begin{aligned}\{H[N],H[M]\} &= -\frac{\epsilon}{2G}\int\mathrm{d}xN'M\frac{E^1}{(E^\varphi)^2}\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi}(2E^\varphi K_\varphi'-K_1(E^1)')-(N\leftrightarrow M)\\ &= -\epsilon D\left[\frac{\mathrm{d}f_2(K_\varphi)}{\mathrm{d}K_\varphi}\frac{E^1}{(E^\varphi)^2}(NM'-N'M)\right].\end{aligned} \tag{127}$$

This modification, following [1,5], differs from the modification of [21] in that it modifies not only the constraints but also their brackets (while the latter remain closed). It therefore implies a new, nonclassical spacetime structure [12,13]. This modification is consistent for all γ and is therefore generic. From this perspective, the modification of [21], which preserves the brackets, requires $\gamma^2=\epsilon$ and is not generic; it does not provide a way to avoid nonclassical spacetime structures without fine-tuning. Our derivations have shown that the different outcomes of [21] versus [1,5] are not a consequence of working with self-dual connections (used in [21]) or real variables (used in [1,5]). The crucial difference is that modified constraints with unmodified brackets, as in [21], can be obtained only for specific γ, while modifications of constraints as well as brackets exist for all γ.

V. CONCLUSION

We have shown that deformations of the classical spacetime structure appear generically in spherically symmetric models of loop quantum gravity. For self-dual variables or Euclidean gravity with $\gamma=\pm1$, we have derived the most general form of the quadratic Hamiltonian constraint free of triad derivatives, such that a system with unmodified closed brackets is obtained. This rigidity result, just as the setting of [21] which it generalizes, relies on the absence of derivative terms of the triad. However, from the point of view of an effective field theory, this result is not generic because it depends on a restriction of derivative terms even within the classical structure of second-order derivatives. Moreover, this rigidity result can be obtained only for specific values of the Barbero-Immirzi parameter γ.

The results of [21] have sometimes been interpreted as saying that deformations arising in the hypersurface-deformation brackets, obtained originally using holonomy modifications in real-valued variables, might be avoided in the self-dual case. Self-dual variables represent a specific choice for the Immirzi parameter, and therefore do not lead to generic results. These variables (or the values of γ they correspond to) are not distinguished intrinsically by symmetries because constraint brackets, which define the symmetries of a canonical theory, can be closed for any γ.

Moreover, we have shown that the possibility of modifications, even within a self-dual setting, formally depends on the derivative structure which can be changed by adding multiples of the Gauss constraint or its spatial derivatives to the Hamiltonian constraint. This ambiguity can be eliminated by solving the Gauss constraint explicitly, following [31], in which case the same derivative structure is obtained in self-dual type variables and in real variables, which agrees with the form originally used in an analysis of modified brackets [1,5]. We therefore conclude that modified brackets and nonclassical spacetime structures are generic in any spherically symmetric model with holonomy modifications, even for self-dual variables. We also pointed out that currently known modifications may not be generic from the point of view of canonical effective theory introduced here: After translating momenta into time derivatives, different derivative orders appear in the terms of a modified Hamiltonian constraint. This observation suggests that there is room for further explorations of possibly new models. A likely candidate for a generic extension is the inclusion of canonical quantum backreaction effects [34–36], which in an action formulation provide higher-curvature terms with generic higher derivatives. However, quantum backreaction on its own does not modify the hypersurface-deformation brackets of constraints [37] and is therefore unlikely to change our conclusions about modified spacetime structures.

Euclidean and self-dual type variables are special also in an analysis of cosmological perturbations [38,39], in which case nongeneric modifications of constraint brackets have been observed as well. Our results present useful indications for operator calculations [40–46] which have demonstrated the possibility of off-shell closure of commutators of constraint operators, mainly in the Euclidean case. So far, these investigations have not yet given rise to indications that the commutators of constraint operators may be modified, in contrast to effective derivations as well as the operator constructions in [6,47]. (However, it is not always clear how to read off modifications of structure functions in the operator setting, which should be some function of a spatial metric or densitized triad and therefore requires a suitable notion of states of a semiclassical geometry which does not yet exist in the operator formulation.) Our results show that the Euclidean setting is, in fact, inconclusive as regards modifications of structure functions because it is a nongeneric case that allows closed brackets with and without modifications. Current effective and operator treatments are therefore consistent with one another. For a complete picture of spacetime structures in loop quantum gravity, it will be important to extend off-shell operator calculations to the full Lorentzian constraint.

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APPENDIX A: RESTRICTIONS ON COEFFICIENTS OF SEMISYMMETRIC GAUSSIAN TERMS

We list the solutions to partial differential equations resulting from the $\mathcal{H}$-$\mathcal{G}$ and $\mathcal{H}$-$\mathcal{D}$ brackets. These will give us the so-called *semisymmetric Gaussian* terms. Denoting $(E^\varphi)^2 = E^{22} + E^{33}$, for β^{ij} we have

$$\begin{cases} \beta^{11} = \beta^{11}(E^1) \\ \beta^{12} = E^3\tilde{C}_\beta(E^1) + E^2\bar{C}_\beta(E^1)\,, \\ \beta^{13} = E^3\bar{C}_\beta(E^1) - E^2\tilde{C}_\beta(E^1) \end{cases}$$

$$\begin{cases} \beta^{22} = 1/2[-8\bar{C}_{\beta^{23}}(E^1)E^{23} + (C_\Sigma(E^1) + \tilde{C}_{\beta^{23}}(E^1))E^{22} + (C_\Sigma(E^1) - \tilde{C}_{\beta^{23}}(E^1))E^{33}] \\ \beta^{33} = 1/2[8\bar{C}_{\beta^{23}}(E^1)E^{23} + (C_\Sigma(E^1) + \tilde{C}_{\beta^{23}}(E^1))E^{33} + (C_\Sigma(E^1) - \tilde{C}_{\beta^{23}}(E^1))E^{22}] \\ \beta^{23} = \tilde{C}_{\beta^{23}}(E^1)E^{23} + 2(E^{22} - E^{33})\bar{C}_{\beta^{23}}(E^1). \end{cases}$$

For γ^i we have

$$\begin{cases} \gamma^1 = \gamma^1(E^1) \\ \gamma^2 = E^3\tilde{C}_\gamma(E^1) + E^2\bar{C}_\gamma(E^1) \\ \gamma^3 = E^3\bar{C}_\gamma(E^1) - E^2\tilde{C}_\gamma(E^1) \end{cases}.$$

For α^i we have

$$\begin{cases} \bar{\alpha}^1 = C_{\alpha^1}(E^1)E^\varphi \\ \bar{\alpha}^2 = (\tilde{C}_{\bar{\alpha}}(E^1)E^3 + \bar{C}_{\bar{\alpha}}(E^1)E^2)E^\varphi \\ \bar{\alpha}^3 = (-\tilde{C}_{\bar{\alpha}}(E^1)E^2 + \bar{C}_{\bar{\alpha}}(E^1)E^3)E^\varphi \end{cases},$$

$$\begin{cases} \alpha_1^1 = \alpha_1^1(E^1) \\ \alpha_1^2 = E^3\tilde{C}_{\alpha_1^2}(E^1) + E^2\bar{C}_{\alpha_1^2}(E^1) \\ \alpha_1^3 = E^3\bar{C}_{\alpha_1^2}(E^1) - E^2\tilde{C}_{\alpha_1^2}(E^1) \\ \alpha_2^1 = (E^2\tilde{C}_{\alpha_2^1}(E^1) + E^3\bar{C}_{\alpha_2^1}(E^1))\frac{1}{(E^\varphi)^2} \\ \alpha_3^1 = (-E^2\bar{C}_{\alpha_2^1}(E^1) + E^3\tilde{C}_{\alpha_2^1}(E^1))\frac{1}{(E^\varphi)^2} \end{cases} \begin{cases} \alpha_2^2 = (-\tilde{C}_{\alpha_2^2}(E^1)E^{23} + \bar{C}_{\alpha_2^2}(E^1)E^{33})\frac{1}{(E^\varphi)^2} \\ \alpha_3^3 = (\tilde{C}_{\alpha_2^2}(E^1)E^{23} + \bar{C}_{\alpha_2^2}(E^1)E^{22})\frac{1}{(E^\varphi)^2} \\ \alpha_3^2 = (-\bar{C}_{\alpha_2^2}(E^1)E^{23} + \tilde{C}_{\alpha_2^2}(E^1)E^{22})\frac{1}{(E^\varphi)^2} \\ \alpha_2^3 = (-\bar{C}_{\alpha_2^2}(E^1)E^{23} - \tilde{C}_{\alpha_2^2}(E^1)E^{33})\frac{1}{(E^\varphi)^2} \end{cases}.$$

For Q we have

$$\begin{cases} \bar{Q} = (E^\varphi)^2 C_{\bar{Q}}(E^1) \\ a_1 = E^\varphi C_{a_1}(E^1) \\ a_2 = \frac{E^3}{E^\varphi}C_{a_2}(E^1) \\ a_3 = -\frac{E^2}{E^\varphi}C_{a_2}(E^1) \end{cases} \begin{cases} c_1 = c_1(E^1) \\ c_2 = \frac{E^3}{(E^\varphi)^2}C_k(E^1) \\ c_3 = -\frac{E^2}{(E^\varphi)^2}C_k(E^1) \end{cases} \begin{cases} b_{11} = b_{11}(E^1) \\ b_{12} = (-c_1(E^1)E^2/2 + E^3C_b(E^1))\frac{1}{(E^\varphi)^2} \\ b_{13} = (-c_1(E^1)E^3/2 - E^2C_b(E^1))\frac{1}{(E^\varphi)^2} \end{cases},$$

$$\begin{cases} b_{22} = (E^{33}C_{b_{22}}(E^1) - 3E^{23}C_k(E^1))\frac{1}{(E^\varphi)^4} \\ b_{33} = (E^{22}C_{b_{22}}(E^1) + 3E^{23}C_k(E^1))\frac{1}{(E^\varphi)^4} \\ b_{23} = [\frac{3}{2}C_k(E^1)(E^{22} - E^{33}) - E^{23}C_{b_{22}}(E^1)]\frac{1}{(E^\varphi)^4} \end{cases}.$$

We also have mixing conditions

$$\begin{cases} C_k(E^1) = -\gamma^1(E^1) = \tilde{C}_{\alpha_2^1}(E^1) \\ C_{a_2}(E^1) = -C_{\alpha^1}(E^1) \\ C_b(E^1) = -\frac{1}{2}\alpha_1^1(E^1) \\ C_{b_{22}}(E^1) = -\frac{1}{2}\bar{C}_{\alpha_2^1}(E^1) \end{cases} \begin{cases} \bar{C}_{\alpha_2^1}(E^1) = -2\beta^{11}(E^1) \\ -\bar{C}_{\alpha_2^2}(E^1) = 2\tilde{C}_\beta(E^1) - \bar{C}_\gamma(E^1) \\ \tilde{C}_{\alpha_2^2}(E^1) = 2\bar{C}_\beta(E^1) + \tilde{C}_\gamma(E^1) \end{cases}.$$

APPENDIX B: SOME USEFUL IDENTITIES

In calculating the $\{H[N(x)], H[M(x)]\}$ bracket, we can often make use of antisymmetry and integration by parts to simplify our calculations. Suppose we have only one canonical pair; then typically we have

$$H[N(x)] \sim \int \mathrm{d}x N(x)[\cdots + f(E(x), K(x))n(x) + \cdots], \tag{B1}$$

where $n(x)$ is a function of phase-space variables depending on x. Plugging this form of the Hamiltonian into the Poission bracket we obtain the nontrivial term

$$\{H[N(x)], H[M(x)]\} \ni \int \mathrm{d}x\mathrm{d}y\{N(x)M(y)[n(x)\{f(E(x), K(x)), \partial_y^n E(y)\}m(y)] - (N \leftrightarrow M)\}. \tag{B2}$$

Denote $\dot{f}(x) \equiv \partial f(E(x), K(x))/\partial K(x)$, and $K_{NM}^{(n)}$ for the above integral term [including the $(N \leftrightarrow M)$], then for $n = 1$ we have

$$K_{NM}^{(1)} = -\int \mathrm{d}x[M'(x)N(x) - N'(x)M(x)]n(x)m(x)\dot{f}(x). \tag{B3}$$

For $n = 2$ we have

$$K_{NM}^{(2)} = \int \mathrm{d}x[M'(x)N(x) - N'(x)M(x)][n(x)\dot{f}(x)m'(x) - m(x)(n(x)\dot{f}(x))']. \tag{B4}$$

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