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Abstract. Detailed applications of minisuperspace methods are presented and compared with results obtained in recent years by means of causal dynamical triangulations (CDTs), mainly in the form of effective actions. The analysis sheds light on conceptual questions such as the treatment of time or the role and scaling behavior of statistical and quantum fluctuations. In the case of fluctuations, several analytical and numerical results show agreement between the two approaches and offer possible explanations of effects that have been seen in causal dynamical triangulations but whose origin remained unclear. The new approach followed here suggests "CDT experiments" in the form of new simulations or evaluations motivated by theoretical predictions, testing CDTs as well as the minisuperspace approximation.

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Contents

1 Introduction

Causal Dynamical Triangulations (CDTs) [1, 2] are an attempt to compute gravitational transition amplitudes by utilizing a discretized version of path integrals. The novelty of the approach with respect to Euclidean Dynamical Triangulations (EDTs) [3] is a built-in notion of causality, which guarantees that branching of the path integrals does not occur — the topology of a spatial slice is preserved in time. In EDTs, by contrast, so-called baby universes can form, leading to divergences and incorrect semiclassical behavior. However, while CDTs enforce causality in every discrete time step, the formulation is still Euclidean in the sense of the Wick rotation: they utilize a path integral of $e^{-S/\hbar}$ rather than $e^{iS/\hbar}$. As a consequence, the solutions obtained within CDT represent *instanton* configurations.

The analytic continuation to Euclidean signature is performed for technical reasons, allowing one to transform the complex propagator into a real partition function. This step opens up the possibility of using numerical methods developed to study statistical systems, such as Monte Carlo simulations. In particular, performing a sequence of Markov moves the system under consideration can be equilibrated. The maximal-entropy state corresponds to the instanton trajectory in the path integral formulation and its thermal fluctuations can be associated with quantum fluctuations around the classical path [4].

Over the last two and a half decades the CDT approach has been investigated in 1+1, 2+1 and 3+1 dimensions. The especially interesting case of 3+1 dimensional CDT has been studied by performing advanced computer simulations with hundreds of thousands of simplices. Most of the simulations were performed assuming the spatial topology to be that of the three-sphere, S^3 [5]. However, studies have recently been extended to the case of toroidal topology, $S^1 \times S^1 \times S^1$ [6–9]. In each case, a contribution from the cosmological constant, Λ, is included in the gravitational action.

The first significant conclusion resulting from the simulations is the nontrivial phase structure of the theory. It has been shown that, depending on the type of order parameter used, three or more phases of the gravitational field can be identified. The different phases are separated by transition lines some of which are of the first and others of higher order [10, 11]. In this article, we will focus our attention on the so-called phase C, which shares properties with classical space-time. In particular, it has been shown that the evolution of the averaged 3-volume is, in this phase, consistent with the results for the 4-dimensional de Sitter instanton [5]. Furthermore, an analysis of both fractal dimension and the so-called spectral dimension revealed the correct large scale dimension of the geometry [12]. It has also been noticed that the spectral dimension undergoes *dimensional reduction* at short scales, reflecting quantum properties of the semiclassical space-time. It is worth stressing at this point that an analogous semiclassical space-time is not present in the EDT formulation, unless some non-trivial measure is introduced [3].

CDTs have therefore been successful in producing results for a full 3+1 dimensional approach to quantum gravity. Minisuperspace models, by comparison, are much more restricted and questions remain as to how reliably they capture properties of full quantum gravity. Nevertheless, there is promise in combining these approaches, both to interpret results from CDTs and potentially to use CDTs to test the validity of minisuperspace truncations and approximations. Here, we initiate a detailed study of the first aspect, using derivations of different properties of quantum fluctuations in minisuperspace models to interpret effective actions extracted from CDTs. We begin with a discussion of one major difference between CDTs and the traditional minisuperspace treatment in the way they deal with the problem of time.

2 Implications of gauge-fixed time

One of the characteristic features of CDTs, as they have mainly been used to find effective actions, is the fixing of a time gauge. (A foliation-independent formulation of CDTs also exists [13, 14], on which we will briefly comment at the end of this section.) For technical reasons, one chooses a preferred time coordinate and the implied constant-time foliation in order to be able to define observables, such as the spatial 3-volume, and to extract their time evolution from numerical simulations. The evolution of homogeneous geometries derived from CDTs then also refers to a preferred choice of time, which should be taken into account in a comparison with minisuperspace models.

As usual, fixing the gauge comes at a price. Breaking time reparameterization invariance reduces the number of constraints in the Hamiltonian formulation of general relativity. In homogeneous models, for instance, one loses the Friedmann equation. More generally, if the scalar (Hamiltonian) constraint is no longer imposed, the theory includes an additional physical (scalar) degree of freedom of the gravitational field. Nevertheless, homogeneous geometries extracted from CDTs seem to be in agreement with minisuperspace considerations in general relativity, see for instance ref. [5]. The purpose of this section is to discuss this issue for the example of a compact (Euclidean) universe with positive cosmological constant Λ, which has been studied in detail with CDTs.

2.1 Minisuperspace solutions with time reparameterization invariance

The minisuperspace counterpart of the model with spatial topology S^3 and positive cosmological constant, $\Lambda > 0$, in CDTs is expected to be the de Sitter space-time. In general relativity, the minisuperspace Lagrangian (without gauge-fixing) is

$$L = \frac{3V_0}{8\pi G}\left(-\frac{a\dot{a}^2}{N} + Na - \frac{\Lambda}{3}a^3 N\right), \tag{2.1}$$

where the coordinate volume is finite and equal to $V_0 = 2\pi^2$, N is the lapse function, a is the scale factor and G is the Newton's constant. The associated canonical momenta are

$$p_N := \frac{\partial L}{\partial \dot{N}} = 0, \tag{2.2}$$

$$p_a := \frac{\partial L}{\partial \dot{a}} = -\frac{3V_0}{4\pi G}\cdot\frac{a\dot{a}}{N}. \tag{2.3}$$

The condition $p_N = 0$ is a primary constraint, which tells us that N is a non-dynamical variable, or a Lagrange multiplier. Performing the Legendre transformation, the Hamiltonian of the model is found to be

$$H = \dot{N}p_N + \dot{a}p_a - L = N\left[-\frac{2\pi G}{3V_0}\frac{p_a^2}{a} - \frac{3V_0}{8\pi G}\left(a - \frac{\Lambda}{3}a^3\right)\right]. \tag{2.4}$$

The Poisson bracket

$$\{f,g\} := \frac{\partial f}{\partial N}\frac{\partial g}{\partial p_N} - \frac{\partial f}{\partial p_N}\frac{\partial g}{\partial N} + \frac{\partial f}{\partial a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial a} \tag{2.5}$$

allows us to write Hamilton's equations $\dot{f} = \{f, H\}$ for arbitrary phase-space functions f. Considering the basic variables a and p_a, we obtain:

$$\dot{a} = \frac{\partial H}{\partial p_a} = -N\frac{4\pi G}{3V_0}\frac{p_a}{a}, \tag{2.6}$$

$$\dot{p}_a = -\frac{\partial H}{\partial a} = -N\left[\frac{2\pi G}{3V_0}\frac{p_a^2}{a^2} - \frac{3V_0}{8\pi G}\left(1 - \Lambda a^2\right)\right]. \tag{2.7}$$

Furthermore, we have the secondary constraint $0 = \dot{p}_N = -\partial H/\partial N$, or

$$-\frac{2\pi G}{3V_0}\frac{p_a^2}{a} - \frac{3V_0}{8\pi G}\left(a - \frac{\Lambda}{3}a^3\right) = 0 \tag{2.8}$$

which, together with eq. (2.6), leads to the Friedmann equation

$$\left(\frac{\dot{a}}{Na}\right)^2 = \frac{\Lambda}{3} - \frac{1}{a^2}. \tag{2.9}$$

If we choose the lapse function $N = 1$ (proper time), the general solution to this equation has the well-known hyperbolic form

$$a(t) = a_0 \cosh(t/a_0), \tag{2.10}$$

where $a_0 := \sqrt{3/\Lambda}$.

We obtain the Euclidean version of this solution by performing a Wick rotation $t \mapsto i\tau$. The Lorentzian Friedmann-Robertson-Walker line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-r^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2) \right)$$

of a closed model with $a(t)$ given in (2.10) is then equivalent to the line element

$$\mathrm{d}s^2 = a_0^2 \left(\mathrm{d}\sigma^2 + \cos^2\sigma \left(\mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2) \right) \right) \tag{2.11}$$

of a 4-sphere with radius a_0, where the angle $\sigma := \tau/a_0$ takes values in the range $-\pi/2 \leq \sigma \leq \pi/2$. Written for the case of the volume variable, the solution is

$$V(\tau) = V_0 a^3(\tau) = 2\pi^2 a_0^3 \cos^3(\tau/a_0) , \tag{2.12}$$

where $\tau \in \left[-\frac{\pi}{2} a_0, \frac{\pi}{2} a_0 \right]$. This cosine cubed solution has been well confirmed as an effective behavior, averaging over a large number of simulations performed within CDTs [15, 16]. Although this result is encouraging, it is also puzzling because (2.12) is a consequence of the constraint equation (2.8), which is not present in a gauge-fixed theory such as CDTs. We will try to explain this outcome in the next subsection.

2.2 Minisuperspace solutions with gauge-fixed time

Let us now revisit the derivation of classical minisuperspace equations, but now fixing the time gauge from the very beginning such that $N = 1$. Using this value in the Lagrangian (2.1), the only remaining variable is the scale factor a, with canonical momentum

$$p_a = -\frac{3V_0}{4\pi G} \cdot a\dot{a} \, . \tag{2.13}$$

The resulting gauge-fixed Hamiltonian is

$$H_{\mathrm{GF}} = -\frac{2\pi G}{3V_0} \frac{p_a^2}{a} - \frac{3V_0}{8\pi G} \left(a - \frac{\Lambda}{3} a^3 \right) . \tag{2.14}$$

Consequently, Hamilton's equations for the phase space variables are

$$\dot{a} = \frac{\partial H_{\mathrm{GF}}}{\partial p_a} = -\frac{4\pi G}{3V_0} \frac{p_a}{a} \tag{2.15}$$

$$\dot{p}_a = -\frac{\partial H_{\mathrm{GF}}}{\partial a} = -\frac{2\pi G}{3V_0} \frac{p_a^2}{a^2} + \frac{3V_0}{8\pi G} \left(1 - \Lambda a^2 \right) . \tag{2.16}$$

There is no Friedmann equation now. However, combining eqs. (2.15) and (2.16), we obtain the second-order Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{1}{2} \left(\frac{\dot{a}}{a} \right)^2 - \frac{1}{2a^2} + \frac{\Lambda}{2} \, . \tag{2.17}$$

A similar equation can, of course, be derived also in the case without gauge fixing. However, by imposing the Friedmann equation, one of the two constants of integration in solutions of the Raychaudhuri equation is fixed. If we have only the second-order Raychaudhuri equation (2.17), the general solution has two unknown constants of integration.

Let us try to find solutions to eq. (2.17). For this purpose, it will be useful to introduce the Hubble parameter

$$\mathbb{H} := \frac{\dot{a}}{a}, \tag{2.18}$$

so that eq. (2.17) can be written as

$$\dot{\mathbb{H}} = a\mathbb{H}\frac{\mathrm{d}\mathbb{H}}{\mathrm{d}a} = -\frac{3}{2}\mathbb{H}^2 - \frac{1}{2a^2} + \frac{\Lambda}{2}. \tag{2.19}$$

In order to find the relation $\mathbb{H}(a)$, we note that any solution of the Friedmann equation must also be a solution to the Raychaudhuri equation. Therefore, we may write the expression for $\mathbb{H}^2$ as a combination of the part that is expected from the Friedmann equation and some unknown remainder which we parametrize by a function $f(a)$:

$$\mathbb{H}^2 = \frac{\Lambda}{3} - \frac{1}{a^2} + f(a). \tag{2.20}$$

We emphasize that this parametrization does not restrict the generality of our considerations, solving (2.17). Inserting (2.20) in (2.19), we find that $f(a)$ has to obey

$$\frac{\mathrm{d}f}{\mathrm{d}a} = -3\frac{f}{a}, \tag{2.21}$$

which can directly be integrated to $f(a) = c/a^3$ with a constant of integration c. Therefore, the first integration of the Raychaudhuri equation (2.17) leads to the Friedmann-like equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{1}{a^2} + \frac{c}{a^3} \tag{2.22}$$

with an extra term that has the same form as the matter contribution from dust. The presence of the scalar constraint in general relativity fixes the free constant of integration to be equal to zero, $c = 0$. However, in the gauge-fixed case relevant for CDTs, the value of c remains undetermined. Is there a way to select a particular value of c within CDTs?

In order to answer this question, we should take into account the fact that computer simulations require finiteness. Instanton configurations simulated in CDTs therefore have a finite spatial extension, in addition to a periodic (or finite-range) time variable. These conditions guarantee that the total 4-volume of the instanton is finite[1] and can be discretized with a finite number of 4-simplices. We may then select solutions to the Euclidean version of eq. (2.22) such that the constraint

$$V_4 = 2\pi^2 \int \mathrm{d}\tau a^3(\tau) < \infty \tag{2.23}$$

is satisfied. Taking into account the periodicity of the time variable in CDTs, the condition (2.23) is satisfied when the value of the 3-volume, or equivalently the scale factor a, is

[1] Furthemore, in CDT the 4-volume is kept fixed.

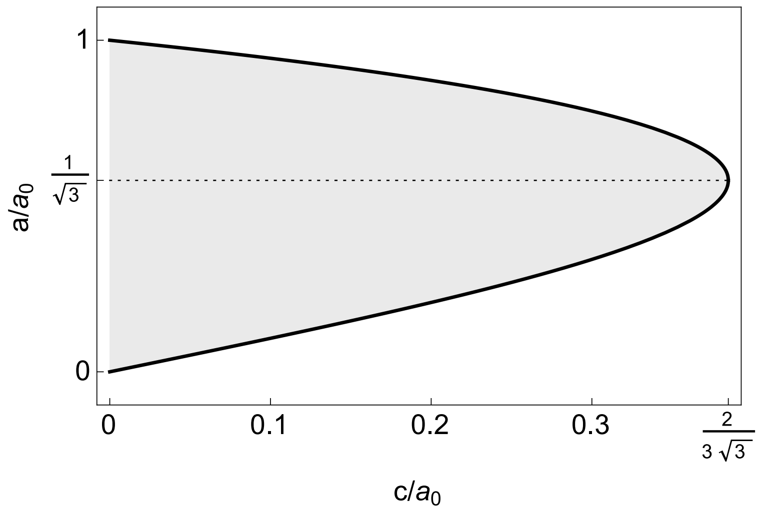

Figure 1. The shadowed region represents allowed values of the scale factor as a function of the integration constant c. For $c = 0$, which overlaps with the case without gauge-fixing, $a \in [0, a_0]$. The range $[a_{\rm min}, a_{\rm max}]$ is decreasing with increasing c, reaching a single point with $a_{\rm min} = a_{\rm max} = a_0/\sqrt{3}$ at $c_{\rm max} := 2a_0/3\sqrt{3}$.

bounded from above. When the maximal value of $a(\tau)$ is reached the Euclidean version of the Hubble parameter, $a^{-1}{\rm d}a/{\rm d}\tau$, as well as the right-hand side of

$$\left(\frac{{\rm d}a}{{\rm d}\tau}\right)^2 = 1 - \frac{\Lambda}{3}a^2 - \frac{c}{a} \tag{2.24}$$

(the Wick rotated version of (2.22)) are equal to zero, leading to the condition

$$\frac{\Lambda}{3}a^3 - a + c = 0\,. \tag{2.25}$$

This equation indicates that the Euclidean version of the Friedmann equation has real solutions for $a \geq 0$ if $c \leq 2/3\sqrt{\Lambda}$. Furthermore, regularity of ${\rm d}a/{\rm d}\tau$ at the boundaries in time is guaranteed if $c \geq 0$. (The term $-c/a$ in eq. (2.24) then tends to $+\infty$ while $a \to 0^+$, a limit which cannot be approached on solutions of eq. (2.24) thanks to positivity of the left-hand side). In summary, well-behaved instanton solutions are obtained for

$$0 \leq c \leq \frac{2}{3}\frac{1}{\sqrt{\Lambda}}\,. \tag{2.26}$$

In this case, the values of the scale factor are confined to the interval $[a_{\rm min}(c), a_{\rm max}(c)]$, as shown in figure 1.

A general solution to eq. (2.24) for $a \geq 0$ is derived in the appendix in terms of Weierstrass's elliptic function, given by eq. (A.12) as a function of a new time variable w related to the time variable τ via

$$\tau = \int_0^w a(w'){\rm d}w'. \tag{2.27}$$

Sample solutions for different values of c are shown in figure 2.

Importantly, while for $c = 0$ the instanton is a single peak, the solutions for $c > 0$ have oscillatory form. However, for a single (central) period there is no qualitative difference

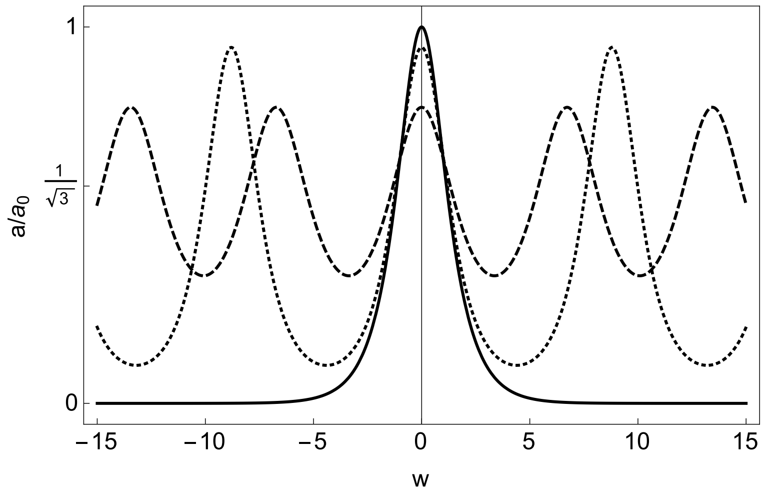

Figure 2. Sample solutions to eq. (2.24) for $c = 0$ (solid line), $c = 0.1a_0$ (dotted line) and $c = 0.2a_0$ (dashed line) as a function of the time parameter w given by eq. (2.27). Here, $w_0 = 0$.

between the different solutions for $c \in \left[0, 2a_0/3\sqrt{3}\right]$. When the value of c is increasing, the amplitude of oscillations is decreasing. For the limiting case $c_{\max} = 2a_0/3\sqrt{3}$ there is only a constant solution $a = a_0/\sqrt{3}$. The minimal value of the scale factor $a_{\min}$ is a monotonic function of c (see the bottom boundary of the shadowed region in figure 1) and tends to zero for $c \to 0$, which corresponds to the solution satisfied by the scalar constraint.

CDT solutions do not exhibit oscillatory behavior of the 3-volume but rather correspond to the case with $c = 0$. One may speculate that this happens for the reason that this solution is selected either by the dynamics of CDTs or by boundary conditions. While passing to Euclidean path integrals and the statistical-physics formulation, only paths that satisfy the condition $\Sigma(\tau_i) = \Sigma(\tau_f)$ are considered, where Σ is a spatial configuration and τ_i and τ_f are initial and final times respectively, so that the time interval is:

$$\tau_f - \tau_i = \frac{\hbar}{k_{\rm B}T} = \text{constant}. \tag{2.28}$$

In CDT simulations, instead of considering the finite time interval $[\tau_i, \tau_f]$ together with the symmetric boundary condition $\Sigma(\tau_i) = \Sigma(\tau_f)$, periodicity of the imaginary time variable τ is introduced. (The time domain becomes S^1.) At the level of the scale factor a (expressed it terms of time w) this implies that regularity conditions for a and its derivatives should be satisfied, which we evaluate here for the first two orders

$$a|_{w_i} = a|_{w_f}, \tag{2.29}$$

$$\left.\frac{{\rm d}a}{{\rm d}w}\right|_{w_i} = \left.\frac{{\rm d}a}{{\rm d}w}\right|_{w_f}, \tag{2.30}$$

where $w_i = w(\tau_i)$ and $w_f = w(\tau_f)$. The first condition can be imposed for all solutions by fixing the value of an integration constant w_0; see the appendix. In figure 2, w_0 is fixed to zero, so that all solutions are even with respect to the point $w = 0$. The condition (2.30) is less trivial to satisfy and requires fixing the second constant of integration, c. Without loss of generality, let us assume now that $w_f = -w_i > 0$. The first observation is that the condition (2.30) cannot be satisfied for small values $w_f \sim 1$, except for the case of a constant solution $a = a_0/\sqrt{3}$ with $c = c_{\max} = 2a_0/3\sqrt{3}$. For $w_f \gg 1$ there is a finite set of values of c for which the boundary conditions (2.29) and (2.30) are satisfied (values of c such that $a(w)$ has local extrema at $\pm w_f$). Furthermore, in general, if one tries different values of w_f, one would have to change the values of c for which the periodic boundary conditions are satisfied. There are two exceptions, the mentioned constant solution for $c = c_{\max}$ and the case with $c = 0$. (In the latter case, (2.30) is not fulfilled exactly but, for large enough w_f, can easily hold within the numerical accuracy of a CDT.) One can summarize this result as follows:

For $w_f \gg 1$ there is only one non-constant solution which satisfies the periodic boundary conditions (2.29) and (2.30) with fixed values of the constants of integration w_0 and c. This solution corresponds to the case with $c = 0$, which is the one satisfying the scalar constraint (Friedmann equation).

This observation may not provide the sole explanation why CDT simulations select the solutions satisfying the scalar constraint. However, the results presented provide an interesting hint as to this enigmatic feature of CDT. (It is interesting to note that CDT simulations in a framework that aims to relax the imposition of a fixed time foliation [13, 14] indicate a behavior with local minima near τ_i and τ_f, closer to one of our background solutions with $c \neq 0$. However, as the authors of [13, 14] point out, differences appear in a region of small volume in which discretization effects may be non-negligible.) Furthermore, in CDT simulations the timespan $\tau_f - \tau_i$ is typically much greater than the characteristic spread scale of the semiclassical solution, which is $\sim \pi a_0$. This property makes the boundary conditions ($a(\tau_i) = 0 = a(\tau_f)$) rather independent from the details of the semiclassical solutions. Therefore, the issue deserves further deepened investigations. Moreover, a comparison with the toroidal model is of interest, in which case we will develop a rather different picture in section 3.2.

3 Fluctuations around the instanton solution

In addition to background solutions, CDTs are able to derive statistical fluctuations around them. These results allow a comparison with quantum minisuperspace models, in which the background volume V can be identified with the expectation value of a volume operator in an evolving state, and statistical fluctuations are replaced by quantum fluctuations. In what follows, the quantum fluctuations are analyzed in the Gaussian approximation. Here, in contrast to the CDT simulations the 4-volume is not kept fixed because imposing such a constraint would make the problem non-linear and much more difficult to handle analytically.[2] However, we will see that in spite of this difference the Gaussian approximation provides a rather good approximation to the CDT results. The validity of the Gaussian approximation in the context of CDT results has also recently been confirmed in ref. [17] for the case of toroidal topology, which will be discussed in more details in section 3.2.

3.1 Spherical model

We will focus on the specific instanton solution associated with the choice $c = 0$ in eq. (2.22). An analytic continuation to complex time, suitable for Euclidean solutions, can be performed in several possible ways. In particular, the two choices $t \to \pm i\tau$ may be considered. The corresponding instanton solution is independent of the choice of the sign: In both cases,

$$\bar{a}(t) = a_0 \cos(\sigma) , \tag{3.1}$$

[2] Numerical studies of the fluctuations imposing the 4-volume constraint have been presented in ref. [1].

where for later convenience we have introduced the parameter

$$\sigma := \tau/a_0 \tag{3.2}$$

such that $\sigma \in [-\pi/2, \pi/2]$.

However, the action $S = \int \mathrm{d}t L$ matters not only for background solutions but also for the derivation of fluctuations. Performing the Wick rotation $t \to \pm i\tau$, the action associated with the gauge-fixed model is

$$S(t = \pm i\tau) = \pm i \frac{3\pi}{4G} \int \mathrm{d}\tau a \left[\left(\frac{\mathrm{d}a}{\mathrm{d}\tau} \right)^2 + 1 - \frac{a^2}{a_0^2} \right] . \tag{3.3}$$

The Euclidean action S_E is then defined such that the Feynman amplitude $e^{iS/\hbar}$ reduces to the Boltzmann weight $e^{-S_\mathrm{E}/\hbar}$, or $S_\mathrm{E} \equiv -iS(t = \pm i\tau)$. The Euclidean action associated with the instanton trajectory (3.1) for $\sigma \in [-\pi/2, \pi/2]$ is then

$$S_\mathrm{E}^\mathrm{C} = \pm \frac{3\pi}{4G} a_0 \int_{-\pi/2}^{\pi/2} \mathrm{d}\sigma \; a \left[\left(\frac{1}{a_0} \frac{\mathrm{d}a}{\mathrm{d}\sigma} \right)^2 + 1 - \left(\frac{a}{a_0} \right)^2 \right] = \pm \frac{\pi}{G} a_0^2 = \pm \frac{3\pi}{\Lambda G} . \tag{3.4}$$

Because the action on half the considered trajectory is associated with the probability of tunneling through the potential barrier from $a = 0$ to $a = a_0$, we can compare (3.4) with the tunneling probability $P \propto e^{-3\pi/(2\hbar\Lambda G)}$ obtained in refs. [18, 19]: $e^{-\frac{1}{2}S_\mathrm{E}^C/\hbar} = e^{\mp 3\pi/(2\hbar\Lambda G)}$. Consistency of the results requires that the Wick rotation $t \to +i\tau$ be chosen.

3.1.1 Fluctuations

We are now ready to consider[3] fluctuations around the instanton trajectory

$$\bar{a}(\sigma) = a_0 \cos\sigma . \tag{3.5}$$

We introduce $y(\sigma)$ such that

$$a(\sigma) = \bar{a}(\sigma) + y(\sigma), \tag{3.6}$$

together with the boundary conditions $y(-\pi/2) = 0 = y(\pi/2)$. Here we are using zero boundary conditions for fluctuations, so that values of the scale factor are fixed at the boundaries. The Dirichlet boundary conditions of this kind are consistent with the scalar constraint (Friedmann equation) and a procedure of derivation of a propagator. However, such choice can be generalized to the case with non-vanishing quantum fluctuations of the scale factor at the boundaries. Furthermore, periodic boundaries with conditions of the type (2.29) and (2.30) can also be considered. Presumably, such choice would be closer to the case of CDTs. Nevertheless, here the standard boundary conditions for fluctuations (which satisfy the scalar constraint) will be considered.

[3] Some of the results presented in this subsection have been discussed by Jakub Wnęk in his Bachelor thesis "Quantum aspects of the de Sitter space" (Jagiellonian University, 2017) who based on calculations and a notebook originally made by his supervisor, one of the authors of this article (JM).

In this case, the Euclidean action can be expanded in y:

$$
\begin{aligned}
S_{\mathrm{E}} &= \frac{3\pi}{4G} a_0^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\sigma \left(\frac{a}{a_0}\right) \left[\left(\frac{1}{a_0}\frac{\mathrm{d}a}{\mathrm{d}\sigma}\right)^2 + 1 - \left(\frac{a}{a_0}\right)^2\right] \\
&= S_{\mathrm{E}}^{\mathrm{C}} - \frac{3\pi}{4G} \int_{-\pi/2}^{\pi/2} \mathrm{d}\sigma \cos^2(\sigma)\; y \left[\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\cos(\sigma)}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right) + \frac{3}{\cos(\sigma)}\right] y \\
&\quad + \frac{3\pi}{4G}\frac{1}{a_0} \int_{-\pi/2}^{\pi/2} \mathrm{d}\sigma\; y \left[\left(\frac{\mathrm{d}y}{\mathrm{d}\sigma}\right)^2 - y^2\right],
\end{aligned}
\tag{3.7}
$$

where $S_{\mathrm{E}}^{\mathrm{C}}$ is the classical action (3.4) with the plus sign. By introducing a new time variable $z = \sin\sigma$, we obtain

$$
S_{\mathrm{E}} = S_{\mathrm{E}}^{\mathrm{C}} + \frac{3\pi}{4G} \int_{-1}^{1} \mathrm{d}z\; y(z) \hat{L} y(z) + \mathcal{O}(y^3) \tag{3.8}
$$

with the differential operator

$$
\hat{L} = (z^2 - 1)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 3\,. \tag{3.9}
$$

We now solve the eigenproblem for the operator $\hat{L}$ defined on $L^2([-1,1], \mathrm{d}\mu)$,

$$
\hat{L}\phi_n = \lambda_n \phi_n \tag{3.10}
$$

together with the boundary conditions $\phi_n(-1) = 0 = \phi_n(1)$. The operator $\hat{L}$ belongs to the class of Sturm-Liouville operators, which guarantees its self-adjointness. The eigenvectors are orthonormal with respect to the measure $\mathrm{d}\mu = w(z)\mathrm{d}z$, where

$$
w(z) = \frac{1}{1 - z^2}\,. \tag{3.11}
$$

The equation $\hat{L}u = \lambda u$ can be written as

$$
(1 - z^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + d\phi = 0, \tag{3.12}
$$

where we defined $d = 3 + \lambda$. By introducing the variable $\xi = \frac{1}{2}(1 + z)$, this equation reduces to a special case of the Heun equation,

$$
\xi(1 - \xi)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + d\phi = 0\,. \tag{3.13}
$$

The general solution to (3.12) can therefore be expressed in terms of hypergeometric functions:

$$
\begin{aligned}
\phi(z) &= {}_2F_1\left(-\frac{1}{4} - \frac{1}{4}\sqrt{1+4d}, -\frac{1}{4} + \frac{1}{4}\sqrt{1+4d}, \frac{1}{2}; z^2\right) c_1 \\
&\quad + {}_2F_1\left(\frac{1}{4} - \frac{1}{4}\sqrt{1+4d}, \frac{1}{4} + \frac{1}{4}\sqrt{1+4d}, \frac{3}{2}; z^2\right) c_2 i z
\end{aligned}
\tag{3.14}
$$

with constants of integration c_1 and c_2. The boundary conditions $\phi(\pm 1) = 0$ lead to

$$
\underbrace{\begin{pmatrix} 2/A & -i/B \\ 2/A & i/B \end{pmatrix}}_{M} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.15}
$$

for

$$A = \Gamma\left[\frac{3}{4} - \frac{1}{4}\sqrt{1+4d}\right]\Gamma\left[\frac{3}{4} + \frac{1}{4}\sqrt{1+4d}\right] \tag{3.16}$$

$$B = \Gamma\left[\frac{5}{4} - \frac{1}{4}\sqrt{1+4d}\right]\Gamma\left[\frac{5}{4} + \frac{1}{4}\sqrt{1+4d}\right]. \tag{3.17}$$

The system of equations (3.15) has a nontrivial solution only if

$$0 = \det M = \frac{4i}{\Gamma\left[\frac{3}{4}-\frac{1}{4}\sqrt{1+4d}\right]\Gamma\left[\frac{3}{4}+\frac{1}{4}\sqrt{1+4d}\right]\Gamma\left[\frac{5}{4}-\frac{1}{4}\sqrt{1+4d}\right]\Gamma\left[\frac{5}{4}+\frac{1}{4}\sqrt{1+4d}\right]}. \tag{3.18}$$

This equation, in turn, is fulfilled if

$$\sqrt{1+4d} = 2n+1 \quad \text{for} \quad n = 1, 2, 3, 4, \dots. \tag{3.19}$$

The eigenvalues of $\hat{L}$ are therefore

$$\lambda_n = -3 + n(n+1) \quad \text{for} \quad n = 1, 2, 3, 4, \dots. \tag{3.20}$$

Interestingly, the first eigenvalue is negative, $\lambda_1 = -1$, while the others are positive. The corresponding eigenfunctions are

$$\phi_n(z) = {}_2F_1\left(-\frac{1}{2}(1+n), \frac{n}{2}, \frac{1}{2}; z^2\right)c_1 + {}_2F_1\left(-\frac{n}{2}, \frac{1}{2}(1+n), \frac{3}{2}; z^2\right)c_2 iz\,. \tag{3.21}$$

From the solutions of (3.15) we obtain $c_1^{(n)} = 0$ for even n and $c_2^{(n)} = 0$ for odd n. This result enables us to classify eigenfunctions according to their parity:

$$\phi^{\rm e}_{n_{\rm e}}(z) = {}_2F_1\left(-\frac{1}{2}(1+n_{\rm e}), \frac{n_{\rm e}}{2}, \frac{1}{2}; z^2\right)c_1 \quad \text{for} \quad n_{\rm e} = 1, 3, 5, 7, \dots, \tag{3.22}$$

$$\phi^{\rm o}_{n_{\rm o}}(z) = {}_2F_1\left(-\frac{n_{\rm o}}{2}, \frac{1}{2}(1+n_{\rm o}), \frac{3}{2}; z^2\right)c_2 iz \quad \text{for} \quad n_{\rm o} = 2, 4, 6, 8, \dots. \tag{3.23}$$

Expressing $n_{\rm e} = 2m+1$ and $n_{\rm o} = 2m$ with $m \in \mathbb{N}$, we have

$$\phi^{\rm e}_m(z) = {}_2F_1\left(-m-1, m+\frac{1}{2}; \frac{1}{2}; z^2\right)c_1 \quad \text{for} \quad m = 0, 1, 2, 3, 4, \dots, \tag{3.24}$$

$$\phi^{\rm o}_m(z) = {}_2F_1\left(-m, m+\frac{1}{2}; \frac{3}{2}; z^2\right)c_2 iz \quad \text{for} \quad m = 1, 2, 3, 4, \dots \tag{3.25}$$

with the corresponding eigenvalues

$$\lambda^{\rm e}_m = -3 + (2m+1)(2m+2) \quad \text{for} \quad m = 0, 1, 2, 3, 4, \dots \tag{3.26}$$

$$\lambda^{\rm o}_m = -3 + 2m(2m+1) \quad \text{for} \quad m = 1, 2, 3, 4, \dots. \tag{3.27}$$

It is possible to express the eigenfunctions in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, employing the relation

$${}_2F_1\left(-n, \alpha+1+\beta+n; \alpha+1; \rho\right) = \frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(1-2\rho) \tag{3.28}$$

where $(\alpha)_n$ is the Pochhammer symbol. Normalization is performed using

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(\rho) P_n^{(\alpha,\beta)}(\rho)(1-\rho)^\alpha(1+\rho)^\beta \mathrm{d}\rho = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \delta_{mn} .$$

The normalized odd eigenfunctions are

$$\phi_m^{\mathrm{o}}(z) = \sqrt{\frac{m(4m+1)}{(2m+1)}}\, z\, P_m^{(\frac{1}{2},-1)}(1-2z^2) \tag{3.29}$$

while the normalized even eigenfunctions are

$$\phi_m^{\mathrm{e}}(z) = \sqrt{\frac{(4m+3)(m+1)}{(2m+1)}}\, P_{m+1}^{(-\frac{1}{2},-1)}(1-2z^2) , \tag{3.30}$$

and they obey the orthonormality relation

$$\int_{-1}^{1} \frac{\phi_n(z)\phi_m(z)}{1-z^2} \mathrm{d}z = \delta_{nm} . \tag{3.31}$$

Using this condition, the normalized version of the eigenfunctions (3.24) and (3.25) can be found:

$$\phi_m^e(z) = \sqrt{\frac{(4m+3)(m+1)}{(2m+1)}}\, P_{m+1}^{(-\frac{1}{2},-1)}(1-2z^2) \quad \text{for} \quad m = 0, 1, 2, 3, 4, \ldots , \tag{3.32}$$

$$\phi_m^o(z) = \sqrt{\frac{m(4m+1)}{(2m+1)}}\, z\, P_m^{(\frac{1}{2},-1)}(1-2z^2) \quad \text{for} \quad m = 1, 2, 3, 4, \ldots . \tag{3.33}$$

There are some important features emerging from the performed analysis. The first is the presence of the negative eigenvalue $\lambda_0^e = -1$ associated with the eigenfunction:

$$\phi_0^e(z) = \frac{\sqrt{3}}{2}(1-z^2) = \frac{\sqrt{3}}{2} \cos^2 s. \tag{3.34}$$

The eigenfunction leads to a negative contribution

$$\frac{3\pi}{4G} \int_{-1}^{1} \mathrm{d}z\ \phi_0^e(z) \hat{L} \phi_0^e(z) = -\frac{3\pi}{4G} \int_{-1}^{1} \mathrm{d}z\ \phi_0^e(z)^2 = -\frac{3\pi}{5G} < 0 \tag{3.35}$$

to the action. Therefore, the second-order variation may lower the value of the action below the value corresponding to the classical trajectory. Furthermore, one can verify that the $\mathcal{O}(y^3)$ term also contributes negatively. This peculiar behavior is a consequence of the negative kinetic term in the gravitational action and may be associated with the presence of the conformal mode [20]. In what follows, the contribution from this eigenvalue will not be taken into account. Secondly, there is no translational mode in the spectrum of the $\hat{L}$ operator, which would be associated with the zero eigenvalue. This is due to the fact that the instanton solution is defined on a finite domain of the imaginary time τ.

3.1.2 Green function

The Euclidean path integral for the model under consideration can be written as

$$Z = \int Dy e^{-S_{\rm E}/\hbar} = e^{-S_{\rm E}^{\rm C}/\hbar} \int Dy \exp\left(-\tfrac{1}{2}\int\limits_{-1}^{1} {\rm d}z\ y(z)\hat{M}y(z) + \mathcal{O}(y^3)\right)\,, \tag{3.36}$$

where we defined

$$\hat{M} = \frac{3\pi}{2G\hbar}\hat{L} = \frac{3\pi}{2l_{\rm Pl}^2}\hat{L}\,, \tag{3.37}$$

so that

$$\hat{M}\phi_n = e_n\phi_n \tag{3.38}$$

with $e_n = \frac{3}{2}\pi l_{\rm Pl}^{-2}\lambda_n$. One can now find the Green function $G(z,z')$ corresponding to the operator $\hat{M}$, which satisfies the relation

$$\hat{M}_z G(z,z') = \delta(z-z')\,. \tag{3.39}$$

The Green function can be expressed in terms of the eigenfunctions and the corresponding eigenvalues using the formula:

$$G(z,z') = \sum_{n=1}^{\infty} \frac{\phi_n(z)\phi_n(z')w(z')}{e_n}\,. \tag{3.40}$$

The contribution from the weight $w(z)$ is due to the fact that the completeness relation is

$$\sum_{n=1}^{\infty} \phi_n(z)\phi_n(z')w(z') = \delta(z-z') \tag{3.41}$$

in the case of Sturm-Liouville operators.

The Green function can be decomposed in odd and even parts according to parity of the eigenvalues:

$$G(z,z') = G^{\rm odd}(z,z') + G^{\rm even}(z,z'), \tag{3.42}$$

$$G^{\rm odd}(z,z') = \frac{2}{3\pi} l_{\rm Pl}^2 \sum_{m=1}^{\infty} \frac{\phi_m^o(z)\phi_m^o(z')w(z')}{-3+2m(2m+1)}\,, \tag{3.43}$$

$$G^{\rm even}(z,z') = \frac{2}{3\pi} l_{\rm Pl}^2 \sum_{m=1}^{\infty} \frac{\phi_m^e(z)\phi_m^e(z')w(z')}{-3+(2m+1)(2m+2)}\,. \tag{3.44}$$

$$\tag{3.45}$$

In figure 3 we show diagonal elements of the Green function.

While it is tempting to relate the Green function with the correlator $\langle y(z)y(z')\rangle$, such identifications cannot be made directly due to the nontrivial measure $w(z)$. Therefore, while the behavior of $\sqrt{G(\sigma,\sigma)}$ shares some qualitative similarity with the quantum fluctuations of the 3-volume measured in CDT (in particular when transformed to volume fluctuations; see section 3.1.3), the two quantities cannot be directly compared.

It is interesting to note that the amplitude of $\sqrt{G(\sigma,\sigma)}$ does not depend on the value of the cosmological constant Λ, and is dimensionally related to the Planck length only. In particular, fluctuations of the *scale factor* are independent of the radius a_0 or the 4-volume V_4 of the spherical universe. In CDTs, by comparison, a non-trivial scaling behavior of *volume* fluctuations with respect to the number N_4 of 4-simplices has been found [15, 16], which implies a non-trivial scaling behavior with respect to V_4.

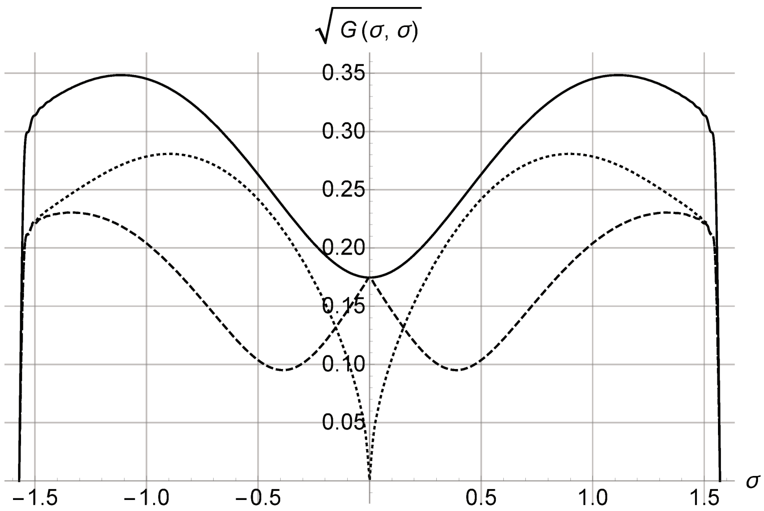

Figure 3. Diagonal elements of the Green function for fluctuations of the scale factor: $\sqrt{G^{\rm odd}(\sigma,\sigma)}$ (dotted line), $\sqrt{G^{\rm even}(\sigma,\sigma)}$ (dashed line) and $\sqrt{G(\sigma,\sigma)}$ (solid line). Contribution from the negative eigenvalue ϕ_0^e was not taken into account in $G^{\rm even}$.

3.1.3 Relations between moments

For a detailed comparison, we should transform the scale-factor fluctuations derived here to volume fluctuations. Because the volume is non-linearly related to the scale factor, relating their fluctuations or moments in an exact form is possible only if we know the quantum state or the statistical ensemble in which they are computed. In a given state, we define moments of an observable O as $\Delta(O^n)=\langle(\Delta\hat{O})^n\rangle$, where $\Delta\hat{O}=\hat{O}-\langle\hat{O}\rangle$. For the observable $V=V_0a^3$, we can use polynomial expansions to obtain

$$\begin{aligned}\frac{\Delta(V^2)}{V_0^2}&=\langle(\Delta\hat{V})^2\rangle=\langle(\hat{a}^3-\langle\hat{a}^3\rangle)^2\rangle\\&=\langle\hat{a}^6\rangle-\langle\hat{a}^3\rangle^2=\langle(\Delta\hat{a}+\langle\hat{a}\rangle)^6\rangle-\langle(\Delta\hat{a}+\langle\hat{a}\rangle)^3\rangle^2\\&=\Delta(a^6)-\Delta(a^3)^2+6\langle\hat{a}\rangle\left(\Delta(a^5)-\Delta(a^3)\Delta(a^2)\right)+3\langle\hat{a}\rangle^2\left(4\Delta(a^4)-3\Delta(a^2)^2\right)\\&\quad+18\langle\hat{a}\rangle^3\Delta(a^3)+9\langle\hat{a}\rangle^4\Delta(a^2)\,.\end{aligned}\tag{3.46}$$

This equation is true in any state, but it requires higher-order moments of a in order to compute fluctuations of V. Since we used a quadratic action for perturbations, it is sufficient to assume a Gaussian state of fluctuations. For a Gaussian wave function, repeated integrations by parts show that

$$\Delta(a^{2n})=(2n-1)\rho^2\Delta(a^{2n-2})=(2n-1)!!\rho^{2n-2}\Delta(a^2)=(2n-1)!!\Delta(a^2)^n\tag{3.47}$$

with the variance $\rho^2=(\Delta a)^2=\Delta(a^2)$. Therefore,

$$\frac{\Delta(V^2)}{V_0^2}=3\left(3\langle\hat{a}\rangle^4+12\langle\hat{a}\rangle^2\Delta(a^2)+5\Delta(a^2)^2\right)\Delta(a^2)\,.\tag{3.48}$$

Applying (3.48) to the fluctuations $\Delta(a^2)=G(\sigma,\sigma)$ given in (3.42), which as noted are independent of Λ or a_0, the leading contribution to volume fluctuations is

$$\frac{\Delta(V^2)}{V_0^2}\approx 9\langle\hat{a}\rangle^4\Delta(a^2)\propto a_0^4\tag{3.49}$$

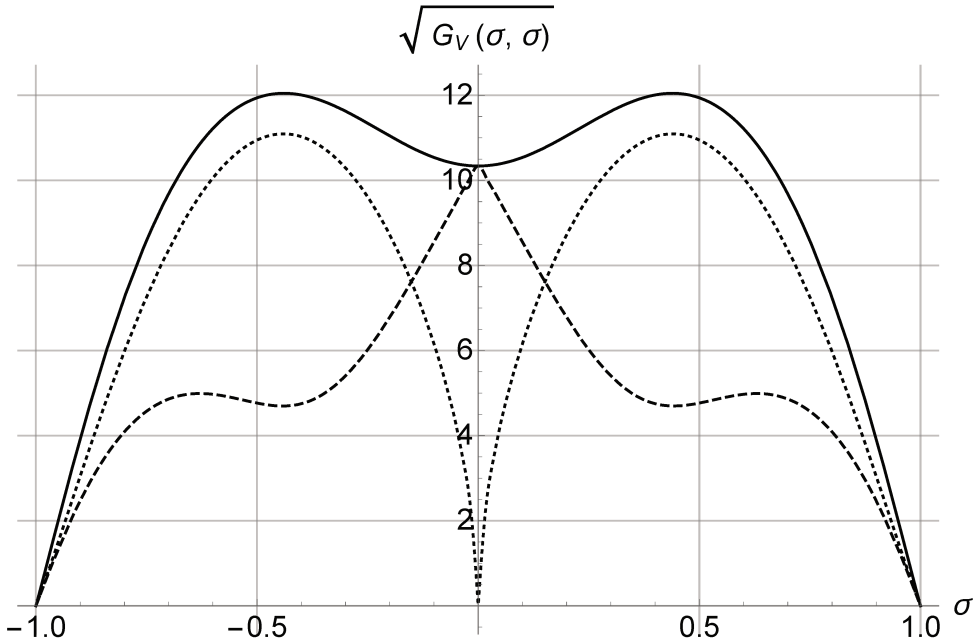

Figure 4. Diagonal elements of the Green function representing volume fluctuations ΔV, using the transformation (3.48) with $V_0 = 2\pi^2$ and $a_0 = 1$. (For this value of a_0, $\sqrt{G_V(\sigma,\sigma)} = \sqrt{\Delta(V^2)/a_0^4}$.) Here, the dotted line is $\sqrt{G_V^{\rm odd}(\sigma,\sigma)}$, the dashed line is $\sqrt{G_V^{\rm even}(\sigma,\sigma)}$ and the solid line is $\sqrt{G_V(\sigma,\sigma)} = \sqrt{G_V^{\rm even}(\sigma,\sigma) + G_V^{\rm odd}(\sigma,\sigma)}$.

if we identify $\langle\hat{a}\rangle$ with the background solution $\bar{a}$ in (3.5). Written as a function of $\sigma = \tau/a_0 \propto \tau/V_4^{1/4}$ in (3.2), $\Delta(V^2)(s)/V_4$ is therefore a universal function independent of the 4-volume V_4 or a_0. This result confirms the scaling behavior found in CDTs [15, 16], where the role of V_4 is played by the total number N_4 of 4-simplices.

In figure 4, we apply the transformation (3.48) to the a-fluctuations shown in figure 3. For comparison, CDT results are usually shown for $\sqrt{\langle(\Delta N_3)^2\rangle/N_4}$ using the correlation function of the number N_3 of 3-simplices as well as the number N_4 of 4-simplices (which is fixed). These variables are related to the 3-volume V and the 4-volume V_4, respectively, by $N_3 = V/C_4$ and $N_4 = V_4/C_4$ with $C_4 = \sqrt{5}/96$ [15]. Moreover, $V_4 = (8/3)\pi^2 a_0^4$. Therefore,

$$\sqrt{\frac{\langle(\Delta N_3)^2\rangle}{N_4}} = \sqrt{\frac{3}{8\pi^2 C_4}}\sqrt{\frac{\Delta(V^2)}{a_0^4}} \approx 1.3\sqrt{\frac{\Delta(V^2)}{a_0^4}}\,, \tag{3.50}$$

which for the maximum in figure 4 is about an order of magnitude greater than the value found in CDTs.

However, compared with CDT results, our fluctuations do not show characteristic "shoulders" where the increase slows down briefly between an endpoint of evolution and the midpoint. (This behavior is closer to the shape shown by the even contribution to the Green function, rather than the full function.) There is therefore an indication that path integral calculations in our minisuperspace model (not imposing the 4-volume constraint) do not capture the correct state selected by CDT simulations. We will now analyze this issue using canonical effective methods, which give us more control on the evolving state.

3.1.4 The spherical minisuperspace model: Canonical effective methods

We will now take the minisuperspace perspective and see what it has to say about fluctuations in the spherical model. For later purposes, it is useful to introduce a general parameterization [21] of canonical variables which goes beyond the examples — a and V — employed so far:

$$Q = \frac{3(\ell_0 a)^{2(1-x)}}{8\pi G(1-x)} \quad \text{and} \quad P = -\ell_0^{2x+1} a^{2x} \dot{a} \tag{3.51}$$

where $\ell_0 = V_0^{1/3}$ with V_0 (for now) fixed to $2\pi^2$. In these canonical variables, the Hamiltonian constraint for Lorentzian signature reads

$$\begin{aligned}\frac{8\pi G}{3} C = &-\left(\frac{8\pi G}{3}(1-x)Q\right)^{(1-4x)/2(1-x)} P^2 - k\ell_0^2 \left(\frac{8\pi G}{3}(1-x)Q\right)^{1/2(1-x)} \\ &+ \frac{1}{3}\Lambda \left(\frac{8\pi G}{3}(1-x)Q\right)^{3/2(1-x)},\end{aligned} \tag{3.52}$$

including a curvature term with $k = 0$ or $k = \pm 1$. We will not always impose the constraint strictly, but rather use $C = -c$ constant, as suitable for path-integral theories such as CDTs in which the lapse function is not varied, as explained in section 2.

As developed in [22, 23] for quantum cosmology, canonical effective methods are based on an extended phase space on which, in addition to Q and P, we consider moments

$$\Delta(Q^a P^b) := \langle (\hat{Q} - \langle \hat{Q} \rangle)^a (\hat{P} - \langle \hat{P} \rangle)^b \rangle_{\text{symm}} \tag{3.53}$$

taking the product in totally symmetric ordering. These moments, together with the expectation values of $\hat{Q}$ and $\hat{P}$, are equipped with a Poisson bracket that is derived from the commutator,

$$\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} = \frac{\langle [\hat{A}, \hat{B}] \rangle}{i\hbar}, \tag{3.54}$$

and extended to all moments using linearity and the Leibniz rule. In particular, for second-order moments we have

$$\{\Delta(Q^2), \Delta(QP)\} = 2\Delta(Q^2) \tag{3.55}$$

$$\{\Delta(Q^2), \Delta(P^2)\} = 4\Delta(QP) \tag{3.56}$$

$$\{\Delta(QP), \Delta(P^2)\} = 2\Delta(P^2), \tag{3.57}$$

while $\{\langle \hat{Q} \rangle, \langle \hat{P} \rangle\} = 1$ and both expectation values have zero Poisson brackets with all moments. For brackets of higher moments, see [22, 24].

Following [25, 26], it is useful to choose Casimir-Darboux coordinates on the phase space of second-order moments, given by the canonical pair

$$s = \sqrt{\Delta(Q^2)}, \qquad p_s = \frac{\Delta(QP)}{\sqrt{\Delta(Q^2)}} \tag{3.58}$$

and the Casimir variable

$$U = \Delta(Q^2)\Delta(P^2) - \Delta(QP)^2 \geq \frac{\hbar^2}{4} \tag{3.59}$$

which is bounded from below by the uncertainty relation. These variables have brackets $\{s, p_s\} = 1$ and $\{s, U\} = 0 = \{p_s, U\}$.

We compute an effective constraint as a function of these canonical variables. To this end, the effective constraint is defined as the expectation value of an operator $\hat{C}$ quantizing C, assuming totally symmetric ordering of all products in the constraint. (If a different ordering is desired, it can always be written as a combination of totally symmetric terms, possibly with an explicit dependence on $\hbar$ from applying commutators.) Expectation values of products of basic operators can then be written as functions of $Q = \langle\hat{Q}\rangle$, $P = \langle\hat{P}\rangle$, s, p_s, and U as follows: We first write the basic operators $\hat{Q}$ and $\hat{P}$ in the form $\hat{A} = A + \Delta\hat{A}$ and then perform a formal Taylor expansion around $A = \langle\hat{A}\rangle$ for small $\Delta\hat{A} = \hat{A} - A$. Applied to the constraint $C(Q, P)$, the expansion yields

$$\begin{aligned}\langle C(\hat{Q},\hat{P})\rangle &= \langle C(Q+\Delta\hat{Q}, P+\Delta\hat{P})\rangle \\ &= C(Q,P) + \sum_{a+b=2}^{\infty} \frac{1}{a!b!}\frac{\partial^{a+b}C(Q,P)}{\partial^a Q\partial^b P}\Delta(Q^aP^b)\,. \end{aligned} \tag{3.60}$$

For the purpose of an approximation to first order in $\hbar$, we then truncate all moment terms to $a + b = 2$, and finally insert the inverse

$$\Delta(Q^2) = s^2\,, \qquad \Delta(QP) = sp_s\,, \qquad \Delta(P^2) = p_s^2 + \frac{U}{s^2} \tag{3.61}$$

of (3.58), (3.59). In its general form, the semiclassical constraint is

$$\begin{aligned}\frac{8\pi G}{3}C_{\rm s} = &-\left(\frac{8\pi G}{3}(1-x)Q\right)^{(1-4x)/2(1-x)}\left(\left(1-\frac{(1-4x)(1+2x)}{8(1-x)^2}\frac{s^2}{Q^2}\right)P^2\right.\\ &\left.\qquad+\frac{1-4x}{1-x}\frac{s}{Q}Pp_s+p_s^2+\frac{U}{s^2}\right)\\ &-k\ell_0^2\left(\frac{8\pi G}{3}(1-x)Q\right)^{1/2(1-x)}\left(1+\frac{2x-1}{8(1-x)^2}\frac{s^2}{Q^2}\right)\\ &+\frac{1}{3}\Lambda\left(\frac{8\pi G}{3}(1-x)Q\right)^{3/2(1-x)}\left(1+\frac{3(1+2x)}{8(1-x)^2}\frac{s^2}{Q^2}\right)\,. \end{aligned} \tag{3.62}$$

It is possible to extend this canonical effective formulation to higher orders of moments [27]. Results obtained up to fourth order [28] suggest the generic behavior $\Delta(Q^{2n}) \sim s^{2n}$ and $\Delta(Q^{2n+1}) = 0$ for a suitable class of states. (There are additional independent degrees of freedom in higher moments, which we do not take into account here.) Using these values, an "all-orders" effective constraint is obtained by replacing any potential-like term $W(Q)$ in the classical constraint by

$$\begin{aligned}\langle W(\hat{Q})\rangle &= W(Q) + \sum_{n=2}^{\infty}\frac{1}{n!}\frac{\partial^n W(Q)}{\partial^n Q}\Delta(Q^n) = W(Q) + \sum_{m=1}^{\infty}\frac{1}{(2m)!}\frac{\partial^{2m}W(Q)}{\partial^{2m}Q}s^{2m} \\ &= \frac{1}{2}\left(W(Q+s)+W(Q-s)\right)\,. \end{aligned} \tag{3.63}$$

These all-orders effective potentials go beyond a finite-order truncation. They have successfully been tested in various tunneling situations [28, 29].

Not all the terms in constraints (3.52) of interest here are potential-like, but expansions similar to (3.63) can be applied to any function of Q and P. The resulting all-orders constraint for (3.52) is given by

$$\begin{aligned}\frac{8\pi G}{3}C_{\rm all}=&-\left(\left(\frac{8\pi G}{3}(1-x)(Q+s)\right)^{(1-4x)/2(1-x)}+\left(\frac{8\pi G}{3}(1-x)(Q-s)\right)^{(1-4x)/2(1-x)}\right)\frac{P^2}{2}\\&-\left(\frac{8\pi G}{3}(1-x)Q\right)^{(1-4x)/2(1-x)}\left(p_s^2+\frac{U}{s^2}+\frac{1-4x}{1-x}\frac{sPp_s}{Q}\right)\\&-\frac{k\ell_0^2}{2}\left(\left(\frac{8\pi G}{3}(1-x)(Q+s)\right)^{1/2(1-x)}+\left(\frac{8\pi G}{3}(1-x)(Q-s)\right)^{1/2(1-x)}\right)\\&+\frac{1}{6}\Lambda\left(\left(\frac{8\pi G}{3}(1-x)(Q+s)\right)^{3/2(1-x)}+\left(\frac{8\pi G}{3}(1-x)(Q-s)\right)^{3/2(1-x)}\right).\end{aligned}\tag{3.64}$$

3.1.5 Minisuperspace fluctuations

We now consider the minisuperspace model of the spherical universe with positive spatial curvature, $k=1$, in which case $Q\propto V$ is convenient ($x=-1/2$). The Wick rotated classical constraint is

$$C=6\pi GVP_V^2+\frac{\Lambda V}{8\pi G}-\frac{3\ell_0^2V^{1/3}}{8\pi G}\tag{3.65}$$

where $V=\ell_0^3a^3$ with $\ell_0=\sqrt[3]{2\pi^2}$, and P_V is the conjugate momentum.

To the first order the semiclassical Hamiltonian is

$$\begin{aligned}C_{\rm s}=&\,6\pi GVP_V^2+\frac{\Lambda V}{8\pi G}-\frac{3\ell_0^2V^{1/3}}{8\pi G}\\&+6\pi GV\Delta(P_V^2)+12\pi G\Delta(VP_V)P_V+\frac{\ell_0^2V^{-5/3}\Delta(V^2)}{24\pi G}.\end{aligned}\tag{3.66}$$

In terms of canonical variables (3.58),

$$\begin{aligned}C_{\rm s}=&\,6\pi GVP_V^2+\frac{\Lambda V}{8\pi G}-\frac{3\ell_0^2V^{1/3}}{8\pi G}\\&+6\pi GV\left(p_s^2+\frac{U}{s^2}\right)+16\pi Gsp_sP_V+\frac{\ell_0^2V^{-5/3}s^2}{24\pi G}.\end{aligned}\tag{3.67}$$

At this order, $\Delta(P_V^2)\Delta(V^2)-\Delta(VP_V)^2=U$ is a constant. For a Gaussian, for instance, we have the minimum value $U=\hbar^2/4$. All four variables, V, P_V, s and p_s, are dynamical. (Since we consider a gauge-fixed treatment, we do not impose effective constraints [30–32] which would otherwise determine a relationship of s and p_s with V and P_V.) Unique evolution requires initial values for all four variables, and therefore partial knowledge about the state through s and p_s. The state derived by CDT simulations is a statistical ensemble, which we can represent by a thermal state with inverse temperature β given by the time period, $\hbar\beta=\pi a_0$. (As shown by (2.11), the Euclidean coordinate τ is related to the 4-sphere angle η by $\tau=a_0\eta$, and η takes values in the range from $-\pi/2$ to $\pi/2$.)

In order to compute fluctuations in such a state, we perturb the Euclidean action

$$S=\frac{3}{8\pi G}\int\left(a\dot{a}^2-\frac{1}{3}\Lambda a^3\right)\mathrm{d}\tau\tag{3.68}$$

by a homogeneous mode, $a = \bar{a} + \tilde{v}$. (We ignore the curvature term in the action because it is not significant near the maximum of $a(\tau)$.) The quadratic perturbation of the action is

$$S_\delta = \frac{3}{8\pi G}\int \left(\dot{\tilde{v}}^2 - \frac{\ddot{\bar{a}}}{\bar{a}}\tilde{v}^2 - \Lambda\tilde{v}^2\right)\bar{a}\,\mathrm{d}\tau \tag{3.69}$$

and implies the Hamiltonian

$$H_\delta = \frac{3}{8\pi G}\left(\left(\frac{4\pi G}{3}\right)^2\frac{\tilde{p}^2}{\bar{a}} + \ddot{\bar{a}}\tilde{v}^2 + \Lambda\bar{a}\tilde{v}^2\right) \tag{3.70}$$

with the momentum

$$\tilde{p} = \frac{3\bar{a}}{4\pi G}\dot{\tilde{v}} \tag{3.71}$$

conjugate to $\tilde{v}$.

To simplify the Hamiltonian we make the canonical transformation

$$v = \sqrt{\frac{3\bar{a}}{4\pi G}}\tilde{v}\,, \qquad p = \sqrt{\frac{4\pi G}{3\bar{a}}}\tilde{p}\,. \tag{3.72}$$

This transformation is generated by the type two generating function,

$$F_2 = \sqrt{\frac{3\bar{a}}{4\pi G}}\bar{v}p\,. \tag{3.73}$$

This implies the transformed Hamiltonian

$$\begin{aligned} K_\delta &= H_\delta + \frac{\partial F_2}{\partial t} \\ &= \frac{1}{2}\left(p^2 + \left(\frac{\ddot{\bar{a}}}{\bar{a}} + \Lambda\right)v^2 + \frac{\dot{\bar{a}}}{\bar{a}}vp\right)\,. \end{aligned} \tag{3.74}$$

Given the background solution $\bar{a}(\tau) = a_0\cos(\tau/a_0)$ (or the Raychaudhuri equation), we have $\ddot{\bar{a}}/\bar{a} = -1/a_0^2 = -\Lambda/3$, and therefore

$$K_\delta = \frac{1}{2}\left(p^2 + \frac{\dot{\bar{a}}}{\bar{a}}vp + \frac{2}{3}\Lambda v^2\right)\,. \tag{3.75}$$

Completing the square we find,

$$K_\delta = \frac{1}{2}\left(\left(p + \frac{\dot{\bar{a}}}{2\bar{a}}v\right)^2 + \left(\frac{2}{3}\Lambda - \frac{\dot{\bar{a}}^2}{4\bar{a}^2}\right)v^2\right)\,. \tag{3.76}$$

We make another time dependent canonical transformation,

$$W = v\,, \qquad p = P - \frac{\dot{\bar{a}}}{2\bar{a}}v\,, \tag{3.77}$$

which is generated by the type two generating function

$$G_2 = vP - \frac{\dot{\bar{a}}}{4\bar{a}}v^2\,. \tag{3.78}$$

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This transformation then implies a Hamiltonian,

$$
\begin{aligned}
M_\delta &= \frac{1}{2}\left(P^2+\left(\frac{2}{3}\Lambda-\frac{\dot{\bar{a}}^2}{4\bar{a}^2}\right)W^2\right)+\frac{\partial G_2}{\partial t}\\
&= \frac{1}{2}\left(P^2+\left(\frac{5}{6}\Lambda+\frac{\dot{\bar{a}}^2}{4\bar{a}^2}\right)W^2\right)\,,
\end{aligned}
\tag{3.79}
$$

equivalent to a harmonic oscillator with a time dependent frequency. On time scales much less than the Hubble time, close to the maximum of $a(\tau)$, the frequency is approximately constant, with

$$
\omega \approx \sqrt{5\Lambda/6}=\frac{\sqrt{5/2}}{a_0}\,. \tag{3.80}
$$

As shown in [28], it is possible to compute statistical quantities in the constant frequency limit by using a semiclassical version of (3.79) in which (W,P) is accompanied by fluctuation variables (s_W, p_{s_W}). In particular, the partition function

$$
\begin{aligned}
\mathcal{Z}(\beta,\omega,\lambda) &= \int_0^\infty\int_{-\infty}^\infty\int_{U_{\rm min}}^\infty \mathrm{d}s_W\,\mathrm{d}p_{s_W}\,\mathrm{d}U\exp\left(-\beta\left(\frac{1}{2}p_{s_W}^2+\lambda\frac{U}{2s_W^2}+\frac{1}{2}\omega^2 s_W^2\right)\right)\\
&= 2\pi\frac{1+\beta\omega\sqrt{U_{\rm min}\lambda}}{\lambda\omega^3\beta^3}\exp\left(-\beta\omega\sqrt{U_{\rm min}\lambda}\right)
\end{aligned}
\tag{3.81}
$$

can be calculated exactly, where $\beta=1/k_{\rm B}T$, $U_{\rm min}=\hbar^2/4$, and λ is a multiplier that allows us to compute the expected uncertainty product

$$
\begin{aligned}
\langle U\rangle &= \frac{2}{\beta^2\omega}\frac{1}{\mathcal{Z}}\left.\frac{\partial^2\mathcal{Z}}{\partial\omega\partial\lambda}\right|_{\lambda=1}\\
&= U_{\rm min}+\frac{6}{\beta^2\omega^2}+\frac{2U_{\rm min}}{1+\sqrt{U_{\rm min}}\beta\omega}\,.
\end{aligned}
\tag{3.82}
$$

Moreover,

$$
\langle s_W^2\rangle = -\frac{1}{\beta\omega}\left.\frac{\partial\log\mathcal{Z}}{\partial\omega}\right|_{\lambda=1}=\frac{3}{\omega^2\beta}+\frac{U_{\rm min}\beta}{1+\sqrt{U_{\rm min}}\omega\beta}\,. \tag{3.83}
$$

We evaluate these expressions by identifying $\hbar\beta$ with the time period of the Euclidean model, $\hbar\beta=\pi a_0$. With $U_{\rm min}=\hbar^2/4$, we obtain

$$
\langle U\rangle = \hbar^2\left(\frac{1}{4}+\frac{12}{5\pi^2}+\frac{1}{2+\sqrt{5/2}\pi}\right)\approx 0.6\hbar^2 \tag{3.84}
$$

and

$$
\langle s_W^2\rangle = \left(\frac{6}{5\pi}+\frac{\pi}{4+\sqrt{10}\pi}\right)\hbar a_0\approx 0.6\hbar a_0\,. \tag{3.85}
$$

Inverting the canonical transformation (3.72), the average v-fluctuations $\langle s_v^2\rangle$ imply fluctuations

$$
\langle\Delta(a^2)\rangle = \frac{4\pi G}{3\bar{a}}\langle s_V^2\rangle\approx\frac{2\pi}{3}\ell_{\rm P}^2 \tag{3.86}
$$

at maximum $\bar{a}(\tau)=a_0$. These fluctuations are independent of a_0, in agreement with the scaling behavior observed in our path-integral derivation. The transformation (3.49) then implies volume fluctuations

$$
\frac{\Delta(V^2)}{V_0^2}=6\pi\ell_{\rm P}^2a_0^4\,. \tag{3.87}
$$

The relationship (3.50) implies that the ratio $\sqrt{\langle\Delta(N_3^2)\rangle/N_4} \approx 5.6$ (in units in which $\ell_{\rm P} = 1$) is closer to the maximum value seen in CDTs [15] than our previous result in figure 4 for a non-thermal state, but still greater by a factor of about two. Nevertheless, it is encouraging to see that our new state may lead to the development of shoulders in a plot of fluctuations: Since these fluctuations were derived at maximum $a(\tau)$, we can use them as initial fluctuations for the volume fluctuation s, in addition to $P_V = 0$ and $p_s = 0$. Figure 5 shows good agreement with the results of our path-integral calculations in section 3.1.2, in particular after applying the transformation (3.49); see figure 4. After setting symmetric initial conditions at $\tau = 0$, the evolution automatically closes the universe in the sense that volume fluctuations approach zero at two opposite points on the time axis. In the effective treatment, this behavior is quite non-trivial because volume fluctuations approaching zero imply diverging momentum fluctuations. It would therefore have been difficult to select a well-defined initial state at one of the two endpoints of evolution. CDTs [15, 16] show fluctuations with a local maximum at the midpoint of evolution, where we set initial conditions, as well as "shoulders" between the local maximum and the two zeros. Although this behavior is not exactly reproduced in quantitative details by our results, figure 5 indicates that there is at least qualitative agreement provided we use the average fluctuations and uncertainty in a thermal state, rather than the minimal values possible in a pure state.

Detailed future studies may reveal additional properties of relevant states. In particular, there is a conceptual difference between the thermal state we have been able to derive, which is thermal at the local maximum of the volume, and a CDT ensemble, whose entire history is in equilibrium with respect to local moves.

3.2 Toroidal model

In models with toroidal topology, a new subtlety arises because CDTs have revealed an additional non-classical term in an effective action that would be masked by the curvature term in a spherical model. The toroidal model also has a more complicated relationship between the scale of fluctuations and the time range. In this subsection, we display path-integral and minisuperspace derivations of correlation functions for actions that include the new term found in CDTs, while the next section will explore possible quantum origins of this term.

3.2.1 Correlation function

We begin with a simple model, defined by isotropy, zero spatial curvature and just a cosmological constant. In terms of the scale factor a, the classical action is given by

$$S[a] = \frac{3V_0}{8\pi G}\int \mathrm{d}\tau N a^3 \left(\left(\frac{\dot{a}}{Na}\right)^2 + \frac{1}{3}\Lambda\right) \tag{3.88}$$

where $\dot{a}$ is the derivative of a by a time coordinate defined implicitly through the lapse function N. (We use here the opposite sign of the cosmological constant compared with (3.3), following the convention in CDT papers on the toroidal model. In this context, Λ is interpreted as a Lagrange multiplier enforcing a fixed 4-volume. It can therefore be derived from a simulation and turns out to have opposite signs in the spherical and toroidal models, respectively. The convention used here is such that $\Lambda > 0$ is always positive. In the toroidal case, the absence of a curvature term then implies that the Euclidean model is equivalent to a Lorentzian model with the opposite sign of Λ. In particular, background solutions

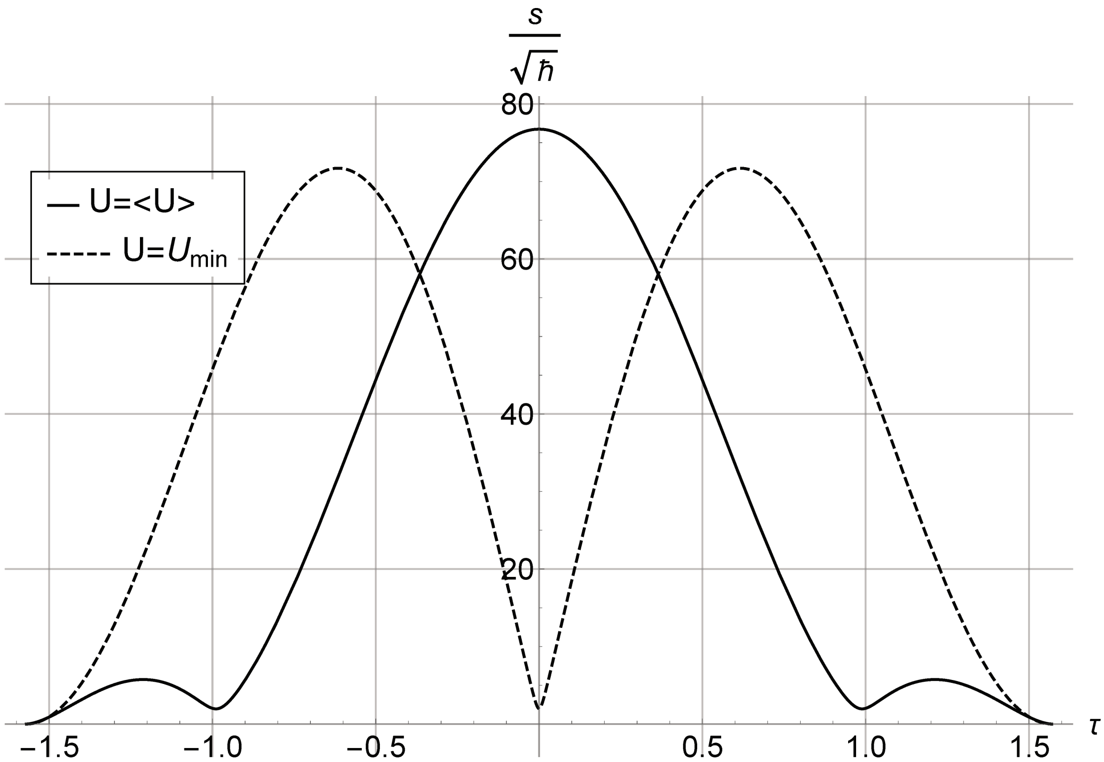

Figure 5. Volume fluctuations in a thermal state (solid), compared with a state with minimal uncertainty (dashed). For the solid curve we determine s_0 from (3.87) and use the ensemble average for the uncertainty (3.84). In the dashed curve we use $s_0 = 2a_0^2$ and $U = U_{\rm min} = \hbar^2/4$. We also take $\Lambda = 3$ and therefore $a_0 = \sqrt{3/\Lambda} = 1$.

will be given by hyperbolic rather than trigonometric functions, as we will see.) Moreover, $V_0 = \int {\rm d}^3x$ is the volume of a finite region chosen to define the averaging of the full action to a homogeneous version. Unlike in the spherical model, there is no preferred value for this parameter, which does not affect classical homogeneous solutions but plays an important role upon quantization; see [33, 34].

Using the variable $w := \sqrt{V}$, with the volume $V = V_0a^3$ of the averaging region, we have a quadratic action

$$S[w] = \frac{3}{8\pi G}\int {\rm d}\tau \left(\frac{4}{9N}\left(\frac{{\rm d}w}{{\rm d}\tau}\right)^2 + \frac{1}{3}N\Lambda w^2\right) \tag{3.89}$$

with a standard kinetic term. The corresponding Friedmann equation, $\delta S/\delta N = 0$, has solutions $\bar{w}(\sigma) = w_0e^\sigma$ with $\sigma = \frac{1}{2}\sqrt{3\Lambda}\tau$, using from now on $N = 1$ and proper time τ.

Perturbing in the form

$$w(\sigma) = w_0(e^\sigma + y(\sigma))\,, \tag{3.90}$$

the action for y is, up to boundary terms, given by

$$S[y] = \frac{\sqrt{\Lambda/3}w_0^2}{4\pi G}\int {\rm d}\sigma\, y\left(-\frac{{\rm d}^2y}{{\rm d}\sigma^2} + y\right)\,. \tag{3.91}$$

Setting zero boundary values at $\sigma = 0$ and $\sigma = \sigma_1$ (to be fixed later), eigenfunctions of the linear operator $\hat{L} = -{\rm d}^2/{\rm d}\sigma^2 + 1$ with eigenvalues $\lambda_n = 1 + n^2\pi^2/\sigma_1^2$ are given by

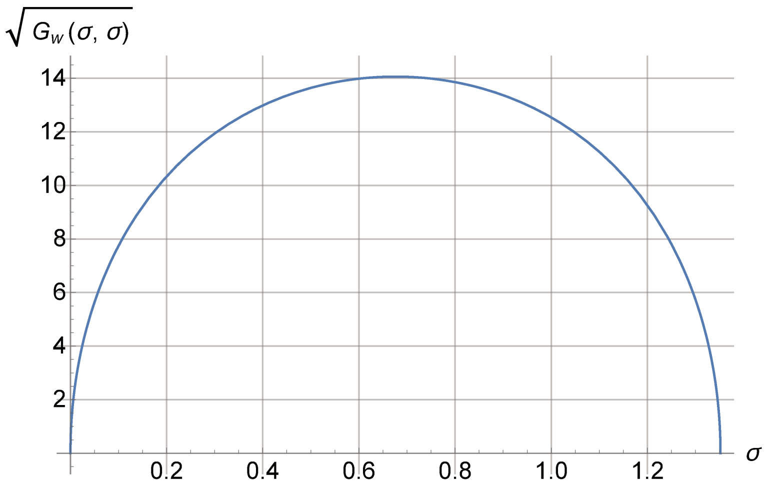

Figure 6. Fluctuations of $w(\sigma)$: $\sqrt{G_w(\sigma,\sigma)}$ for an action (3.89).

$y_n(\sigma) = \sqrt{2/\sigma_1}\sin(n\pi\sigma/\sigma_1)$. We can therefore compute y-fluctuations from the Green's function as in section 3.1.2:

$$G_y(\sigma,\sigma) = \frac{8\pi G\hbar}{\sqrt{\Lambda/3}w_0^2\sigma_1}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\sigma/\sigma_1)}{1+n^2\pi^2/\sigma_1^2} \tag{3.92}$$

$$= -\frac{\pi G\hbar}{\sqrt{\Lambda/3}w_0^2}\left(\frac{2}{\sigma_1} - 2\coth(\sigma_1) + e^{-2i\pi\sigma/\sigma_1}\left(\frac{{}_2F_1(1,1-i\sigma_1/\pi,2-i\sigma_1/\pi,e^{-2\pi i\sigma/\sigma_1})}{\sigma_1+i\pi} + \frac{{}_2F_1(1,1+i\sigma_1/\pi,2+i\sigma_1/\pi,e^{-2\pi i\sigma/\sigma_1})}{\sigma_1-i\pi}\right) + e^{2i\pi\sigma/\sigma_1}\left(\frac{{}_2F_1(1,1-i\sigma_1/\pi,2-i\sigma_1/\pi,e^{2\pi i\sigma/\sigma_1})}{\sigma_1+i\pi} + \frac{{}_2F_1(1,1+i\sigma_1/\pi,2+i\sigma_1/\pi,e^{2\pi i\sigma/\sigma_1})}{\sigma_1-i\pi}\right)\right). \tag{3.93}$$

(This series has also been used in [17], but applied only to solutions with $\Lambda = 0$ and $\mu = 0$ in the following equations.)

Fluctuations of w are related to y-fluctuations by

$$G_w(\sigma,\sigma) = w_0^2 G_y(\sigma,\sigma) \tag{3.94}$$

shown in figure 6. The result, unlike w, is independent of w_0. Therefore, the ratio $(\Delta w)/w$ (shown in figure 7 for one example) depends on w_0 and the averaging volume V_0.

In [8, 9], an isotropic model with $k = 0$ has been studied using CDTs. As one of the results, the effective minisuperspace action has an additional term proportional to V^γ where

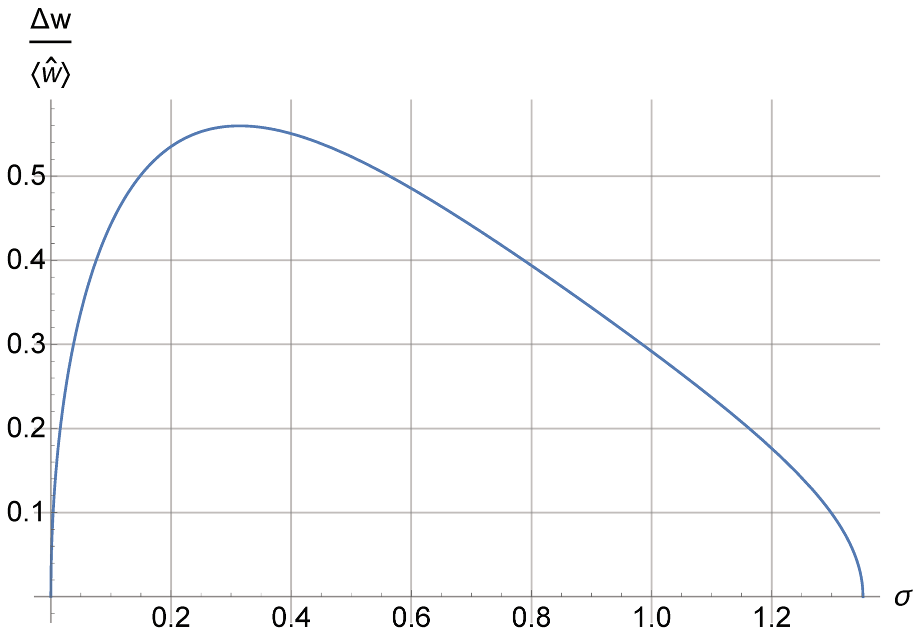

Figure 7. Relative fluctuations of $w(\sigma)$ for an action (3.89).

γ is close to $-3/2$. With such a term in the action,

$$S[w] = \frac{3}{8\pi G}\int \mathrm{d}\tau \left(\frac{4}{9N}\left(\frac{\mathrm{d}w}{\mathrm{d}\tau}\right)^2 + \frac{1}{3}N\left(\mu w^{2\gamma} + \Lambda w^2\right)\right)\,, \tag{3.95}$$

it is still possible to solve the corresponding Friedmann equation analytically, by

$$\bar{w}(\sigma) = (\mu/\Lambda)^{1/2(1-\gamma)}\sinh(\sigma+\sigma_0)^{1/(1-\gamma)}\,, \tag{3.96}$$

now with $\sigma = \frac{3}{2}(1-\gamma)\sqrt{\Lambda/3}\tau$. Notice that the action is no longer homogeneous in w, and therefore μ should scale with the averaging volume V_0 used to define homogeneous variables such that the volume is $V = V_0 a^3$. Since $\mu w^{2\gamma}$ should scale like $w^2 = V = V_0 a^3$ if we change V_0, μ should be proportional to $V_0^{1-\gamma}$. The solution (3.96) is then proportional to $\sqrt{V_0}$, which is the correct scaling behavior of w.

The integration constant σ_0 is determined by the value of $\bar{w}$ at the first boundary point, $\sigma = 0$:

$$\sinh(\sigma_0) = \sqrt{\frac{\Lambda}{\mu}}\bar{w}(0)^{1-\gamma}\,. \tag{3.97}$$

The second boundary point, $\sigma = \sigma_1$, is then fixed so as to obtain a desired value of $\bar{w}(\sigma_1)$:

$$\sinh(\sigma_1+\sigma_0) = \sqrt{\frac{\Lambda}{\mu}}\bar{w}(\sigma_1)^{1-\gamma} \tag{3.98}$$

or

$$\begin{aligned}\sinh(\sigma_1) &= \sinh(\sigma_1+\sigma_0)\cosh(\sigma_0) - \cosh(\sigma_1+\sigma_0)\sinh(\sigma_0)\\ &= \frac{\Lambda}{\mu}\bar{w}(0)^{1-\gamma}\bar{w}(\sigma_1)^{1-\gamma}\left(\sqrt{1+\frac{\mu}{\Lambda}\bar{w}(0)^{2(\gamma-1)}} - \sqrt{1+\frac{\mu}{\Lambda}\bar{w}(\sigma_1)^{2(\gamma-1)}}\right)\,.\end{aligned} \tag{3.99}$$

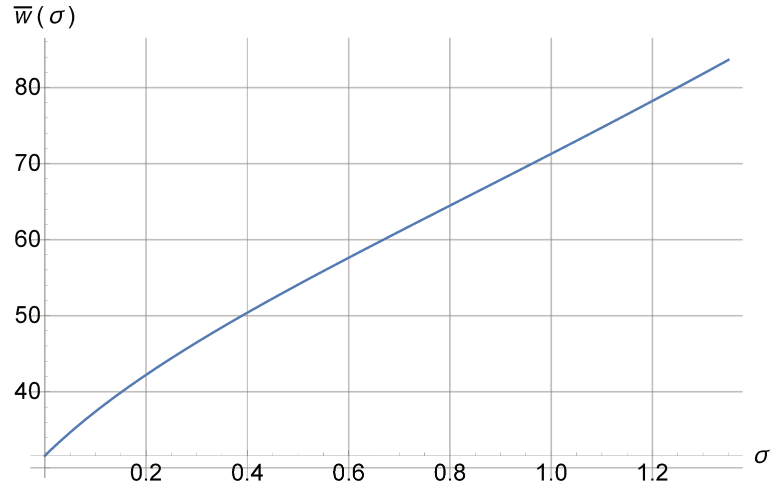

Figure 8. The function $\bar{w}(\sigma)$ for boundary values as used in toroidal models of CDTs.

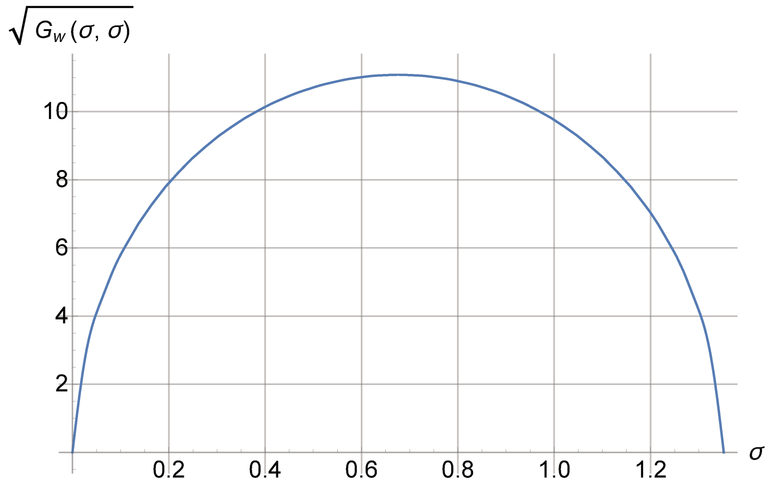

Figure 9. Fluctuations of $w(\sigma)$: $\sqrt{G_w(\sigma,\sigma)}$ for $\gamma = -3/2$, based on a numerical integration of (3.102) and summing the first 25 terms in the corresponding series (3.103).

For a comparison with results from CDTs [8, 9], we will use $\bar{w}(0) = \sqrt{10^3} = 31.6$ and $\bar{w}(\sigma_1) = \sqrt{7000} = 83.7$, and thus $\sigma_0 = 0.195$ and $\sigma_1 = 1.352$. Moreover, $\mu = 2.86 \cdot 10^5$ and $\Lambda = 3.5 \cdot 10^{-4}$, such that $\sqrt{\mu/\Lambda} = 2.86 \cdot 10^4$. For these values, $\bar{w}(s)$ is shown in figure 8.

We perturb (3.96) by

$$w(\sigma) = (\mu/\Lambda)^{1/2(1-\gamma)} \left(\sinh(\sigma+\sigma_0)^{1/(1-\gamma)} + y(\sigma)\right) \tag{3.100}$$

such that y is independent of V_0. (The V_0-scaling in (3.90) is now provided by $\mu^{1/2(1+\gamma)}$.) Up to boundary terms, the perturbed action (with $N = 1$) expanded to second order is

$$S[y] = \frac{\sqrt{\Lambda/3}}{4\pi G(1-\gamma)} \left(\frac{\mu}{\Lambda}\right)^{1/(1-\gamma)} \int \mathrm{d}\sigma\, y \left(-(1-\gamma)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\sigma^2} + 1 + \gamma(\gamma-1)\sinh(\sigma+\sigma_0)^{2(2\gamma-1)/(1-\gamma)}\right). \tag{3.101}$$

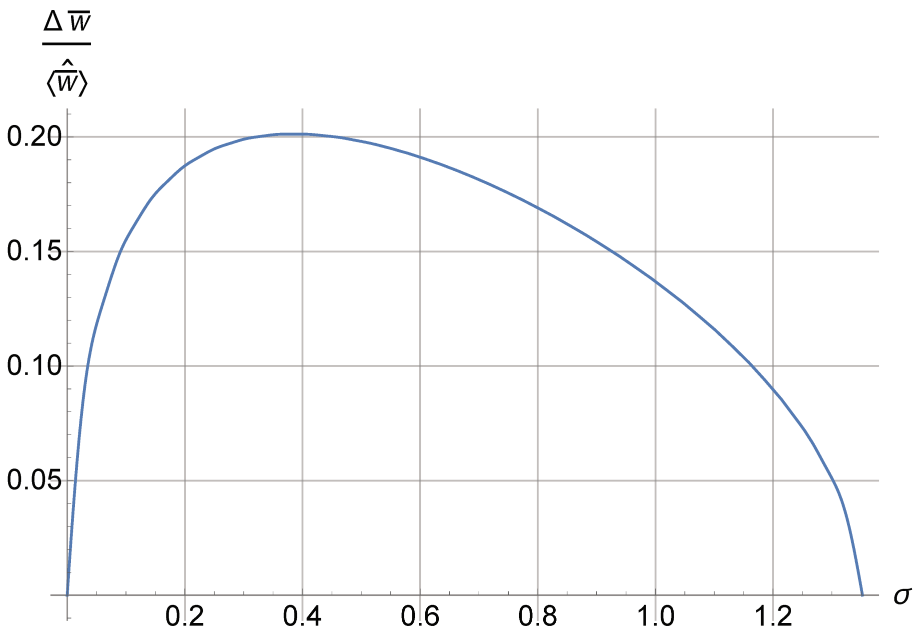

Figure 10. Relative fluctuations of $w(\sigma)$ for $\gamma = -3/2$, based on a numerical integration of (3.102) and summing the first 25 terms in the corresponding series (3.103).

The new linear operator leads to an eigenvalue problem with a potential given by a power of the sinh function, which in general is difficult to solve. However, for the values of γ indicated by CDTs, the σ-dependent potential in (3.101) can be treated as a perturbation. For instance, for $\gamma \approx -1$, the potential is proportional to $\bar{w}^{2(2\gamma-1)} = V^{2\gamma-1} = V^{-3}$, while for $\gamma \approx -3/2$ it is proportional to V^{-4} in terms of the volume V, which is large in the parameter range considered in toroidal models of CDTs. Using quantum-mechanical perturbation theory, the eigenvalues are then approximately given by

$$\tilde{\lambda}_n = 1 + (\gamma-1)^2 \frac{n^2\pi^2}{\sigma_1^2} + \frac{2}{\sigma_1}\frac{\gamma}{\gamma-1}\int_0^{\sigma_1} \mathrm{d}\sigma \sin^2(n\pi\sigma/\sigma_1)\sinh(\sigma+\sigma_0)^{2(2\gamma-1)/(1-\gamma)}\,. \quad (3.102)$$

In this case, we have to multiply y-fluctuations with $(\mu/\Lambda)^{1/(1-\gamma)}$ in order to obtain w-fluctuations:

$$G_w(\sigma,\sigma) = \frac{8\pi G(1-\gamma)\hbar}{\sqrt{\Lambda/3}\sigma_1}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\sigma/\sigma_1)}{\tilde{\lambda}_n}\,, \quad (3.103)$$

see figure 9. These fluctuations are independent of μ, but they are sensitive to the $\mu w^{2\gamma}$-term in the action through the value of γ. Formally, the previous result (3.94) is obtained for $\gamma = 0$, in which case μ just adds a constant to the action (3.95). Relative fluctuations, shown in figure 10, depend on μ through (3.100).

We note that the curvature term of the spherical model cannot be treated as a perturbation around the cosmological-constant model. In terms of w, the action is then

$$S = \frac{3\pi}{4G}\int \mathrm{d}\tau \left(\frac{4}{9}\dot{w}^2 - \frac{1}{3}\Lambda w^2 + \ell_0^2 w^{2/3}\right) \quad (3.104)$$

with $\Lambda > 0$. Therefore, the cosmological-constant model in this case does not lead to a discrete spectrum of λ_n. Moreover, the curvature term, which corresponds to $\mu = 1$ and

$\gamma = 1/3$ when written as $\mu w^{2\gamma}$, implies a perturbative correction in the analog of (3.101) which is proportional to $\bar{w}^{-1/3}$. Although the exponent is still negative, small w are important in the spherical model near the end points of its evolution. For such values, the curvature term is not just a perturbation.

3.2.2 Volume fluctuations

We have computed fluctuations of $w = \sqrt{V}$. In order to obtain volume fluctuations as in section 3.1.3, we expand

$$\begin{aligned}\Delta(V^2) &= \langle(\Delta\hat{V})^2\rangle = \langle(\hat{w}^2 - \langle\hat{w}^2\rangle)^2\rangle \\ &= \langle\hat{w}^4\rangle - \langle\hat{w}^2\rangle^2 = \langle(\Delta\hat{w} + \langle\hat{w}\rangle)^4\rangle - \langle(\Delta\hat{w} + \langle\hat{w}\rangle^2\rangle^2 \\ &= \Delta(w^4) - \Delta(w^2)^2 + 4\langle\hat{w}\rangle\Delta(w^3) + 4\langle\hat{w}\rangle^2\Delta(w^2)\,. \end{aligned} \tag{3.105}$$

For a Gaussian state,

$$\Delta(w^{2n}) = (2n-1)!!\Delta(w^2)^n\,, \tag{3.106}$$

as before, and (3.105) can be simplified to

$$\Delta(V^2) = 2\left(2\langle\hat{w}\rangle^2 + \Delta(w^2)\right)\Delta(w^2)\,. \tag{3.107}$$

For relative fluctuations, we obtain

$$\begin{aligned}\frac{\Delta V}{\langle\hat{V}\rangle} &= \frac{\sqrt{\Delta(V^2)}}{\langle\hat{w}^2\rangle} = \frac{\sqrt{\Delta(V^2)}}{\Delta(w^2) + \langle\hat{w}\rangle^2} \\ &= 2\frac{\sqrt{\langle\hat{w}\rangle^2 + \frac{1}{2}\Delta(w^2)}}{\langle\hat{w}\rangle^2 + \Delta(w^2)}\Delta w \approx 2\frac{\Delta w}{\langle\hat{w}\rangle}\,. \end{aligned} \tag{3.108}$$

The volume fluctuations obtained from w-fluctuations are shown in figures 11 and 12, demonstrating qualitative agreement with CDT results [8, 9]. Moreover, we can confirm the same scaling behavior of fluctuations as seen in the spherical model, section 3.1.3, except that there is a different universal function that describes the shape of the fluctuation curve. To this end, we first compute the Λ-dependence of the 4-volume

$$\begin{aligned}V_4 = V_0\int \bar{w}(\tau)^2\mathrm{d}\tau &= \frac{2V_0}{(1-\gamma)\sqrt{3\Lambda}}\int_0^{\sigma_1}\bar{w}(\sigma)^2\mathrm{d}\sigma \\ &= \frac{2V_0}{(1-\gamma)\sqrt{3\Lambda}}\left(\frac{\mu}{\Lambda}\right)^{1/(1-\gamma)}\int_0^{\sigma_1}\sinh(\sigma+\sigma_0)^{2/(1-\gamma)}\mathrm{d}\sigma \propto \Lambda^{(3-\gamma)/(2(1-\gamma))}\,. \end{aligned} \tag{3.109}$$

Using (3.107), volume fluctuations

$$\begin{aligned}G_V(\sigma,\sigma) &\approx 4\bar{w}(\sigma)^2 G_w(\sigma,\sigma) \\ &= \frac{32\sqrt{3}\pi G(1-\gamma)\hbar}{\sqrt{\Lambda}\sigma_1}\left(\frac{\mu}{\Lambda}\right)^{1/(1-\gamma)}\sinh(\sigma+\sigma_0)^{1/(1-\gamma)}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\sigma/\sigma_1)}{\tilde{\lambda}_n} \\ &\propto \Lambda^{(3-\gamma)/(2(1-\gamma))} \propto V_4 \end{aligned} \tag{3.110}$$

have the same Λ-dependence, such that $\Delta(V^2)/V_4$ as a function of

$$\sigma \propto \sqrt{\Lambda}\tau \propto V_4^{(\gamma-1)/(3-\gamma)}\tau \tag{3.111}$$

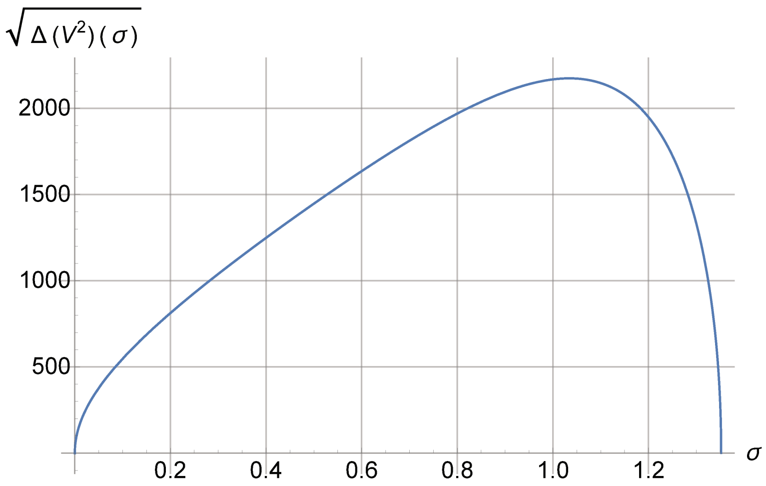

Figure 11. Fluctuations of $V(\sigma)$: $\sqrt{\Delta(V^2)(\sigma)}$ for $\mu = 0$, where the initial volume is $V(0) = 10^3$.

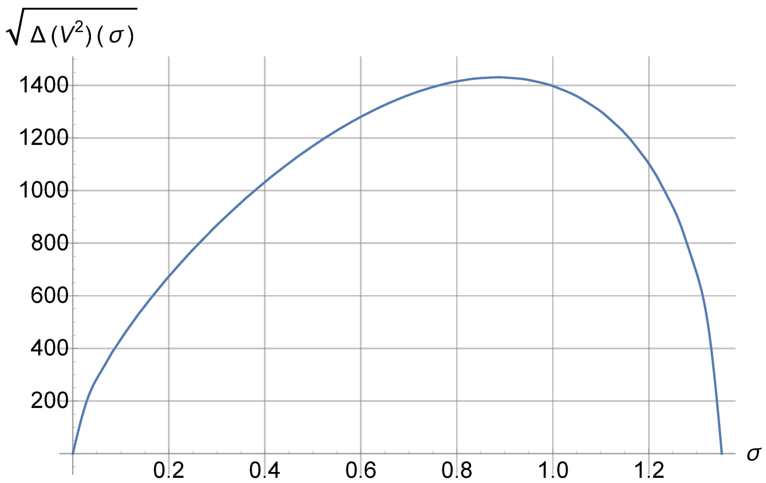

Figure 12. Fluctuations of $V(\sigma)$: $\sqrt{\Delta(V^2)(\sigma)}$ for $\gamma = -3/2$, based on a numerical integration of (3.102) and summing the first 25 terms in the corresponding series (3.103).

is independent of V_4. Provided the rescaling of the time coordinate is adjusted from $\tau/V_4^{1/4}$ in the spherical model to $V_4^{(\gamma-1)/(3-\gamma)}\tau$, a universal scaling behavior is obtained. This result assumes a fixed γ, while some CDT fits suggest that γ may be running [8, 9] in which case there could be violations of the universal behavior. Comparing the scaling behavior (3.110), (3.111) derived here with CDT simulations may therefore shed further light on the role of γ.

3.2.3 Fluctuations

We now apply the minisuperspace methods used in section 3.1.4 to the toroidal model. First ignoring the term μV^γ, the constraint

$$C = -6\pi G V P_V^2 + \frac{\Lambda V}{8\pi G} \tag{3.112}$$

with $\Lambda > 0$ generates equations of motion

$$\dot{V} = -12\pi G V P_V \tag{3.113}$$

$$\dot{P}_V = 6\pi G P_V^2 - \frac{\Lambda}{8\pi G} = -\frac{C}{V}\,. \tag{3.114}$$

(Again, we have a different sign convention compared with (3.65), following [8, 9].)

For $C = 0$, $4\pi G P_V = \pm\sqrt{\Lambda/3}$ is constant, such that $V(\tau) \propto \exp(\mp\sqrt{3\Lambda}\,\tau)$. For $C \neq 0$, we combine the two first-order equations of motion to obtain

$$\ddot{V} = -12\pi G \frac{\mathrm{d}(V P_V)}{\mathrm{d}\tau} = 12\pi G(12\pi G V P_V^2 + C) = 3(\Lambda V - 4\pi G C)\,. \tag{3.115}$$

This inhomogeneous equation is solved by

$$V(\tau) = 4\pi G \frac{C}{\Lambda} + A \exp\left(\sqrt{3\Lambda}\,\tau\right) + B \exp\left(-\sqrt{3\Lambda}\,\tau\right)\,. \tag{3.116}$$

Therefore,

$$P_V(\tau) = -\frac{\dot{V}}{12\pi G V} = -\frac{1}{4\pi G}\sqrt{\frac{\Lambda}{3}} \frac{A \exp\left(\sqrt{3\Lambda}\,\tau\right) - B \exp\left(-\sqrt{3\Lambda}\,\tau\right)}{A \exp\left(\sqrt{3\Lambda}\,\tau\right) + B \exp\left(-\sqrt{3\Lambda}\,\tau\right) + 4\pi G C/\Lambda}\,. \tag{3.117}$$

The constraint equation then implies $AB = 4\pi^2 G^2 C^2/\Lambda^2$, solutions of which can be parameterized by a single constant D such that

$$A = 2\pi G \frac{CD}{\Lambda}\,, \qquad B = 2\pi G \frac{C}{\Lambda D}\,. \tag{3.118}$$

In order for $V(\tau)$ to have positive late-time regimes, we need $A > 0$ and $B > 0$. Therefore, $\mathrm{sgn} D = \mathrm{sgn} C$. With these conditions, we can write the solutions as

$$\frac{V(\tau)}{8\pi G} = \begin{cases} C\Lambda^{-1}\cosh^2\left(\frac{1}{2}\sqrt{3\Lambda}\,\tau + \log|D|\right) & \text{if } C > 0 \\ |C|\Lambda^{-1}\sinh^2\left(\frac{1}{2}\sqrt{3\Lambda}\,\tau + \log|D|\right) & \text{if } C < 0 \end{cases} \tag{3.119}$$

$$4\pi G P_V(\tau) = \begin{cases} -\sqrt{\Lambda/3}\tanh\left(\frac{1}{2}\sqrt{3\Lambda}\,\tau + \log|D|\right) & \text{if } C > 0 \\ -\sqrt{\Lambda/3}\coth\left(\frac{1}{2}\sqrt{3\Lambda}\,\tau + \log|D|\right) & \text{if } C < 0 \end{cases} \tag{3.120}$$

Notice that $P_V(\tau)$ depends on C only through $\mathrm{sgn} D = \mathrm{sgn} C$. Moreover, the two independent solutions in the limiting case of $C = 0$ are obtained for $D = 0$ and $D \to \infty$, respectively.

We introduce fluctuation terms to second order, using the quantum constraint

$$C_{\mathrm{s}} = -6\pi G V P_V^2 + \frac{\Lambda V}{8\pi G} - 6\pi G V \left(p_s^2 + \frac{U}{s^2}\right) - 12\pi G P_V s p_s\,. \tag{3.121}$$

The dependence on s is not quadratic owing to the term U/s^2, where $U \geq \hbar^2/4$. However, we can formally interpret this term as a centrifugal potential of a system with two auxiliary fluctuation variables, X and Y, such that (3.121) is the spherically symmetric reduction with

$$s^2 = X^2 + Y^2\,. \tag{3.122}$$

The extended constraint,

$$C_{\rm s} = -6\pi GVP_V^2 + \frac{\Lambda V}{8\pi G} - 6\pi GV\left(p_X^2 + p_Y^2\right) - 12\pi GP_V\left(Xp_X + Yp_Y\right)\,, \tag{3.123}$$

is quadratic in the new variables and such that X and Y decouple. Focusing on X and its momentum p_X, we have equations of motion

$$\dot{p}_X = 12\pi GP_Vp_X \tag{3.124}$$

$$\dot{X} = -12\pi G(Vp_X - P_VX) \tag{3.125}$$

linear in these variables, but coupled to the expectation values V and P_V.

If we first assume small quantum back-reaction, we can solve for X and p_X by using the classical equations of motion and solutions for V and P_V. In particular, replacing $12\pi GP_V$ in (3.124) by $-\dot{V}/V$, using (3.113), this equation turns into

$$\dot{p}_X = -\frac{\dot{V}}{V}p_X \tag{3.126}$$

and is solved by

$$p_X = \frac{B_X}{V} \tag{3.127}$$

with constant B_X. The second equation can then be written as

$$\dot{X} = -32\pi^2G^2B_X + \frac{\dot{V}}{V}X \tag{3.128}$$

with solution

$$X(\tau) = A_XV(\tau) - 12\pi GB_XV(\tau)\int_0^\tau \frac{1}{V(t)}{\rm d}t \tag{3.129}$$

with another constant A_X. Again using a classical equation, (3.114), this solution can be simplified to

$$X(\tau) = V(\tau)\left(A_X + 12\pi G\frac{B_X}{C}P_V(\tau)\right) \tag{3.130}$$

using (3.114), assuming $C \neq 0$. For $C = 0$, we have

$$X(\tau) = A_XV \mp 4\pi G\sqrt{3/\Lambda}B_X\,. \tag{3.131}$$

In terms of auxiliary variables, we have related the semiclassical constant of motion U to angular momentum in the XY-plane. The additional condition

$$U = (Xp_Y - Yp_X)^2 \geq \frac{\hbar^2}{4} \tag{3.132}$$

should therefore be imposed, which evaluates to

$$(A_XB_Y - A_YB_X)^2 \geq \frac{\hbar^2}{4} \tag{3.133}$$

Moreover, the phase of the auxiliary variables, related to

$$\cot\left(\phi(\tau)\right) = \frac{X}{Y} = \frac{A_XC + 12\pi GB_XP_V(\tau)}{A_YC + 12\pi GB_YP_V(\tau)}\,, \tag{3.134}$$

is spurious. At any time τ at which $P_V(\tau) \neq 0$, we can eliminate the spurious phase by fixing the ratio of B_X/B_Y. For simplicity, we choose $B_X = B_Y = B$ and therefore

$$U = B^2 (A_X - A_Y)^2 \geq \frac{\hbar^2}{4} . \tag{3.135}$$

(This ratio can formally be identified with the spurious phase at a singularity: The background solutions contain a τ_∞ where $P_V(\tau_\infty) \to \infty$, and

$$\cot(\phi(\tau_\infty)) = \frac{B_X}{B_Y} . \tag{3.136}$$

However, around a singularity it may not be safe to assume weak quantum back-reaction.)

For a comparison with CDT results or our preceding path-integral calculations, we are interested in solutions that have zero fluctuations at two different times, the endpoints of a CDT universe. Using (3.122), $s = 0$ if and only if $X = 0$ and $Y = 0$. Equation (3.130) and its analog for $Y(\tau)$ then imply that at least at one τ, we have

$$P_V(\tau) = -\frac{A_X C}{12\pi G B} = -\frac{A_Y C}{12\pi G B} . \tag{3.137}$$

Since the resulting $A_X = A_Y$ is in violation of (3.135), it is impossible to have zero fluctuations even at a single time, let alone two times for the endpoints of a CDT universe. Similarly, if $C = 0$ the solution (3.131) implies that $A_X = A_Y$ following the same arguments. Quantum back-reaction therefore seems relevant. An analysis then requires numerical solutions, in which we will consider also a possible term μV^γ.

Including the term μV^γ, the constraint

$$C = -6\pi G V P_V^2 + \frac{\Lambda V}{8\pi G} + \frac{\mu V^\gamma}{8\pi G} \tag{3.138}$$

generates the equations of motion

$$\dot{V} = -12\pi G V P_V \tag{3.139}$$

$$\dot{P}_V = 6\pi G P_V^2 - \frac{\Lambda}{8\pi G} - \frac{\gamma \mu V^{\gamma-1}}{8\pi G} = -\frac{C}{V} + \frac{(1-\gamma)\mu}{8\pi G} V^{\gamma-1} . \tag{3.140}$$

In terms of $w = \sqrt{V}$, the constraint equation can be written as

$$\frac{1}{6\pi G} \left(\frac{\mathrm{d}w}{\mathrm{d}\tau}\right)^2 = -C + \frac{\Lambda w^2}{8\pi G} + \frac{\mu w^{2\gamma}}{8\pi G} . \tag{3.141}$$

As in the example of the appendix, solutions can be written in terms of Weierstrass' function, but we will not use them explicitly.

Including fluctuations to second order, the constraint is

$$C_\gamma = C_{\mathrm{s}} + \frac{\mu V^\gamma}{8\pi G} + \frac{\mu\gamma(\gamma-1)}{16\pi G} V^{\gamma-2} s^2 \tag{3.142}$$

where C_{s} is given in (3.121). Alternatively, we may use (3.123) with $s^2 = X^2 + Y^2$. In the latter case, the equations of motion are

$$\dot{p}_X = 12\pi G P_V p_X - \frac{\mu\gamma(\gamma-1)}{8\pi G} V^{\gamma-2} X \tag{3.143}$$

$$\dot{X} = -12\pi G (V p_X + P_V X) \tag{3.144}$$

and, using the background equations (not ignoring quantum back-reaction at this point) imply the second-order equation

$$\ddot{X} = (12\pi G)^2 \left(V p_X + \frac{1}{2} P_V X \right) P_V + \frac{3}{2} \left(\Lambda + \mu\gamma^2 V^{\gamma-1} \right) X \,. \tag{3.145}$$

As in the spherical model, we need a suitable state to produce appropriate X-solutions. For the sign of Λ realized in the toroidal model, there is no stable thermal state according to the methods used in the spherical model. Instead, we use the existence of a local maximum of fluctuations in order to restrict possible initial values of X and its momentum: For fluctuations as shown by CDTs as well as our path-integral calculations, using Dirichlet boundary conditions at two times, we need a local maximum of $X(\tau)$ and therefore a range with $\ddot{X} < 0$. (Without loss of generality, we assume $X > 0$ at this point. The sign of X is irrelevant for fluctuations s, and our argument here works equally for negative X. In this case we would need a local minimum of X and therefore $\ddot{X} > 0$ for a local maximum of s.) For the relevant background solutions, $P_V \propto -\dot{V}/V < 0$. Equation (3.145) then implies that p_X has to be positive and sufficiently large for $\ddot{X}$ to be negative, such that $\dot{X} \approx 0$ is possible near a local maximum. Using (3.144), we need $p_X \approx -XP_V/V$ which is possible for positive p_X thanks to $P_V < 0$. In this range, therefore,

$$\ddot{X} \approx -\frac{1}{2}(12\pi G)^2 V P_V p_X + \frac{3}{2} \left(\Lambda + \mu\gamma^2 V^{\gamma-1} \right) X > 0 \,. \tag{3.146}$$

We can find generic constraints on our initial conditions in order to get a local maximum. Supposing we are at the local maximum, we need $p_X = -XP_V/V$. Using (3.146), this equation implies

$$(12\pi G)^2 P_V^2 = (12\pi G)^2 \left(\frac{V p_X}{X} \right)^2 > 3 \left(\Lambda + \mu\gamma^2 V^{\gamma-1} \right) \,. \tag{3.147}$$

Therefore, the classical constraint

$$C = -6\pi G V P_V^2 + \frac{\Lambda V}{8\pi G} + \frac{\mu V^\gamma}{8\pi G} < \frac{\mu(1-\gamma^2)V^\gamma}{8\pi G} \tag{3.148}$$

cannot be zero unless $\gamma = \pm 1$. Because X does not have a simple algebraic relationship with V, the semiclassical constraint $C_{\rm s}$ cannot be satisfied either. The existence of a local maximum of fluctuations in the toroidal model is therefore an indication that the constraint, unlike in the spherical model, is not imposed by CDT simulations.

In figure 13 we show an example of a solution with a local maximum of X. For suitable fluctuations, it is possible to have two zeros and be in agreement with Dirichlet boundary conditions as imposed in CDT simulations of the toroidal model. The large maximum value found in our example, $s_0 = 300\sqrt{\hbar}$, is in good agreement with fluctuations derived from CDT simulations [8, 9]. The influence of γ on the existence of two zeros is, however, minor; see figure 14. Imposing Dirichlet boundary conditions therefore cannot explain the origin of a term μV^γ in an effective action.

4 Quantum origin

The new term μV^γ with $\gamma \sim -3/2$ in an effective action derived from CDTs has no classical analog, and so far it has been difficult to find possible interpretations in terms of known

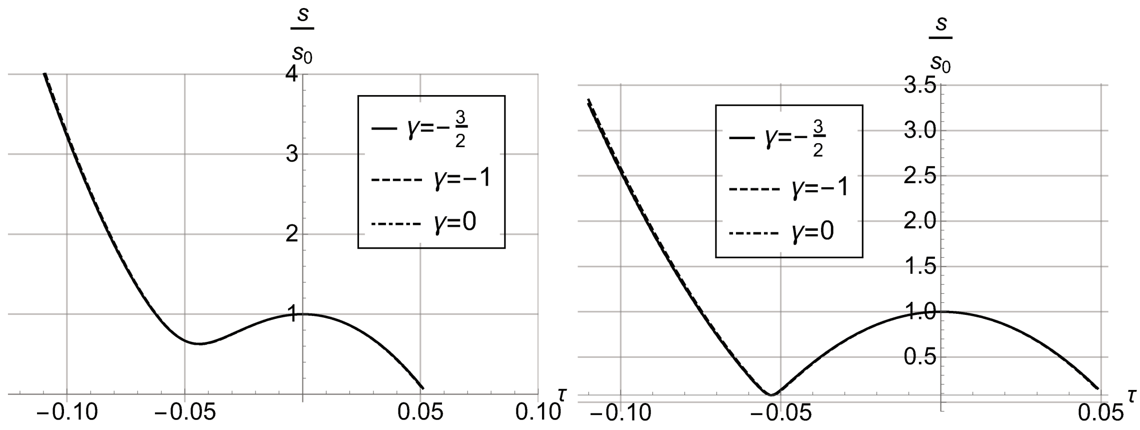

Figure 13. Volume fluctuations as a function of time in a situation where the inequality (3.147) is satisfied. The parameters $\Lambda = 3.5 \cdot 10^{-4}$ and $\mu = 2.86 \cdot 10^5$ are as they appear in the CDT toroidal model. Moreover, according to figures 8 and 9, the volume close to the local maximum of fluctuations is approximated as $V(0.7) = \bar{w}(0.7)^2 \approx 60^2$, which we use in (3.147) to obtain an estimate of P_V at our initial time here, $\tau = 0$. For initial fluctuations, we chose $s_0 = 100\sqrt{\hbar}$ (left) and $s_0 = 300\sqrt{\hbar}$ (right), respectively, and then computed $p_{s,0} = -s_0 P_V(0)/V(0)$. For larger fluctuations, it is possible to have vanishing fluctuations at two different times. The value of γ, however, has only little influence on local extrema or zeros, see also figure 14, making an origin of the μV^γ-term in the boundary conditions unlikely.

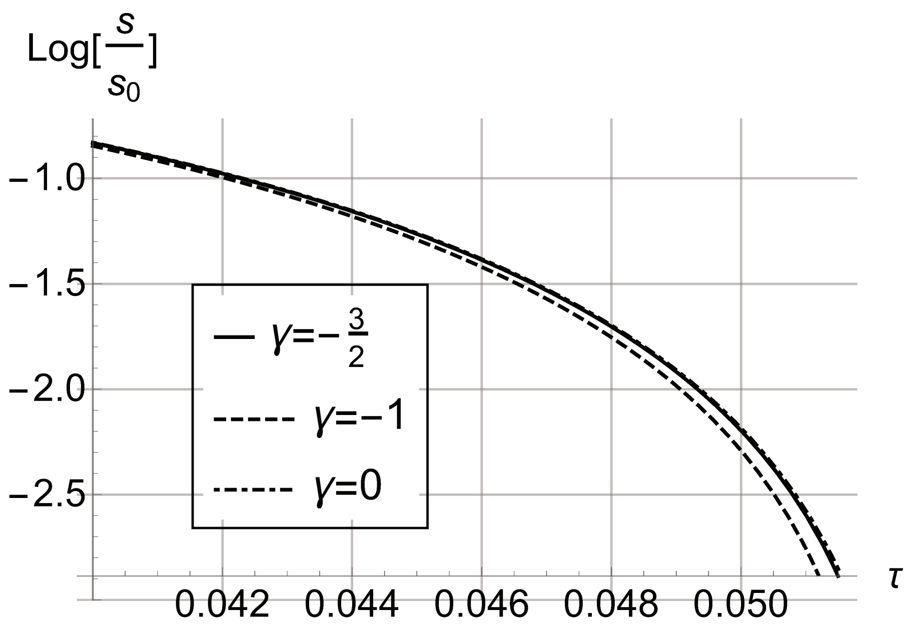

Figure 14. Logarithmic plot of fluctuations as in figure 13, zoomed in on the zero at positive times.

quantum effects. We will now address possible origins of this new term. We will discuss mainly two general possibilities, quantum back-reaction of fluctuations on expectation values, or factor-ordering choices. It turns out that each could explain the new term, but not in an easy manner. The term therefore seems to contain much information about specific effects, which could be exploited in further studies.

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We note that not setting the Hamiltonian constraint equal to zero leads to an additional term, but, as derived in section 2, it is dust-like in the Friedmann equation (2.22) and does not have the $a^{-15/2}$-behavior shown by the CDT term.

4.1 Non-Gaussianity

Before we propose our two main proposals to explain the origin of the new term in the action, we briefly mention the option that such a term could indicate the influence of a non-Gaussian state put into the straightjacket of a Gaussian distribution by using quadratic expansions to extract the effective action.

We start with the cosmological term proportional to $Q^{3/2(1-x)}$ in a Hamiltonian, using the generic canonical variables (3.51). It contributes a semiclassical correction through its second derivative, which is also proportional to s^2, defined in (3.58). If there is a non-zero third-order moment, there would be an additional contribution proportional to the third derivative of $Q^{3/2(1-x)}$. Assuming that a third-order moment of Q is proportional to s^3 (as suggested by [27, 28]), this new term could be of the form $V^{-3/2}$ provided the third derivative of $Q^{3/2(1-x)}$ has the same form as the second derivative of $V^{-3/2} \propto Q^{-9/4(1-x)}$, or

$$\frac{2x+1}{2(1-x)} = -\frac{9}{4(1-x)} \tag{4.1}$$

which implies $x = -11/4$. The basic configuration variable would then be $Q \propto a^{15/2} \propto V^{5/2}$, a rather unusual choice.

4.2 Fluctuation couplings

We have already seen in section 3.2.3 that quantum back-reaction is likely relevant in the toroidal model. An effective potential or force as extracted from CDTs could be a consequence of fluctuations back-reacting on the volume expectation value. Such terms depend sensitively on quantization choices, such as what one considers the basic canonical variables corresponding to some fixed x in Q and P, defined in (3.51). Here, we illustrate fluctuation couplings based on a few sample solutions.

In (at least) two cases, the system of Q, s and their canonical momenta can be decoupled and solved completely. For $x = 1/4$, we have the canonical pair

$$Q = \frac{\sqrt{V_0}a^{3/2}}{2\pi G} \propto w\,, \qquad P = -\sqrt{V_0 a}\,\dot{a} \tag{4.2}$$

and the constraint

$$\frac{8\pi G}{3}C = -P^2 - (2\pi GQ)^{2/3}k\ell_0^2 + \frac{1}{3}\Lambda(2\pi GQ)^2\,. \tag{4.3}$$

For $k = 0$, the classical constraint is quadratic and therefore the all-orders constraint agrees with the second-order quantum constraint. For Euclidean signature and the opposite sign of the cosmological-constant term (in accordance with [8, 9]), we have

$$\frac{8\pi G}{3}C = P^2 - \frac{1}{3}\Lambda(2\pi GQ)^2\,. \tag{4.4}$$

The semiclassical (or all-orders) constraint is then

$$\frac{8\pi G}{3}C_{\rm s} = P^2 + p_s^2 + \frac{U}{s^2} - \frac{4\pi^2G^2}{3}\Lambda(Q^2+s^2)\,. \tag{4.5}$$

The background equations of motion

$$\dot{Q} = \frac{3}{4\pi G} P\,, \qquad \dot{P} = \pi G \Lambda Q \tag{4.6}$$

can be solved in a standard way. They imply that the contribution

$$\frac{3}{8\pi G} P^2 - \frac{\pi G}{2} \Lambda Q^2 = c_1 \tag{4.7}$$

in C is conserved separately of $C = c$ itself. Moreover, the equations

$$\dot{s} = \frac{3}{4\pi G} p_s \tag{4.8}$$

$$\dot{p}_s = \frac{3}{4\pi G} \frac{U}{s^2} + \pi G \Lambda s \tag{4.9}$$

imply that

$$\begin{aligned}(sp_s)^\bullet &= \frac{3}{4\pi G}\left(p_s^2 + \frac{U}{s^2}\right) + \pi G \Lambda s^2 = -\frac{3}{4\pi G}(P^2 + 2C_s) + \pi G \Lambda (Q^2 + 2s^2) \\ &= 2(C_s - c_1) + 2\pi G \Lambda s^2\end{aligned} \tag{4.10}$$

and therefore

$$(sp_s)^{\bullet\bullet} = 4\pi G \Lambda p_s \dot{p}_s = 3\Lambda s p_s \tag{4.11}$$

can be solved for sp_s. We obtain

$$sp_s = A \sinh\left(\sqrt{3\Lambda}(\tau - \tau_0)\right)\,, \tag{4.12}$$

and

$$p_s = \frac{A \sinh\left(\sqrt{3\Lambda}(\tau - \tau_0)\right)}{s}\,, \tag{4.13}$$

inserted in (4.8), implies

$$s^2 = \frac{A}{2\pi G} \sqrt{3/\Lambda} \left(\cosh\left(\sqrt{3\Lambda}(\tau - \tau_0)\right) + B\right)\,. \tag{4.14}$$

Combining our solutions for sp_s and p_s, we obtain

$$p_s = \sqrt{2\pi G A}(\Lambda/3)^{1/4} \frac{\sinh\left(\sqrt{3\Lambda}(\tau - \tau_0)\right)}{\sqrt{\cosh\left(\sqrt{3\Lambda}(\tau - \tau_0)\right) + B}} \tag{4.15}$$

where

$$B = \sqrt{1 - \frac{U}{A^2}} \tag{4.16}$$

in order to fulfill (4.9). Owing to the decoupling, these solutions do not have back-reaction of moments on expectation values, and are therefore not consistent with an effective force other than that implied by the cosmological constant.

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The situation is only slightly different for $x = -1/2$. In this case,

$$Q = \frac{V_0 a^3}{4\pi G} \quad \text{and} \quad P = -\frac{\dot{a}}{a} \tag{4.17}$$

and

$$\frac{8\pi G}{3} C = -4\pi G Q P^2 - k\ell_0^2 (4\pi G)^{1/3} Q^{1/3} + \frac{4\pi G}{3} \Lambda Q \,. \tag{4.18}$$

The all-orders quantum constraint (3.64) is

$$C_{\rm all} = -\frac{3}{2}\left(QP^2 + 2sPp_s + Qp_s^2\right) - \frac{3}{2}\frac{Q}{s^2} \tag{4.19}$$

$$-\frac{3}{16\pi G} k\ell_0^2 (4\pi G)^{1/3} \left((Q+s)^{1/3} + (Q-s)^{1/3}\right) + \frac{1}{2}\Lambda Q \,. \tag{4.20}$$

By the canonical trasformation to

$$X = \sqrt{2(Q+s)}, \qquad P_X = \sqrt{Q+s}\,(P_Q + p_s)\,, \tag{4.21}$$

$$Y = \sqrt{2(Q-s)}, \qquad P_Y = \sqrt{Q-s}\,(P_Q - p_s)\,, \tag{4.22}$$

the kinetic term is brought to standard form:

$$C_{XY} = -\frac{3}{2}(P_X^2 + P_Y^2) - 6U \frac{X^2+Y^2}{(X^2-Y^2)^2} \tag{4.23}$$

$$-\frac{3}{16\sqrt[3]{2}\pi G} k\ell_0^2 (4\pi G)^{1/3} (X^{2/3} + Y^{2/3}) + \frac{1}{4}\Lambda (X^2+Y^2)\,. \tag{4.24}$$

The flat Euclidean version

$$C = \frac{3}{2}(P_X^2 + P_Y^2) + 6\frac{X^2+Y^2}{(X^2-Y^2)^2} - \frac{1}{4}\Lambda(X^2+Y^2) \tag{4.25}$$

generates equations of motion

$$\dot{X} = 3P_X \tag{4.26}$$

$$\dot{Y} = 3GP_Y \tag{4.27}$$

$$\dot{P}_X = 12UX \frac{X^2+3Y^2}{(X^2-Y^2)^3} + \frac{1}{2}\Lambda X \tag{4.28}$$

$$\dot{P}_Y = -12UY \frac{3X^2+Y^2}{(X^2-Y^2)^3} + \frac{1}{2}\Lambda Y\,. \tag{4.29}$$

For $\frac{1}{2}(X^2+Y^2) = 2Q = V/12\pi G$, we obtain the decoupled equation

$$\frac{1}{2}(X^2+Y^2)^{\bullet\bullet} = 3\left(2C_s + \Lambda(X^2+Y^2)\right)\,. \tag{4.30}$$

The volume

$$\frac{1}{2}(X^2+Y^2) = \frac{C_s}{\Lambda} + A\sinh(\sqrt{6\Lambda}\tau) + B\cosh(\sqrt{6\Lambda}\tau) \tag{4.31}$$

therefore does not couple to fluctuations. However, this system is not fully decoupled because the fluctuation, $\frac{1}{2}(X^2-Y^2)=2s$, obeys the equation

$$\begin{aligned}\frac{1}{2}(X^2-Y^2)^{\bullet\bullet} &= 9\left(8U\frac{(X^2+Y^2)^2}{(X^2-Y^2)^3}-\frac{4U}{X^2-Y^2}+\frac{1}{6}\Lambda(X^2-Y^2)\right)\\ &\quad+9(P_X^2-P_Y^2)\end{aligned} \tag{4.32}$$

coupled to the volume and the momenta.

As a more complicated example, we consider the case of $x=1/2$, or $Q=3\ell_0 a/4\pi G$ proportional to the scale factor. The semiclassical constraint

$$C_{\rm s}=\frac{9}{32\pi^2G^2}\left(\frac{1}{Q}\left(1+\frac{s^2}{Q^2}\right)P^2-2\frac{s}{Q^2}Pp_s+\frac{1}{Q}p_s^2+\frac{U}{Qs^2}\right)-\frac{8\pi^2G^2}{27}\Lambda(Q^3+3Qs^2) \tag{4.33}$$

now generates the equations of motion

$$\dot{Q}=\frac{9}{16\pi^2G^2}\left(\frac{1}{Q}\left(1+\frac{s^2}{Q^2}\right)P-\frac{s}{Q^2}p_s\right) \tag{4.34}$$

$$\dot{s}=\frac{9}{16\pi^2G^2}\left(-\frac{s}{Q^2}P+\frac{1}{Q}p_s\right) \tag{4.35}$$

$$\begin{aligned}\dot{P}&=\frac{9}{32\pi^2G^2}\left(\frac{1}{Q^2}\left(1+3\frac{s^2}{Q^2}\right)P^2-4\frac{s}{Q^3}Pp_s+\frac{1}{Q^2}p_s^2+\frac{U}{Q^2s^2}\right)\\ &\quad+\frac{8\pi^2G^2}{9}\Lambda(Q^2+s^2)\end{aligned} \tag{4.36}$$

$$\dot{p}_s=-\frac{9}{16\pi^2G^2}\left(\frac{s}{Q^3}P^2-\frac{1}{Q^2}Pp_s-\frac{U}{Qs^3}\right)+\frac{16\pi^2G^2}{9}\Lambda Qs\,. \tag{4.37}$$

A combination of these equations leads to the rather messy equation

$$\begin{aligned}\ddot{Q}&=\left(\frac{81}{256\pi^4G^4}\right)^2\left(-\frac{1}{2Q^3}\left(1+6\frac{s^2}{Q^2}+3\frac{s^4}{Q^4}\right)P^2+\frac{3s}{Q^4}\left(1+\frac{s^2}{Q^2}\right)Pp_s\right.\\ &\quad\left.-\frac{1}{2Q^3}\left(1+3\frac{s^2}{Q^2}\right)p_s^2-\frac{1}{2Q^3s^2}\left(1-\frac{s^2}{Q^2}\right)U\right)+\frac{1}{2}\Lambda\frac{Q^4+s^4}{Q^3}\,.\end{aligned} \tag{4.38}$$

For Q^n, it implies

$$\begin{aligned}(Q^n)^{\bullet\bullet}&=nQ^{n-4}(Q^3\ddot{Q}+(n-1)Q^2\dot{Q}^2)\\ &=nQ^{n-4}\left(\frac{9}{16\pi^2G^2}\right)^2\left(\frac{1}{2}\left(2n-3+2(2n-5)\frac{s^2}{Q^2}+(2n-5)\frac{s^4}{Q^4}\right)P^2\right.\\ &\quad-\frac{s}{Q}(2n-5)\left(1+\frac{s^2}{Q^2}\right)Pp_s-\frac{1}{2}\left(1-(2n-5)\frac{s^2}{Q^2}\right)p_s^2\\ &\quad\left.-\frac{1}{2s^2}\left(1-\frac{s^2}{Q^2}\right)U\right)+\frac{1}{2}n\Lambda(Q^4+s^4)Q^{n-4}\,.\end{aligned} \tag{4.39}$$

In particular, for $n=2$, the resulting equation for the square of the scale factor can be simplified because it has the same combination of Pp_s and p_s^2 as the constraint:

$$(Q^2)^{\bullet\bullet}=2\frac{9}{16\pi^2G^2}\left(\frac{P^2}{Q^2}+\frac{U}{Q^4}\right)-\frac{9}{8\pi^2G^2}\left(1+\frac{s^2}{Q^2}\right)\frac{c}{Q}+\frac{2}{3}\Lambda(Q^2-2s^2)\,. \tag{4.40}$$

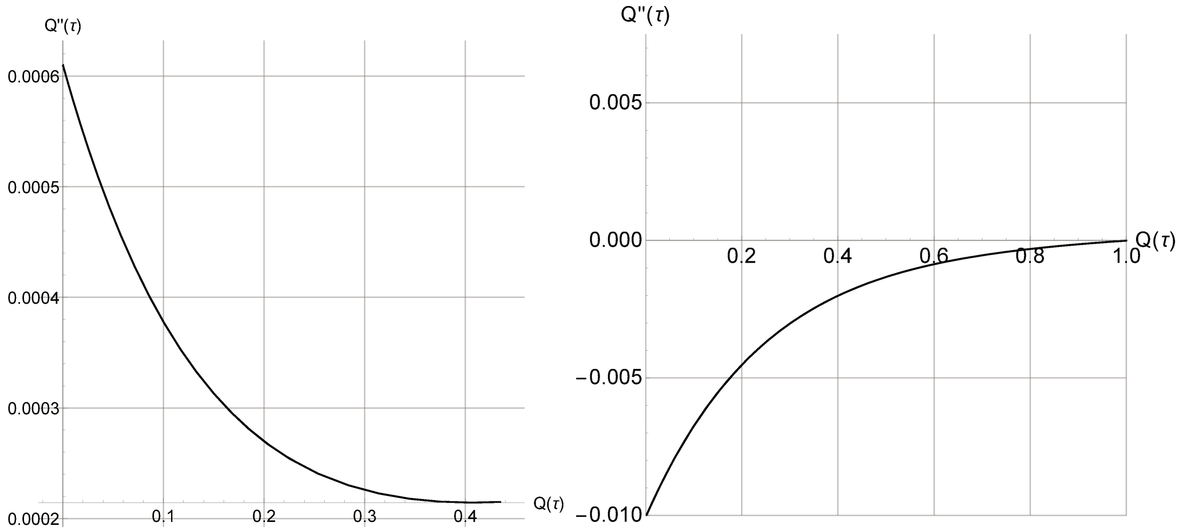

Figure 15. The effective force (4.41) in terms of Q for $U = \hbar^2/4$ with $C_{\rm s} = 0$ (left) and $C_{\rm s} = 0.05$ (right). The cosmological constant $\Lambda = 3.5 \cdot 10^{-4}$ is chosen as in the toroidal model of CDTs.

Here, we can identify several effective force terms. Most of them are positive, but for small Q with $c > 0$ and $s^2 c > 2QU$ the dominant effective force

$$F_C = -\frac{9}{8\pi^2 G^2}\frac{s^2 C_{\rm s}}{Q^3} \tag{4.41}$$

is negative. Qualitatively, this expectation agrees with the effective potential obtained from CDTs. Numerically, the effective force $\mathrm{d}^2Q/\mathrm{d}\tau^2$ can be deduced from $Q(\tau)$. It is plotted parameterically in figure 15 for both $C_{\rm s} = 0$ and $C_{\rm s} = c \neq 0$. Intriguingly, $\ddot{Q}$ is negative for small Q, as would be expected for a force from an inverse-power potential such as the effective force $-U'(V) = -\Lambda - \mu\gamma V^{\gamma-1}$ deduced in [9]. In figure 16, we show the evolution of Q and its fluctuations s with both options: strictly imposing the constraint $C_{\rm s} = 0$ and with $C_{\rm s} = c \neq 0$, respectively. This plot further demonstrates that an effective force from quantum back-reaction can lead to negative $\ddot{Q}$ at small Q, but only if the constraint is not imposed. The phase during which $\ddot{Q} < 0$ is accompanied by decreasing fluctuations s. This behavior may be difficult to resolve in CDT simulations where discretization effects are larger at small Q. Such decreasing fluctuations in a toroidal model have not been seen in [8, 9].

A negative term only exists if we do not strictly impose the constraint, $C_{\rm s} = c \neq 0$. The term is then proportional to $\hbar$, which is the typical behavior of the variance s^2. Moreover, if s does not depend on V_0, as indicated by our preceding path-integral results (3.103), the new term in an effective potential for (4.40) scales like

$$s^2 \frac{c}{Q^2} \propto \frac{\hbar V_0}{V_0^{2/3}} = \hbar \ell_0\,. \tag{4.42}$$

There is an interesting contrast with the scaling in our second potential origin of the μV^γ-term, discussed in the next subsection.

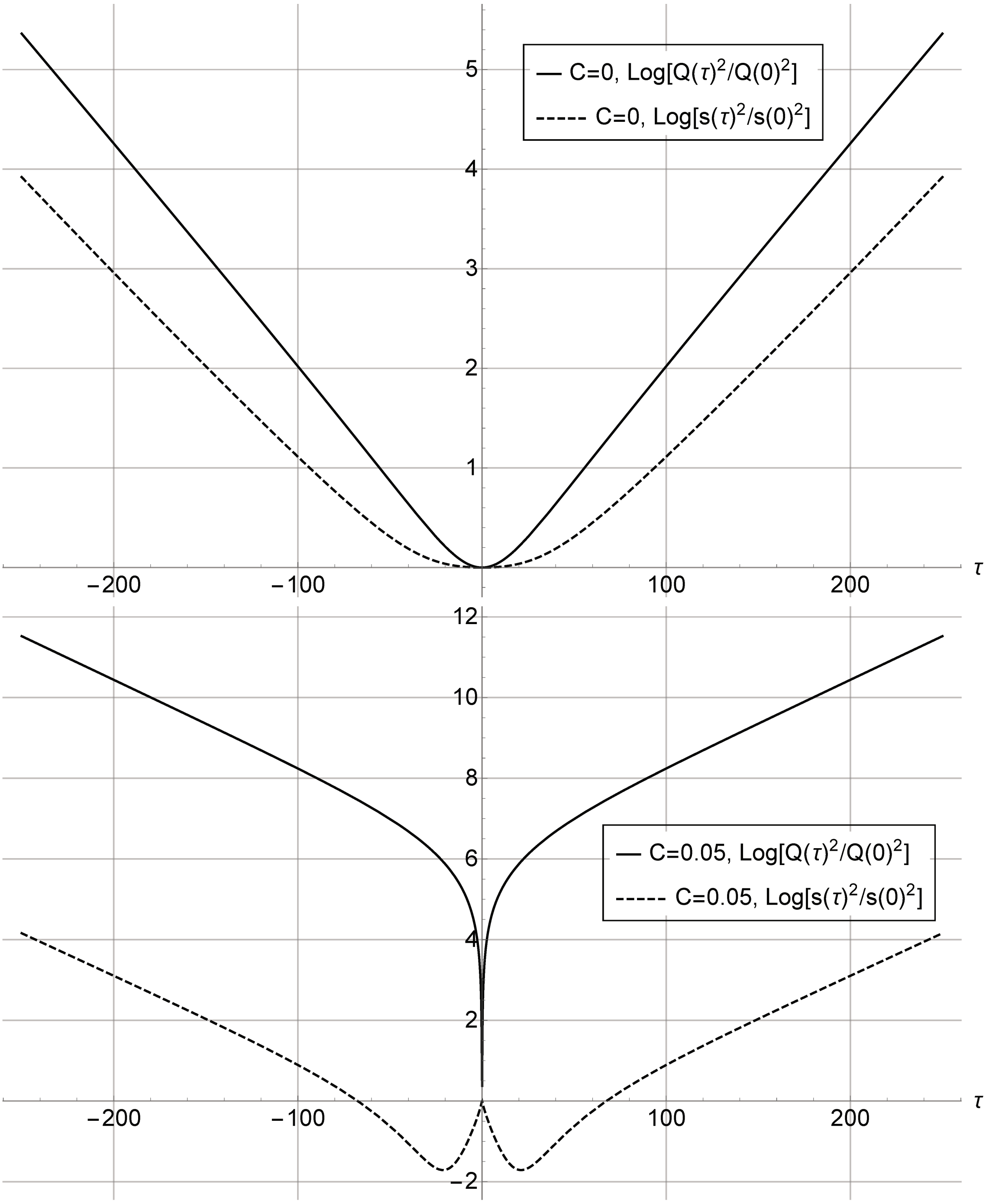

Figure 16. Logarithmic plots of $Q(\tau)$ (solid) and $s(\tau)$ (dashed) for $C_{\mathrm{s}} = 0$ (top) and $C_{\mathrm{s}} = 0.05$ (bottom). For small Q, $\ddot{Q}$ is negative for $C_{\mathrm{s}} = 0.05$ but not for $C_{\mathrm{s}} = 0$. Not imposing the constraint could therefore be related to a new potential such as μV^{γ}.

4.3 Ordering terms

Factor-ordering choices are another potential source of non-classical terms in an effective action. For easier reference to the volume, we now write canonical variables as

$$Z = V^p \,, \qquad P_Z = \frac{1}{p} V^{1-p} P_V \tag{4.43}$$

where $V = V_0 a^3$ and $P_V = -(4\pi G)^{-1}\dot{a}/a$. For a symmetric Hamiltonian quadratic in momenta, the main factor ordering ambiguity arises from the fact that there are two standard symmetric orderings, $Z^q P_Z^2 Z^q$ and $\frac{1}{2}(Z^{2q}P_Z^2 + P_Z^2 Z^{2q})$, that can be used to quantize $Z^{2q}P_Z^2$ for some q determined by the classical constraint. Any linear combination of these two orderings can be obtained by adding a multiple of

$$2Z^q P_Z^2 Z^q - (Z^{2q}P_Z^2 + P_Z^2 Z^{2q}) = 2q^2\hbar^2 Z^{2q-2} \tag{4.44}$$

to a given ordering. A new ordering term $2q^2\hbar^2 Z^{2q-2}$ is then implied, which can be compared with an effective potential such as μV^γ.

Other, less standard ordering choices, such as using

$$P_Z Z^{2\epsilon} P_Z - Z^\epsilon P_Z^2 Z^\epsilon = -\epsilon(\epsilon-1)\hbar^2 Z^{2\epsilon-2} \tag{4.45}$$

or

$$Z^{q+\epsilon}P_Z^2 Z^{q-\epsilon} + Z^{q-\epsilon}P_Z^2 Z^{q+\epsilon} - (Z^{2q}P_Z^2 + P_Z^2 Z^{2q}) = 2(q^2-\epsilon^2)\hbar^2 Z^{2q-2} \tag{4.46}$$

lead to the same power law as in (4.44). For a constraint of the form (3.52) with $x = -1/2$, we need

$$Z^{2q}P_Z^2 = \frac{1}{p^2} V^{2pq-2p+2} P_V^2 \sim V P_V^2 \,, \tag{4.47}$$

which implies $2p(q-1) = -1$. However, the ordering term is then

$$Z^{2q-2} = V^{2p(q-1)} = V^{-1} \tag{4.48}$$

rather than V^γ with γ close to $-3/2$ as suggested by [9].

If the constraint is not quadratic in P_Z, owing to higher-order corrections in the momentum which are possibly indicated by [16], there is more freedom in ordering terms. For instance, we have

$$2Z^q P_Z^4 Z^q - (Z^{2q}P_Z^4 + P_Z^4 Z^{2q}) = 12q^2\hbar^2 Z^{q-1}P_Z^2 Z^{q-1} + 2q^2(q^2-1)\hbar^2 Z^{2q-4} \,. \tag{4.49}$$

The first term shows quantum corrections to the kinetic term. The last term has a different power from the quadratic result in (4.46) and can lead to

$$Z^{2q-4} = V^{2p(q-2)} = V^{-3/2} \tag{4.50}$$

if $p = 1/4$ and $q = -1$. In this estimate, we still use $2p(q-1) = -1$ assuming that the kinetic term is quantum corrected as in $Z^q(P_Z^2 + \ell^2 P_Z^4 + \cdots)Z^q$. This value of p implies $x = 1 - \frac{3}{2}p = 5/8$ in the previous parameterization. However, the coefficient $q^2 - 1$ in (4.49) is then equal to zero.

A final modification gives the desired term: We have

$$\begin{aligned} &Z^{q+\epsilon}P_Z^4 Z^{q-\epsilon} + Z^{q-\epsilon}P_Z^4 Z^{q+\epsilon} - (Z^{2q}P_Z^4 + P_Z^4 Z^{2q}) \\ &= 12\hbar^2(q^2-\epsilon^2)Z^{q-1}P_Z^2 Z^{q-1} - 2\hbar^2(q^2-\epsilon^2)(1-q^2+\epsilon^2+12\epsilon)Z^{2q-4} \,. \end{aligned} \tag{4.51}$$

The power of Z in the last term has not changed, but we have a new coefficient. Still using $q = -1$, the final ordering term is

$$-2\hbar^2(1-\epsilon^2)(\epsilon+12)\epsilon Z^{-6}\,. \tag{4.52}$$

It is non-zero for generic ϵ, and can be rather large for large ϵ. In order to obtain a value around $\mu = 2.86\cdot 10^5$ (with $\hbar = 1$), we need $\epsilon \approx 17$. This value may seem unnaturally large, but it is encouraging that it is close to an integer.

A possible Hamiltonian operator could have the kinetic term

$$\begin{aligned} H_{\rm kin} =& -\frac{3}{2}Z^{-1}(P_Z^2+\ell^2P_Z^4)Z^{-1}\\ &-\frac{3}{2}\ell^2\left(Z^{-1+\epsilon}P_Z^4Z^{-1-\epsilon}+Z^{-1-\epsilon}P_Z^4Z^{-1+\epsilon}-Z^{-2}P_Z^2-P_Z^2Z^{-2}\right)\\ =& -\frac{3}{2}Z^{-1}(P_Z^2+\ell^2P_Z^4)Z^{-1}-18\ell^2\hbar^2(1-\epsilon^2)Z^{-2}P_Z^2Z^{-2}\\ &-3\ell^2\hbar^2(1-\epsilon^2)(\epsilon+12)\epsilon Z^{-6}\,. \end{aligned} \tag{4.53}$$

Using $p = 1/4$, we have

$$\begin{aligned} H_{\rm kin} =& -\frac{3}{2}\left(V^{-1/4}\left(4(V^{3/4}P_V+P_VV^{3/4})^2+16\ell^2(V^{3/4}P_V+P_VV^{3/4})^4\right)V^{-1/4}\right)\\ &-72\ell^2\hbar^2(1-\epsilon^2)V^{-1/2}(V^{3/4}P_V+P_VV^{3/4})^2V^{-1/2}\\ &-3\ell^2\hbar^2(1-\epsilon^2)(\epsilon+12)\epsilon V^{-3/2} \end{aligned} \tag{4.54}$$

in terms of V. Here, the symmetric ordering of P_Z^2 is equal to

$$(V^{3/4}P_V+P_VV^{3/4})^2 = 4V^{3/4}P_V^2V^{3/4}-\frac{15}{16}\hbar^2V^{-1/2}\,. \tag{4.55}$$

The factor-ordering term to be compared with the new contribution to the action is the last contribution in (4.54), proportional to $\hbar^2$ and scaling like $V_0^{-3/2} = \ell_0^{-9/2}$. This behavior is rather different from the potential term (4.42) seen from quantum back-reaction. Varying $\hbar$ and V_0 in causal dynamical triangulations can therefore distinguish between these two options.

5 Conclusions

We have derived several minisuperspace results for fluctuations in models studied previously in CDTs. We have found qualitative agreement and potential explanations of subtle features such as the issue of fixing time and imposing the constraints, the scaling behavior of fluctuations with respect to Λ, or the possible origin of new non-classical terms in effective actions. However, in all these issues there is room for further explorations.

In the spherical model, section 2.2, we have been able to identify a crucial difference between background solutions of the constraint (the Friedmann equation) compared with solutions in which the constraint is assumed to be non-zero (but then remains constant). This difference may explain why the background solution of the volume extracted from CDTs agrees with solutions of the Friedmann equation, even though fixing the time gauge would seem to relax the Friedmann equation and only impose the less restrictive Raychaudhuri

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equation. In the toroidal model, however, the difference between imposing the constraint and not doing so is much less pronounced. We have found several indications that it may not be imposed, in contrast to the spherical model.

In fact, not imposing the constraint may be one reason why CDTs in toroidal models have indicated the presence of an unexpected non-classical term, μV^γ with γ close to $-3/2$, in an effective action. Unfortunately, the detailed derivation through quantum back-reaction in minisuperspace models, shown in section 4.2, indicates that such a term, though possible, does not seem natural. Another potential origin, through factor-ordering choices shown in section 4.3, appears perhaps more natural but also requires some work to obtain the required power-law behavior. In conclusion, it seems difficult to explain the term in a unique fashion. Our derivations indicate how this issue could be explored further: Quantum corrections that could account for terms seen in CDTs have different dependencies, (4.42) compared with (4.55), on the averaging volume V_0 or $\hbar$. These constants are usually fixed in CDT simulations, but, as we suggest, running several simulations with different choices for these values can shed additional light on possible quantum corrections.

Another suggestion based on the scaling behavior, this time of volume fluctuations with respect to the cosmological constant, follows from our derivations of minisuperspace fluctuations using two different methods: path-integral calculations and moment dynamics. In the spherical model, we have been able to rederive the universal behavior found in CDTs. In the toroidal model, we have found a new universal behavior that suggests plotting relative volume fluctuations as a function of time multiplied by a specific power of the cosmological constant, (3.111). The exponent depends on γ, which we assumed to be constant in our calculation. More detailed investigations of the scaling behavior in CDTs could therefore show additional features such as a potential running of γ.

Our results have therefore suggested several "CDT experiments" which, motivated by detailed analytical calculations, have the potential of further illuminating some of the main open questions in this framework. Open questions also remain on the minisuperspace side, for instance related to finer details in the plots of CDT fluctuations that we have not been able to reproduce in the spherical model, or to the imposition of Dirichlet boundary conditions which appears somewhat unnatural in the toroidal minisuperspace model. With further studies on both sides of the correspondence between CDTs and minisuperspace models analyzed here, it may become possible to use CDTs to test the minisuperspace approximation, or to extend calculations to midisuperspace models [35].

A Solving the Raychaudhuri equation

The aim of this appendix is to derive the general solution to the instanton equation

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 = 1 - \frac{\Lambda}{3}a^2 - \frac{c}{a}, \tag{A.1}$$

for $a \geq 0$. For this purpose, let us introduce a new variable

$$u := \frac{a_0}{a}\,, \tag{A.2}$$

together with the rescaled imaginary time $s := \tau/a_0$, which transforms eq. (A.1) to

$$\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 = u^4 - u^2 - \tilde{c}u^5\,, \tag{A.3}$$

where $\tilde{c} := c/a_0$. The next step is to introduce a new time variable w, defined such that

$$\mathrm{d}s = \frac{\mathrm{d}w}{u}, \tag{A.4}$$

which is well-defined since we consider u being positive definite, and when applied to (A.3) gives

$$\left(\frac{\mathrm{d}u}{\mathrm{d}w}\right)^2 = u^2 - 1 - \tilde{c}u^3\,, \tag{A.5}$$

which resembles the case of an oscillator with cubic anharmonicity. A further change of variable,

$$u = -\frac{4}{\tilde{c}}v + \frac{1}{3\tilde{c}}\,, \tag{A.6}$$

transforms eq. (A.5) into the Weierstrass equation [36]:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}w}\right)^2 = 4v^3 - g_2 v - g_3 = 4(v-e_1)(v-e_2)(v-e_3)\,, \tag{A.7}$$

where

$$g_2 = \frac{1}{12} \quad \text{and} \quad g_3 = \frac{\tilde{c}^2}{16} - \frac{1}{216}\,. \tag{A.8}$$

The constants e_1, e_2 and e_3 are roots of the polynomial equation $4e_i^3 - g_2 e_i - g_3 = 0$, which for further convenience are ordered such that $e_1 > e_2 > e_3$. Equation (A.7) has solutions in the form of Weierstrass elliptic $\wp$ function

$$v(w) = \wp(w - w_0; g_2, g_3)\,, \tag{A.9}$$

where w_0 is a constant of integration. The Weierstrass $\wp$ function is a doubly periodic function with the two half-periods ω_1 and ω_2:

$$\omega_1 = \int_{e_1}^{+\infty} \frac{\mathrm{d}v}{\sqrt{4v^3 - g_2 v - g_3}} \quad \text{and} \quad \omega_1 = i\int_{-\infty}^{e_3} \frac{\mathrm{d}v}{\sqrt{4v^3 - g_2 v - g_3}}\,. \tag{A.10}$$

Due to the third-order form of the polynomial in eq. (A.7), there are in general two branches of solutions, for positive and negative values of v. In the considered case for $\tilde{c} \in \left[0, \frac{2}{3\sqrt{3}}\right]$ there are two types of solutions: (i) unbounded solutions in the range $v \in [e_1, +\infty)$ and (ii) oscillatory solutions in the range $v \in [e_3, e_2]$. The solution corresponding to the branch (i) is the one given by eq. (A.9), while the solution corresponding to the second branch is obtained by taking $v(w + \omega_3)$, where $\omega_3 := \omega_1 + \omega_2$:

$$v(w+\omega_3) = e_2 + \frac{2e_2^2 + e_3 e_1}{v(w) - e_2} = e_2 + \frac{2e_2^2 + e_3 e_1}{\wp(w - w_0; g_2, g_3) - e_2}\,, \tag{A.11}$$

where the addition theorem for the Weierstrass elliptic function has been used [36]. The second solution is the one we are interested in since it corresponds to the solution in the positive domain of the scale factor. Namely, applying eq. (A.11) to eq. (A.6) and then to eq. (A.2) we find the solution:

$$a(w) = \frac{3\tilde{c}a_0}{1 - 12v(w+\omega_3)} = \frac{3\tilde{c}a_0}{1 - 12e_2 - 12(2e_2^2 + e_3 e_1)/(\wp(w - w_0; g_2, g_3) - e_2)}\,. \tag{A.12}$$

In the special case when $\tilde{c} = 0$, the solution to eq. (A.5) can easily be found to be $u(w) = \cosh(w - w_0)$, which leads to

$$a(w) = \frac{a_0}{\cosh(w - w_0)}. \tag{A.13}$$

The new time variable w used above can be related with the $s = \tau/a_0$ time variable through the integral

$$s = \int_0^w \frac{\mathrm{d}w'}{u(w')}. \tag{A.14}$$

In the case of $\tilde{c} = 0$ the integration can be performed in a straightforward manner, giving (for $w_0 = 0$):

$$s = 2 \arctan\left(\tanh(w/2)\right), \tag{A.15}$$

which can be rewritten into the form $\cosh(w) = 1/\cos(s)$, with the use of which eq. (A.13) can be expressed as

$$a(s) = a_0 \cos(s), \tag{A.16}$$

which is correctly the Wick rotated version of eq. (2.10).

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