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BMO-estimates for non-commutative vector valued Lipschitz functions ☆

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We construct Markov semi-groups $\mathcal{T}$ and associated BMO-spaces on a finite von Neumann algebra $(\mathcal{M}, \tau)$ and obtain results for perturbations of commutators and non-commutative Lipschitz estimates. In particular, we prove that for any $A \in \mathcal{M}$ self-adjoint and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz there is a Markov semi-group $\mathcal{T}$ such that for $x \in \mathcal{M}$,

$$\|[f(A), x]\|_{\mathrm{bmo}(\mathcal{M},\mathcal{T})} \leq c_{abs} \|f'\|_\infty \|[A, x]\|_\infty.$$

We obtain an analogue of this result for more general von Neumann valued-functions $f : \mathbb{R}^n \to \mathcal{N}$ by imposing Hörmander-Mikhlin type assumptions on f.
In establishing these result we show that Markov dilations of Markov semi-groups have certain automatic continuity properties. We also show that Markov semi-groups of double operator integrals admit (standard and reversed) Markov dilations.

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1. Introduction

Non-commutative Lipschitz properties of functions have been studied for a long time and go back at least to the work of M.G. Krein [26]. One question raised in [26] in this direction is whether every Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ is also a non-commutative Lipschitz function in the sense that the mapping

$$B(H)_{sa} \to B(H) : A \mapsto f(A), \tag{1.1}$$

is Lipschitz. Here $B(H)_{sa}$ is the self-adjoint part of the bounded operators on a Hilbert space $B(H)$. In its original statement, Krein's question has a negative answer as was shown in [16], [17], [18]. In fact already for f the absolute value map the statement fails [10], [23]. Only after imposing additional smoothness/differentiability properties on f the mapping (1.1) is Lipschitz. Indeed, in [3], [4] Birman and Solomyak showed that for $f' \in \mathrm{Lip}_\varepsilon(\mathbb{R}) \cap L^p(\mathbb{R}) \cap L_\infty(\mathbb{R})$ with $\varepsilon > 0, p \geq 1$ we have that (1.1) is Lipschitz. The result was improved on by Peller in [28], [29] who showed that it suffices to take f in the Besov space $B^1_{\infty 1}$, see [19] for Besov spaces.

Krein's question can be altered by replacing the uniform operator norms in (1.1) by non-commutative L_p-norms with $1 < p < \infty$ associated with the Schatten-von Neumann classes $\mathcal{S}_p$. In this case a complete answer to the non-commutative differentiability properties of (1.1) was found [30], namely any Lipschitz function is a non-commutative Lipschitz function in the sense that there is a constant c_p such that for any self-adjoint operators $A, B \in \mathcal{S}_p$ we have,

$$\|f(A) - f(B)\|_p \leq c_p \|f'\|_\infty \|A - B\|_p.$$

The constant c_p grows to ∞ if either $p \to 1$ or $p \to \infty$. In fact the asymptotic behaviour was found in [7] (see also [8]) where it was shown that asymptotically $c_p \simeq p^2(p-1)^{-1}$.

In this paper we start the investigation of perturbation of commutators and non-commutative Lipschitz functions from two new view points: BMO-spaces and vector valued estimates.

We use the theory of BMO-spaces to obtain 'end-point estimates' of Krein's problem. The optimal behaviour for the constant c_p hints towards the existence of such an end-point estimate but so far the proof was not obtained. In this context we use the theory of semi-group BMO-spaces, in the commutative case extensively studied by e.g. [33], [36], and much more recently in [14], [15]. For non-commutative BMO-spaces the theory was developed in [20], see also [22].

BMO-spaces depend on the choice of a semi-group. This is just as for other definitions of BMO, which depend on the filtration of a von Neumann algebra or in the classical setting the choice of cubes/shapes over which means are taken. This choice gives a flexibility in finding the appropriate BMO-space for Krein's problem. In the current paper we introduce a natural BMO-space to resolve such problems in perturbation theory. In particular, we prove the result announced in the abstract. Our main theorem which makes this all work, proved in Section 6, yields as follows.

Theorem 1.1. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz with $\|f'\|_\infty \leq 1$. For every $A = A^* \in \mathcal{M}$,*

(i) *The semi-group of double operator integrals $\mathcal{I}^A = (\mathcal{I}^A_{e^{-tF}})_{t\geq 0}$ with symbol*

$$F(\lambda, \mu) = |\lambda - \mu|^2 + |f(\lambda) - f(\mu)|^2, \quad \lambda, \mu \in \mathbb{R},$$

is Markov (i.e. a strongly continuous semi-group of trace preserving unital completely positive maps);

(ii) *The double operator integral $\mathcal{I}^A_{f^{[1]}}$ with $f^{[1]}$ the divided difference of f maps $\mathcal{M}$ to $\mathrm{bmo}_{\mathcal{I}^A}(\mathcal{M})$ and its norm is bounded by an absolute constant c_{abs}.*

As a corollary of Theorem 1.1 we retrieve many existing results in perturbation theory, in particular the ones from [23], [10], [25], [12], [13], [30], [7], and partly [24]. We also retrieve the optimal estimates in case $p = \infty$ for finite dimensional Schatten classes in Theorem 7.6, see [1]. Together with the weak $(1, 1)$ estimate of [8] (see also [27]), which is complementary to our paper, they complete the study of the end-point estimates. At the same time, we emphasize that our results do not cover the case of infinite von Neumann algebras, due to the fact that BMO-spaces, even in the case $\mathcal{M} = B(H)$, are not realized as spaces of operators. On the other hands, a lot of techniques and proofs developed in this paper continue to hold for general semifinite von Neumann algebras almost verbatim (see also Section 2.4) and this is a cause for careful optimism that our approach can be extended to the latter case as well.

To apply Theorem 1.1 and obtain these corollaries we shall further develop the theory of Markov dilations and we obtain some results of independent interest. In particular we show that Markov semi-groups can be studied through their discrete subsemi-groups and get automatic continuity of a Markov dilation. The following is proved in Theorem 3.2.

Theorem 1.2. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group. If $(T_t)_{t\in\epsilon\mathbb{N}_{\geq 0}}$ admits a standard (resp. reversed) Markov dilation for every $\epsilon > 0$, then also $\mathcal{T}$ admits a standard (resp. reversed) Markov dilation. Moreover, the dilation has continuous path.*

We then apply this to Markov semi-groups of double operator integrals and through Ricard's results [31] on dilations of Schur multipliers we prove that they also admit a (standard and reversed) Markov dilation.

In the final part of the paper, Section 9, we initiate the study of vector-valued Lipschitz functions (in fact von Neumann algebra valued to be precise). As we show in Corollary 9.4 Khintchine type inequalities and free probability estimates can be recasted in terms of perturbations of vector valued commutators. Section 9 is strongly based on non-commutative Calderón-Zygmund theory as developed in [21]; in particular we obtain our results through the non-commutative Hörmander-Mikhlin theorem of [21].

Structure of the paper. Section 2 recalls all preliminaries and settles notation. Section 3 proves our discretization result for reversed Markov dilations, i.e. Theorem 1.2. Then in Section 4 we show that Markov semi-groups of double operator integrals admit a reversed Markov dilation. We collect the corresponding results on standard Markov dilations in Section 5; these results are not used in this paper but we believe they are of independent interest and state them for convenience of the reader. Section 6 proves Theorem 1.2 and we derive all its corollaries for perturbation theory in Section 7. In Section 8 and in Section 9 we retrieve the von Neumann-valued Lipschitz estimates.

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2. Preliminaries

2.1. General notations

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $|\alpha| = \sum_{k=1}^n \alpha_k$. For a finite von Neumann algebra $\mathcal{M}$ with faithful normal trace τ we write $L_2(\mathcal{M})$ for the non-commutative L_2-space with respect to τ. We let $\Omega_\tau = 1_\mathcal{M} \in L_2(\mathcal{M})$ be the cyclic vector. We identify elements of $\mathcal{M}$ as vectors in $L_2(\mathcal{M})$ if necessary. We write $L_p(\mathcal{M})$ for the non-commutative L_p-space, $1 \leq p < \infty$, associated with $\mathcal{M}$ and τ. It is the space of all closed densely defined operators x affiliated with $\mathcal{M}$ such that $\|x\|_p = \tau(|x|^p)^{1/p}$ is finite. Naturally $\mathcal{M} \subseteq L_p(\mathcal{M})$. We set $L_\infty(\mathcal{M}) = \mathcal{M}$. The L_2-topology on $\mathcal{M}$ is then the topology of the norm $\| \, \|_2$.

2.2. Non-commutative finite BMO-spaces

We recall the following from [20]. Fix a finite von Neumann algebra $(\mathcal{M}, \tau)$. We restrict ourselves here to the finite case in order to avoid several technicalities. We then treat the (non-finite) Euclidian case separately in Section 2.4.

Definition 2.1. We say that a semi-group $\mathcal{T} = (T_t)_{t \geq 0}$ of linear maps $\mathcal{M} \to \mathcal{M}$ is a Markov semi-group if:

(i) $T_t(1) = 1$ and T_t completely positive for every $t \geq 0$;
(ii) for every $x, y \in \mathcal{M}$ and for every $t \geq 0$ we have $\tau(xT_t(y)) = \tau(T_t(x)y)$;
(iii) for every $x \in \mathcal{M}$, we have $t \mapsto T_t(x)$ is continuous in measure.

Fix such a Markov semi-group $\mathcal{T} = (T_t)_{t\geq 0}$. By a standard interpolation argument for every $t \geq 0$ the map T_t extends to a completely contractive map,

$$T_t^p : L_p(\mathcal{M}) \to L_p(\mathcal{M}) : x \mapsto T_t(x), \qquad \forall x \in \mathcal{M} \subseteq L_p(\mathcal{M}).$$

We set

$$\mathcal{M}^\circ = \left\{ x \in \mathcal{M} \mid \lim_{t\to\infty} T_t(x) = 0 \right\},$$

where the limit is a σ-weak limit. For $1 \leq p < \infty$ we set by a norm limit,

$$L_p^\circ(\mathcal{M}) = \left\{ x \in L_p(\mathcal{M}) \mid \lim_{t\to\infty} T_t^p(x) = 0 \right\}.$$

It is a straightforward verification that $L_p^\circ(\mathcal{M})$ is a Banach space in the induced norm. For $x \in \mathcal{M}^\circ$ we set the column BMO-norm,

$$\|x\|_{\mathrm{bmo}_{\mathcal{T}}^c} = \sup_{t\geq 0} \|T_t(x^*x) - T_t(x)^*T_t(x)\|_\infty^{\frac{1}{2}}. \tag{2.1}$$

Further set,

$$\|x\|_{\mathrm{bmo}_{\mathcal{T}}^r} = \|x^*\|_{\mathrm{bmo}_{\mathcal{T}}^c}, \qquad \|x\|_{\mathrm{bmo}_{\mathcal{T}}} = \max(\|x\|_{\mathrm{bmo}_{\mathcal{T}}^r}, \|x^*\|_{\mathrm{bmo}_{\mathcal{T}}^c}). \tag{2.2}$$

We define $\mathrm{bmo}_{\mathcal{T}} = \mathrm{bmo}(\mathcal{M}, \mathcal{T})$ as the completion of the space of $x \in \mathcal{M}^\circ$ with $\|x\|_{\mathrm{bmo}_{\mathcal{T}}} < \infty$; it carries norm $\| \, \|_{\mathrm{bmo}_{\mathcal{T}}}$. We have contractive inclusions, see [6, Lemma 3.6],

$$\mathcal{M}^\circ \subseteq \mathrm{bmo}(\mathcal{M}, \mathcal{T}) \subseteq L_1(\mathcal{M}).$$

This allows us to represent elements of bmo as concrete operators that are affiliated with $\mathcal{M}$ and which are L_1 and in particular τ-measurable; this is again a reason to prefer working in the finite setting. In particular $L_1(\mathcal{M})$ and $\mathrm{bmo}(\mathcal{M}, \mathcal{T})$ form a compatible couple of Banach spaces. Also we impose the operator space structure,

$$M_n(\mathrm{bmo}(\mathcal{M}, \mathcal{T})) = \mathrm{bmo}(M_n \otimes \mathcal{M}, \mathrm{id}_n \otimes \mathcal{T}).$$

We will also make use of the following alternative BMO-norm. For $x \in \mathcal{M}^\circ$ we set

$$\|x\|_{\mathrm{BMO}_{\mathcal{T}}^c} = \sup_{t\geq 0} \|T_t\left(|x - T_t(x)|^2\right)\|_\infty^{\frac{1}{2}}.$$

Then put $\|x\|_{\mathrm{BMO}_{\mathcal{T}}^r} = \|x^*\|_{\mathrm{BMO}_{\mathcal{T}}^c}$ and $\|x\|_{\mathrm{BMO}_{\mathcal{T}}} = \max(\|x\|_{\mathrm{BMO}_{\mathcal{T}}^c}, \|x\|_{\mathrm{BMO}_{\mathcal{T}}^r})$. The completion of $\mathcal{M}$ for $\| \cdot \|_{\mathrm{BMO}_{\mathcal{T}}}$ is then defined as $\mathrm{BMO}_{\mathcal{T}} := \mathrm{BMO}(\mathcal{M}, \mathcal{T})$. We observe that $L_1(\mathcal{M})$ and $\mathrm{BMO}(\mathcal{M}, \mathcal{T})$ form a compatible couple of Banach spaces.

For later use, we record the following lemma here.

Lemma 2.2. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $\mathcal{T} = (T_t)_{t\geq 0}$ and $\mathcal{T}^l = (T_t^l)_{t\geq 0}, l \in \mathbb{N}$ be semigroups on $\mathcal{M}$. Suppose $\mathcal{T}$ is continuous in measure in the sense of Definition 2.1 (iii). If $\mathcal{T}_l$ is Markov for each l and if $T_t^l(x) \to T_t(x)$ in measure for $x \in \mathcal{M}$ as $l \to \infty$, then $\mathcal{T}$ is Markov.*

2.3. Markov dilations

Recall the following definition from [20, Page 717].

Definition 2.3. We say that a Markov semi-group $\mathcal{T} = (T_t)_{t\geq 0}$ on a finite von Neumann algebra $(\mathcal{M}, \tau)$ admits a standard Markov dilation if there exist:

(i) a finite von Neumann algebra $(\mathcal{B}, \tau_{\mathcal{B}})$;
(ii) an increasing filtration $\mathcal{B}_s, s \geq 0$ of $\mathcal{B}$;
(iii) trace preserving $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{B}_s$;

satisfying the property:

$$\mathbb{E}_{\mathcal{B}_s} \circ \pi_t = \pi_s \circ T_{t-s}, \qquad t \geq s,$$

where $\mathbb{E}_{\mathcal{B}_s} : \mathcal{B} \to \mathcal{B}_s$ are the $\tau_{\mathcal{B}}$-preserving conditional expectations.

Definition 2.4. We say that the dilation has continuous path if, for every $x \in \mathcal{M}$ the mapping $\mathbb{R}_{\geq 0} \to \mathcal{B} : t \to \pi_t(x)$ is continuous in measure.

In [20, Theorem 5.2 (i)] the following interpolation result was obtained.

Theorem 2.5. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $\mathcal{T}$ be a Markov semi-group on $\mathcal{M}$ that admits a standard Markov dilation. Then the complex interpolation space $[\mathrm{BMO}_{\mathcal{T}}, L_2^\circ(\mathcal{M})]_{\frac{2}{p}}$ equals $L_p^\circ(\mathcal{M})$ with equivalence of norms up to a constant $\simeq p$.*

2.4. The Heat semi-group and Euclidean BMO-spaces

In the Euclidean (non-finite) case we describe BMO-spaces separately. For $f \in L_2(\mathbb{R}^n)$ let

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) \exp(i\langle \xi, \eta \rangle) d\xi,$$

be its unitary Fourier transform. Define the gradient and Laplace operator

$$\nabla = \frac{1}{i}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad \Delta = -\nabla \cdot \nabla = \sum_{j=1}^n \frac{\partial^2}{\partial^2 x_j}.$$

So $\Delta \leq 0$. For $t \geq 0$ let $e^{t\Delta} : L_\infty(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$ be the normal unital completely positive map, which is also described by

$$\widehat{e^{t\Delta} f} = H_t^n \widehat{f}, \qquad f \in L_\infty(\mathbb{R}^n) \cap L_2(\mathbb{R}^n),$$

with positive definite function $H_t^n(\xi) = \exp(-t\|\xi\|_2^2), \xi \in \mathbb{R}^n$, i.e. the Heat kernel. Then $\mathcal{S} = (e^{t\Delta})_{t \geq 0}$ is a semi-group of completely positive maps that preserve the Haar integral on $L_\infty(\mathbb{R}^n)$. Moreover, for $f \in L_\infty(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ we have that $e^{t\Delta}(f) \in L_2(\mathbb{R}^n)$ and $t \mapsto e^{t\Delta}(f)$ is continuous for the norm of $L_2(\mathbb{R}^n)$.

We may define BMO-spaces with respect to the Heat semi-group as operator spaces as follows. Let $\mathcal{M}$ be a von Neumann algebra (not necessarily finite) and let $L_\infty(\mathbb{R}^n, \mathcal{M}) \simeq \mathcal{M} \otimes L_\infty(\mathbb{R}^n)$ be the space of all σ-weakly measurable essentially bounded functions $f : \mathbb{R}^n \to \mathcal{M}$. We may tensor amplify to get a new Markov semi-group $\mathcal{S}^{\otimes \mathcal{M}} := (\mathrm{id}_\mathcal{M} \otimes e^{t\Delta})_{t \geq 0}$. Consider the subspace $L_\infty^\circ(\mathbb{R}^n, \mathcal{M})$ of all functions $f \in L_\infty(\mathbb{R}^n, \mathcal{M})$ such that $(\mathrm{id}_\mathcal{M} \otimes e^{t\Delta})(f) \to 0$ in the σ-weak topology as $t \to \infty$. On $L_\infty^\circ(\mathbb{R}^n, \mathcal{M})$ we may define a column BMO-norm by,

$$\|f\|_{\mathrm{bmo}^c_{\mathcal{S}\otimes\mathcal{M}}} = \sup_{t \geq 0} \|(\mathrm{id}_\mathcal{M} \otimes e^{t\Delta})(f^* f) - (\mathrm{id}_\mathcal{M} \otimes e^{t\Delta})(f)^* (\mathrm{id}_\mathcal{M} \otimes e^{t\Delta})(f)\|^{\frac{1}{2}}.$$

Then set the row BMO- and the BMO-norm by,

$$\|f\|_{\mathrm{bmo}^r_{\mathcal{S}\otimes\mathcal{M}}} = \|f^*\|_{\mathrm{bmo}^c_{\mathcal{S}\otimes\mathcal{M}}}, \qquad \|f\|_{\mathrm{bmo}_{\mathcal{S}\otimes\mathcal{M}}} = \max(\|f\|_{\mathrm{bmo}^c_{\mathcal{S}\otimes\mathcal{M}}}, \|f\|_{\mathrm{bmo}^r_{\mathcal{S}\otimes\mathcal{M}}}).$$

The completion of the elements in $L_\infty^\circ(\mathbb{R}^n, \mathcal{M}) \cap L_2(\mathbb{R}^n, \mathcal{M})$ with finite $\|f\|_{\mathrm{bmo}_{\mathcal{S}\otimes\mathcal{M}}}$-norm with respect to $\|f\|_{\mathrm{bmo}_{\mathcal{S}\otimes\mathcal{M}}}$ is then denoted by $\mathrm{bmo}(\mathbb{R}^n, \mathcal{S}^{\otimes\mathcal{M}})$ or simply $\mathrm{bmo}_{\mathcal{S}\otimes\mathcal{M}}$. $\mathrm{bmo}(\mathbb{R}^n, \mathcal{S}^{\otimes\mathcal{M}})$ has the operator space structure given by the natural identification,

$$M_n(\mathrm{bmo}(\mathbb{R}^n, \mathcal{S}^{\otimes\mathcal{M}})) = \mathrm{bmo}(\mathbb{R}^n, \mathcal{S}^{\otimes M_n(\mathcal{M})}).$$

2.5. Completely bounded Fourier multipliers

Definition 2.6. A symbol $m : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ is called homogeneous if for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and $\lambda \in \mathbb{R}_{\geq 0}$ we have $m(\lambda\xi) = m(\xi)$. For such a symbol we extend it by $m(0) = 0$.

By spectral calculus $m(\nabla)$ is the Fourier multiplier with symbol m; more precisely $\widehat{m(\nabla)(f)} = \widehat{mf}$ where we recall that $f \mapsto \widehat{f}$ is the unitary Fourier transform. The following proposition with just bounds instead of complete bounds is a consequence of the Hörmander-Mikhlin multiplier theorem. For the complete bounds we base ourselves on [21]. Recall that $\mathcal{S} = (e^{t\Delta})_{t \geq 0}$ is the Heat semi-group on $\mathbb{R}^n$.

We call a function $f \in L_\infty(\mathbb{R}^n, \mathcal{N})$ trigonometric if it is in the linear span of functions $e_{\eta,x}(\xi) = e^{i\langle \eta, \xi \rangle} x, \xi, \eta \in \mathbb{R}^n, x \in \mathcal{N}$. Let $\mathcal{A}$ be the $*$-algebra of trigonometric functions in $L_\infty(\mathbb{R}^n, \mathcal{N})$.

Proposition 2.7. *Let $\mathcal{N}$ be a semi-finite von Neumann algebra. Let $m : \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ be a smooth homogeneous symbol and set $m(0) = 0$. For every trigonometric function $f \in L_\infty(\mathbb{R}^n, \mathcal{N})$, we have*

$$\|(m(\nabla) \otimes \mathrm{id}_{\mathcal{N}})(f)\|_{\mathrm{bmo}(L_\infty(\mathbb{R}^n)\otimes\mathcal{N},\mathcal{S}\otimes\mathrm{id}_{\mathcal{N}})} \le c_m \|f\|_\infty,$$

where the constant c_m depends only on the function m (that is, it does not depend either on $\mathcal{N}$ or f).

Proof. We first note that as $m(0) = 0$ we find that $m(\nabla)(f) \in L^\circ_\infty(\mathbb{R}^n, \mathcal{N})$ for every $f \in \mathcal{A}$. We check Conditions (i) and (ii) of [21, Lemma 2.3]. As m is bounded as a function (by homogeneity) [21, Remark 2.4] immediately gives Condition (i). Next homogeneity of m implies that there exists a constant c_n such that for all multi-indices β with $|\beta| \le n + 2$,

$$|(\partial_\beta\, m)(\xi)| \le c_n \|\xi\|_2^{-|\beta|}, \qquad \xi \in \mathbb{R}^n\backslash\{0\}.$$

This implies Condition (ii) for the Fourier transform $k = \widehat{m}$ of [21, Lemma 2.3] by [32, p. 75, Theorem 6]. Then [21, Lemma 2.3] shows that there is a constant c_m, only depending on m, such that for $f \in \mathcal{A}$ we have

$$\|(m(\nabla) \otimes \mathrm{id}_{\mathcal{N}})(f)\|_{\mathrm{bmo}^c(L_\infty(\mathbb{R}^n)\otimes\mathcal{N},\mathcal{S}\otimes\mathrm{id}_{\mathcal{N}})} \le c_m \|f\|_\infty. \tag{2.3}$$

This yields the column estimate. Further, as we have, for $f \in \mathcal{A}$,

$$\begin{aligned}&\|(m(\nabla) \otimes \mathrm{id}_{\mathcal{N}})(f)\|_{\mathrm{bmo}^r(L_\infty(\mathbb{R}^n)\otimes\mathcal{N},\mathcal{S}\otimes\mathrm{id}_{\mathcal{N}})} \\ &\qquad = \|((m^\vee)(\nabla) \otimes \mathrm{id}_{\mathcal{N}})(\overline{f})\|_{\mathrm{bmo}^c(L_\infty(\mathbb{R}^n)\otimes\mathcal{N},\mathcal{S}\otimes\mathrm{id}_{\mathcal{N}})},\end{aligned}$$

with $m^\vee(\xi) = \overline{m(-\xi)}$ we also get the row estimate; in combination with the column estimate (2.3) we see that there is a constant c_m such that for every $f \in \mathcal{A}$ we have

$$\|(m(\nabla) \otimes \mathrm{id}_{\mathcal{N}})(f)\|_{\mathrm{bmo}(L_\infty(\mathbb{R}^n)\otimes\mathcal{N},\mathcal{S}\otimes\mathrm{id}_{\mathcal{N}})} \le c_m \|f\|_\infty. \quad \square$$

2.6. Double operator integrals

We recall the following from [11]. Let $\mathcal{M}$ be a von Neumann algebra (not necessarily finite). Let $A_l \in \mathcal{M}, 1 \le l \le n$ be commuting self-adjoint operators. Briefly set $\mathbf{A} = (A_1, \ldots, A_n)$. Let $E : \mathcal{B}(\mathbb{R}^n) \to \mathcal{M}$ be the joint spectral measure of $\mathbf{A}$ on the Borel sets $\mathcal{B}(\mathbb{R}^n)$. So that we have spectral decompositions $A_l = \int_{\mathbb{R}^n} \xi_l dE(\xi)$ with ξ_l the l-th coordinate function. We define a spectral measure $F : \mathcal{B}(\mathbb{R}^{2n}) \to B(L_2(\mathcal{M}))$ by $F(X \times Y)(x) = E(X)xE(Y)$ where $X, Y \subseteq \mathbb{R}^n$ are Borel sets and $x \in L_2(\mathcal{M})$. So F

takes values in the projections on $L_2(\mathcal{M})$. Then for $\phi : \mathbb{R}^{2n} \to \mathbb{C}$ a bounded Borel function we set the double operator integral,

$$\mathcal{I}_\phi^{\mathbf{A}} = \int_{\mathbb{R}^{2n}} \phi(\eta_1, \eta_2) dF(\eta_1, \eta_2) \in B(L_2(\mathcal{M})),$$

we shall also use the notation,

$$\mathcal{I}_\phi^{\mathbf{A}}(x) = \int_{\mathbb{R}^{2n}} \phi(\eta_1, \eta_2) dE(\eta_1) x E(\eta_2), \qquad x \in L_2(\mathcal{M}).$$

In case $\mathbf{A}$ is just a single operator A we write $\mathcal{I}_\phi^A$.

In this paper we shall be interested in extensions of $\mathcal{I}_\phi^A$ to BMO- and L_p-spaces associated with $\mathcal{M}$. Here we record the relation that if $\mathcal{M}$ is finite and $A = \sum_{\lambda \in \sigma(A)} \lambda p_\lambda$ has discrete spectrum with $p_\lambda = E(\{\lambda\})$ then,

$$\mathcal{I}_\phi^A(x) = \sum_{\lambda, \mu \in \sigma(A)} \phi(\lambda, \mu) p_\lambda x p_\mu, \quad x \in \mathcal{M}.$$

For $B \in \mathcal{M}$ self-adjoint set $\lfloor B \rfloor := \sum_{i \in \mathbb{Z}} i \chi_{[i, i+1)}(B)$ with χ the indicator function. We shall repeatedly make use of the following Lemma 2.8 without further reference.

Lemma 2.8. *Let $A \in \mathcal{M}$ be self-adjoint and for $l \in \mathbb{N}_{\geq 1}$ let $A_l = l^{-1} \lfloor lA \rfloor$. Let $\phi : \mathbb{R}^2 \to \mathbb{C}$ be continuous. For every $x \in L_2(\mathcal{M})$ we have $\| \mathcal{I}_\phi^{A_l}(x) - \mathcal{I}_\phi^A(x) \|_2 \to 0$ as $l \longrightarrow \infty$.*

Proof. We have $\mathcal{I}_\phi^{A_l} = \mathcal{I}_{\phi_l}^A$ with $\phi_l(\xi) = \phi(l^{-1} \lfloor l\xi \rfloor)$. Then $\| \mathcal{I}_{\phi_l}^A(x) - \mathcal{I}_\phi^A(x) \|_2 \to 0$, cf. [9, Lemma 5.1]. □

2.7. Vector valued double operator integrals

We define vector valued analogues of double operator integrals. To this end suppose that $\mathcal{M}$ and $\mathcal{N}$ are finite von Neumann algebras. Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{N}$ be an essentially bounded σ-weakly continuous function. Then in particular we also have the same map $\phi : \mathbb{R}^n \times \mathbb{R}^n \to L_2(\mathcal{N})$ and this mapping is norm continuous for $L_2(\mathcal{N})$. As before let $\mathbf{A}$ be an n-tuple of mutually commuting self-adjoint operators. For $x \in L_2(\mathcal{M})$ we define the double operator integral $\mathcal{I}_\phi^{\mathbf{A}}(x)$ as the unique element in $L_2(\mathcal{N}) \otimes L_2(\mathcal{M})$ that is characterized by,

$$\langle \mathcal{I}_\phi^{\mathbf{A}}(x), \xi \otimes \eta \rangle = \langle \mathcal{I}_{\xi^* \circ \phi}^{\mathbf{A}}(x), \eta \rangle_{L_2(\mathcal{M}), L_2(\mathcal{M})}, \qquad \xi \in L_2(\mathcal{N}), \eta \in L_2(\mathcal{M}). \tag{2.4}$$

Here $\xi^*(\eta) = \langle \eta, \xi \rangle$ so that $\xi^* \circ \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous bounded function and the right hand side of (2.4) is the usual double operator integral.

In case the spectrum of $\mathbf{A}$ is finite our constructions simplify. We may view $\mathcal{I}_\phi^{\mathbf{A}}(x), x \in L_2(\mathcal{M}) \cap \mathcal{M}$ as an element of $\mathcal{N} \otimes \mathcal{M}$ given by,

$$\mathcal{I}_\phi^{\mathbf{A}}(x) = \sum_{i,j \in \sigma(\mathbf{A})} \phi(i,j) \otimes p_i x p_j,$$

where $p_i = \prod_{k=1}^n \chi_{\{i_k\}}(A_k)$ is a spectral projection of $\mathbf{A}$.

2.8. Exterior algebra

Let $H_\mathbb{R}$ be a real Hilbert space and let $H = H_\mathbb{R} \otimes \mathbb{C}$ be its complexified Hilbert space. Let $F^\circ = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^\infty H^{\otimes n}$ with unit vector Ω (the vacuum vector). The pre-inner product on F° is set by,

$$\langle \xi_1 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \ldots \otimes \eta_k \rangle = \delta_{n,k} \sum_{\sigma \in S_n} (-1)^{i(\sigma)} \langle \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)}, \eta_1 \otimes \ldots \otimes \eta_k \rangle$$

where $i(\sigma)$ is the number of inversions on σ, i.e. then number of pairs (a,b) with $a < b$ such that $\sigma(b) < \sigma(a)$. Let F be the completion of F° modulo its degenerate part. We denote $\xi_1 \wedge \ldots \wedge \xi_n \in F$ for the equivalence class of $\xi_1 \otimes \ldots \otimes \xi_n \in F^\circ$. So with the wedge product F is the usual exterior algebra (or Clifford algebra with the zero quadratic form; note if $\dim(H) < \infty$ also $\dim(F) < \infty$). For $\xi \in H$ we set

$$l(\xi)\eta = \xi \wedge \eta, \qquad l^*(\xi)(\eta_1 \wedge \ldots \wedge \eta_n) = \sum_{k=1}^n (-1)^k \langle \eta_k, \xi \rangle \eta_1 \wedge \ldots \wedge \widehat{\eta_k} \wedge \ldots \wedge \eta_n,$$

and extend them to bounded operators on F. Here $\widehat{\eta_k}$ means that the k-th wedge term is excluded from the term. $l^*(\xi)$ is the adjoint of $l(\xi)$. We set $s(\xi) = l(\xi) + l^*(\xi)$. And further $\Gamma := \Gamma(H_\mathbb{R}) := \{s(\xi) \mid \xi \in H_\mathbb{R}\}''$. We record the fundamental property of the exterior algebra:

$$s(\xi)s(\eta) + s(\eta)s(\xi) = 2\langle \xi, \eta \rangle, \quad \xi, \eta \in H_\mathbb{R}. \tag{2.5}$$

The von Neumann algebra Γ has faithful normal tracial state $\tau_\Omega(x) = \langle x\Omega, \Omega \rangle$, the vacuum state, cf. [5] for these results in greater generality.

3. Discrete Markov dilations

We show how Markov dilations of discrete semi-groups can be used to get Markov dilations of a continuous one through ultraproduct techniques. In particular we show that we can always guarantee path continuity (in measure topology) of Markov dilations for finite von Neumann algebras.

In the special case, when $\mathsf{G} = \mathbb{R}_{\geq 0}$, the definition below coincides with Definition 2.3 above.

Definition 3.1. Let G be a subsemi-group of $\mathbb{R}_{\geq 0}$. We say that a Markov semi-group $(T_t)_{t\in\mathsf{G}}$ acting on the probability space $(\mathcal{M}, \tau)$ admits a standard Markov dilation if there exist:

(i) a finite von Neumann algebra $(\mathcal{B}, \tau_{\mathcal{B}})$;
(ii) an increasing filtration $(\mathcal{B}_t)_{t\in\mathsf{G}}$ of $\mathcal{B}$;
(iii) trace preserving $*$-homomorphisms $\pi_t : \mathcal{M} \to \mathcal{B}_t$, $t \in \mathsf{G}$;

such that for $t, s \in \mathsf{G}$ with $s \geq t$, we have

$$\mathbb{E}_{\mathcal{B}_t} \circ \pi_s = \pi_t \circ T_{s-t}.$$

Here, $\mathbb{E}_{\mathcal{B}_t} : \mathcal{B} \to \mathcal{B}_t$ are the $\tau_{\mathcal{B}}$-preserving conditional expectations.

Theorem 3.2. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group. If the Markov semi-group $(T_t)_{t\in\epsilon\mathbb{N}_{\geq 0}}$ admits a standard Markov dilation for every $\epsilon > 0$, then so does $\mathcal{T}$. Moreover, the dilation has continuous path.*

We prove this theorem in the next couple of lemmas. We shall repeatedly make use of the fact that the measure topology and the L_2-topology coincide on the unit ball of a finite von Neumann algebra.

Lemma 3.3. *Set the semi-group $\mathsf{G} = \cup_{l\in\mathbb{N}_{\geq 0}} 2^{-l}\mathbb{N}_{\geq 0}$. Under the assumptions of Theorem 3.2, $(T_t)_{t\in\mathsf{G}}$ admits a standard Markov dilation.*

Proof. For $l \geq 0$, let $\mathsf{G}_l = 2^{-l}\mathbb{N}_{\geq 0}$ so that $\mathsf{G} = \cup_{l\geq 0}\mathsf{G}_l$. We see G as a subsemi-group of $\mathbb{R}_{\geq 0}$ and equip it with the Euclidean topology. By assumption (with $\epsilon = 2^{-l}$), there exists:

(i) a finite von Neumann algebra $(\mathcal{B}^l, \tau_{\mathcal{B}^l})$;
(ii) an increasing filtration $(\mathcal{B}^l_m)_{m\in\mathsf{G}_l}$ of $\mathcal{B}^l$;
(iii) trace preserving $*$-homomorphisms $\pi^l_m : \mathcal{M} \to \mathcal{B}^l_m$;

such that for $m, k \in \mathsf{G}_l$ with $k \geq m$, we have

$$\mathbb{E}_{\mathcal{B}^l_m} \circ \pi^l_k = \pi^l_m \circ T_{k-m}.$$

Set Ocneanu ultrapowers (see e.g. [2]) $(\mathcal{B}, \tau_{\mathcal{B}}) = \prod_{l,\omega}(\mathcal{B}^l, \tau_{\mathcal{B}^l})$ and $\mathcal{B}_m = \prod_{l,\omega}(\mathcal{B}^l_m, \tau_{\mathcal{B}_l})$ for $m \in \mathsf{G}$. The second ultraproduct runs over large enough l, namely such that $m \in \mathsf{G}_l$.

Fix $m_1, m_2 \in \mathsf{G}$ such that $m_2 \leq m_1$ and choose l_0 such that $m_1, m_2 \in \mathsf{G}_l$ for all $l \geq l_0$. We have

$$\mathcal{B}_{m_1} = \prod_{l,\omega}(\mathcal{B}^l_{m_1}, \tau_{\mathcal{B}_l}), \quad \mathcal{B}_{m_2} = \prod_{l,\omega}(\mathcal{B}^l_{m_2}, \tau_{\mathcal{B}_l}),$$

where ultrafilter runs over all $l \geq l_0$. Since for every $l \geq l_0$ we have

$$\mathcal{B}^l_{m_2} \subset \mathcal{B}^l_{m_1},$$

it follows that

$$\mathcal{B}_{m_2} \subset \mathcal{B}_{m_1}.$$

Therefore, we have an increasing filtration. Let $\mathbb{E}_{\mathcal{B}_m}$ be the trace preserving conditional expectation of $\mathcal{B}$ onto $\mathcal{B}_m$. Note that

$$\mathbb{E}_{\mathcal{B}_m}((x_l)_{l,\omega}) = (\mathbb{E}_{\mathcal{B}^l_m}(x_l))_{l,\omega}.$$

For $m \in \mathsf{G}$, define a trace preserving $*$-homomorphism $\pi_m : \mathcal{M} \to \mathcal{B}_m$ by the formula

$$\pi_m(x) = (\pi^l_m(x))_{l,\omega}.$$

For $m, k \in \mathsf{G}$ with $k \geq m$, we have

$$(\mathbb{E}_{\mathcal{B}_m})(\pi_k(x)) = (\mathbb{E}_{\mathcal{B}^l_m}(\pi^l_k(x)))_{l,\omega} = (\pi^l_m(T_{k-m}(x)))_{l,\omega} = \pi_m(T_{k-m}(x)). \quad \square$$

Lemma 3.4. *Let G be a subsemi-group in $\mathbb{R}_+$ and let $(T_t)_{t\geq 0}$ be a Markov semi-group. If $(T_t)_{t\in\mathsf{G}}$ admits a standard Markov dilation, then for every $x \in \mathcal{M}$,*

$$\|\pi_t(x) - \pi_s(x)\|_2^2 \leq 2\|x\|_2 \|x - T_{|s-t|}(x)\|_2, \quad t, s \in \mathsf{G}.$$

Proof. Without loss of generality, $s \geq t$. For $x \in \mathcal{M}$, we have by [34, p. 211 (3) and (4)],

$$\begin{aligned}\tau_{\mathcal{B}}(\pi_t(x)^*\pi_s(x)) &= (\tau_{\mathcal{B}} \circ \mathbb{E}_{\mathcal{B}_t})(\pi_t(x)^*\pi_s(x)) \\ &= \tau_{\mathcal{B}}(\pi_t(x)^*\mathbb{E}_{\mathcal{B}_t}(\pi_s(x))) = \tau_{\mathcal{B}}(\pi_t(x)^*\pi_t(T_{s-t}(x))).\end{aligned}$$

Since π_t is trace preserving, it follows that

$$\tau_{\mathcal{B}}(\pi_t(x)^*\pi_s(x)) = \tau(x^*T_{s-t}(x)), \quad t, s \in \mathsf{G}, \quad s \geq t.$$

Similarly,

$$\tau_{\mathcal{B}}(\pi_s(x)^*\pi_t(x)) = \tau(xT_{s-t}(x^*)), \quad t, s \in \mathsf{G}, \quad s \geq t.$$

Therefore, we have

$$\begin{aligned}\|\pi_t(x) - \pi_s(x)\|_2^2 =& \tau_{\mathcal{B}}(\pi_t(x)^*\pi_t(x)) + \tau_{\mathcal{B}}(\pi_s(x)^*\pi_s(x)) \\ &- \tau_{\mathcal{B}}(\pi_t(x)^*\pi_s(x)) - \tau_{\mathcal{B}}(\pi_s(x)^*\pi_t(x)) \\ =& 2\tau(x^*x) - \tau(x^*T_{s-t}(x)) - \tau(xT_{s-t}(x^*)) \\ =& \tau(x^*(x - T_{s-t}(x))) + \tau(x(x - T_{s-t}(x))^*).\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we conclude the argument. □

We call a family $(\pi_t)_{t\geq 0}$ of $*$-homomorphisms $\mathcal{M} \to \mathcal{B}$ continuous in the point-measure topology if for every $x \in \mathcal{M}$ we have that $t \mapsto \pi_t(x)$ is continuous in measure.

Lemma 3.5. *Let G be a dense subsemi-group in $\mathbb{R}_+$ and let $(T_t)_{t\geq 0}$ be a Markov semi-group. If $(T_t)_{t\in\mathsf{G}}$ admits a standard Markov dilation, then $(\pi_t)_{t\in\mathsf{G}}$ extends to a family $(\pi_t)_{t\geq 0}$ of trace preserving *-homomorphisms so that, for every $x \in \mathcal{M}$, the mapping $t \to \pi_t(x)$ is continuous in measure.*

Proof. From the fact that Markov semi-groups are by definition continuous in measure this is a direct consequence of Lemma 3.4. □

Proof of Theorem 3.2. Let $\mathsf{G} = \cup_{l\in\mathbb{N}_{\geq 0}} 2^{-l}\mathbb{N}_{\geq 0}$ be the set of all non-negative binary rationals. By Lemma 3.3, $(T_t)_{t\in\mathsf{G}}$ admits a standard Markov dilation. Set

$$\mathcal{B}_t = \Big(\bigcup_{\substack{u\leq t\\ u\in\mathsf{G}}} \mathcal{B}_u \Big)''.$$

In the following equations we shall take the limit $u \to t$ over the sets in the subscript of the limit. By construction, we have

$$\mathbb{E}_{\mathcal{B}_t}(w) = \lim_{\substack{u\leq t\\ u\in\mathsf{G}}} \mathbb{E}_{\mathcal{B}_u}(w), \quad w \in \mathcal{B},$$

in the L_2-topology. Let $(\pi_t)_{t\geq 0}$ be an L_2-continuous family of trace preserving $*$-homomorphisms constructed in Lemma 3.5.

If $s \geq t$ and $s \in \mathsf{G}$, then

$$\mathbb{E}_{\mathcal{B}_t}(\pi_s(x)) = \lim_{\substack{u\leq t\\ u\in\mathsf{G}}} \mathbb{E}_{\mathcal{B}_u}(\pi_s(x)) = \lim_{\substack{u\leq t\\ u\in\mathsf{G}}} \pi_u(T_{s-u}(x)).$$

By assumption, we have

$$\lim_{\substack{u\leq t\\ u\in\mathsf{G}}} T_{s-u}(x) = \lim_{\substack{u\leq t\\ u\in\mathsf{G}}} T_{t-u}(T_{s-t}(x)) = T_{s-t}(x)$$

in the L_2-norm. Each π_u, $u \geq 0$, is a trace preserving $*$-homomorphism and therefore contracts the L_2-norm. Hence, as $(\pi_t)_{t\in\mathsf{G}}$ are $*$-homomorphisms of a Markov dilation we see by Lemma 3.5 that we have a limit in measure,

$$\mathbb{E}_{\mathcal{B}_t}(\pi_s(x)) = \lim_{\substack{u\leq t\\ u\in\mathsf{G}}} \pi_u(T_{s-t}(x)) \overset{L.3.5}{=} \pi_t(T_{s-t}(x)).$$

Let now $s \geq t \geq 0$. If $s_k \in \mathsf{G}$, $s_k \searrow s$, then $\pi_{s_k}(x) \to \pi_s(x)$ in the L_2-norm. Since $\mathbb{E}_{\mathcal{B}_t}$ contracts the L_2-norm, it follows that

$$\mathbb{E}_{\mathcal{B}_t}(\pi_s(x)) = \lim_{k\to\infty} \mathbb{E}_{\mathcal{B}_t}(\pi_{s_k}(x)) = \lim_{k\to\infty} \pi_t(T_{s_k-t}(x))$$

We have $T_{s_k-t}(x) = T_{s_k-s}(T_{s-t}(x))$. By assumption, $T_{s_k-s}(x) \to x$ in the L_2-norm and hence,

$$\mathbb{E}_{\mathcal{B}_t}(\pi_s(x)) = \lim_{k\to\infty} \pi_t(T_{s_k-t}(x)) = \pi_t(T_{s-t}(x)).$$

This completes the proof. □

As a direct corollary of Lemma 3.4 we obtain the following automatic continuity property for finite von Neumann algebras.

Corollary 3.6. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group that admits a standard Markov dilation. Then the dilation has continuous path.*

4. Markov dilations for semi-groups of double operator integrals

In [31] Ricard proved that discrete semi-groups of Schur multipliers admit a standard Markov dilation. In this section we show that also semi-groups of double operator integrals have a standard Markov dilation. In what follows, this fact, together with Theorem 2.5 allows us to interpolate between respective BMO-space and L_2-space.

Theorem 4.1. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group such that there exists $A = A^* \in \mathcal{M}$ and $\phi_t : \mathbb{R}^2 \to \mathbb{C}$ continuous with $T_t = \mathcal{I}^A_{\phi_t}, t \geq 0$. Then the semi-group $\mathcal{T}$ admits a standard Markov dilation.*

The crucial part of the argument is similar to that of Ricard [31].

Proposition 4.2. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $A = A^* \in \mathcal{M}$ be such that* $\mathrm{spec}(A) \subset \mathbb{Z}$. *Let $\phi : \mathbb{Z}^2 \to \mathbb{R}$ be a positive matrix such that for all i we have $\phi(i,i) = 1$. The semi-group $((\mathcal{I}^A_\phi)^n)_{n\in\mathbb{N}_{\geq 0}}$ acting on $\mathcal{M}$ admits a standard Markov dilation.*

Proof. As A is bounded, we have $\mathrm{spec}(A) \subset \{-n, 1-n, \cdots, n-1, n\}$ for some $n \in \mathbb{N}$. Denote for brevity $p_i = \chi_{\{i\}}(A)$, $-n \leq i \leq n$. As ϕ is positive, the expression

$$\langle \xi, \eta \rangle = \sum_{i,j=-n}^{n} \phi(i,j)\xi_i\eta_j, \tag{4.1}$$

defines a positive (possibly degenerate) inner product on $\mathbb{R}^{2n+1}$. Let $H_{\mathbb{R}}$ be $\mathbb{R}^{2n+1}$ *equipped with inner product* (4.1) and quotienting out the degenerate part. Construct the associated exterior algebra $\Gamma = \Gamma(H_{\mathbb{R}})$ from it.

Let $\{e_i\}_{i=-n}^{n}$ be the standard orthonormal basis of $\mathbb{R}^{2n+1}$ viewed as elements (e.g. equivalence classes) of $H_{\mathbb{R}}$. Let $\mathcal{B} = \mathcal{M} \otimes \Gamma^{\otimes\infty}$ with tensor product trace $\tau_{\mathcal{B}} = \tau \otimes \tau_{\Omega}^{\otimes\infty}$ (tensor products constructed from the vacuum state, see [35] for infinite tensor powers). We infer from (2.5) that $s(e_i)^2 = 1$. Define a unitary

$$u = \sum_{i=-n}^{n} p_i \otimes s(e_i) \in \mathcal{M} \otimes \Gamma,$$

which we view as a unitary in the first two tensor legs of $\mathcal{B} = \mathcal{M} \otimes \Gamma^{\otimes\infty} = \mathcal{M} \otimes \Gamma \otimes \Gamma^{\otimes\infty}$ by identifying it with $u \otimes 1_{\Gamma}^{\otimes\infty}$. Let S be the tensor shift on $\Gamma^{\otimes\infty}$ determined by,

$$S(x_1 \otimes \ldots \otimes x_m \otimes 1 \otimes \ldots) = 1 \otimes x_1 \otimes \ldots \otimes x_m \otimes 1 \otimes \ldots,$$

and then set the $*$-homomorphism $\beta(x) = u^*(\mathrm{id}_{\mathcal{M}} \otimes S)(x)u$. For $k \geq 0$, define a trace-preserving $*$-homomorphism $\pi_k : \mathcal{M} \to \mathcal{B}$ as follows:

$$\pi_0 : x \mapsto x \otimes 1 \otimes 1 \ldots$$

and

$$\pi_k : x \mapsto (\beta^k \circ \pi_0)(x), \qquad k \in \mathbb{N}_{\geq 0}.$$

Using induction we obtain that,

$$\pi_k(x) = \sum_{i,j=-n}^{n} p_i x p_j \otimes (s(e_i)s(e_j))^{\otimes k} \otimes 1_{\Gamma}^{\otimes\infty} \in \mathcal{B}. \tag{4.2}$$

Indeed,

$$\begin{aligned}
\pi_{k+1}(x) =& u^*(\mathrm{id}_{\mathcal{M}} \otimes S)\left(\sum_{i,j=-n}^{n} p_i x p_j \otimes (s(e_i)s(e_j))^{\otimes k} \otimes 1_{\Gamma}^{\otimes\infty}\right) u \\
=& u^*\left(\sum_{i,j=-n}^{n} p_i x p_j \otimes 1_{\Gamma} \otimes (s(e_i)s(e_j))^{\otimes k} \otimes 1_{\Gamma}^{\otimes\infty}\right) u \\
=& \sum_{i,j=-n}^{n} p_i x p_j \otimes (s(e_i)s(e_j))^{\otimes k+1} \otimes 1_{\Gamma}^{\otimes\infty}.
\end{aligned}$$

Define the increasing family of subalgebras $\mathcal{B}_m$ as the von Neumann algebras $\mathcal{M} \otimes \Gamma^{\otimes m} \otimes 1_{\Gamma}^{\infty} \subseteq \mathcal{B}$. If $k \geq m$ and if $x \in \mathcal{M}$ is such that $p_i x p_j = x$, then a direct computation yields

$$\mathbb{E}_m(x \otimes (s(e_i)s(e_j))^{\otimes k} \otimes 1_\Gamma^{\otimes \infty}) = \tau_\Omega(s(e_i)s(e_j))^{k-m} x \otimes (s(e_i)s(e_j))^{\otimes m} \otimes 1_\Gamma^{\otimes \infty}.$$

We get that for $x \in \mathcal{M}$ and for $k \geq m$,

$$(\mathbb{E}_m \circ \pi_k)(x) = \sum_{i,j=-n}^{n} \tau_\Omega(s(e_i)s(e_j))^{k-m}\, p_i x p_j \otimes (s(e_i)s(e_j))^{\otimes m} \otimes 1_\Gamma^{\infty}.$$

By (2.5) and (4.1),

$$\tau_\Omega(s(e_i)s(e_j)) = \langle e_i, e_j \rangle = \phi(i,j).$$

Therefore,

$$\tau_\Omega(s(e_i)s(e_j))^{k-m}\, p_i x p_j = \phi(i,j)^{k-m} p_i x p_j = (\mathcal{I}_\phi^A)^{k-m}(p_i x p_j) = p_i((\mathcal{I}_\phi^A)^{k-m}(x))p_j.$$

Hence,

$$(\mathbb{E}_m \circ \pi_k)(x) = \sum_{i,j=-n}^{n} p_i((\mathcal{I}_\phi^A)^{k-m}(x))p_j \otimes (s(e_i)s(e_j))^{\otimes m} \otimes 1_\Gamma^{\infty}. \tag{4.3}$$

By (4.2) and (4.3) we get,

$$(\mathbb{E}_m \circ \pi_k)(x) = \pi_m((\mathcal{I}_\phi^A)^{k-m}(x)).$$

This completes the proof. □

The passage to operators A with arbitrary spectrum (not just integral) requires the approximation result below.

Proposition 4.3. *Let T and T_l be unital trace preserving maps on $\mathcal{M}$ such that $T_l(x) \to T(x)$ strongly for all $x \in \mathcal{M}$. If every semi-group $(T_l^n)_{n \in \mathbb{N}_{\geq 0}}$ admits a standard Markov dilation, then so does the semi-group $(T^n)_{n \in \mathbb{N}_{\geq 0}}$.*

Proof. By assumption, there exists

(i) a finite von Neumann algebra $(\mathcal{B}^l, \tau_{\mathcal{B}^l})$;
(ii) an increasing filtration $(\mathcal{B}_m^l)_{m \in \mathbb{N}_{\geq 0}}$;
(iii) trace preserving $*$-homomorphisms $\pi_m^l : \mathcal{M} \to \mathcal{B}_m^l$;

such that for $m, k \in \mathbb{N}_{\geq 0}$ with $k \geq m$, we have

$$\mathbb{E}_{\mathcal{B}_m^l} \circ \pi_k^l = \pi_m^l \circ T_l^{k-m}.$$

Fix a non-principal ultrafilter ω on $\mathbb{N}_{\geq 1}$ and let $\mathcal{B} = \prod_{l,\omega}(\mathcal{B}^l, \tau_l)$ and $\mathcal{B}_m = \prod_{\omega}(\mathcal{B}^l_m, \tau_l)$ be the Ocneanu ultrapowers, see [2]. Let $\mathbb{E}_m : \mathcal{B} \to \mathcal{B}_m$ be the expectation preserving the ultraproduct trace $\tau_{\mathcal{B}}$ on $\mathcal{B}$. We have that $\{\mathcal{B}_m\}_{m\geq 0}$ is an increasing filtration of subalgebras of $\mathcal{B}$ and that

$$\mathbb{E}_m((x_l)_{l,\omega}) = (\mathbb{E}^l_m(x_l))_{l,\omega}.$$

For every $m \geq 0$, define a trace-preserving $*$-homomorphism $\pi_m : \mathcal{M} \to \mathcal{B}_m$ by setting

$$\pi_m : \mathcal{M} \to \mathcal{B}_m : x \mapsto (\pi^l_m(x))_{l,\omega}.$$

We find that for $x \in \mathcal{M}$, $k, m \in \mathbb{N}_{\geq 0}$ and $k \geq m$,

$$(\mathbb{E}_m \circ \pi_k)(x) = \Big((\mathbb{E}^l_m \circ \pi^l_k)(x)\Big)_{l,\omega} = \Big((\pi^l_m \circ T_l^{k-m})(x)\Big)_{l,\omega}. \tag{4.4}$$

Since π^l_m is trace preserving, it follows that

$$\left\| \Big((\pi^l_m \circ T_l^{k-m})(x)\Big)_{l,\omega} - \Big((\pi^l_m \circ T^{k-m})(x)\Big)_{l,\omega} \right\|_2 = \lim_{l\to\omega} \left\| T_l^{k-m}(x) - T^{k-m}(x) \right\|_2.$$

By the triangle inequality, we have

$$\begin{aligned}\left\| T_l^{k-m}(x) - T^{k-m}(x) \right\|_2 &\leq \sum_{j=0}^{k-m-1} \left\| \Big(T_l^{k-m-j-1} \circ (T_l - T) \circ T^j\Big)(x) \right\|_2 \\ \leq \sum_{j=0}^{k-m-1} \|T_l\|_{L_2\to L_2}^{k-m-1-j} \left\| (T_l - T)(T^j x) \right\|_2 &\leq \sum_{j=0}^{k-m-1} \left\| (T_l - T)(T^j x) \right\|_2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\| \Big((\pi^l_m \circ T_l^{k-m})(x)\Big)_{l,\omega} - \Big((\pi^l_m \circ T^{k-m})(x)\Big)_{l,\omega} \right\|_2 \\ \leq &(k-m) \max_{0\leq j<k-m} \limsup_{l\to\infty} \left\| (T_l - T)(T^j x) \right\|_2.\end{aligned}$$

By assumption, we have that

$$(T_l - T)(T^j x) \to 0, \quad l \to \infty,$$

in L_2-norm. We conclude that

$$\Big((\pi^l_m \circ T_l^{k-m})(x)\Big)_{l,\omega} - \Big((\pi^l_m \circ T^{k-m})(x)\Big)_{l,\omega} = 0.$$

Substituting into (4.4), we obtain

$$(\mathbb{E}_m \circ \pi_k)(x) = \Big((\pi_m^l \circ T^{k-m})(x) \Big)_{l,\omega} = \pi_m(T^{k-m}(x)). \quad \square$$

Proof of Theorem 4.1. Fix $l \in \mathbb{N}_{\geq 1}$ and $\varepsilon > 0$. The matrix $(\phi_\epsilon(\frac{i}{l}, \frac{j}{l}))_{i,j\in\mathbb{Z}}$ and the operator

$$\lfloor lA \rfloor := \sum_{i\in\mathbb{Z}} i\chi_{[i,i+1)}(lA)$$

satisfy the condition of Proposition 4.2. Hence, the semi-group $((\mathcal{I}_{\phi_\epsilon}^{\frac{1}{l}\lfloor lA \rfloor})^n)_{n\in\mathbb{N}_{\geq 0}}$ admits a standard Markov dilation. Since ϕ_ϵ is continuous, it follows that

$$\mathcal{I}_{\phi_\epsilon}^{\frac{1}{l}\lfloor lA \rfloor}(x) \to \mathcal{I}_{\phi_\epsilon}^{A}(x), \quad l \to \infty,$$

in L_2. By Proposition 4.3, the semi-group $((\mathcal{I}_{\phi_\epsilon}^{A})^n)_{n\in\mathbb{N}_{\geq 0}}$ admits a standard Markov dilation. By Theorem 3.2 so does the semi-group $(\mathcal{I}_{\phi_t}^{A})_{t\geq 0}$. $\square$

5. Complements on reversed Markov dilations

As we believe these results are of independent use, we also state the corresponding results for reversed Markov dilations. These shall not be used in the current paper. The proofs are completely analogous to the proofs in Sections 3 and 4.

Definition 5.1. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group on a finite von Neumann algebra $(\mathcal{M}, \tau)$. Let G be a subsemi-group of $\mathbb{R}_{\geq 0}$. We say that $(T_t)_{t\in\mathsf{G}}$ admits a reversed Markov dilation if there exist:

(i) a finite von Neumann algebra $(\mathcal{B}, \tau_{\mathcal{B}})$;
(ii) a decreasing filtration $\mathcal{B}_s, s \geq 0$ with conditional expectations $\mathbb{E}_{\mathcal{B}_s} : \mathcal{B} \to \mathcal{B}_s$;
(iii) trace preserving $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{B}_s$

such that the following property holds

$$\mathbb{E}_{\mathcal{B}_t} \circ \pi_s = \pi_t \circ T_{t-s}, \qquad s, t \in \mathsf{G}, t \geq s.$$

Definition 5.2. We say that a reversed Markov dilation has continuous path if for every $x \in \mathcal{M}$ the mapping $\mathbb{R}_{\geq 0} \to \mathcal{B} : t \to \pi_t(x)$ is continuous in measure.

In the same way as we proved Theorem 3.2 we may now obtain the following result.

Theorem 5.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a Markov semi-group. If $(T_t)_{t\in\epsilon\mathbb{N}_{\geq 0}}$ admits a reversed Markov dilation for every $\epsilon > 0$, then so does $\mathcal{T}$. Moreover, the dilation has continuous path.*

In [31, p. 4370] Ricard shows that a semi-group of Schur multipliers $(T_\phi^k)_{k\in\mathbb{N}_{\geq 0}}$ admits a reversed Markov dilation. By essentially the same argument as in Theorem 4.1 we also get reversed Markov dilations for double operator integrals.

Theorem 5.4. *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $A = A^* \in \mathcal{M}$. Let $(\mathcal{I}^A_{\phi_t})_{t\geq 0}$ be a Markov semi-group of double operator integrals. If each ϕ_t is continuous, then this semi-group admits a reversed Markov dilation.*

6. Transference of multipliers and BMO-spaces

Fix a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and assume $\|f'\|_\infty \leq 1$. For $f : \mathbb{R} \to \mathbb{R}$ we set the divided difference function $f^{[1]} : \mathbb{R}^2 \to \mathbb{R}$ by

$$f^{[1]}(\lambda,\mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \lambda \neq \mu, \\ 0, & \lambda = \mu. \end{cases}$$

The main result we prove in this section is the following theorem.

Theorem 6.1. *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz with $\|f'\|_\infty \leq 1$. For every $A = A^* \in \mathcal{M}$,*

(i) *the semi-group $\mathcal{I}^A = (\mathcal{I}^A_{e^{-tF}})_{t\geq 0}$ with*

$$F(\lambda,\mu) = |\lambda-\mu|^2 + |f(\lambda)-f(\mu)|^2, \quad \lambda,\mu \in \mathbb{R},$$

is Markov;

(ii) *the operator $\mathcal{I}^A_{f^{[1]}}$ maps $\mathcal{M}$ to $\mathrm{bmo}(\mathcal{M},\mathcal{I}^A)$ and its norm is bounded by an absolute constant c_{abs}.*

For $\eta \in \mathbb{R}^2$, let $e_\eta \in L_\infty(\mathbb{R}^2)$ be defined as

$$e_\eta(\xi) = e^{i\langle \xi,\eta\rangle}.$$

For $A = A^* \in \mathcal{M}$ with finite spectrum, define a unitary element $U^A \in L_\infty(\mathbb{R}^2) \otimes \mathcal{M}$ by setting

$$U^A = \int_{\mathbb{R}} e_{(\lambda,f(\lambda))} \otimes dE_A(\lambda),$$

where $\{E_A(\lambda)\}_{\lambda\in\mathbb{R}}$ is a spectral family of A. Due to the finiteness assumption on the spectrum, the convergence of the integral follows automatically (in fact, integral is a finite sum of operators).

Define the $*$-monomorphism $\varphi_A : \mathcal{M} \to L_\infty(\mathbb{R}^2) \otimes \mathcal{M}$ by setting

$$\varphi_A(x) = U^A(1 \otimes x)(U^A)^*, \quad x \in \mathcal{M}.$$

Let m_0 be a smooth homogeneous multiplier such that

$$m_0(\xi_1, \xi_2) = \frac{\xi_2}{\xi_1} \text{ when } |\xi_2| \leq |\xi_1|.$$

Both statements of Theorem 6.1 are proved through the following transference lemma.

Lemma 6.2. *If, in the setting of Theorem 6.1, A has finite spectrum, then*

$$\varphi_A \circ \mathcal{I}^A_{e^{-tF}} = (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}}) \circ \varphi_A, \quad \varphi_A \circ \mathcal{I}^A_{f^{[1]}} = (m_0(\nabla) \otimes \mathrm{id}_{\mathcal{M}}) \circ \varphi_A.$$

Proof. By definition of φ_A, we have

$$\varphi_A(x) = \iint_{\mathbb{R}^2} e_{(\lambda-\mu, f(\lambda)-f(\mu))} \otimes dE_A(\lambda) x dE_A(\mu).$$

Clearly,

$$\begin{aligned} e^{t\Delta}(e_{(\lambda-\mu,f(\lambda)-f(\mu))}) &= e^{-tF(\lambda,\mu)} e_{(\lambda-\mu,f(\lambda)-f(\mu))} \\ (m_0(\nabla))(e_{(\lambda-\mu,f(\lambda)-f(\mu))}) &= f^{[1]}(\lambda,\mu) e_{(\lambda-\mu,f(\lambda)-f(\mu))}. \end{aligned}$$

Therefore,

$$(e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}})(\varphi_A(x)) = \iint_{\mathbb{R}^2} e^{-tF(\lambda,\mu)} e_{(\lambda-\mu,f(\lambda)-f(\mu))} \otimes dE_A(\lambda) x dE_A(\mu)$$

$$= \iint_{\mathbb{R}^2} e_{(\lambda-\mu,f(\lambda)-f(\mu))} \otimes dE_A(\lambda)(\mathcal{I}^A_{e^{-tF}}(x)) dE_A(\mu) = \varphi_A(\mathcal{I}^A_{e^{-tF}}(x))$$

and

$$(m_0(\nabla) \otimes \mathrm{id}_{\mathcal{M}})(\varphi_A(x)) = \iint_{\mathbb{R}^2} f^{[1]}(\lambda,\mu) e_{(\lambda-\mu,f(\lambda)-f(\mu))} \otimes dE_A(\lambda) x dE_A(\mu)$$

$$= \iint_{\mathbb{R}^2} e_{(\lambda-\mu,f(\lambda)-f(\mu))} \otimes dE_A(\lambda)(\mathcal{I}^A_{f^{[1]}}(x)) dE_A(\mu) = \varphi_A(\mathcal{I}^A_{f^{[1]}}(x)). \quad \square$$

Next we prove each of the statements 6.1 (i) and (ii) in the following subsections.

6.1. Proof of Theorem 6.1 (i)

Lemma 6.3. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If $(\mathcal{I}^A_{e^{-tG}})_{t\geq 0}$ is Markov for every $A = A^* \in \mathcal{M}$ with finite spectrum, then $(\mathcal{I}^A_{e^{-tG}})_{t\geq 0}$ is Markov for every $A = A^* \in \mathcal{M}$.*

Proof. Let $A = A^* \in \mathcal{M}$ and let $A_l = l^{-1}\lfloor lA \rfloor$ for $l \geq 1$. If $\mathcal{N}$ is a finite von Neumann algebra and if $x \in \mathcal{N} \otimes \mathcal{M}$ is such that $0 \leq x \leq 1$, then

$$0 \leq (\mathrm{id}_{\mathcal{N}} \otimes \mathcal{I}^{A_l}_{e^{-tG}})(x) \leq 1.$$

Clearly,

$$(\mathrm{id}_{\mathcal{N}} \otimes \mathcal{I}^{A_l}_{e^{-tG}})(x) \to (\mathrm{id}_{\mathcal{N}} \otimes \mathcal{I}^{A}_{e^{-tG}})(x)$$

in $L_2(\mathcal{N} \otimes \mathcal{M})$ and therefore in measure. Hence,

$$0 \leq (\mathrm{id}_{\mathcal{N}} \otimes \mathcal{I}^{A}_{e^{-tG}})(x) \leq 1.$$

Thus, $(\mathcal{I}^A_{e^{-tG}})_{t\geq 0}$ is completely positive. Since $(\mathcal{I}^A_{e^{-tG}})_{t\geq 0}$ is obviously unital, the condition (i) follows.

By assumption, $\mathcal{I}^{A_l}_{e^{-tG}}$ is self-adjoint on $L_2(\mathcal{M})$. Clearly, $\mathcal{I}^{A_l}_{e^{-tG}} \to \mathcal{I}^{A}_{e^{-tG}}$ strongly. Therefore, $\mathcal{I}^{A}_{e^{-tG}}$ is self-adjoint on $L_2(\mathcal{M})$. This yields the condition (ii). The condition (iii) is obvious. □

Proof of Theorem 6.1 (i). If A has finite spectrum, then the assertion follows by Lemma 6.2 and the fact that the Heat semi-group is Markov. For generic A, the assertion follows by Lemma 6.3. □

6.2. Proof of Theorem 6.1 (ii)

For $A = A^* \in \mathcal{M}$, let $A_l = l^{-1}\lfloor lA \rfloor$ for $l \geq 1$.

Lemma 6.4. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $(\mathcal{I}^A_{e^{-tG}})_{t\geq 0}$ is Markov for every $A = A^* \in \mathcal{M}$. We have as $l \to \infty$ that,*

$$\mathcal{I}^{A_l}_{e^{-tG}}\Big(\mathcal{I}^{A_l}_{f^{[1]}}(x)^*\mathcal{I}^{A_l}_{f^{[1]}}(x)\Big) \to \mathcal{I}^{A}_{e^{-tG}}\Big(\mathcal{I}^{A}_{f^{[1]}}(x)^*\mathcal{I}^{A}_{f^{[1]}}(x)\Big),$$
$$\mathcal{I}^{A_l}_{e^{-tG}}\Big(\mathcal{I}^{A_l}_{f^{[1]}}(x)\Big) \to \mathcal{I}^{A}_{e^{-tG}}\Big(\mathcal{I}^{A}_{f^{[1]}}(x)\Big),$$

in measure for every $x \in L_2(\mathcal{M})$.

Proof. Denote, for brevity,

$$y_l = \mathcal{I}^{A_l}_{f^{[1]}}(x), \quad y = \mathcal{I}^{A}_{f^{[1]}}(x).$$

We have that $y_l \to y$ in L_2-norm and, therefore, $y_l^* y_l \to y^* y$ in L_1-norm as $l \to \infty$. We have,

$$\mathcal{I}^{A_l}_{e^{-tG}}(y_l^* y_l) = \mathcal{I}^{A_l}_{e^{-tG}}(y_l^* y_l - y^* y) + \mathcal{I}^{A_l}_{e^{-tG}}(y^* y). \tag{6.1}$$

Since every Markov semi-group is an L_1-contraction, it follows that

$$\mathcal{I}^{A_l}_{e^{-tG}}(y_l^* y_l - y^* y) \to 0, \quad l \to \infty, \tag{6.2}$$

in the L_1-norm and, therefore in measure.

Let $z \in L_1(\mathcal{M})$ be arbitrary and fix $\epsilon > 0$. Recall that $L_2(\mathcal{M})$ is dense in $L_1(\mathcal{M})$, [34, Theorem IX.2.13]. Therefore choose a decomposition $z = z_1 + z_2$ such that $\|z_1\|_1 < \epsilon$ and such that $z_2 \in L_2(\mathcal{M})$. We have,

$$\mathcal{I}^{A_l}_{e^{-tG}}(z) - \mathcal{I}^{A}_{e^{-tG}}(z) = \Big(\mathcal{I}^{A_l}_{e^{-tG}}(z_2) - \mathcal{I}^{A}_{e^{-tG}}(z_2)\Big) + \mathcal{I}^{A_l}_{e^{-tG}}(z_1) - \mathcal{I}^{A}_{e^{-tG}}(z_1).$$

Clearly,

$$\mathcal{I}^{A_l}_{e^{-tG}}(z_2) - \mathcal{I}^{A}_{e^{-tG}}(z_2) \to 0, \quad l \to \infty,$$

in L_2-norm. Hence, there exists $l(\epsilon)$ such that, for $l > l(\epsilon)$,

$$\|\mathcal{I}^{A_l}_{e^{-tG}}(z_2) - \mathcal{I}^{A}_{e^{-tG}}(z_2)\|_1 \leq \|\mathcal{I}^{A_l}_{e^{-tG}}(z_2) - \mathcal{I}^{A}_{e^{-tG}}(z_2)\|_2 < \epsilon.$$

Since,

$$\|\mathcal{I}^{A_l}_{e^{-tG}}(z_1)\|_1 < \|z_1\|_1 \leq \epsilon, \quad \|\mathcal{I}^{A}_{e^{-tG}}(z_1)\|_1 < \|z_1\|_1 \leq \epsilon,$$

it follows that

$$\|\mathcal{I}^{A_l}_{e^{-tG}}(z) - \mathcal{I}^{A}_{e^{-tG}}(z)\|_1 < 3\epsilon, \quad l \geq l(\epsilon).$$

So we conclude that for $z \in L_1(\mathcal{M})$ we have

$$\mathcal{I}^{A_l}_{e^{-tG}}(z) \to \mathcal{I}^{A}_{e^{-tG}}(z), \quad l \to \infty, \tag{6.3}$$

in L_1-norm. Applying this to $z = y^* y$ and combining this with (6.1) and (6.2), we infer the first assertion. The second (easier) assertion follows as the convergence actually holds in L_2-norm. □

Lemma 6.5. *If $x \in L_2(\mathcal{M})$, then $\mathcal{I}^A_{f^{[1]}}(x) \in L_2^\circ(\mathcal{M})$.*

Proof. Let $\mathcal{D}_A$ be the von Neumann algebra generated by the spectral projections of A. Let $\mathcal{D}' = \mathcal{D}_A' \cap \mathcal{M}$ be its relative commutant with trace preserving conditional expectation $\mathbb{E}_{\mathcal{D}'} : \mathcal{M} \to \mathcal{D}$. If $z \in L_2(\mathcal{M})$, then

$$\mathcal{I}^A_{e^{-tF}}(z) \to \mathbb{E}_{\mathcal{D}'}(z), \quad t \to \infty,$$

in measure. Therefore, $z \in L_2^\circ(\mathcal{M})$ if and only if $\mathbb{E}_{\mathcal{D}'}(z) = 0$. We claim that $\mathbb{E}_{\mathcal{D}'}(\mathcal{I}^A_{f^{[1]}}(x)) = 0$. Set

$$p_{m,k} = \chi_{[\frac{k}{m}, \frac{k+1}{m})}(A), \quad x_m = \sum_{k \in \mathbb{Z}} p_{m,k} x p_{m,k}.$$

We have

$$\mathcal{I}^A_{f^{[1]}}\Big(p_{m,k} x p_{m,l}\Big) = p_{m,k} \cdot \mathcal{I}^A_{f^{[1]}}(x) \cdot p_{m,l}.$$

Therefore,

$$\mathbb{E}_{\mathcal{D}'}\Big(\mathcal{I}^A_{f^{[1]}}\Big(p_{m,k} x p_{m,l}\Big)\Big) = p_{m,k} \cdot \mathbb{E}_{\mathcal{D}'}(\mathcal{I}^A_{f^{[1]}}(x)) \cdot p_{m,l}.$$

If $k \neq l$, then

$$\mathbb{E}_{\mathcal{D}'}\Big(\mathcal{I}^A_{f^{[1]}}\Big(p_{m,k} x p_{m,l}\Big)\Big) = p_{m,k} \cdot p_{m,l} \cdot \mathbb{E}_{\mathcal{D}'}(\mathcal{I}^A_{f^{[1]}}(x)) = 0.$$

Therefore,

$$\mathbb{E}_{\mathcal{D}'}(\mathcal{I}^A_{f^{[1]}}(x)) = \sum_{k,l \in \mathbb{Z}} \mathbb{E}_{\mathcal{D}'}\Big(\mathcal{I}^A_{f^{[1]}}\Big(p_{m,k} x p_{m,l}\Big)\Big)$$
$$= \sum_{k \in \mathbb{Z}} \mathbb{E}_{\mathcal{D}'}\Big(\mathcal{I}^A_{f^{[1]}}\Big(p_{m,k} x p_{m,k}\Big)\Big) = \mathbb{E}_{\mathcal{D}'}(\mathcal{I}^A_{f^{[1]}}(x_m)).$$

As $m \to \infty$, we have convergence in measure

$$x_m \to \mathbb{E}_{\mathcal{D}'}(x), \quad \mathcal{I}^A_{f^{[1]}}(x_m) \to \mathcal{I}^A_{f^{[1]}}(\mathbb{E}_{\mathcal{D}'}(x)) = 0.$$

This concludes the proof. □

Proof of Theorem 6.1 (ii)**.** By the first equality of Lemma 6.2 we see that φ_{A_l} maps $\mathrm{bmo}(\mathcal{M}, \mathcal{I}^{A_l})$ to $\mathrm{bmo}(L_\infty(\mathbb{R}^2) \otimes \mathcal{M}, \mathcal{S} \otimes \mathrm{id}_{\mathcal{M}})$ isometrically with $\mathcal{S}$ the Heat semi-group. By the second equality of Lemma 6.2 we then further have,

$$\begin{aligned}\|\mathcal{I}^{A_l}_{f^{[1]}}(x)\|_{\mathrm{bmo}(\mathcal{M},\mathcal{I}^{A_l})} &= \|\varphi_{A_l}\circ\mathcal{I}^{A_l}_{f^{[1]}}(x)\|_{\mathrm{bmo}(\mathcal{M},\mathcal{I}^{A_l})}\\ &= \|(m_0(\nabla)\otimes\mathrm{id}_{\mathcal{M}})(\varphi_{A_l}(x))\|_{\mathrm{bmo}(L_\infty(\mathbb{R}^2)\otimes\mathcal{M},\mathcal{S}\otimes\mathrm{id}_{\mathcal{M}})}.\end{aligned}$$

As $\varphi_{A_l}(x)$ is trigonometric, by Proposition 2.7,

$$\|(m_0(\nabla)\otimes\mathrm{id}_{\mathcal{M}})(\varphi_{A_l}(x))\|_{\mathrm{bmo}(L_\infty(\mathbb{R}^2)\otimes\mathcal{M},\mathcal{S}\otimes\mathrm{id}_{\mathcal{M}})}\le c_{abs}\|\varphi_{A_l}(x)\|_\infty = c_{abs}\|x\|_\infty.$$

Therefore, we have

$$\|\mathcal{I}^{A_l}_{f^{[1]}}(x)\|_{\mathrm{bmo}(\mathcal{M},\mathcal{I}^{A_l})}\le c_{\mathrm{abs}}\|x\|_\infty.$$

Thus, for every $t\ge 0$, we have

$$-c_{\mathrm{abs}}^2\|x\|_\infty^2\le B_l(t)\le c_{\mathrm{abs}}^2\|x\|_\infty^2,$$

where

$$B_l(t)=\mathcal{I}^{A_l}_{e^{-tF}}\left(\mathcal{I}^{A_l}_{f^{[1]}}(x)^*\mathcal{I}^{A_l}_{f^{[1]}}(x)\right)-\mathcal{I}^{A_l}_{e^{-tF}}\left(\mathcal{I}^{A_l}_{f^{[1]}}(x)\right)^*\mathcal{I}^{A_l}_{e^{-tF}}\left(\mathcal{I}^{A_l}_{f^{[1]}}(x)\right).$$

By Lemma 6.4, we have $B_l(t)\to B(t)$ in measure as $l\to\infty$. Here,

$$B(t)=\mathcal{I}^{A}_{e^{-tF}}\left(\mathcal{I}^{A}_{f^{[1]}}(x)^*\mathcal{I}^{A}_{f^{[1]}}(x)\right)-\mathcal{I}^{A}_{e^{-tF}}\left(\mathcal{I}^{A}_{f^{[1]}}(x)\right)^*\mathcal{I}^{A}_{e^{-tF}}\left(\mathcal{I}^{A}_{f^{[1]}}(x)\right).$$

Therefore,

$$-c_{\mathrm{abs}}^2\|x\|_\infty^2\le B(t)\le c_{\mathrm{abs}}^2\|x\|_\infty^2$$

for every $t\ge 0$. In other words,

$$\|\mathcal{I}^{A}_{f^{[1]}}(x)\|_{\mathrm{bmo}(\mathcal{M},\mathcal{I}^{A})}\le c_{\mathrm{abs}}\|x\|_\infty.$$

By Lemma 6.5, we also have $\mathcal{I}^A_{f^{[1]}}(x)\in L_2^\circ(\mathcal{M})$. A combination of this fact and the norm estimate complete the proof. □

We shall need the following auxiliary lemma in the next section.

Lemma 6.6. *Suppose that A has finite spectrum. We have that* $\mathrm{bmo}(\mathcal{M},\mathcal{I}^A)=\mathrm{BMO}(\mathcal{M},\mathcal{I}^A)$ *as vector spaces with equality of norms.*

Proof. We have an equality [21, Proof of Lemma 1.3] for $f\in L_\infty(\mathbb{R}^n)\otimes\mathcal{M}$,

$$\begin{aligned}&(e^{t\Delta}\otimes\mathrm{id}_{\mathcal{M}})(f^*f)-(e^{t\Delta}\otimes\mathrm{id}_{\mathcal{M}})(f)^*(e^{t\Delta}\otimes\mathrm{id}_{\mathcal{M}})(f)\\ =&(e^{t\Delta}\otimes\mathrm{id}_{\mathcal{M}})\left(|f-(e^{t\Delta}\otimes\mathrm{id}_{\mathcal{M}})(f)|^2\right).\end{aligned}$$

For $x \in \mathcal{M}$ set $f = \varphi_A(x)$. We get by Lemma 6.2,

$$\begin{aligned}\|x\|_{\mathrm{BMO}(\mathcal{M},\mathcal{I}^A)} &= \sup_{t>0} \left\| (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}}) \left(|f - (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}})(f)|^2 \right) \right\|_\infty^{\frac{1}{2}} \\ &= \sup_{t>0} \left\| (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}})(f^* f) - (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}})(f)^* (e^{t\Delta} \otimes \mathrm{id}_{\mathcal{M}})(f) \right\|_\infty^{\frac{1}{2}} \\ &= \|x\|_{\mathrm{bmo}(\mathcal{M},\mathcal{I}^A)}. \quad \square\end{aligned}$$

7. Conclusions for BMO-estimates for commutators

We now collect several results in perturbation theory of commutators as a consequence of Theorem 6.1. In particular we recover the main results from [30] and [7]. We in fact improve of them in terms of BMO-estimates.

As before we fix a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and we assume that $\|f'\|_\infty \leq 1$. We set,

$$\psi(\lambda, \mu) = \lambda - \mu, \qquad \psi_f(\lambda, \mu) = f(\lambda) - f(\mu).$$

Note that $\mathcal{I}^A_\psi(x) = Ax - xA = [A, x]$ and $\mathcal{I}^A_{\psi_f}(x) = [f(A), x]$ for $A \in \mathcal{M}$ self-adjoint. We start with the following corollary.

Corollary 7.1. *In the setting of Theorem 6.1, there exists a constant c_{abs} such that for every $A \in \mathcal{M}$ self-adjoint and every $x \in \mathcal{M}$ we have,*

$$\|[f(A), x]\|_{\mathrm{bmo}_{\mathcal{I}^A}} \leq c_{abs} \|[A, x]\|_\infty.$$

Proof of Corollary 7.1. We have,

$$[f(A), x] = \mathcal{I}^A_{\psi_f}(x) = (\mathcal{I}^A_{f^{[1]}} \circ \mathcal{I}^A_\psi)(x) = \mathcal{I}^A_{f^{[1]}}([A, x]).$$

Hence, by Theorem 6.1,

$$\|[f(A), x]\|_{\mathrm{bmo}_{\mathcal{I}^A}} = \|\mathcal{I}^A_{f^{[1]}}([A, x])\|_{\mathrm{bmo}_{\mathcal{I}^A}} \leq \|\mathcal{I}^A_{f^{[1]}} : \mathcal{M} \to \mathrm{bmo}_{\mathcal{I}^A}\| \|[A, x]\|_\infty. \quad \square$$

Remark 7.2. For a general von Neumann algebra $\mathcal{M}$ one cannot define a canonical Markov semi-group without further structure. This is why in Corollary 7.1 the semi-group depends on the self-adjoint operator $A \in \mathcal{M}$ and the Lipschitz function f and we believe this is the suitable end-point estimate. After interpolation the dependence of A and f vanishes in Theorem 7.4 and we obtain best constant estimates.

Next, through our BMO approach we collect many optimal results in perturbation theory. Firstly we retrieve the main result of [7].

Theorem 7.3. *In the setting of Theorem 6.1, let $A \in \mathcal{M}$ be self-adjoint. There exists a constant c_{abs} such that for every $1 < p < \infty$,*

$$\|\mathcal{I}^{A}_{f^{[1]}} : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| \leq c_{abs} \frac{p^2}{p-1}.$$

Proof. Setting as before $A_l = \frac{1}{l}\lfloor lA \rfloor$, we infer from Lemma 6.6 that $\mathrm{bmo}(\mathcal{M}, \mathcal{I}^{A_l}) = \mathrm{BMO}(\mathcal{M}, \mathcal{I}^{A_l})$.

By Theorem 6.1 and its proof we have

$$\|\mathcal{I}^{A_l}_{f^{[1]}} : \mathcal{M} \to \mathrm{bmo}(\mathcal{M}, \mathcal{I}^{A_l})\| \leq \|m_0(\nabla) : L_\infty \to \mathrm{bmo}_{\mathcal{S}}\|_{cb}.$$

Also,

$$\|\mathcal{I}^{A_l}_{f^{[1]}} : L_2(\mathcal{M}) \to L_2(\mathcal{M})\| \leq \|f'\|_\infty \leq 1.$$

By Theorem 5.4 we see that $\mathcal{I}^{A_l}$ has a standard Markov dilation. Therefore, by Theorem 2.5 for $2 \leq p < \infty$ we have

$$\|\mathcal{I}^{A_l}_{f^{[1]}} : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| \leq c_{abs} p.$$

Further, for $x \in \mathcal{M}$ we have $\mathcal{I}^{A_l}_{f^{[1]}}(x) \to \mathcal{I}^{A}_{f^{[1]}}(x)$ in measure as $l \to \infty$. Hence it follows that also

$$\|\mathcal{I}^{A}_{f^{[1]}} : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| \leq c_{abs} p.$$

Next let $1 < p \leq 2$ and let q be conjugate, i.e. $p^{-1} + q^{-1} = 1$. By duality we find that

$$(\mathcal{I}^{A}_{f^{[1]}})^* : L_p(\mathcal{M}) \to L_p(\mathcal{M}),$$

is the extension of the double operator integral $\mathcal{I}^{A}_{\overline{f}^{[1]}}$. So that,

$$\mathcal{I}^{A}_{f^{[1]}} : L_p(\mathcal{M}) \to L_p(\mathcal{M}) = (\mathcal{I}^{A}_{\overline{f}^{[1]}} : L_q(\mathcal{M}) \to L_q(\mathcal{M}))^*$$

is bounded with $\|\mathcal{I}^{A}_{f^{[1]}} : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| \leq c_{abs}(p-1)^{-1}$. □

Theorem 7.4. *In the setting of Theorem 6.1, there exists an absolute constant c_{abs} such that for any operators $A \in \mathcal{M}$ self-adjoint and $x \in \mathcal{M}$, and any $1 < p < \infty$, we have*

$$\|[f(A), x]\|_p \leq c_{abs} \frac{p^2}{p-1} \|[A, x]\|_\infty.$$

Proof. We derive the proof from Theorem 7.3 as in Corollary 7.1. We find that for $x \in \mathcal{M}$,

$$\begin{aligned} &\|[f(A), x]\|_p = \|\mathcal{I}^A_{f^{[1]}}([A, x])\|_p \\ \leq &\|\mathcal{I}^A_{f^{[1]}} : L_p \to L_p\| \|[A, x]\|_p \leq c_{abs} \frac{p^2}{p-1} \|[A, x]\|_p. \quad \square \end{aligned} \tag{7.1}$$

Corollary 7.5. *There exists a constant c_{abs} such that for any self-adjoint operators $B, C \in \mathcal{M}$ we have*

$$\|f(B) - f(C)\|_p \leq c_{abs} \frac{p^2}{p-1} \|B - C\|_p.$$

Proof. Apply Theorem 7.4 to

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

As

$$[A, x] = \begin{pmatrix} 0 & B - C \\ C - B & 0 \end{pmatrix},$$

we find $\|[A, x]\|_p = 2^{\frac{1}{p}} \|B - C\|_p$ and similarly $\|[f(A), x]\|_p = 2^{\frac{1}{p}} \|f(B) - f(C)\|_p$.

Theorem 7.4 gives

$$2^{\frac{1}{p}} \|f(B) - f(C)\|_p = \|[f(A), x]\|_p \leq c_{abs} \frac{p^2}{p-1} \|[A, x]\|_p = 2^{\frac{1}{p}} c_{abs} \frac{p^2}{p-1} \|B - C\|_p. \quad \square$$

As another corollary we get a proof of the Aleksandrov-Peller results in [1, Theorem 11.4].

Theorem 7.6. *There exists a constant c_{abs} such that for any two self-adjoint operators $A, B \in \mathcal{B}(\mathbb{C}^n)$ and any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ we have*

$$\|f(B) - f(C)\|_\infty \leq C_{abs}(1 + \log(n)) \|B - C\|_\infty.$$

Proof. Let $\mathcal{S}^n_p$ be the Schatten class associated with $\mathcal{B}(\mathbb{C}^n)$. We have that $\mathcal{B}(\mathbb{C}^n) \subseteq \mathcal{S}^n_p$ contractively. The converse inclusion $\mathcal{S}^n_p \subseteq \mathcal{B}(\mathbb{C}^n)$ has norm at most $n^{\frac{2}{p}}$ by complex interpolation between $p = 1$ and $p = \infty$.

We find that for $\log(2) \leq p < \infty$,

$$\|f(B) - f(C)\|_\infty \leq \|f(B) - f(C)\|_p \leq c_{abs}\, p \|B - C\|_p \leq c_{abs} p n^{\frac{2}{p}} \|B - C\|_\infty.$$

Now for $n \geq 2$ take $p = \log(n)$ so that we get,

$$\|f(B) - f(C)\|_\infty \leq c_{abs} \log(n) e^{\frac{2}{p}\log(n)} \|B - C\|_\infty = c_{abs} \log(n) e^2 \|B - C\|_\infty$$

This yields the theorem as in case $n = 1$ it is obvious. □

8. Estimates for vector-valued double operator integrals

8.1. Assumptions and statements

As before we let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Throughout the entire section we fix a finite von Neumann algebra $\mathcal{N}$ whose trace shall not be used explicitly. We write $\mathcal{S}^n$ for the Heat semi-group on $\mathbb{R}^n$ to stipulate the dimension. We let $C(\mathbb{R}^n, \mathcal{N})$ be the space of norm continuous functions $\mathbb{R}^n \to \mathcal{N}$.

Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an n-tuple of commuting self-adjoint elements in $\mathcal{M}$. Let $E^{\mathbf{A}} : \mathbb{R}^n \to \mathcal{M}$ be their joint spectral measure.

Definition 8.1. The semi-group $\mathcal{J}^{\mathbf{A}} : \mathcal{M} \otimes M_2(\mathbb{C}) \to \mathcal{M} \otimes M_2(\mathbb{C})$ is defined by the formula

$$\mathcal{J}_t^{\mathbf{A}}(x \otimes e_{ij}) = \mathcal{I}^{\mathbf{A}}_{e^{-tF_{ij}}}(x) \otimes e_{ij}, \quad x \in \mathcal{M},$$

where

$$F_{ij}(\lambda, \mu) = \begin{cases} |\lambda - \mu|^2, & i = j, \\ |\lambda|^2 + |\mu|^2, & i \neq j. \end{cases}$$

Let $L : L_\infty(\mathbb{R}^n, \mathcal{N}) \to L_\infty(\mathbb{R}^{2n}, \mathcal{N})$ be defined by the formula

$$(Lh)(t, s) = \int_0^1 h(\theta s + (1 - \theta)t) d\theta. \tag{8.1}$$

We need the following Hörmander-Mikhlin-type condition.

Condition 8.2. The function $h \in C(\mathbb{R}^n, \mathcal{N})$ is a compactly supported C^{n+2}-function such that

$$\|h\|_{HM_n} \stackrel{\text{def}}{=} \max_{0 \leq |\alpha| \leq n+2} \sup_{t \neq 0} \|t\|_2^{|\alpha|} \|(\partial_\alpha h)(t)\|_{\mathcal{N}} \leq 1. \tag{8.2}$$

The following Theorem 8.3 is the main result we prove in this sections from which we derive vector valued commutator estimates.

Theorem 8.3. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n-tuple of commuting self-adjoint elements in $\mathcal{M}$. The following statements hold.*

(i) *The semi-group $\mathcal{J}^{\mathbf{A}}$ in Definition 8.1 is Markov.*
(ii) *If $h \in C(\mathbb{R}^n, \mathcal{N})$ satisfies the Condition 8.2 and $h(0) = 0$, then*

$$\left\| \mathcal{I}^{\mathbf{A}}_{Lh}(x) \otimes e_{12} \right\|_{\mathrm{bmo}(\mathcal{N} \otimes \mathcal{M} \otimes M_2(\mathbb{C}), \mathrm{id}_{\mathcal{N}} \otimes \mathcal{J}^{\mathbf{A}})} \leq c_n \|x\|_{\mathcal{M}}, \quad x \in \mathcal{M}.$$

8.2. Transference of multipliers

Let again $e_s \in L_\infty(\mathbb{R}^n)$ be given by $e_s(t) = e^{i\langle s,t \rangle}$. When $\mathbf{A}$ has finite spectrum, define unitary operators $U, V \in L_\infty(\mathbb{R}^n) \otimes L_\infty(\mathbb{R}^n) \otimes \mathcal{M}$ by the formula

$$U = \int_{\mathbb{R}^n} e_s \otimes 1_{L_\infty(\mathbb{R}^n)} \otimes dE^{\mathbf{A}}(s), \quad V = \int_{\mathbb{R}^n} 1_{L_\infty(\mathbb{R}^n)} \otimes e_{-s} \otimes dE^{\mathbf{A}}(s).$$

Set

$$W = U \otimes e_{11} + V \otimes e_{22},$$

and further

$$\pi_{\mathbf{A}}(x) = W(1_{L_\infty(\mathbb{R}^n)} \otimes 1_{L_\infty(\mathbb{R}^n)} \otimes x)W^*, \quad x \in \mathcal{M} \otimes M_2(\mathbb{C}).$$

The map

$$\mathrm{corner} : \mathcal{M} \to \mathcal{M} \otimes M_2(\mathbb{C})$$

is defined by the formula $x \to x \otimes e_{12}$.

Proposition 8.4. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n-tuple of commuting self-adjoint elements in $\mathcal{M}$ with finite spectrum.*

(i) *for every $t \geq 0$, we have*

$$(S^{2n}_t \otimes \mathrm{id}_{\mathcal{M} \otimes M_2(\mathbb{C})}) \circ \pi_{\mathbf{A}} = \pi_{\mathbf{A}} \circ \mathcal{J}^{\mathbf{A}}_t.$$

(ii) *for every $k \in C(\mathbb{R}^{2n}, \mathcal{N})$, we have*

$$(k(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M} \otimes M_2(\mathbb{C})}) \circ \pi_{\mathbf{A}} \circ \mathrm{corner} = (\mathrm{id}_{\mathcal{N}} \otimes \pi_{\mathbf{A}}) \circ (\mathrm{id}_{\mathcal{N}} \otimes \mathrm{corner}) \circ \mathcal{I}^{\mathbf{A}}_k.$$

Proof. If $x \in \mathcal{M}$, then

$$\pi_{\mathbf{A}}(x \otimes e_{11}) = UxU^* \otimes e_{11} = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e_{s-u} \otimes 1_{L_\infty(\mathbb{R}^n)} \otimes dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(u) \otimes e_{11}.$$

Therefore,

$$\begin{aligned}
&\Big(S_t^{2n} \otimes \mathrm{id}_{\mathcal{M}\otimes M_2(\mathbb{C})}\Big)(\pi_{\mathbf{A}}(x \otimes e_{11})) \\
=&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-t|s-u|^2} e_{s-u} \otimes 1_{L_\infty(\mathbb{R}^n)} \otimes dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(u) \otimes e_{11} \\
=&\pi_{\mathbf{A}}\Big(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-t|s-u|^2} dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(t) \otimes e_{11}\Big) \\
=&(\pi_{\mathbf{A}} \circ \mathcal{J}_t^{\mathbf{A}})(x \otimes e_{11}).
\end{aligned}$$

Also,

$$\pi_{\mathbf{A}}(x \otimes e_{12}) = UxV^* \otimes e_{12} = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e_s \otimes e_t \otimes dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(t) \otimes e_{12}.$$

Therefore,

$$\begin{aligned}
&\Big(S_t^{2n} \otimes \mathrm{id}_{\mathcal{M}\otimes M_2(\mathbb{C})}\Big)(\pi_{\mathbf{A}}(x \otimes e_{12})) \\
=&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-t|s|^2} e_s \otimes e^{-t|u|^2} e_u \otimes dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(u) \otimes e_{12} \\
=&\pi_{\mathbf{A}}\Big(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-t|s|^2 - t|u|^2} dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(u) \otimes e_{12}\Big) \\
=&(\pi_{\mathbf{A}} \circ \mathcal{J}_t^{\mathbf{A}})(x \otimes e_{12}).
\end{aligned}$$

The argument for $x \otimes e_{21}$ and for $x \otimes e_{22}$ goes *mutatis mutandi.* This proves the first assertion.

We have,

$$\pi_{\mathbf{A}}(x \otimes e_{12}) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e_s \otimes e_t \otimes dE^{\mathbf{A}}(s) x dE^{\mathbf{A}}(t) \otimes e_{12}.$$

Therefore,

$$\Big(k(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}\otimes M_2(\mathbb{C})}\Big)(\pi_{\mathbf{A}}(x \otimes e_{12}))$$

$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(s,u)\otimes e_s\otimes e_u\otimes dE^{\mathbf{A}}(s)xdE^{\mathbf{A}}(u)\otimes e_{12}$$

$$=(\mathrm{id}_{\mathcal{N}}\otimes\pi_{\mathbf{A}})\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(s,u)\otimes dE^{\mathbf{A}}(s)xdE^{\mathbf{A}}(u)\otimes e_{12}\Big)$$

$$=(\mathrm{id}_{\mathcal{N}}\otimes\pi_{\mathbf{A}})\Big(\mathcal{I}_k^{\mathbf{A}}(x)\otimes e_{12}\Big).$$

This proves the second assertion. □

8.3. Smooth vector valued multipliers

Fix a convolution kernel $K:\mathbb{R}^n\backslash\{0\}\to\mathcal{N}$. Assume K determines a convolution operator by the principal value integral,

$$(K*g)(x)=\int_{\mathbb{R}^n} K(y)g(x-y)dy,\qquad x\in\mathbb{R}^n,$$

and $g:\mathbb{R}^n\to\mathbb{C}$ smooth and compactly supported; the domain of $K*$ will be extended shortly. We shall also write K for $K*$, the convolution operator (Calderón-Zygmund operator). We say that a function $K:\mathbb{R}^n\backslash\{0\}\to\mathcal{N}$ satisfies the Hörmander-Mikhlin condition if,

$$\sup_{x\in\mathbb{R}^{2n}}\int_{y\in\mathbb{R}^{2n},\|y\|_2>2\|x\|_2}\|K(x-y)-K(y)\|_{\mathcal{N}}dy<\infty. \tag{8.3}$$

Note that if $h\in C(\mathbb{R}^n,\mathcal{N})$ is integrable it has a Fourier transform $\widehat{h}:\mathbb{R}^n\to\mathcal{N}$ that is uniquely determined by $\omega\circ\widehat{h}=\widehat{\omega\circ h}$ for all $\omega\in\mathcal{N}_*$. The following is essentially proved in [21, Lemma 2.3 and Lemma 3.3]; we explain how it can be derived.

Proposition 8.5. *If $h\in C(\mathbb{R}^n,\mathcal{N})$ satisfies Condition 8.2 and $h(0)=0$, then*

$$h(\nabla_{\mathbb{R}^n})\otimes\mathrm{id}_{\mathcal{M}}:L_\infty(\mathbb{R}^n)\otimes\mathcal{M}\to\mathrm{bmo}(\mathcal{N}\otimes L_\infty(\mathbb{R}^n)\otimes\mathcal{M},\mathrm{id}_{\mathcal{N}}\otimes\mathcal{S}^n\otimes\mathrm{id}_{\mathcal{M}})$$

and its norm is bounded by an absolute constant c^n_{abs} only depending on the dimension n.

Proof. Let $\omega\in\mathcal{N}_*$ with $\|\omega\|=1$ then $h^\omega:=\omega\circ h$ is a C^{n+2}-function whose associated Fourier transform is given by $K^\omega=\omega\circ K$. We still have $\sup_{|\alpha|\leq n+2}\|\xi\|_2^{|\alpha|}\|\partial_\alpha\omega\circ h\|_{\mathcal{N}}\leq 1$. The proof of [21, Lemma 3.3] (more precisely, the statement in its first line) shows that we have the gradient estimate, $|\|s\|^{+n+1}(\nabla K^\omega)(s)|\leq c^n_{abs}, s\in\mathbb{R}^n\backslash\{0\}$, for an absolute constant c^n_{abs} independent of ω. Taking the supremum over all ω in the unit ball of $\mathcal{N}_*$ concludes that in fact $\|(\nabla K)(s)\|_{\mathcal{N}}\leq c^n_{abs}\|s\|^{-n-1}, s\in\mathbb{R}^n\backslash\{0\}$. So that certainly

(8.3) holds. So [21, Lemma 2.3, Condition (ii)] is fulfilled. Further [21, Lemma 2.3, Condition (i)] is satisfied by Remark [21, Remark 2.4]. Hence, [21, Lemma 2.3] gives the result for the column estimate. The row estimate follows by taking adjoints. Note that as in Proposition 2.7 the condition $h(0) = 0$ guarantees that $h(\nabla_{\mathbb{R}^n})(1) = 0$ so that $h(\nabla_{\mathbb{R}^n})(f) \in \mathcal{N} \odot L_\infty^\circ(\mathbb{R}^n)$ with $f \in L_\infty^\circ(\mathbb{R}^n)$ trigonometric. □

Proposition 8.6. *If $h \in C(\mathbb{R}^n, \mathcal{N})$ satisfies Condition 8.2 and $h(0) = 0$, then for $y \in L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}$,*

$$\left\| \Big((Lh)(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}_2} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})} \leq c_n \|y\|_{L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}},$$

Proof. Set $h_\theta(t,s) = h(\theta s + (1-\theta)t)$. Set $g_\theta(t) = h(t \cdot \sqrt{\theta^2 + (1-\theta)^2})$. By definition, we have

$$\Big((Lh)(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) = \int_0^1 \Big(h_\theta(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) d\theta,$$

where the integral is a Bochner integral in L_2. Therefore, we have

$$\begin{aligned} &\left\| \Big((Lh)(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})} \\ \leq &\int_0^1 \left\| \Big(h_\theta(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})} d\theta. \end{aligned}$$

By the rotation invariance, we have

$$\begin{aligned} &\left\| \Big(h_\theta(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})} \\ = &\left\| \Big(g_\theta(\nabla_{\mathbb{R}^n}) \otimes \mathrm{id}_{L_\infty(\mathbb{R}^n)} \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})}. \end{aligned}$$

By Proposition 8.5, we have

$$\begin{aligned} &\left\| \Big(g_\theta(\nabla_{\mathbb{R}^n}) \otimes \mathrm{id}_{L_\infty(\mathbb{R}^n)} \otimes \mathrm{id}_{\mathcal{M}} \Big)(y) \right\|_{\mathrm{bmo}(\mathcal{N} \otimes L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M}, \mathrm{id}_{\mathcal{N}} \otimes \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}})} \\ \leq &c_n \|y\|_{\mathcal{M}} = c_n \|y\|_{\mathcal{M}}. \end{aligned}$$

Combining these inequalities, we complete the proof. □

8.4. Proof of Theorem 8.3

In order to explicitly give the proof of Lemma 8.8 below we single out the following fact.

Fact 8.7. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. Suppose that

(i) $\mathcal{T}^1 = (T_t^1)_{t\geq 0}$ is a semi-group of positive unital operators on $\mathcal{M}_1$;
(ii) $\mathcal{T}^2 = (T_t^2)_{t\geq 0}$ is a semi-group of positive unital operators on $\mathcal{M}_2$;
(iii) *-monomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is such that

$$T_t^1 \circ \pi = \pi \circ T_t^2, \quad t \geq 0.$$

We have

(a) If $\mathcal{T}^1$ is completely positive, then so is $\mathcal{T}^2$ and

$$(\mathrm{id}_{\mathcal{N}} \otimes \mathcal{T}^1) \circ (\mathrm{id}_{\mathcal{N}} \otimes \pi) = (\mathrm{id}_{\mathcal{N}} \otimes \pi) \circ (\mathrm{id}_{\mathcal{N}} \otimes \mathcal{T}^2).$$

(b) If $\mathcal{T}^1$ and $\mathcal{T}^2$ are Markov, then

$$\|(\mathrm{id}_{\mathcal{N}} \otimes \pi)(z)\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}_1, \mathrm{id}_{\mathcal{N}}\otimes\mathcal{T}^1)} = \|z\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}_2, \mathrm{id}_{\mathcal{N}}\otimes\mathcal{T}^2)}, \quad z \in \mathcal{N} \otimes \mathcal{M}_2.$$

Lemma 8.8. *The assertion of Theorem 8.3 holds provided that* $\mathbf{A}$ *has finite spectrum.*

Proof. Let

$$\mathcal{M}_1 = L_\infty(\mathbb{R}^{2n}) \otimes \mathcal{M} \otimes M_2(\mathbb{C}), \quad \mathcal{M}_2 = \mathcal{M} \otimes M_2(\mathbb{C}),$$

and

$$\mathcal{T}^1 = \mathcal{S}^{2n} \otimes \mathrm{id}_{\mathcal{M}\otimes M_2(\mathbb{C})}, \quad \mathcal{T}^2 = \mathcal{J}^{\mathbf{A}}, \quad \pi = \pi_{\mathbf{A}}.$$

Denote for brevity

$$y = \pi(x \otimes e_{12}), \quad z = \mathcal{I}^{\mathbf{A}}_{Lh}(x) \otimes e_{12}.$$

By Lemma 8.4, we have

$$T_t^1 \circ \pi = \pi \circ T_t^2, \quad t \geq 0.$$

Since $\mathcal{T}^1$ is completely positive semi-group, then, by Fact 8.7, so is $\mathcal{T}^2$. Since $\mathbf{A}$ has finite spectrum, it follows immediately that $\mathcal{T}^2$ is symmetric and strongly continuous at 0. In other words, $\mathcal{T}^2$ is Markov. This proves the first assertion of Theorem 8.3.

By Proposition 8.6 (applied to the algebra $\mathcal{M}_2$), we have

$$\begin{aligned}&\Big\|\Big((Lh)(\nabla_{\mathbb{R}^{2n}}) \otimes \mathrm{id}_{\mathcal{M}_2}\Big)(y)\Big\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}_1, \mathrm{id}_{\mathcal{N}}\otimes\mathcal{T}_1)}\\ \leq &c_n\|y\|_{\mathcal{M}_1} = c_n\|x\|_{\mathcal{M}}.\end{aligned}$$

By Lemma 8.4, we have

$$\Big((Lh)(\nabla_{\mathbb{R}^{2n}})\otimes \mathrm{id}_{\mathcal{M}_2}\Big)(y)=(\mathrm{id}_{\mathcal{N}}\otimes\pi)(z).$$

Therefore, we have

$$\Big\|(\mathrm{id}_{\mathcal{N}}\otimes\pi)(z)\Big\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}_1,\mathrm{id}_{\mathcal{N}}\otimes\mathcal{T}_1)}\le c_n\|x\|_{\mathcal{M}}.$$

By Fact 8.7, we have

$$\|z\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}_2,\mathrm{id}_{\mathcal{N}}\otimes\mathcal{T}_2)}\le c_n\|x\|_{\mathcal{M}}.$$

This proves the second assertion of Theorem 8.3. □

Proof of Theorem 8.3. Suppose now $\mathbf{A}$ is arbitrary. Set $\mathbf{A}^l=(\frac{1}{l}\lfloor lA_1\rfloor,\cdots,\frac{1}{l}\lfloor lA_n\rfloor)$. By Lemma 8.8, $\mathcal{J}^{\mathbf{A}^l}$ is Markov. Clearly, $\mathcal{J}_t^{\mathbf{A}^l}(x)\to\mathcal{J}_t^{\mathbf{A}}(x)$ in L_2-norm and hence in measure for $x\in\mathcal{M}$ as $l\to\infty$. By Lemma 2.2 $\mathcal{J}^{\mathbf{A}}$ is also Markov. This proves the first assertion.

We briefly sketch the proof of the second assertion. By Lemma 8.8, we have

$$\Big\|\mathcal{I}_{Lh}^{\mathbf{A}^l}(x)\otimes e_{12}\Big\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}\otimes M_2(\mathbb{C}),\mathrm{id}_{\mathcal{N}}\otimes\mathcal{J}^{\mathbf{A}^l})}\le c_n\|x\|_{\mathcal{M}},\quad x\in\mathcal{M}.$$

In other words, we have

$$-(c_n\|x\|_{\mathcal{M}})^2\le B_l(t)\le(c_n\|x\|_{\mathcal{M}})^2,$$

where

$$B_l(t)\stackrel{\mathrm{def}}{=}\mathcal{I}^{\mathbf{A}^l}_{e^{-tF_{22}}}\Big(\mathcal{I}^{\mathbf{A}^l}_{Lh}(x)^*\mathcal{I}^{\mathbf{A}^l}_{Lh}(x)\Big)-\Big(\mathcal{I}^{\mathbf{A}^l}_{e^{-tF_{12}}}(\mathcal{I}^{\mathbf{A}^l}_{Lh}(x))\Big)^*\Big(\mathcal{I}^{\mathbf{A}^l}_{e^{-tF_{12}}}(\mathcal{I}^{\mathbf{A}^l}_{Lh}(x))\Big).$$

An argument identical to that in Lemma 6.4 yields $B_l(t)\to B(t)$ in measure, where

$$B(t)\stackrel{\mathrm{def}}{=}\mathcal{I}^{\mathbf{A}}_{e^{-tF_{22}}}\Big(\mathcal{I}^{\mathbf{A}}_{Lh}(x)^*\mathcal{I}^{\mathbf{A}}_{Lh}(x)\Big)-\Big(\mathcal{I}^{\mathbf{A}}_{e^{-tF_{12}}}(\mathcal{I}^{\mathbf{A}}_{Lh}(x))\Big)^*\Big(\mathcal{I}^{\mathbf{A}}_{e^{-tF_{12}}}(\mathcal{I}^{\mathbf{A}}_{Lh}(x))\Big).$$

Therefore, we have

$$-(c_n^n\|x\|_{\mathcal{M}})^2\le B(t)\le(c_n\|x\|_{\mathcal{M}})^2.$$

In other words,

$$\Big\|\mathcal{I}_{Lh}^{\mathbf{A}}(x)\otimes e_{12}\Big\|_{\mathrm{bmo}(\mathcal{N}\otimes\mathcal{M}\otimes M_2(\mathbb{C}),\mathrm{id}_{\mathcal{N}}\otimes\mathcal{J}^{\mathbf{A}})}\le c_n\|x\|_{\mathcal{M}},\quad x\in\mathcal{M}.\quad\square$$

9. Vector valued perturbations and Lipschitz estimates

In this section we consider vector valued commutator estimates. Consider a function

$$f : \mathbb{R}^n \to \mathcal{N},$$

which we assume to be differentiable. Shortly, we shall require additional smoothness assumptions on f. The function f plays the role of the Lipschitz function in Section 6. For a differentiable function $g : \mathbb{R} \to \mathbb{C}$ and $s \le t$ we have,

$$g(t) - g(s) = (t-s) \int_0^1 g'((1-\theta)s + \theta t) d\theta.$$

Therefore taking directional derivatives in the direction of the unit vector $\|t-s\|_2^{-1}(t-s)$ we find,

$$\begin{aligned} f(t) - f(s) =& \|t-s\|_2 \int_0^1 (\nabla|_{(1-\theta)s+\theta t} f) \cdot \frac{t-s}{\|t-s\|_2} d\theta \\ =& \sum_{k=1}^n \int_0^1 (\partial_k f)((1-\theta)s + \theta t)(s_k - t_k) d\theta \\ =& \sum_{k=1}^n (L\partial_k f)(s,t)(s_k - t_k). \end{aligned} \tag{9.1}$$

Lemma 9.1. *Let $c > 0$.*

(i) *There exist Schwartz functions $\varphi_l : \mathbb{R}^n \to [0,1]$ that are compactly supported with $\varphi_l(\xi) = 1$ for $\|\xi\|_2 \le l$ and $\|\xi\|_2^{|\alpha|} |\partial_\alpha \varphi_l)(\xi)| \le c$ for all $1 \le |\alpha| \le n+2$.*

(ii) *If $h \in C(\mathbb{R}^n, \mathcal{N})$ is a C^{n+2}-function that satisfies (8.2) then $(1 + c \cdot 2^{n+2})^{-1} \varphi_l h$ satisfies Condition 8.2.*

Proof. In case $n = 1$ and $l = 1$ let $\varphi_1^1 : \mathbb{R} \to [0,1]$ be a function satisfying the conditions and then set $\varphi_l^1(\xi) = \varphi_l^1(l^{-1}\xi)$ which proves the lemma for $n = 1$. For general n set the rotational invariant function $\varphi_l^n(\xi) = \varphi_l^1(\|\xi\|_2)$ which are Schwartz and satisfy (i) and (ii). We have for $\xi_1 \in \mathbb{R}$ that $|(\partial_\alpha \varphi_l)(\xi_1, 0, \ldots, 0)| = \delta_{|\alpha|=\alpha_1} |(\partial_{\alpha_1} \varphi_l^1)(\xi_1)| \le c\|\xi\|_2^{\alpha_1}$. By rotation of variables this gives $|(\partial_\alpha \varphi_l)(\xi)| \le c\|\xi\|_2^{-|\alpha|}$. By the Leibniz rule,

$$\partial_\alpha(\varphi_l h) = \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} (\partial_\beta \varphi_l)\, (\partial_\gamma f),$$

for certain combinatorical coefficients $c_{\beta,\gamma} \in \mathbb{N}$ which satisfy $\sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} = 2^{|\alpha|}$. So that,

$$
\begin{aligned}
|(\partial_\alpha \varphi_l h)(\xi)| \leq & \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} |(\partial_\beta \varphi_l)(\xi)|\, |(\partial_\gamma h)(\xi)| \\
\leq & \|\xi\|_2^{-|\alpha|} + \sum_{\beta+\gamma=\alpha,\beta\neq 0} c \cdot c_{\beta,\gamma} \|\xi\|_2^{-|\beta|} \|\xi\|_2^{-|\gamma|} \leq (1 + c\cdot 2^{|\alpha|})\|\xi\|_2^{-|\alpha|}.
\end{aligned}
$$

So for all $|\alpha| \leq n+2$ we obtain that $|(\partial_\alpha \varphi_l f)(\xi)| \leq (1 + c\cdot 2^{n+2})\|\xi\|_2^{-|\alpha|}$, i.e. Condition 8.2. □

For a function $f \in C(\mathbb{R}^n, \mathcal{N})$ and an n-tuple $\mathbf{A}$ of commuting self-adjoint operators in $\mathcal{M}$ we define,

$$
f(\mathbf{A}) = \int_{\mathbb{R}^n} f(\xi) \otimes dE^{\mathbf{A}}(\xi) \in \mathcal{N} \otimes \mathcal{M},
$$

where $E^{\mathbf{A}}$ was the spectral measure of the n-tuple $\mathbf{A}$. It is the unique element in $\mathcal{N} \otimes \mathcal{M}$ such that for every $\omega \in \mathcal{N}_*$ we have

$$
(\omega \otimes \mathrm{id})(f(\mathbf{A})) = \int_{\mathbb{R}^n} \omega \circ f(\xi) dE^{\mathbf{A}}(\xi) \in \mathcal{M}.
$$

Theorem 9.2. *Let $f : \mathbb{R}^n \to \mathcal{N}$ be a C^{n+3}-function such that each of the functions $h_k = \partial_k f, k = 1, \ldots, n$ satisfy (8.2). There exists a constant c_n only depending on the dimension n such that for every $x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M})$ and every n-tuple $\mathbf{A} = (A_1, \ldots, A_n)$ of commuting self-adjoint operators in $\mathcal{M}$ we have $[f(\mathbf{A}), 1 \otimes x] \in L_p(\mathcal{N} \otimes \mathcal{M})$. Moreover,*

$$
\|[f(\mathbf{A}), 1 \otimes x]\|_p \leq c_n \frac{p^2}{p-1} \max_k (\|h_k\|_{HM_n}) \sum_{k=1}^n \|[A_k, x]\|_p.
$$

Proof. As the theorem is true for the coordinate functions $g_k : \mathbb{R}^n \to \mathbb{R} : \xi \mapsto \xi_k, 1 \leq k \leq n$, we may replace f by $f - \sum_{k=1}^n (\partial_k f)(0) g_k$ and assume without loss of generality that $(\partial_k f)(0) = 0, 1 \leq k \leq n$.

By Lemma 9.1 let $\varphi_l : \mathbb{R}^n \to [0,1], l \in \mathbb{N}_{\geq 0}$ be as in Lemma 9.1 with $c = 2^{-n-2}$. By Lemma 9.1 we have that $2^{-1}\varphi_l h_k$ satisfies Condition 8.2. Let $l_0 \in \mathbb{N}$ be larger than $\max_k \|A_k\|$ and set $\varphi = \varphi_{l_0}$.

Consider the function $\psi_k(s,t) = s_k - t_k$ and $\psi_f(s,t) = f(s) - f(t)$. We have by (9.1) that

$$
\psi_f(\xi) = \sum_{k=1}^n L(h_k)\psi_k(\xi) = \sum_{k=1}^n L(\varphi_l h_k)\psi_k(\xi),
$$

for all $\xi \in \mathbb{R}^n$ with $\xi_i \leq \|A_k\|$.

As in the proof of Theorem 7.3, by Theorem 8.3 and Theorem 2.5 we find through a discretization of $\mathbf{A}$ and complex interpolation that for $2 \leq p \leq \infty$,

$$\|\mathcal{I}^{\mathbf{A}}_{L(\varphi h_k)} : L_p(\mathcal{M}) \to L_p(\mathcal{N} \otimes \mathcal{M})\| \le c_n p.$$

So that,

$$[f(\mathbf{A}), x] = \mathcal{I}^{\mathbf{A}}_{\psi_f}(x) = \sum_{k=1}^{n} \mathcal{I}^{\mathbf{A}}_{L(\varphi h_k)} \circ \mathcal{I}^{\mathbf{A}}_{\psi_k}(x) = \sum_{k=1}^{n} \mathcal{I}^{\mathbf{A}}_{L(\varphi h_k)}([A_k, x]).$$

Then,

$$\|[f(\mathbf{A}), x]\|_p \le \max_k(\|\mathcal{I}^{\mathbf{A}}_{L(\varphi h_k)} : L_p \to L_p\|) \sum_{k=1}^{n} \|[A_k, x]\|_p \le c_n\, p \sum_{k=1}^{n} \|[A_k, x]\|_p.$$

This concludes the proof for $2 \le p < \infty$. For $1 < p \le 2$ the proof follows by duality just as in Theorem 7.3. □

Theorem 9.3. *Let $f : \mathbb{R}^n \to \mathcal{N}$ be a C^{n+3}-function such that each of the functions $h_k = \partial_k f, k = 1, \ldots, n$ satisfy (8.2). There exists a constant c_n such that for every n-tuples of self-adjoint operators $\boldsymbol{B} = (A_1, \ldots, A_n)$ and $\boldsymbol{C} = (C_1, \ldots, C_n)$ of commuting self-adjoint operators in $\mathcal{M}$ we have*

$$\|f(\boldsymbol{B}) - f(\boldsymbol{C})\|_p \le c_n \frac{p^2}{p-1} \sum_{k=1}^{n} \|B_k - C_k\|_p.$$

Proof. Apply Theorem 9.2 to the n-tuple $\begin{pmatrix} B_k & 0 \\ 0 & C_k \end{pmatrix}$ with $k = 1, \ldots, n$ and $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. See Corollary 7.5 for details. □

We apply our results to the particular case that $\mathcal{N}$ is an algebra of freely independent semi-circular elements.

Corollary 9.4. *Let $s_i, i \in \mathbb{N}$ be freely independent semi-circular random variables and let $f_i : \mathbb{R} \to \mathbb{C}$ be C^4-functions. Put $F_l = \sum_{i=1}^{l} s_i \otimes f_i$ and assume that F_l satisfies (8.2). We have for every l that,*

$$\|\sum_{i=1}^{l} s_i \otimes f_i(B) - \sum_{i=1}^{l} s_i \otimes f_i(C)\|_p \le c_n \frac{p^2}{p-1} \|B - C\|_p.$$

Proof. This follows from Theorem 9.3 with $n = 1$ and $\mathbf{B} = B$ and $\mathbf{A} = a$ a single operator and further $f = F_l$. □

References

[1] A. Aleksandrov, V. Peller, Estimates of operator moduli of continuity, J. Funct. Anal. 261 (10) (2011) 2741–2796.

[2] H. Ando, U. Haagerup, Ultraproducts of von Neumann algebras, J. Funct. Anal. 266 (12) (2014) 6842–6913.

[3] M. Birman, M. Solomyak, Double Stieltjes operator integrals, in: Probl. Math. Phys., Izdat. Leningrad. Univ., Leningrad, 1966, pp. 33–67 (in Russian). English translation in: Topics in Mathematical Physics, vol. 1, Consultants Bureau Plenum Publishing Corporation, New York, 1967, pp. 25–54.

[4] M. Birman, M. Solomyak, Double Stieltjes operator integrals, III, Probl. Mat. Fiz. 6 (1973) 27–53 (in Russian).

[5] M. Bozejko, R. Speicher, An example of a generalized Brownian motion, Comm. Math. Phys. 137 (3) (1991) 519–531.

[6] M. Caspers, Harmonic analysis and BMO-spaces of free Araki-Woods factors, Studia Math. 246 (1) (2019) 71–107, https://doi.org/10.4064/sm170904-14-1.

[7] M. Caspers, S. Montgomery-Smith, D. Potapov, F. Sukochev, The best constants for operator Lipschitz functions on Schatten classes, J. Funct. Anal. 267 (10) (2014) 3557–3579.

[8] M. Caspers, D. Potapov, F. Sukochev, D. Zanin, Weak type commutator and Lipschitz estimates: resolution of the Nazarov-Peller conjecture, Amer. J. Math. 141 (3) (2019) 593–610, https://doi.org/10.1353/ajm.2019.0019.

[9] M. Caspers, F. Sukochev, D. Zanin, Weak type operator Lipschitz and commutator estimates for commuting tuples, Ann. Inst. Fourier (Grenoble) 68 (4) (2018) 1643–1669, https://doi.org/10.5802/aif.3195.

[10] E. Davies, Lipschitz continuity of functions of operators in the Schatten classes, J. Lond. Math. Soc. 37 (1988) 148–157.

[11] B. de Pagter, F. Sukochev, H. Witvliet, Double operator integrals, J. Funct. Anal. 192 (1) (2002) 52–111.

[12] P. Dodds, T. Dodds, B. de Pagter, F. Sukochev, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, J. Funct. Anal. 148 (1997) 28–69.

[13] P. Dodds, T. Dodds, B. de Pagter, F. Sukochev, Lipschitz continuity of the absolute value in preduals of semifinite factors, Integral Equations Operator Theory 34 (1999) 28–44.

[14] X. Duong, L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, J. Amer. Math. Soc. 18 (4) (2005) 943–973 (electronic).

[15] X. Duong, L. Yan, New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications, Comm. Pure Appl. Math. 58 (10) (2005) 1375–1420 (electronic).

[16] Y. Farforovskaya, An estimate of the nearness of the spectral decompositions of self-adjoint operators in the Kantorovich-Rubinstein metric, Vestn. Leningr. Univ. 22 (19) (1967) 155–156.

[17] Y. Farforovskaya, The connection of the Kantorovich-Rubinstein metric for spectral resolutions of selfadjoint operators with functions of operators, Vestn. Leningr. Univ. 23 (19) (1968) 94–97.

[18] Y. Farforovskaya, An example of a Lipschitz function of self-adjoint operators with non-nuclear difference under a nuclear perturbation, Zap. Nauchn. Sem. Leningr. Otdel. Mat. Inst. Steklov. (LOMI) 30 (1972) 146–153.

[19] L. Grafakos, Classical Fourier Analysis, third edition, Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.

[20] M. Junge, T. Mei, BMO spaces associated with semigroups of operators, Math. Ann. 352 (3) (2012) 691–743.

[21] M. Junge, T. Mei, J. Parcet, Smooth Fourier multipliers in group von Neumann algebras, Geom. Funct. Anal. 24 (2014) 1913–1980.

[22] M. Junge, T. Mei, J. Parcet, An invitation to harmonic analysis associated with semigroups of operators, in: Harmonic Analysis and Partial Differential Equations, in: Contemp. Math., vol. 612, Amer. Math. Soc., Providence, RI, 2014, pp. 107–122.

[23] T. Kato, Continuity of the map $S \mapsto |S|$ for linear operators, Proc. Japan Acad. 49 (1973) 157–160.

[24] E. Kissin, D. Potapov, V. Shulman, F. Sukochev, Operator smoothness in Schatten norms for functions of several variables: Lipschitz conditions, differentiability and unbounded derivation, Proc. Lond. Math. Soc. (3) 105 (4) (2012) 661–702.

[25] M. Kosaki, Unitarily invariant norms under which the map $A \to |A|$ is continuous, Publ. Res. Inst. Math. Sci. 28 (1992) 299–313.

[26] M. Krein, Some New Studies in the Theory of Perturbations of Self-Adjoint Operators, First Math. Summer School, Part I (in Russian), Izdat. "Naukova Dumka", Kiev, 1964, pp. 103–187.
[27] F. Nazarov, V. Peller, Lipschitz functions of perturbed operators, C. R. Math. Acad. Sci. Paris 347 (2009) 857–862.
[28] V. Peller, Hankel operators in the theory of perturbations of unitary and selfadjoint operators, Funktsional. Anal. i Prilozhen. 19 (2) (1985) 37–51, 96.
[29] V. Peller, Hankel operators in the perturbation theory of unbounded selfadjoint operators, in: Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990, pp. 529–544.
[30] D. Potapov, F. Sukochev, Operator-Lipschitz functions in Schatten-von Neumann classes, Acta Math. 207 (2) (2011) 375–389.
[31] E. Ricard, A Markov dilation for self-adjoint Schur multipliers, Proc. Amer. Math. Soc. 136 (12) (2008) 4365–4372.
[32] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, NJ, 1970.
[33] S. Stroock, D. Varadhan, A probabilistic approach to $H^p(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 192 (1974) 245–260.
[34] M. Takesaki, Theory of Operator Algebras. II, Encyclopaedia of Mathematical Sciences, vol. 125, Operator Algebras and Non-Commutative Geometry, vol. 6, Springer-Verlag, Berlin, 2003.
[35] M. Takesaki, Theory of Operator Algebras. III, Encyclopaedia of Mathematical Sciences, vol. 127, Operator Algebras and Non-Commutative Geometry, vol. 8, Springer-Verlag, Berlin, 2003.
[36] N. Varopoulos, Hardy-Littlewood theory for semigroups, J. Funct. Anal. 63 (2) (1985) 240–260.