SIMPLY-CONNECTED, SPINELESS 4-MANIFOLDS

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Abstract. We construct infinitely many compact, smooth 4-manifolds which are homotopy equivalent to S^2 but do not admit a spine, i.e., a piecewise-linear embedding of S^2 which realizes the homotopy equivalence. This is the remaining case in the existence problem for codimension-2 spines in simply-connected manifolds. The obstruction comes from the Heegaard Floer d invariants.

1. Introduction

Given an m-dimensional, piecewise-linear, compact manifold M which is homotopy equivalent to some closed manifold N of dimension $n < m$, a *spine* of M is a piecewise-linear embedding $N \to M$ which is a homotopy equivalence. Such an embedding is not required to be locally flat. We call M *spineless* if it does not admit a spine.

In this paper, we prove:

Theorem 1.1. *There exist infinitely many smooth, compact, spineless 4-manifolds which are homotopy equivalent to S^2.*

By way of background, Browder [Bro68], Casson, Haefliger [Hae68], Sullivan, and Wall [Wal70] showed that when $m - n > 2$, any homotopy equivalence from N to M can be perturbed into a spine. When $m - n = 2$, Cappell and Shaneson [CS76] showed that the same is true for any odd $m \geq 5$, and for any even $m \geq 6$ provided that M and N are simply-connected; they also produced examples of non-simply-connected, spineless manifolds for any even $m \geq 6$ [CS77]. (See [Sha75] for a summary of their results.) In dimension 4, Matsumoto [Mat75] produced an example of a compact spineless 4-manifold homotopy equivalent to the torus; the proof relies on higher-dimensional surgery theory. However, the question of finding spineless, compact, simply-connected 4-manifolds has remained open until now; it appears in Kirby's problem list [Kir97, Problem 4.25]. (Removing the compactness hypothesis, Matsumoto and Venema [MV79] used Casson handles to construct a simply-connected, spineless 4-manifold. By removing the boundary from the examples in Theorem 1.1, we recover such manifolds as well.)

Remark 1.2. Any compact, smooth, simply-connected 4-manifold X admitting a handlebody decomposition with no 1-handles admits a basis for H_2 represented by PL spheres. Consequently, the 4-manifolds from Theorem 1.1 cannot be constructed without 1-handles.

The proof of the theorem proceeds in two parts. The first is to give an obstruction to a spine in a compact PL 4-manifold homotopy equivalent to S^2 coming from Heegaard Floer homology. This obstruction only depends on the boundary of the 4-manifold and the sign of the intersection form. The second step is to construct the manifolds homotopy equivalent to S^2 that fail the obstruction.

The first author was partially supported by NSF grant DMS-1707795.
The second author was partially supported by NSF grant DMS-1709702.

Acknowledgments. We are grateful to Weimin Chen, Çağrı Karakurt, and Yukio Matsumoto for bringing this problem to our attention, and to Marco Golla, Josh Greene, Jen Hom, and Danny Ruberman for helpful conversations.

2. Obstruction

In order to prove Theorem 1.1, we use an obstruction coming from Heegaard Floer homology. Recall that for any rational homology sphere Y and any spin^c structure $\mathfrak{s}$ on Y, Ozsváth and Szabó [OS03] define the *correction term* $d(Y, \mathfrak{s}) \in \mathbb{Q}$, which is invariant under spin^c rational homology cobordism. To state our obstruction, we first establish the following notational convention.

Convention 2.1. Suppose X is a smooth, compact, oriented 4-manifold with $H_*(X) \cong H_*(S^2)$, and let n denote the self-intersection number of a generator of $H_2(X)$. Let $Y = \partial X$, which has $H_1(Y) \cong H^2(Y) \cong \mathbb{Z}/n$. Fix a generator $\alpha \in H_2(X)$. For $i \in \mathbb{Z}$, let $\mathfrak{t}_i$ denote the unique spin^c structure on X with

$$\langle c_1(\mathfrak{t}_i), \alpha \rangle + n = 2i.$$

Let $\mathfrak{s}_i = \mathfrak{t}_i|_Y$; this depends only on the class of i mod n. We will often treat the subscript of $\mathfrak{s}_i$ as an element of $\mathbb{Z}/n$.

Conjugation of spin^c structures swaps $\mathfrak{t}_i$ with $\mathfrak{t}_{n-i}$ and $\mathfrak{s}_i$ with $\mathfrak{s}_{n-i} = \mathfrak{s}_{-i}$. In particular, $\mathfrak{s}_0$ is self-conjugate, as is $\mathfrak{s}_{n/2}$ if n is even. Choosing the opposite generator for $H_2(X)$ likewise replaces each $\mathfrak{t}_i$ or $\mathfrak{s}_i$ with its conjugate. Because of the conjugation symmetry of Heegaard Floer homology, all statements below are insensitive to this choice.

Finally, when $n \neq 0$, we have

$$d(Y, \mathfrak{s}_i) \equiv \frac{(2i-n)^2 - |n|}{4n} \pmod{2\mathbb{Z}} \tag{2.1}$$

by [OS03, Theorem 1.2].

Our obstruction to the existence of a spine comes from the following theorem:

Theorem 2.2. *Let X be any smooth, compact, oriented 4-manifold with $H_*(X) \cong H_*(S^2)$, with a generator of $H_2(X)$ having self-intersection $n > 1$, and let $Y = \partial X$. If a generator of $H_2(X)$ can be represented by a piecewise-linear embedded 2-sphere (e.g., if X admits an S^2 spine), then for each $i \in \{0, \dots, n-1\}$,*

$$d(Y, \mathfrak{s}_i) - d(Y, \mathfrak{s}_{i+1}) = \begin{cases} \dfrac{n-2i-1}{n} \ \text{or} \ \dfrac{-n-2i-1}{n} & \text{if } 0 \leq i \leq \dfrac{n-2}{2} \\ 0 & \text{if } n \text{ is odd and } i = \dfrac{n-1}{2} \\ \dfrac{n-2i-1}{n} \ \text{or} \ \dfrac{3n-2i-1}{n} & \text{if } \dfrac{n}{2} \leq i \leq n-1. \end{cases} \tag{2.2}$$

In particular, for any i, we have

$$|d(Y, \mathfrak{s}_i) - d(Y, \mathfrak{s}_{i+1})| \leq \frac{2n-1}{n}. \tag{2.3}$$

It is easy to verify that (2.3) follows as an easy consequence of (2.2).

For any knot $K \subset S^3$, let $X_n(K)$ denote the trace of n-surgery on S^3, i.e., the manifold obtained by attaching an n-framed 2-handle to the 4-ball along a knot $K \subset S^3$. Note that $X_n(K)$ is homotopy equivalent to S^2 and has a spine obtained as the union of the cone over K in B^4 with the core of the 2-handle.

Lemma 2.3. *For any knot $K \subset S^3$ and any $n > 0$, the manifold $Y = S^3_n(K)$ satisfies the conclusions of Theorem 2.2.*

Proof. Associated to any knot $K \subset S^3$, Ni and Wu [NW15, Section 2.2] defined a sequence of nonnegative integers $V_i(K)$, which are derived from the knot Floer complex of K. (See also [Ras03].) By [HW16, Equation 2.3], these numbers have the property that

$$V_i(K) - 1 \leq V_{i+1}(K) \leq V_i(K); \tag{2.4}$$

that is, the sequence $(V_i(K))$ is non-increasing and only decreases in increments of 1. Ni and Wu proved that for each $i = 0, \ldots, n-1$, we have

$$d(S^3_n(K), \mathfrak{s}_i) = \frac{(2i-n)^2 - n}{4n} - 2\max\{V_i(K), V_{n-i}(K)\}. \tag{2.5}$$

(The first term in (2.5) is the d invariant of the lens space $L(n,1)$ in a particular spin^c structure; see [OS03, Proposition 4.8].)

For $0 \leq i \leq \frac{n-2}{2}$, we then compute:

$$\begin{aligned} d(S^3_n(K), \mathfrak{s}_i) - d(S^3_n(K), \mathfrak{s}_{i+1}) &= \frac{(2i-n)^2 - (2i+2-n)^2}{4n} - 2(V_i(K) - V_{i+1}(K)) \\ &= \frac{n-2i-1}{n} - 2(V_i(K) - V_{i+1}(K)) \\ &= \frac{n-2i-1}{n} \text{ or } \frac{-n-2i-1}{n}. \end{aligned}$$

(The last line follows from the fact that $V_i(K) - V_{i+1}(K)$ equals either 0 or 1.)

If $\frac{n}{2} \leq i \leq n-1$, then

$$d(S^3_n(K), \mathfrak{s}_i) - d(S^3_n(K), \mathfrak{s}_{i+1}) = d(S^3_n(K), \mathfrak{s}_{n-i}) - d(S^3_n(K), \mathfrak{s}_{n-i-1}),$$

and we may apply the previous case using $n-i-1$ in place of i.

In the special case where n is odd and $i = \frac{n-1}{2}$, the difference $d(S^3_n(K), \mathfrak{s}_i) - d(S^3_n(K), \mathfrak{s}_{i+1})$ is 0 since the two spin^c structures are conjugate. □

Proof of Theorem 2.2. Suppose S is a PL embedded sphere representing a generator of $H_2(X)$. We may assume that S has a single singularity modeled on the cone of a knot $K \subset S^3$ and is otherwise smooth. Therefore, S has a tubular neighborhood diffeomorphic to $X_n(K)$. To see this, observe that a neighborhood of the cone point is a copy of B^4 and the rest of the neighborhood then makes up a 2-handle attached along K. That the framing is n follows from the fact that the intersection form of X is (n). The complement of the interior of this neighborhood is a homology cobordism between $S^3_n(K)$ and Y; moreover, for each $i \in \mathbb{Z}/n$, the spin^c structures labeled $\mathfrak{s}_i$ on $S^3_n(K)$ and Y as in Convention 2.1 are identified through this cobordism. In particular, $d(Y, \mathfrak{s}_i) = d(S^3_n(K), \mathfrak{s}_i)$. By Lemma 2.3, we deduce that the conclusions of the theorem hold for Y. □

Remark 2.4. For surgery on a knot K in an arbitrary homology sphere Y, the analogue of the Ni–Wu formula (2.5) need not hold. Instead, just as in our paper with Hom [HLL18, Lemma 2.2], one can prove an inequality

$$-2N_Y \leq d(Y_n(K), \mathfrak{s}_i) - d(Y) - \frac{(2i-n)^2 - n}{4n} + 2\max\{V_i(K), V_{n-i}(K)\} \leq 0 \tag{2.6}$$

where

$$N_Y = \min\{k \geq 0 \mid U^k \cdot \mathrm{HF}_{\mathrm{red}}(Y) = 0\}.$$

It is precisely the failure of (2.5) to hold in general that makes it possible to obstruct the existence of PL disks and spheres.

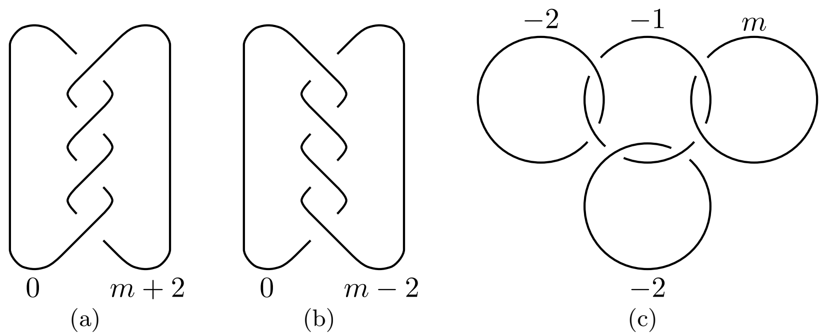

FIGURE 1. Three surgery descriptions of Q_m.

Remark 2.5. There is also an obstruction to the existence of a PL sphere in the case where $n = 0$, although we do not know of any actual example where it is effective. If Y is any 3-manifold with vanishing triple cup product on $H^1(Y)$, and $\mathfrak{s}$ is any torsion spin^c structure on Y, then there are two relevant invariants to consider: the untwisted "bottom" d invariant $d_b(Y, \mathfrak{s})$ defined by Ozsváth and Szabó [OS03] (see also [LRS15]), and the totally twisted d invariant $\underline{d}(Y, \mathfrak{s})$ defined by Behrens and Golla [BG18]. These invariants are both preserved under spin^c homology cobordism, and they satisfy $\underline{d}(Y, \mathfrak{s}) \leq d_b(Y, \mathfrak{s})$ [BG18, Proposition 3.8]. We do not know of any 3-manifold for which this inequality is strict.

For any knot $K \subset S^3$, Behrens and Golla showed that $\underline{d}(S^3_0(K), \mathfrak{s}_0) = d_b(S^3_0(K), \mathfrak{s}_0)$, where $\mathfrak{s}_0$ denotes the unique torsion spin^c structure [BG18, Example 3.9]. Just as in the proof of Theorem 2.2, it follows that if X is a smooth 4-manifold with the homology of S^2 and vanishing intersection form, and the generator of $H_2(X)$ can be represented by a PL sphere, then $\underline{d}(\partial X, \mathfrak{s}_0) = d_b(\partial X, \mathfrak{s}_0)$.

3. CONSTRUCTION

We now describe a family of 4-manifolds homotopy equivalent to S^2 which fail to satisfy the conclusion of Theorem 2.2.

For any integer m, let Q_m denote the total space of a circle bundle over $\mathbb{RP}^2$ with normal Euler number m. For more detail on these manifolds, see for instance [LRS15]. This is a rational homology sphere with

$$H_1(Q_m) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & m \text{ even} \\ \mathbb{Z}/4 & m \text{ odd.} \end{cases}$$

The manifold Q_m can be described by any of the surgery diagrams in Figure 1.

For any m, Doig [Doi15, Section 3] proved that the d invariants of Q_m in the four spin^c structures are

$$\left\{ \frac{m+2}{4}, \frac{m-2}{4}, 0, 0 \right\}. \tag{3.1}$$

(See also the work of Ruberman, Strle, and the first author [LRS15, Theorem 5.1].)

For each integer p, let Y_p be the 3-manifold given by the surgery diagram in Figure 2, which naturally bounds a plumbed 4-manifold. It is easy to check that Y_p is the Seifert fibered homology

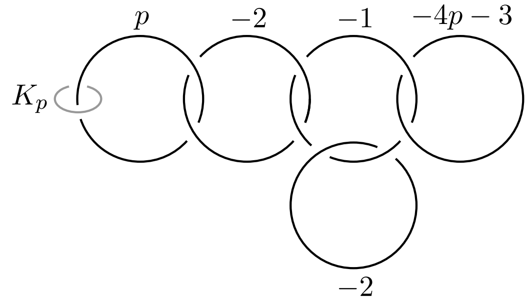

FIGURE 2. Surgery description of the Brieskorn sphere Y_p. The knot K_p represents a singular fiber in a Seifert fibration on Y_p.

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sphere

$$Y_p \cong \begin{cases} \Sigma(2, -(2p+1), -(4p+3)) & p < -1 \\ S^3 & p = -1, 0 \\ -\Sigma(2, 2p+1, 4p+3) & p > 0. \end{cases}$$

(Our convention is that for pairwise relatively prime integers $a, b, c > 0$, the Brieskorn sphere $\Sigma(a, b, c)$ is oriented as the boundary of a positive-definite plumbing. Note, however, that the plumbing shown in Figure 2 is indefinite.)

Let $K_p \subset Y_p$ be the knot obtained as a meridian of the p-framed surgery curve, shown in Figure 2. In the cases $p = -1$ or $p = 0$, where $Y_p \cong S^3$, K_p is the unknot or the right-handed trefoil, respectively; otherwise, K_p is the singular fiber of order $2p + 1$. The 0-framing on this curve (viewed as a knot in S^3) corresponds to the $+4$ framing on K_p (as a knot in Y_p). Performing surgery using this framing produces Q_{-4p-3}, since we can cancel the p-framed component with its 0-framed meridian to produce Figure 1(c) with $m = -4p - 3$.

We are now able to construct the spineless four-manifolds claimed in Theorem 1.1. Define the four-manifold W_p obtained by taking $(Y_p - B^3) \times [0, 1]$, which has boundary $Y_p \# -Y_p$, and attaching a $+4$-framed 2-handle along the knot $K_p \times \{1\}$. The boundary of W_p is $Q_{-4p-3} \# -Y_p$; denote this three-manifold by M_p.

Proposition 3.1. *For each p, the manifold W_p is homotopy equivalent to S^2.*

Proof. First, notice that $(Y_p - B^3) \times [0, 1]$ is an integer homology ball, so after attaching the 2-handle, W_p has the same homology as S^2. To show that W_p is simply-connected (and hence homotopy equivalent to S^2), it is sufficient to show that the homotopy class of K_p normally generates $\pi_1(Y_p)$. This is obvious in the case that $p = -1, 0$ as $Y_p = S^3$. The following lemma proves this claim in the remaining cases. □

Lemma 3.2. *For any pairwise relatively prime integers p, q, r, the fundamental group of the Brieskorn sphere $\Sigma(p, q, r)$ is normally generated by any of the singular fibers.*

Proof. Write $\Sigma(p, q, r) = S^2(e; (p, p'), (q, q'), (r, r'))$, where $gcd(p, p') = gcd(q, q') = gcd(r, r') = 1$. Then,

$$\pi_1(\Sigma(p, q, r)) = \langle x, y, z, h \mid h \text{ central}, x^p h^{p'} = y^q h^{q'} = z^r h^{r'} = xyzh^e = 1\rangle. \tag{3.2}$$

To see this presentation, we consider the standard surgery description for $\Sigma(p, q, r)$ as in Figure 3. The complement of the surgery link L has

$$\pi_1(S^3 - L) = \langle x, y, z, h \mid h \text{ central}\rangle.$$

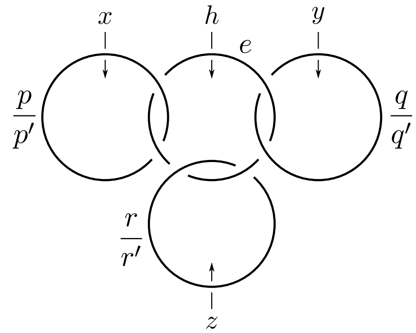

FIGURE 3. Surgery description of $\Sigma(p,q,r)$, along with generators for π_1.

Here, x, y, z represent meridians of the three parallel curves while h represents the fiber direction. The four additional relators in (3.2) represent the longitudes filled by the Dehn surgeries.

Without loss of generality, we consider the singular fiber of order p, which is the core of the Dehn surgery on the leftmost component in Figure 3. This curve is represented in $\pi_1(\Sigma(p,q,r))$ by $x^a h^b$, where a, b are any integers such that $|bp - ap'| = 1$. Thus, we must show that the quotient $G = \pi_1(\Sigma(p,q,r))/\langle\langle x^a h^b \rangle\rangle$ is trivial. Because x and h commute and $|bp - ap'| = 1$, the subgroup of G generated by x and h is the same as the subgroup generated by $x^a h^b$ and $x^p h^{p'}$. Therefore, $x = h = 1$ in G, so

$$G \cong \langle y, z \mid y^q = z^r = yz = 1 \rangle.$$

Since q and r are relatively prime, this implies that G is the trivial group. Consequently, the singular fibers normally generate the fundamental group of $\Sigma(p,q,r)$. $\square$

The following proposition now establishes Theorem 1.1; specifically, it shows that the manifolds W_p are spineless for $p \notin \{-2,-1,0\}$. (Both W_{-1} and W_0 contain spines since they are obtained by attaching a 2-handle to the 4-ball; we do not know whether W_{-2} has a spine.)

Proposition 3.3. *If M_p bounds a compact, smooth, oriented 4-manifold X with $H_*(X) \cong H_*(S^2)$ in which a generator of $H_2(X)$ can be represented by a PL 2-sphere, then $p \in \{-2,-1,0\}$.*

Proof. Suppose M_p bounds a compact, smooth, oriented 4-manifold X with $H_*(X) \cong H_*(S^2)$. Observe that the four d invariants of M_p are equal to those of Q_{-4p-3} minus the even integer $d(Y_p)$. To be precise, label the four spin^c structures on M_p by $\mathfrak{s}_0, \dots, \mathfrak{s}_3$ according to Convention 2.1. By (2.1), we deduce that the intersection form of X must be positive-definite, and

$$d(M_p, \mathfrak{s}_0) \equiv \frac{3}{4}, \quad d(M_p, \mathfrak{s}_1) = d(M_p, \mathfrak{s}_3) \equiv 0, \quad d(M_p, \mathfrak{s}_2) \equiv \frac{7}{4} \pmod{2\mathbb{Z}}.$$

(If the intersection form were negative-definite, the d invariants of $\mathfrak{s}_0$ and $\mathfrak{s}_2$ would be congruent to $\frac{5}{4}$ and $\frac{1}{4}$ respectively, which would violate (3.1).) These congruences enable us to identify which of the two self-conjugate spin^c structures is $\mathfrak{s}_0$ and which is $\mathfrak{s}_2$. Specifically, when p is odd, we have

$$\begin{aligned} d(M_p, \mathfrak{s}_0) &= -d(Y_p) - \frac{4p+1}{4} \\ d(M_p, \mathfrak{s}_1) = d(M_p, \mathfrak{s}_3) &= -d(Y_p) \\ d(M_p, \mathfrak{s}_2) &= -d(Y_p) - \frac{4p+5}{4}. \end{aligned}$$

By Theorem 2.2, if there is a PL sphere representing a generator of $H_2(X)$, then:

$$-\frac{4p+1}{4} = d(M_p, \mathfrak{s}_0) - d(M_p, \mathfrak{s}_1) = \frac{3}{4} \text{ or } -\frac{5}{4}$$
$$\frac{4p+5}{4} = d(M_p, \mathfrak{s}_1) - d(M_p, \mathfrak{s}_2) = \frac{1}{4} \text{ or } -\frac{7}{4}$$

These two equations imply that $p = -1$.

Similarly, when p is even, the roles of $\mathfrak{s}_0$ and $\mathfrak{s}_2$ are exchanged, and we deduce that p equals either -2 or 0. □

Remark 3.4. In [Doi15], Doig computed the d invariants of Q_m and used these to show that many of the Q_m cannot be obtained by surgery on a knot in S^3. Our arguments further show that Q_m cannot be integrally homology cobordant to surgery on a knot. While Doig's arguments use d invariants, which are homology cobordism invariants, they also rely on the fact that the Q_m are L-spaces, which is not a property that is preserved under homology cobordism.

Remark 3.5. For any $k > 1$, one can modify the construction above to obtain spineless 4-manifolds X with $H_1(\partial X) \cong \mathbb{Z}/k^2$. Let $Q_{k,m}$ be the manifold obtained by $(0, m+k)$ surgery on the $(2, 2k)$ torus link. (Using our previous notation, $Q_m = Q_{2,m}$, as seen in Figure 1(a).) Then $\left|H^2(Q_{k,m})\right| = k^2$, and $H^2(Q_{k,m})$ is cyclic iff $\gcd(k, m) = 1$. Since $Q_{k,m}$ bounds a rational homology ball, the d invariants of k of the k^2 spin^c structures on $Q_{k,m}$ vanish. On the other hand, the exact triangle relating the Heegaard Floer homologies of $S^1 \times S^2$, $Q_{k,m}$, and $Q_{k,m+1}$ shows that the d invariants of the remaining spin^c structures vary roughly linearly in m. In particular, the differences between d invariants of adjacent spin^c structures can be arbitrarily large. Moreover, one can realize $Q_{k,m}$ (for appropriate m) as surgery on a fiber in a Brieskorn sphere; the result then follows as above.

We do not know of any instances where Theorem 2.2 obstructs the existence of a PL sphere when n is not a perfect square.

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