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ON RELATIVE ENTROPY AND GLOBAL INDEX

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ABSTRACT. Certain duality of relative entropy can fail for a chiral conformal net with nontrivial representations. In this paper we quantify such statement by defining a quantity which measures the failure of such duality, and identify this quantity with relative entropy and global index associated with multi-interval subfactors for a large class of conformal nets. As a consequence of such new formulation we show that the duality holds for a large class of conformal nets if and only if they are holomorphic. The same argument also applies to CFT in two dimensions. In particular we show that the duality holds for a large class of CFT in two dimensions if and only if they are modular invariant. We also obtain various limiting properties of relative entropies which naturally follow from our formula.

1. INTRODUCTION

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [9] for a partial list of references. See [8], [19], [18], [20], [21], [26], [32], and [33] for a partial list of recent mathematical work.

For a nice introduction to various aspects of entropy, we refer the reader to Chapters 5 and 6 of [25]. We recall some definitions there. A von Neumann entropy for a state associated with a density matrix ρ on the space of bounded operators on a Hilbert space $\mathcal{H}$ is given by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \ .$$

A von Neumann entropy can be viewed as a measure of the lack of information about a system to which one has ascribed the state. This interpretation is in accord for instance with the facts that $S(\rho) \geq 0$ and that a pure state $\rho = |\Psi\rangle\langle\Psi|$ has vanishing von Neumann entropy. A related notion is that of the relative entropy. It is defined for two density matrices ρ, ρ' by

$$S(\rho, \rho') = \mathrm{Tr}(\rho \log \rho - \rho \log \rho') \ . \tag{1}$$

Like $S(\rho)$, $S(\rho, \rho')$ is nonnegative, and can be infinite.

A generalization of the relative entropy in the context of von Neumann algebras of arbitrary-type was found by Araki [1] and is formulated using modular theory. Given two faithful, normal states ω, ω' on a von Neumann algebra $\mathcal{A}$ in standard form, we choose the vector representatives in the natural cone $\mathcal{P}^\sharp$, called $|\Omega\rangle, |\Omega'\rangle$. The anti-linear operator $S_{\omega,\omega'} a|\Omega'\rangle = a^*|\Omega\rangle$, $a \in \mathcal{A}$, is closable and one considers

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again the polar decomposition of its closure $\bar{S}_{\omega,\omega'} = J\Delta^{1/2}_{\omega,\omega'}$. Here J is the modular conjugation of $\mathcal{A}$ associated with $\mathcal{P}^\sharp$ and $\Delta_{\omega,\omega'} = S^*_{\omega,\omega'}\bar{S}_{\omega,\omega'}$ is the relative modular operator w.r.t. $|\Omega\rangle, |\Omega'\rangle$. Of course, if $\omega = \omega'$, then $\Delta_\omega = \Delta_{\omega,\omega'}$ is the usual modular operator. The relative entropy w.r.t. ω and ω' is defined by

$$S(\omega, \omega') = \langle \Omega | \log \Delta_{\omega,\omega'} \ \Omega\rangle. \tag{2}$$

S is extended to positive linear functionals that are not necessarily normalized by the formula $S(\lambda\omega, \lambda'\omega') = \lambda S(\omega, \omega') + \lambda \log(\lambda/\lambda')$, where $\lambda, \lambda' > 0$ and ω, ω' are normalized. If ω' is not normal, then one sets $S(\omega, \omega') = \infty$.

One can use relative entropy to define entropy for states on general von Neumann algebras as in (6.9) of [25], but for type III von Nemann algebras which is the type of von Neumann algebras we will consider in this paper, this will always be infinity (cf. Lemma 6.9 of [25]).

This paper is motivated by a very simple fact about von Neumann entropy. In the finite dimensional case the von Neumann entropy of a pure state for a matrix algebra M and its commutant M' are equal, a simple exercise in linear algebra. In the case of conformal net $\mathcal{A}$ the algebra M is replaced by the algebra of observables localized on disjoint union of intervals I denoted by $\mathcal{A}(I)$. The vacuum state is a pure state. Hence one may expect that the von Neumann entropy of vacuum state for $\mathcal{A}(I)$ and its commutant are equal. But for type III factors von Neumann entropy is always infinity so this is not very interesting. By the work of [3] and [21] one can define a regularized von Neumann entropy (cf. Definition 2.23) for $\mathcal{A}(I)$, denoted by $G(I)$, which is finite but not positive, yet verifies equations similar to von Neumann entropy in the finite dimensional case. When the global index of $\mathcal{A}$ is one (such a conformal net is also called *holomorphic*), $\mathcal{A}(I)' = \mathcal{A}(I')$, one can therefore ask if the regularized von Neumann entropy for $\mathcal{A}(I)$ and $\mathcal{A}(I)' = \mathcal{A}(I')$ is the same. This is what we called a **duality relation**.

It was observed in §3 of [21] that the regularized von Neumann entropy for $\mathcal{A}(I)$ and $\mathcal{A}(I')$ are different when the global index of $\mathcal{A}$ is greater than one in some cases, and it is natural to conjecture that duality relation above holds if and only if the conformal net has global index equal to 1 (cf. Conjecture 2.24 for a precise statement). The only currently known example that verifies such a relation is the free fermion net for which we have explicit formulas for mutual information in general as in [21]. One of the goals of this paper is to prove that this conjecture is true for a large class of chiral CFT (Corollary 2.34) and also CFT in two dimensions which are modular invariant (Corollary 3.10). For an example, it follows from Corollary 2.34 that such a duality relation is true for conformal nets associated with any even positive unimodular lattices. The number of such lattices grow very fast as their rank increases (for an example there are more than a billion such lattices with rank 32, and the number grows even greater when the rank is greater than 32 according to [12]), demonstrating the power of our new results.

To prove such results we are led to consider a quantity called **deficit**, which is simply the difference $D_{\mathcal{A}}(I) = G(I) - G(I')$, and conjecture (cf. Conjecture 2.27) that $D_{\mathcal{A}}(I)$ is equal to another quantity $\hat{D}_{\mathcal{A}}$ which is defined by using the data associated with the inclusion $\mathcal{A}(I) \subset \mathcal{A}(I')'$ (cf. Definition 2.13). Even though Conjecture 2.24 is the motivating problem for this paper, the more refined Conjecture 2.27 is in fact one of the key new ideas in this paper. We will (cf. Proposition 2.28) see that Conjecture 2.27 implies Conjecture 2.24. Our main observation is Theorem 2.29 where $D_{\mathcal{A}}(I) - \hat{D}_{\mathcal{A}}(I)$ remain the same for a pair of conformal nets $\mathcal{A} \subset \mathcal{B}$ with finite index. Note that $D_{\mathcal{A}}(I) - \hat{D}_{\mathcal{A}}(I) = 0$ for free fermion nets can be verified by explicit formulas of [21]. It follows that any conformal net $\mathcal{A}$ that is **chain related** to free fermion net $\mathcal{A}_r$, i.e., there exists a sequence of conformal nets $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $\mathcal{B}_1 = \mathcal{A}, \mathcal{B}_n = \mathcal{A}_r$ and either $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ or $\mathcal{B}_{i+1} \subset \mathcal{B}_i, 1 \leq i \leq n-1$, and all inclusions of finite index must verify our conjecture (cf. Corollaries 2.30 and 3.9).

To give the reader an idea what kind of equalities are proved in this paper let us consider a special case of Corollary 2.30 for a conformal net $\mathcal{A}$ that is chain related to free fermion net $\mathcal{A}_r$. Then for $I = I_1 \cup I_2, I' = J_1 \cup J_2$ we have

$$S(\omega, \omega_{J_1} \otimes \omega_{J_2}) - S(\omega, \omega_{I_1} \otimes \omega_{I_2}) - \frac{c}{6} \ln \eta = S(\omega, \omega F_I) - \frac{1}{2} \ln \mu_{\mathcal{A}},$$

where S is the relative entropy, ω is the vacuum state, c is the central charge, $\mu_{\mathcal{A}}$ is the global index of $\mathcal{A}$, $\eta = \frac{r_{J_1} r_{J_2}}{r_{I_1} r_{I_2}}$ is a cross ratio, and $F_I : \mathcal{A}(J_1 \cup J_2)' \to \mathcal{A}(I_1 \cup I_2)$ is the conditional expectation.

Previously relations among relative entropies, central charge, and global index were given in asymptotic form in Th. 4.2 of [21]. The above relation is an identity. The duality condition as described above holds when the righthand side is 0. As a simple consequence we note that since $S(\omega, \omega F_I) \geq 0$ and is monotonic increasing with respect to I, it follows immediately that

$$S(\omega, \omega_{J_1} \otimes \omega_{J_2}) - S(\omega, \omega_{I_1} \otimes \omega_{I_2}) - \frac{c}{6} \ln \eta + \frac{1}{2} \ln \mu_{\mathcal{A}} \geq 0$$

and is monotonic increasing with respect to I, an interesting result in its own right.

For the relation between our results and physicists computation using the replica trick, see Remark 3.11.

The rest of this paper is as follows: In section 2, after introducing relative entropy, spatial derivatives, and index for general von Neumann algebras, we prove a property of relative entropy in Proposition 2.4 which is motivated by our conjecture above. Then we consider a chiral conformal net. We include two preliminary sections on conformal nets from [2] and [11], and some results from [21] which will be crucial for this paper. Then we first define a quantity called deficit to measure the failure of duality and we prove our main theorem, Theorem 2.29, and deduce its consequences. In sections 2.4 and 2.5 we apply Theorem 2.29 to study a number of natural problems on relative entropy.

In section 3 we consider the two dimensional CFT cases while essentially all results of section 2 hold with small modifications.

2. Preliminaries

2.1. **Spatial derivatives, relative entropy, and index theory for general subfactors.** Let ψ be a normal state on a von Neumann algebra M acting on a Hilbert space H and let ϕ' be a normal faithful state on the von Neumann algebra M'. The Connes spatial derivative, usually denoted by $\frac{d\psi}{d\phi'}$, is a positive operator (cf. [5]). We will use the simplified notation of [25] and write $\frac{d\psi}{d\phi'} = \Delta(\frac{\psi}{\phi'})$. If ψ is faithful, we have

$$\Delta(\frac{\psi}{\phi'})^{it} m \Delta(\frac{\psi}{\phi'})^{-it} = \sigma_t^{\psi}(m) \ \forall m \in M, \Delta(\frac{\psi}{\phi'})^{it} m \Delta(\frac{\psi}{\phi'})^{-it} = \sigma_{-t}^{\phi'}(m) \ \forall m \in M',$$

where $\sigma_t^{\psi}, \sigma_{-t}^{\phi'}$ are modular automorphisms

$$[D\psi_1 : \psi_2]_t := \Delta(\frac{\psi_1}{\phi'})^{it}\Delta(\frac{\psi_2}{\phi'})^{-it}$$

is independent of the choice of ϕ' and is called a **Connes cocycle**.

Also, if $\psi_1 \geq \psi_2$, then

$$\Delta(\frac{\psi_1}{\phi'}) \geq \Delta(\frac{\psi_2}{\phi'}).$$

By page 476 of [30] this is equivalent to

$$\frac{1}{1+\Delta(\frac{\psi_1}{\phi'})} \leq \frac{1}{1+\Delta(\frac{\psi_2}{\phi'})}$$

as bounded operators.

Suppose M acts on a Hilbert space H and ω is a vector state given by $\Omega \in H$. The relative entropy (cf. 5.1 of [25]) in this case is $S(\omega, \phi) = -\langle \ln \Delta(\phi/\omega')\Omega, \Omega\rangle$ where ω' is the vector state on M' defined by vector Ω and $\Delta(\phi/\omega') := \frac{d\phi}{d\omega'}$ is a Connes spatial derivative. When Ω is not in the support of ϕ we set $S(\omega, \phi) = \infty$.

A list of properties of relative entropies that will be used later can be found in [25] (cf. Th. 5.3, Th. 5.15, and Cor. 5.12 of [25]).

Theorem 2.1. (1) *Let M be a von Neumann algebra and let M_1 be a von Neumann subalgebra of M. Assume that there exists a faithful normal conditional expectation E of M onto M_1. If ψ and ω are states of M_1 and M, respectively, then $S(\omega, \psi\cdot E) = S(\omega\restriction M_1, \psi) + S(\omega, \omega \cdot E)$.*

(2) *Let M_i be an increasing net of von Neumann subalgebras of M with the property $(\bigcup_i M_i)'' = M$. Then $S(\omega_1 \restriction M_i, \omega_2 \restriction M_i)$ converges to $S(\omega_1, \omega_2)$ where ω_1, ω_2 are two normal states on M.*

(3) *Let ω and ω_1 be two normal states on a von Neumann algebra M. If $\omega_1 \geq \mu\omega$, then $S(\omega, \omega_1) \leq \ln \mu^{-1}$.*

(4) *Let ω and ϕ be two normal states on a von Neumann algebra M, and denote by ω_1 and ϕ_1 the restrictions of ω and ϕ to a von Neumann subalgebra $M_1 \subset M$, respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$.*

(5) *Let ϕ be a normal faithful state on $M_1 \otimes M_2$. Denote by ϕ_i the restriction of ϕ to $M_i, i = 1, 2$. Let ψ_i be normal faithful states on $M_i, i = 1, 2$. Then*

$$S(\phi, \psi_1 \otimes \psi_2) = S(\phi_1, \psi_1) + S(\phi_2, \psi_2) + S(\phi, \phi_1 \otimes \phi_2).$$

Let $E : M \to N$ be a normal faithful conditional expectation onto a subalgebra N. $E^{-1} : N' \to M'$ is in general an operator valued weight which verifies the following equation: for any pair of normal faithful weights ψ on N and ϕ' on M' we have

$$\Delta(\frac{\psi E}{\phi'}) = \Delta(\frac{\psi}{\phi' E^{-1}}).$$

Kosaki (cf. [13]) defined an index of E, denoted by Ind$E$ to be $E^{-1}(1)$. When 1 is in the domain of E^{-1}, we say that E has finite index. When both N, M are factors and E has finite index, we have the (cf. [27]) Pimsner-Popa inequality

$$E(m) \geq \lambda m \ \forall m \in M_+,$$

where $\lambda = (\mathrm{Ind}E)^{-1}$. The action of the modular group $\sigma_t^{\psi E}$ on $N' \cap M$ is independent of the choice of ψ. When E is the minimal conditional expectation such action is trivial on $N' \cap M$. Also the compositions of minimal conditional expectations are minimal (cf. [14]).

2.2. **A result on relative entropy.**

Lemma 2.2. *Let A, B be positive unbounded operators on a Hilbert space such that $A \geq B$, and let Ω be a unit vector such that $B\Omega = c\Omega$ where $c > 0$ is a constant, $\langle A\Omega, \Omega\rangle = 1$. Let m_A be the spectral measure of A associated with Ω. Then $\int_0^\infty (\ln \lambda)^2 dm_A(\lambda) < \infty$.*

Proof. By page 476 of [30] we have that $\frac{1}{1/n+A} \leq \frac{1}{1/n+B}$ $\forall n > 0$ and it follows

$$\int_0^\infty \frac{1}{1/n + \lambda} dm_A(\lambda) \leq \frac{1}{1/n + c} \ \forall n > 0.$$

Let n go to infinity and by the monotone convergence theorem we have

$$\int_0^\infty \frac{1}{\lambda} dm_A(\lambda) \leq \frac{1}{c} \ \forall n > 0.$$

We note that $(\ln \lambda)^2$ is bounded by a constant times $1/\lambda$ on $(0, 1)$, and a constant times λ on $[1, \infty)$. Since by assumption $\int_0^\infty \lambda dm_A(\lambda) = 1$, we have shown that

$\int_0^1 (\ln \lambda)^2 dm_A(\lambda) < \infty$,

$\int_1^\infty (\ln \lambda)^2 dm_A(\lambda) < \infty$, and the proof is complete. □

Lemma 2.3. *Let A be a self-adjoint operator on a Hilbert space, and let Ω be a vector in the domain of A. Let $f(t)$ be a strong operator continuous function in a neighborhood of 0 with value in the space of bounded operators such that $f(0)$ is the identity. Then*

$$\lim_{t\to 0} \frac{-i}{t} \langle (e^{itA} - 1) f(t)\Omega, \Omega\rangle = \langle A\Omega, \Omega\rangle.$$

Proof. By assumption it is enough to check that

$$\lim_{t\to 0} \frac{-i}{t} \langle (e^{itA} - 1)(f(t) - 1)\Omega, \Omega\rangle = 0.$$

We note that

$$||\frac{-i}{t}(e^{itA} - 1)\Omega||^2 = \int |\frac{1}{t}(e^{it\lambda} - 1)|^2 dm_A(\lambda) \leq \int |\lambda|^2 dm_A(\lambda) < \infty,$$

$$||(f(t) - f(0))\omega||,$$

goes to 0 as t goes to 0, and the lemma is proved. □

Proposition 2.4. *Let M be a factor and let ω be a normal faithful state on M acting on the standard representation space H, and let Ω be the corresponding vector such that $\langle m\Omega, \Omega\rangle = \omega(m)$ $\forall m \in M$. We shall use the same notation ω to denote the vector state on $B(H)$ and its restriction to subalgebras of $B(H)$.*

Let $E_1 : M \to M_1, E_2 : M' \to M_2$ be normal conditional expectation with finite index, where M_1, M_2 are also factors. Then

$$S(\omega, \omega E_1) - S(\omega, \omega E_2) = S(\omega, \omega E_1 E_2^{-1})$$

and this equation can also be written as

$$S(\omega, \omega E_1) + S(\omega, \omega E_2^{-1}) = S(\omega, \omega E_1 E_2^{-1}).$$

Proof. Ad (1): By definition we have

$$S(\omega,\omega E_1) - S(\omega,\omega E_2) = \lim_{t\to 0}\frac{-i}{t}\langle(\Delta(\frac{\omega E_2}{\omega})^{it} - (\Delta(\frac{\omega E_1}{\omega'})^{it})\Omega,\Omega\rangle.$$

We note that

$$\Delta(\frac{\omega E_1}{\omega'})^{it}\Omega = \Delta(\frac{\omega E_1}{\omega'})^{it}\Delta(\frac{\omega}{\omega'})^{-it}\Omega = [D\omega E_1 : D\omega]_t\Omega,$$

$$\Delta(\frac{\omega E_1}{\omega E_2})^{it}\Delta(\frac{\omega E_2}{\omega})^{it} = \Delta(\frac{\omega E_1}{\omega'})^{it}\Delta(\frac{\omega}{\omega'})^{-it}.$$

It follows that

$$S(\omega,\omega E_1) - S(\omega,\omega E_2) = \lim_{t\to 0}\frac{-i}{t}\langle((\Delta(\frac{\omega E_1}{\omega E_2})^{-it} - 1)\Delta(\frac{\omega E_1}{\omega'})^{it}\Omega,\Omega\rangle.$$

Note that $\Delta(\frac{\omega E_1}{\omega E_2}) = \Delta(\frac{\omega E_1 E_2^{-1}}{\omega'}) \geq \mu\Delta(\frac{\omega}{\omega'})$ for some $\mu > 0$. Here the spatial derivative $\Delta(\frac{\omega}{\omega'})$ is determined by state ω on M_2' and M_2, respectively.

By Lemmas 2.2 and 2.3 we have proved the first equation. If we apply this equation with E_1 equal to identity we get

$$S(\omega,\omega) - S(\omega,\omega E_2) = S(\omega,\omega E_2^{-1})$$

and the second equation follows. □

It is convenient to formulate the second equation of the above proposition in the following form.

Corollary 2.5. *Let $N_3 \subset N_2 \subset N_1$ be factors on a Hilbert space H and let ω be a vector state on $B(H)$ given by a vector $\Omega \in H$. Let $F_i, N_i \to N_{i+1}, i = 1,2$ be a conditional expectation with finite index. Assume that Ω is cyclic and separating for N_2. Then*

$$S(\omega,\omega F_2F_1) = S(\omega,\omega F_2) + S(\omega,\omega F_1).$$

Proof. This is just a reformulation of the second equation of Proposition 2.4 by noting that we can rename $N_1 = M_2', N_2 = M, N_3 = M_1, F_1 = (\mathrm{Ind}E_2)^{-1}E_2^{-1}, F_2 = E_1$. □

Remark 2.6. Under the conditions of the above corollary. $S(\omega,\omega F)$ is additive under compositions of conditional expectations, just like $\ln \mathrm{Ind}E$. But of course $S(\omega,\omega F)$ also depends on the state ω. This fact plays an important role in the proof of Theorem 2.29 and Theorem 2.39 in the following.

2.3. **Chiral CFT case.**

2.3.1. *Graded nets.* This section is contained in [2] and [11]. We refer to [2] and [11] for more details and proofs.

We shall denote by Möb the Möbius group, which is isomorphic to $SL(2,\mathbb{R})/\mathbb{Z}_2$ and acts naturally and faithfully on the circle S^1.

By an interval of S^1 we mean, as usual, a nonempty, nondense, open, connected subset of S^1 and we denote by $\mathcal{I}$ the set of all intervals. If I is an interval on the circle on a complex plane with two end points a, b, $r_I := |b-a|$ is called the length of I. If $I \in \mathcal{I}$, then also $I' \in \mathcal{I}$ where I' is the interior of the complement of I. Two intervals are *disjoint* if their closures are disjoint. A finite set of intervals are disjoint if any two different intervals from the set are disjoint.

We will denote by $\mathcal{PI}$ **the set which consists of the union of a finite set of disjoint intervals**. Such a set will play an important role in the next section.

A *net* $\mathcal{A}$ *of von Neumann algebras on* S^1 is a map

$$I \in \mathcal{I} \mapsto \mathcal{A}(I)$$

from the set of intervals to the set of von Neumann algebras on a (fixed) Hilbert space $\mathcal{H}$ which verifies the *isotony property*:

$$I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2),$$

where $I_1, I_2 \in \mathcal{I}$.

A *Möbius covariant net* $\mathcal{A}$ of von Neumann algebras on S^1 is a net of von Neumann algebras on S^1 such that the following properties hold:

1. MÖBIUS COVARIANCE**:** *There is a strongly continuous unitary representation* U *of* Möb *on* $\mathcal{H}$ *such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \ , \qquad g \in \text{Möb}, \ I \in \mathcal{I} \ .$$

2. POSITIVITY OF THE ENERGY**:** *The generator of the rotation one-parameter subgroup* $\theta \mapsto U(\mathrm{rot}(\theta))$ *(conformal Hamiltonian) is positive*, namely U is a positive energy representation.
3. EXISTENCE AND UNIQUENESS OF THE VACUUM**:** *There exists a unit* U*-invariant vector* Ω *(vacuum vector), unique up to a phase, and* Ω *is cyclic for the von Neumann algebra* $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.

A *conformal net* $\mathcal{A}$ of von Neumann algebras on S^1 is a net of von Neumann algebras on S^1 such that the above properties 2 and 3 hold, and 1 is replaced by conformal covariance:

> *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I), \ g \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphisms of S^1 and $\mathrm{Diff}(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$.

A $\mathbb{Z}_2$-grading on $\mathcal{A}$ is an involutive automorphism $\mathbf{g} = \mathrm{Ad}\Gamma$ of $\mathcal{A}$, such that $\Gamma^2 = 1$, $\Gamma\Omega = \Omega, \Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$ for all I.

Given the grading $\mathbf{g}$, an element x of $\mathcal{A}$ such that $\mathbf{g}(x) = \pm x$ is called homogeneous, indeed a Bose or Fermi element according to the $\pm$ alternative, or simply even or odd elements. We shall say that the degree ∂x of the homogeneous element x is 0 in the Bose case and 1 in the Fermi case.

A *Möbius covariant graded net* $\mathcal{A}$ on S^1 is a $\mathbb{Z}_2$-graded Möbius covariant net satisfying graded locality, namely a Möbius covariant net of von Neumann algebras on S^1 such that the following holds.

4. GRADED LOCALITY**:** *There exists a grading automorphism* $\mathbf{g}$ *of* $\mathcal{A}$ *such that if* I_1 *and* I_2 *are disjoint intervals,*

$$[x, y] = 0, \quad x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2) \ .$$

Here $[x, y]$ is the graded commutator with respect to the grading automorphism $\mathbf{g}$ defined as follows: if x, y are homogeneous, then

$$[x, y] \equiv xy - (-1)^{\partial x \cdot \partial y} yx$$

and, for the general elements x, y, it is extended by linearity. When the grading is trivial, i.e., when $\Gamma = 1$, we shall refer to $\mathcal{A}$ as a *local net*.

Note the *Bose subnet* $\mathcal{A}_b$, namely the $\mathbf{g}$-fixed point subnet $\mathcal{A}^{\mathbf{g}}$ of degree zero elements, is local.

Moreover, setting

$$Z \equiv \frac{1 - i\Gamma}{1 - i} \ ,$$

we have that the unitary Z fixes Ω and

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*$$

(twisted locality w.r.t. Z).

Theorem 2.7. *Let $\mathcal{A}$ be a Möbius covariant Fermi net on S^1. Then Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.*

If $I \in \mathcal{I}$, we shall denote by Λ_I the one-parameter subgroup of $\mathsf{M\ddot{o}b}$ of "dilation associated with I".

Theorem 2.8. *Let $I \in \mathcal{I}$ and Δ_I, J_I be the modular operator and the modular conjugation of $(\mathcal{A}(I), \Omega)$. Then we have:*

(i):

$$\Delta_I^{it} = U(\Lambda_I(-2\pi t)), \ t \in \mathbb{R}, \tag{3}$$

(ii): *U extends to an (anti-)unitary representation of $\mathsf{M\ddot{o}b} \ltimes \mathbb{Z}_2$ determined by*

$$U(r_I) = ZJ_I, \ I \in \mathcal{I},$$

acting covariantly on $\mathcal{A}$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(\dot{g}I) \quad g \in \mathsf{M\ddot{o}b} \ltimes \mathbb{Z}_2 \ I \in \mathcal{I} \ .$$

Here $r_I : S^1 \to S^1$ is the reflection mapping I onto I'.

Corollary 2.9. (Additivity) *Let I and I_i be intervals with $I \subset \cup_i I_i$. Then $\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$.*

Theorem 2.10. *For every $I \in \mathcal{I}$, we have:*

$$\mathcal{A}(I') = Z\mathcal{A}(I)'Z^* \ .$$

In the following corollary, the grading and the graded commutator is considered on $B(\mathcal{H})$ w.r.t. $\mathrm{Ad}\Gamma$.

Corollary 2.11. $\mathcal{A}(I') = \big\{x \in B(\mathcal{H}) : \ [x, y] = 0 \ \forall y \in \mathcal{A}(I)\big\}$.

Now let G be a simply connected compact Lie group. By Th. 3.2 of [7], the vacuum positive energy representation of the loop group LG (cf. [28]) at level k gives rise to an irreducible local net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [7], every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$. **When no confusion arises we will write $\mathcal{A}_{G_k}$ simply as G_k.**

As another class of conformal nets, for any positive even lattice L, one can associate with a rational conformal net $\mathcal{A}_L$ as in [6].

Now we recall some definitions from [11]. Recall that $\mathcal{I}$ denotes the set of intervals of S^1. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1. When I_1, I_2 are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [34]. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible Möbius covariant net as in section 2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \bigvee A(I_2), \quad \hat{\mathcal{A}}(E) := (A(I_3) \bigvee A(I_4))'.$$

Note that for a local net $\mathcal{A}$ $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \bigvee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \bigvee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I.

Definition 2.12 ([11])**.** $\mathcal{A}$ is said to be completely rational if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of E by Prop. 5 of [11]) is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the μ-index of $\mathcal{A}$ to be infinity.

Note that by [17] every irreducible, split, local conformal net with finite μ-index is automatically strongly additive, and when $\mathcal{A}$ is a local conformal net, it follows that $\mathcal{A}$ is always split (cf. [22]). Hence a local conformal net $\mathcal{A}$ is completely rational or simply **rational** if the index $\mu_{\mathcal{A}}$ is finite. $\mu_{\mathcal{A}}$ is also known as a **global index** of $\mathcal{A}$.

Definition 2.13. Let $\mathcal{A}$ be a local rational conformal net, and let $I \in \mathcal{PI}$ be a union of n disjoint intervals. We define $F_I : \mathcal{A}(I')' \to \mathcal{A}(I)$ as the condition expectation of index $\mu_{\mathcal{A}}^{n-1}$ (cf. [11] and [34]). When there is a pair of nets involved we shall use the notation $F_{I,\mathcal{A}}$ to avoid confusion.

Let $\mathcal{A}$ be a graded Möbius covariant net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_2), \quad \hat{\mathcal{A}}(E) := (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))'.$$

Note that $\mathcal{A}(E) \subset Z\hat{\mathcal{A}}(E)Z^{-1}$, and its index will be denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$ or the global index of $\mathcal{A}$. This generalizes the usual μ-index of $\mathcal{A}$ when $\mathcal{A}$ is local.

Let $\mathcal{A}$ be a graded Möbius net. By a *Möbius subnet* (cf. [16]) we shall mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotonic

$$\mathcal{B}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and Möbius covariant with respect to the representation U, namely

$$U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$$

for all $g \in \mathsf{M\ddot{o}b}$ and $I \in \mathcal{I}$, and we also require that $\mathrm{Ad}\Gamma$ preserves $\mathcal{B}$ as a set. Note that by Lemma 13 of [16] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{A}(I) \to \mathcal{B}(I)$ such that E_I preserves the vector state given by the vacuum of $\mathcal{A}$. Let P be the projection onto the closed subspace spanned by $\mathcal{B}(I)\Omega$.

Definition 2.14. Let $\mathcal{A}$ be a graded Möbius covariant net and let $\mathcal{B} \subset \mathcal{A}$ be a subnet. We say $\mathcal{B} \subset \mathcal{A}$ is of finite index if $\mathcal{B}(I) \subset \mathcal{A}(I)$ is of finite index for some (and hence all) interval I. The index will be denoted by $[\mathcal{A} : \mathcal{B}]$.

The following is proved in exactly the same way as in [11] or [34].

Lemma 2.15. *If $\mathcal{B} \subset \mathcal{A}$ is a Möbius subnet such that $\mu_{\mathcal{A}}$ is finite and $[\mathcal{A} : \mathcal{B}] < \infty$, then $\mu_{\mathcal{B}} = \mu_{\mathcal{A}}[\mathcal{A} : \mathcal{B}]^2$.*

2.3.2. *Some key results from* [21]. In this section we recall some results from [21] which play key roles in this paper, and also to fix our notation. Let H denote the Hilbert space $L^2(S^1; \mathbb{C}^r)$ of square-summable $\mathbb{C}^r$-valued functions on the circle. The group LU_r of smooth maps $S^1 \to U_r$, with U_r the unitary group on $\mathbb{C}^r$, acts on H multiplication operators.

Let us decompose $H = H_+ \oplus H_-$, where

$$H_+ = \{\text{functions whose negative Fourier coeffients vanish}\}.$$

We denote by p the Hardy projection from H onto H_+.

Denote by $U_{\mathrm{res}}(H)$ the group consisting of unitary operator A on H such that the commutator $[p, A]$ is a Hilbert-Schmidt operator. Denote by $\mathrm{Diff}^+(S^1)$ the group of orientation preserving diffeomorphism of the circle. It follows from Propositions 6.3.1 and 6.8.2 in [28] that LU_r and $\mathrm{Diff}^+(S^1)$ are subgroups of $U_{\mathrm{res}}(H)$. The basic representation of LU_r is the representation on Fermionic Fock space $F_p = \Lambda(pH) \otimes \Lambda((1-p)H)^*$ as defined in §10.6 of [28]. For more details, see [28] or [31]. Such a representation gives rise to a graded net as follows. Denote by $\mathcal{A}_r(I)$ the von Neumann algebra generated by $c(\xi)'s$, with $\xi \in L^2(I, \mathbb{C}^r)$. Here $c(\xi) = a(\xi) + a(\xi)^*$ and $a(\xi)$ is the creation operator defined as in Chapter 1 of [31]. Let $Z : F_p \to F_p$ be the Klein transformation given by multiplication by 1 on even forms and by i on odd forms. It follows from §15 of chapter 2 of [31] that $\mathcal{A}_r$ is a graded Möbius covariant net, and $\mathcal{A}_r$ will be called the *net of r free fermions*. It follows from Prop. 1.3.2 of [15] that $\mathcal{A}_r$ is strongly additive and §15 of chapter 2 of [31] that $\mu_{\mathcal{A}_r} = 1$.

Fix $I_i \in \mathcal{PI}, i = 1, 2$, and I_1, I_2 disjoint, that is, $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, and $I = I_1 \cup I_2$.

For bounded operators $A, B : F_p \to F_p$, we define $A^+ = \frac{1}{2}(A + \Gamma A \Gamma)$, $A^- = A - A^+$, where Γ is an operator on F_p given by multiplication by 1 on even forms and -1 on odd forms. An operator A is called even (resp., odd) if $A = A^+$ (resp., $A = A^-$).

We define a graded tensor product $\otimes_2$ by the following formula:

$$A \otimes_2 B = A \otimes B^+ + A\Gamma \otimes B^- ,$$

where $A \otimes_2 B$ is considered as an operator on Hilbert space tensor product $F_p \otimes F_p$.

Let A_1, A_2, B_1, B_2 be even or odd operators, i.e., $\Gamma A_i \Gamma = A_i$ or $-A_i$, $\Gamma B_i \Gamma = B_i$, or $-B_i$, $i = 1, 2$. Define the degree $d(A) = 0$ or 1 if A is even or odd.

It follows from the definition of $\otimes_2$ that:

$$(A_1 \otimes_2 B_1)^* = (-1)^{d(A_1)d(B_1)} A_1^* \otimes_2 B_1^* ,$$

$$(A_1 \otimes_2 B_1) \cdot (A_2 \otimes_2 B_2) = (-1)^{d(B_1)d(A_2)} A_1 A_2 \otimes_2 B_1 B_2 .$$

For $A \in \mathcal{A}_r(I_1)$, $B \in \mathcal{A}_r(I_2)$, we define

$$\omega(A \otimes_2 B) = \langle \Omega, AB\,\Omega \rangle,$$

where Ω is the vacuum vector in F_p.

The following is (1) of Lemma 3.1 in [21].

Lemma 2.16. *ω extends to a normal faithful state on the von Neumann algebra $\{A \otimes_2 B,\ A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)\}''$ (denoted by $\mathcal{A}_r(I_1)\hat{\otimes}_2\mathcal{A}_r(I_2)$) on $F_p \otimes F_p$. There exists a unitary operator $U_1 : F_p \to F_p \otimes F_p$ such that:*

$$U_1 ABU_1^* = A \otimes_2 B \qquad \text{for every} \qquad A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)\ .$$

Definition 2.17. We set

$$\omega_1 \otimes_2 \omega_2(AB) = \langle \Omega \otimes \Omega, A \otimes_2 B\, \Omega \otimes \Omega \rangle \quad \forall A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)\ .$$

By Lemma 2.16 $\omega_1 \otimes_2 \omega_2$ defines a normal state on $\mathcal{A}_r(I)$. We note that the restriction of $\omega_1 \otimes_2 \omega_2$ to $\mathcal{A}_r(I_1)$ and $\mathcal{A}_r(I_2)$ is the same as ω.

The mutual information we will compute is $S(\omega, \omega_1 \otimes_2 \omega_2)$. When we wish to emphasize the underlying net, we will also write the mutual information as $S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2)$. When $\mathcal{B} \subset \mathcal{A}_r$ is a subnet, we write $S_{\mathcal{B}}(\omega, \omega_1 \otimes_2 \omega_2)$, the mutual information for the net $\mathcal{B}$ obtained by restricting $\omega, \omega_1 \otimes_2 \omega_2$ from $\mathcal{A}_r$ to $\mathcal{B}$. Note that by (4) of Theorem 2.1 $S_{\mathcal{B}}(\omega, \omega_1 \otimes_2 \omega_2) \leq S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2)$.

Definition 2.18. If $I = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_n, b_n) \in \mathcal{PI}$ in counterclockwise order, define

$$G(I) := \frac{1}{6}\left(\sum_{i,j} \log|b_i - a_j| - \sum_{i<j} \log|a_i - a_j| - \sum_{i<j} \log|b_i - b_j|\right)\ .$$

The following is Theorem 3.18 of [21].

Theorem 2.19. *Let $I \in \mathcal{PI}, I_1 \in \mathcal{PI}, I_2 \in \mathcal{PI}$ and $I_1 \cup I_2 = I, \bar{I}_1 \cap \bar{I}_2 = \emptyset$. Then*

$$S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2) = r\big(G(I_1) + G(I_2) - G(I_1 \cup I_2)\big)\ .$$

Now we determine the exact limit of relative entropies which are necessary for analyzing the singularity structures of entropies. The following is Theorem 4.4 of [21] (We note that there is a missing log in Theorem 4.4 of [21]).

Theorem 2.20. *Assume that subnet $\mathcal{B} \subset \mathcal{A}$ has finite index and $\mathcal{B}$ is strongly additive. Let I_1 and I_2 be two intervals obtained from an interval I by removing an interior point, and let $J_n \subset I_2, n \geq 1$ be an increasing sequence of intervals such that*

$$\bigcup_n J_n = I_2, \quad \bar{J}_n \cap \bar{I}_1 = \emptyset\ .$$

Let E_n be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ to $\mathcal{A}(I_1) \vee \mathcal{B}(J_n)$ such that $E_n(xy) = xE_I(y)\ \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$. Then

$$\lim_{n \to \infty} S(\omega, \omega \cdot E_n) = \ln[\mathcal{A} : \mathcal{B}]\ .$$

2.3.3. *Relating relative entropy and global index.* Let $\mathcal{A}$ be a local conformal net. $\mathcal{A}$ is always split (cf. [22]). For any $I = I_1 \cup I_2 \cup \ldots \cup I_n, I \in \mathcal{PI}$, ω_I denotes the restriction of ω to $\mathcal{A}(I)$. It follows that $\omega_{I_1} \otimes \ldots \otimes \omega_{I_n}$ is a normal state on $\mathcal{A}(I)$.

Since we will be concerned with relative entropy of various states, we introduce some definitions to simplify notation. For $I = I_1 \cup I_2 \ldots \cup I_n \in \mathcal{PI}$ where I_i are disjoint intervals,

$$\omega^{\otimes} := \omega_{I_1} \otimes \omega_{I_2} \otimes \ldots \otimes \omega_{I_n}.$$

When $\mathcal{A}$ is free fermion net $\mathcal{A}_r$ the tensor product is defined as graded tensor product before Lemma 2.16.

A state ψ on $\mathcal{A}(I)$ is said to be **related** to vacuum state ω if we can partition I into disjoint union $I = J_1 \cup J_2 \ldots \cup J_m, J_i \in \mathcal{PI}, 1 \leq i \leq m$, such that $\psi = \omega_{J_1} \otimes \omega_{J_2} \otimes \ldots \otimes \omega_{J_m}$.

We shall consider conformal net whose mutual information for vacuum state are always finite.

Definition 2.21. A conformal net $\mathcal{A}$ is said to have finite mutual information if $S(\omega, \omega_I^{\otimes}) < \infty \ \forall I \in \mathcal{PI}$.

Suppose $\mathcal{A} \subset \mathcal{B}$ is an inclusion of conformal nets with finite index. We shall denote by $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ the unique conditional expectation which preserves the vacuum state when I is an interval. When $I = I_1 \cup I_2 \cup \ldots \cup I_n$ is a union of disjoint n intervals, **we shall use E_I to denote $E_{I_1} \otimes \ldots \otimes E_{I_n}$ which is the unique conditional expectation from $\mathcal{B}(I)$ to $\mathcal{A}(I)$ which preserves $\omega_{I_1} \otimes \ldots \otimes \omega_{I_n}$.**

Lemma 2.22. (1) *If $\mathcal{A}$ has finite mutual information, then $S(\omega, \psi) < \infty$ for all ψ on $\mathcal{A}(I)$ that is related to vacuum state ω.*

(2) *If $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ has finite mutual information, then $\mathcal{A}$ also has finite mutual information.*

(3) *If $\mathcal{A} \subset \mathcal{B}$ has finite index and $\mathcal{A}$ has finite mutual information, then $\mathcal{B}$ also has finite mutual information.*

Proof. By (5) of Theorem 2.1 we have

$$S(\omega, \omega_{I \cup J}^{\otimes}) = S(\omega, \omega_I^{\otimes}) + S(\omega, \omega_J^{\otimes}) + S(\omega, \omega_I \otimes \omega_I)$$

and

$$S(\omega, \psi_I \otimes \phi_J) = S(\omega, \psi_I) + S(\omega, \phi_J) + S(\omega, \omega_I \otimes \omega_I).$$

It follows that any $S(\omega, \psi)$ can be expressed as a linear combination of $S(\omega, \omega_J^{\otimes})$ for suitable intervals $J \subset I$ and (1) is proved.

(2) follows from the definition and monoticity of relative entropy in Theorem 2.1.

By Theorem 2.1 $S_{\mathcal{B}}(\omega, \omega_I^{\otimes}) - S_{\mathcal{A}}(\omega, \omega_I^{\otimes}) = S(\omega, \omega E_I)$. Since $S(\omega, \omega E_I) \leq \ln(\mathrm{Ind} E_I)$, (3) is proved. □

It is proved on page 13 of [33] that most of known conformal net (and probably all) has finite mutual information. We refer the reader to page 13 of [33] for precise statements.

Two conformal nets A and B are said to be **chain related** if there exists a sequence of conformal nets $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $\mathcal{B}_1 = \mathcal{A}, \mathcal{B}_n = \mathcal{B}$ and either $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ or $\mathcal{B}_{i+1} \subset \mathcal{B}_i, 1 \leq i \leq n-1$, and all inclusions are of finite index. See §4 of [21] for a large class of conformal nets that are chain related to free fermion nets.

Definition 2.23. For a conformal net $\mathcal{A}$ with central charge c and finite mutual information, the *regularized von Neumann entropy* of vacuum state for $\mathcal{A}(I), I \in \mathcal{PI}$ is defined as follows: For an interval I we let $G(I) := c/6 \ln r_I$, r_I is the length of interval I, and if $I_1 \cup I_2 \cup \ldots \cup I_n$ is a union of disjoint intervals

$$G(I_1 \cup I_2 \cup \ldots \cup I_n) = G(I_1) + \ldots + G(I_n) - S(\omega, \omega_{I_1} \otimes \omega_{I_2} \otimes \ldots \otimes \omega_{I_n}).$$

Note that von Neumann entropy for type III factors are always infinity, and regularized von Neumann entropy as defined are motivated by the results of [3] and §4.1 of [21]. Note unlike relative entropy, the regularized von Neumann entropy is not always nonnegative and not invariant under the conformal transformations on intervals, but otherwise verifies many interesting properties of von Neumann entropy as discussed in detail in §4.1 of [21].

When $\mu_{\mathcal{A}} = 1$, $\mathcal{A}(I) = \mathcal{A}(I')'$ $\forall I \in \mathcal{PI}$, and the vacuum state ω is a pure vector state, we expect that the von Neumann entropy of ω for $\mathcal{A}(I), I \in \mathcal{PI}$ and $\mathcal{A}(I'), I \in \mathcal{PI}$ should be the same. Of course both are infinity, but what is more interesting is to conjecture the following.

Conjecture 2.24. *For a rational conformal net $\mathcal{A}$ with finite mutual information*

$$G(I) = G(I') \ \forall I \in \mathcal{PI}$$

if and only if $\mu_A = 1$.

One reason for such a conjecture is that in §4.2.1 of [21] we have shown that in some cases

$$G(I) \neq G(I')$$

if $\mu_{\mathcal{A}} > 1$. Hence we expect that

$$G(I) = G(I') \ \forall I \in \mathcal{PI}$$

if and only if $\mu_{\mathcal{A}} = 1$. At present the only known example which verifies $\mu_{\mathcal{A}} = 1$ and

$$G(I) = G(I') \ \forall I \in \mathcal{PI}$$

is the free fermion net for which $G(I)$ $\forall I \in \mathcal{PI}$ is known as in Theorem 2.19. To investigate the general cases we define the following.

Definition 2.25. Let $\mathcal{A}$ be a conformal net with finite mutual information. We define the deficit for $\mathcal{A}(I), I \in \mathcal{PI}$ to be $D_{\mathcal{A}}(I) := G_{\mathcal{A}}(I) - G_{\mathcal{A}}(I')$.

Definition 2.26. Let $\mathcal{A}$ be a local rational conformal net, and let $I \in \mathcal{PI}$ be a disjoint union of n intervals. Define

$$\hat{D}_{\mathcal{A}}(I) := S(\omega, \omega F_I) - \frac{n-1}{2} \ln \mu_{\mathcal{A}}.$$

Where F_I is defined as in Definition 2.13. For free fermion net $\mathcal{A}_r$ we will set

$$\hat{D}_{\mathcal{A}_r(I)} = 0.$$

The main conjecture of this paper is the following.

Conjecture 2.27. *For a rational conformal net $\mathcal{A}$ with finite mutual information*

$$D_{\mathcal{A}}(I) = \hat{D}_{\mathcal{A}}(I).$$

In the rest of this section we will prove that in many cases the above conjecture is true. For simplicity **we will assume that in the rest of this section all conformal nets are local rational with finite mutual information unless stated otherwise. The only nonlocal conformal net will be free fermion net $\mathcal{A}_r$ which is treated separately**.

Proposition 2.28. *For a rational conformal net $\mathcal{A}$ with finite mutual information we have Conjecture* 2.27 *implies Conjecture* 2.24.

Proof. Note that when $\mu_{\mathcal{A}} = 1$, Conjecture 2.27 implies that

$$G(I) = G(I') \ \forall I \in \mathcal{PI}.$$

Now suppose that

$$G(I) = G(I') \ \forall I \in \mathcal{PI}.$$

Conjecture 2.27 implies that

$$S(\omega, \omega F_I) = \frac{n-1}{2} \ln \mu_{\mathcal{A}} \; \forall I \in \mathcal{PI}.$$

Take I^k to be a disjoint union of two intervals $I_1(k), I_2(k)$, and consider a contacting sequence along an interval between $I_1(k)$ and $I_2(k)$ as in Theorem 2.39. By Theorem 2.39 we obtain

$$\lim_k S(\omega, \omega F_{I_k}) = \ln \mu_{\mathcal{A}}.$$

Hence

$$\ln \mu_{\mathcal{A}} = \frac{1}{2} \ln \mu_{\mathcal{A}}$$

which implies $\mu_{\mathcal{A}} = 1$. □

Suppose $\mathcal{A} \subset \mathcal{B}$ is an inclusion of conformal nets with finite index. Recall that $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ is the unique conditional expectation which preserves the vacuum state when I is an interval. When $I = I_1 \cup I_2 \cup \ldots \cup I_n$ is a disjoint union of n intervals, E_I denotes $E_{I_1} \otimes \ldots \otimes E_{I_n}$ which is the unique conditional expectation from $\mathcal{B}(I)$ to $\mathcal{A}(I)$ which preserves $\omega_{I_1} \otimes \ldots \otimes \omega_{I_n}$.

We will prove Conjecture 2.27 for a large class of conformal nets. The idea is the following : Since we have an important example of free fermion net $\mathcal{A}_r$ for which we already know

$$D_{\mathcal{A}_r}(I) = \hat{D}_{\mathcal{A}_r}(I) = 0$$

by Theorem 2.19, and there are many conformal nets that are chain related to $\mathcal{A}_r$, if we can show that for a pair of conformal nets $\mathcal{A} \subset \mathcal{B}$ with finite index that

$$D_{\mathcal{A}}(I) - \hat{D}_{\mathcal{A}}(I) = D_{\mathcal{B}}(I) - \hat{D}_{\mathcal{B}}(I),$$

then it follows that Conjecture 2.27 is true for conformal nets that are chain related to $\mathcal{A}_r$. To state the theorem in more general terms, we note that

$$D_{\mathcal{A}}(I) - \hat{D}_{\mathcal{A}}(I) = D_{\mathcal{B}}(I) - \hat{D}_{\mathcal{B}}(I)$$

is equivalent to

$$D_{\mathcal{A}}(I) - D_{\mathcal{B}}(I) = \hat{D}_{\mathcal{A}}(I) - \hat{D}_{\mathcal{B}}(I).$$

But by definition and (1) of Theorem 2.1

$$D_{\mathcal{A}}(I) - D_{\mathcal{B}}(I) = S(\omega, \omega E_I) - S(\omega, \omega E_{I'}).$$

Hence it is enough to show

$$S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = \hat{D}_{\mathcal{A}}(I) - \hat{D}_{\mathcal{B}}(I).$$

Then the following theorem does exactly this (we note that in the theorem we are **not assuming** that $\mathcal{A}, \mathcal{B}$ have finite mutual information).

Theorem 2.29. (1) *Let $\mathcal{A} \subset \mathcal{B}$ be rational local conformal nets with finite index; then*

$$S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = \hat{D}_{\mathcal{A}}(I) - \hat{D}_{\mathcal{B}}(I).$$

(2) (1) *also holds when $\mathcal{B}$ is free fermion net $\mathcal{A}_r$, more precisely in this case we have*

$$S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = \hat{D}_{\mathcal{A}}(I).$$

Proof. Fix $I \in \mathcal{PI}$ which is a disjoint union of n intervals.

Ad (1): Let $E := (\mathrm{Ind}E_{I'}\mathrm{Ind}F_{I',\mathcal{B}})^{-1}E_I F^{-1}_{I',\mathcal{B}}E^{-1}_{I'}$ be the condition expectation from $\mathcal{A}(I')' \to \mathcal{A}(I)$.

Set $E_1 := E_I F^{-1}_{I',\mathcal{B}}E^{-1}_{I'}$.

Let us compute $S(\omega,\omega E_I)-S(\omega,\omega F_{I',\mathcal{B}})-S(\omega,\omega E_{I'})$. Note that Ω is separating and cyclic for $\mathcal{B}(I)'$. By Proposition 2.4 we have

$$S(\omega,\omega E_I)-S(\omega,\omega F_{I',\mathcal{B}})-S(\omega,\omega E_{I'}) = S(\omega,\omega E_1).$$

By §4 of [13] and [14] E restricts to trace on $\mathcal{A}(I)'\cap\mathcal{A}(I')'$. Let $P_\mathcal{A}$ be the projection in $\mathcal{A}(I)'\cap\mathcal{A}(I')'$ which projects onto the closure of $\mathcal{A}(I)\Omega$. Then we have

$$\Delta(\frac{\omega E}{\omega'})^{it}P_\mathcal{A}\Delta(\frac{\omega E}{\omega'})^{-it} = P_\mathcal{A}\ \forall t,$$

where ω' is the state on $\mathcal{A}(I')$ given by Ω. It follows that $\Delta(\frac{\omega E}{\omega'})$ commutes with $P_\mathcal{A}$. We note that when restricted to $P_\mathcal{A}\mathcal{A}(I')'P_\mathcal{A}$, ωE is given by $E(P_\mathcal{A})\omega E_{P_\mathcal{A}}$ where

$$E_{P_\mathcal{A}} : P_\mathcal{A}\mathcal{A}(I')'P_\mathcal{A} \to P_\mathcal{A}\mathcal{A}(I)$$

is the unique conditional expectation and can be identified with $F_{I,\mathcal{A}} : \mathcal{A}(I')' \to \mathcal{A}(I)$ where the algebras are on $P_\mathcal{A}H_\mathcal{B} = H_\mathcal{A}$. Note that $E(P_\mathcal{A}) = [\mathcal{B}:\mathcal{A}]^{-1} = \frac{\mu_\mathcal{B}^{1/2}}{\mu_\mathcal{A}^{1/2}}$. Hence

$$\langle \ln\Delta(\frac{\omega E}{\omega'})\Omega,\Omega\rangle = \ln E(P_\mathcal{A}) + \langle \ln\Delta(\frac{\omega E_{P_\mathcal{A}}}{\omega'})\Omega,\Omega\rangle = \ln E(P_\mathcal{A}) + \langle \ln\Delta(\frac{\omega F_{I,\mathcal{A}}}{\omega'})\Omega,\Omega\rangle.$$

Note that

$$\mathrm{Ind}E_I = (\frac{\mu_\mathcal{A}}{\mu_\mathcal{B}})^{n/2}, \mathrm{Ind}F_{I',\mathcal{B}} = \mu_\mathcal{B}^{n-1}.$$

Putting the above pieces together we have shown that

$$S(\omega,\omega E_I)-S(\omega,\omega F_{I',\mathcal{B}})-S(\omega,\omega E_{I'}) = S(\omega,\omega F_{I,\mathcal{A}}) - \frac{n-1}{2}(\ln\mu_\mathcal{A}+\ln\mu_\mathcal{B}).$$

Finally, by Proposition 2.4 we have

$$-S(\omega,\omega F_{I,\mathcal{B}}) = S(\omega,\omega) - S(\omega,\omega F_{I,\mathcal{B}}) = S(\omega,\omega F^{-1}_{I,\mathcal{B}}) = S(\omega,\omega F_{I',\mathcal{B}}) - (n-1)\ln\mu_\mathcal{B}$$

and the proof of the theorem is complete.

Ad (2): We need to evaluate

$$S(\omega,\omega E_I) - S(\omega,\omega E_{I'}).$$

Note that $E^{-1}_{I'} : \mathcal{A}(I')' \to \mathcal{A}_r(I')' = k\mathcal{A}_r(I)k^{-1}$ where k is the Klein transform. Let us define

$$\hat{E}_I(kak^{-1}) = E_I(a)\ \forall a \in \mathcal{A}_r(I).$$

Since $k\Omega = \Omega$, it follows that

$$\omega(\hat{E}_I(kak^{-1})) = \omega(E_I(a)), \omega(kak^{-1}) = \omega(a)$$

and $S(\omega,\omega E_I) = S(\omega,\omega\hat{E}_I)$. Hence by (2) of Prop. 2.4

$$S(\omega,\omega E_I) - S(\omega,\omega E_{I'}) = S(\omega,\omega\hat{E}_I) - S(\omega,\omega E_{I'}) = S(\omega,\omega\hat{E}_I E^{-1}_{I'}).$$

The rest of the proof is the same as in (1) above. □

By Theorem 2.29 we immediately have the following.

Corollary 2.30. *If a local conformal net $\mathcal{A}$ is chain related to $\mathcal{A}_r$, then Conjecture 2.27 is true for $\mathcal{A}$.*

We also have the following.

Corollary 2.31. *If a local conformal net $\mathcal{A}$ is chain related to a rational local conformal net $\mathcal{B}$, and $\mathcal{B}$ has finite mutual information and verifies Conjecture 2.27, then Conjecture 2.27 is true for $\mathcal{A}$.*

Proof. Under the assumption $\mathcal{B}$ has finite mutual information and $\mathcal{A}$ is chain related to $\mathcal{B}$, by (2) and (3) of Lemma 2.22,

$$D_{\mathcal{A}}(I), \hat{D}_{\mathcal{A}}(I), D_{\mathcal{B}}(I), \hat{D}_{\mathcal{B}}(I)$$

are finite; then

$$D_{\mathcal{A}}(I) - \hat{D}_{\mathcal{A}}(I) = D_{\mathcal{B}}(I) - \hat{D}_{\mathcal{B}}(I)$$

is equivalent to

$$D_{\mathcal{A}}(I) - D_{\mathcal{B}}(I) = \hat{D}_{\mathcal{A}}(I) - \hat{D}_{\mathcal{B}}(I).$$

But by definition and (1) of Theorem 2.1

$$D_{\mathcal{A}}(I) - D_{\mathcal{B}}(I) = S(\omega, \omega E_I) - S(\omega, \omega E_{I'})$$

and the corollary follows from Theorem 2.29. □

Corollary 2.32. *Conjecture 2.27 is true for conformal nets associated with even positive definite lattices.*

Proof. First we prove this for rank 1 lattices. Such a lattice has basis vector α with inner product $\langle \alpha, \alpha \rangle = a$, where a is a positive even integer. The corresponding conformal net is usually denoted by $\mathcal{A}_{U(1)_a}$ or simply by $U(1)_a$ if no confusion arises. Denote by $D_1(a) := D_{\mathcal{A}_{U(1)_a}}(I) - \hat{D}_{\mathcal{A}_{U(1)_a}}(I)$. We prove by induction on k that

$$D_1(ka) = D_1(a) \ \forall k \geq 1.$$

When $k = 1$ this is trivial. Also $\mathcal{A}_{U(1)_1}$ is one free fermion net and by Theorem 2.19 $D_1(1) = 0$.

Assume the above equation is true for k. Consider the following finite index inclusions (for simplicity we will use $U(1)_a$ to denote the corresponding conformal net $\mathcal{A}_{U(1)_a}$):

$$U(1)_{(k+1)a} \times U(1)_{(k+1)ka} \subset U(1)_{ka} \times U(1)_a,$$

where $U(1)_{(k+1)a}$ is diagonally embedded in $U(1)_{ka} \times U(1)_a$ and its commutant in $U(1)_{ka} \times U(1)_a$ is $U(1)_{(k+1)ka}$. This can be seen as follows: Suppose the lattices correspond to $U(1)_a$ and $U(1)_{ka}$ are spanned by α and β, respectively, with inner products $\langle \alpha, \alpha \rangle = a, \langle \beta, \beta \rangle = ka, \langle \alpha, \beta \rangle = 0$. Then the diagonal rank 1 lattice spanned by $\alpha + \beta$ corresponds to $U(1)_{(k+1)a}$. The orthogonal complement of rank 1 lattice spanned by $\alpha + \beta$ in the rank 2 lattice spanned by α, β is spanned by $-k\alpha + \beta$, and since $\langle -k\alpha + \beta, -k\alpha + \beta \rangle = k(k+1)a$, this corresponds to $U(1)_{(k+1)ka}$.

By Theorem 2.29 and induction hypothesis we have

$$2D_1((k+1)a) = 2D_1(a)$$

and it follows by induction we have proved

$$D_1(ka) = D_1(a) \ \forall k \geq 1.$$

Now from the inclusion

$$U(1)_2 \times U(1)_2 \subset U(1)_1 \times U(1)_1$$

and Theorem 2.29 we conclude that $2D_1(2) = 2D_1(1) = 0$. It follows that $D_1(a) = 0$ for all even a.

Now assume that the corollary is proved for rank k even positive definite lattices. If L is an even positive definite lattice, choose a nonzero element $e \in L$ and consider sublattices $L_1 = \mathbb{Z}e$ of L and L_2 of L which is orthogonal to L_1 with rank equal to k. Denote by $L_1 \times L_2$ the sublattice of L generated by L_1, L_2. Note that the vacuum representation of $\mathcal{A}_L$ decomposes into finitely many irreducible representations of $\mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2}$ which are in one-to-one correspondence with finite abelian group $L/L_1 \times L_2$, and each has index 1 by Section 3 of [6].

Applying Theorem 2.29 to the finite index inclusions

$$\mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2} \subset \mathcal{A}_L$$

and induction hypothesis, we have proved the corollary. □

By Theorem 2.29 and Corollary 2.32 we have the following.

Corollary 2.33. *If a local conformal net $\mathcal{A}$ is chain related to $\mathcal{A}_L$ where L is a positive even lattice, then Conjecture 2.27 is true for $\mathcal{A}$.*

By §3 of [6] $\mu_{\mathcal{A}_L} = 1$ if and only if L is an even positive definite unimodular lattice, hence Corollary 2.32 implies the following.

Corollary 2.34. *$D_{\mathcal{A}_L} = 0$ if and only if L is an even positive definite unimodular lattice.*

2.4. **Some continuous properties. Let us first fix a rational conformal net $\mathcal{A}$ with finite mutual information.**

By (2) of Theorem 2.1 relative entropies are continuous from "inside". As an application of Theorem 2.29, we will prove that relative entropies in Theorem 2.29 are also continuous from "outside". First we have the following.

Lemma 2.35. *If $I \subset J, I, J \in \mathcal{PI}$, then F_J restrict to F_I on $\mathcal{A}(I)$ and hence $S(\omega, \omega F_I)$ increase with I.*

Proof. This is proved in §2 of [11] for $n = 2$, but the same argument works for any n. □

Corollary 2.36. *Let $\mathcal{A} \subset \mathcal{B}$ be as in Theorem 2.29. Then $S(\omega, \omega E_I)$ is continuous from "outside", i.e., if I_n is a decreasing sequence of intervals such that $\bigcap I_n = I$, and $E_{I'}$ restrict to $E_{I'_n}$, then*

$$\lim_{n \to \infty} S(\omega, \omega E_{I_n}) = S(\omega, \omega E_I).$$

Proof. This follows from Theorem 2.29 and Lemma 2.35. □

2.5. **Singular limits. Let us again fix a rational conformal net $\mathcal{A}$ with finite mutual information.**

It is usually an interesting problem to study the limiting properties of relative entropies when intervals get close together. One can find such studies in §§3 and 4 of [21]. In the same spirit we will consider such singular limits for related entropy $S(\omega, \omega F_I)$ for a conformal net $\mathcal{A}$.

The following proposition is a reformulation of Proposition 4.5 of [21].

Proposition 2.37. *Assume that M_n is an increasing sequence of factors acting on a fixed Hilbert space, $N_n \subset M_n$ are subfactors, and ω is a vector state associated with a vector Ω. Suppose that $E_n : M_n \to N_n, n \geq 1$ is a sequence of conditional expectations such that when restricting to M_n, $E_{n+1} = E_n, n \geq 1$, and $\mathrm{Ind}E_n = \lambda^{-1}$ is a positive real number independent of n. Set $\phi_n := \omega E_n$. If strong operator closure of $\bigcup_n N_n$ contains M_1, then given any $\epsilon > 0$, we can find $e \in M_n$ for sufficiently large n, such that*

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, \ |\omega(e^*e) - 1| < \epsilon, \ |\phi_n(ee^*) - \lambda| < \epsilon \ .$$

Proof. Let $e_1 \in M_1$ be the Jones projection for $E_1 : M_1 \to N_1$, and let $v \in N_1$ be the isometry such that $\lambda^{-1}v^*e_1v = 1$. By assumptions we can find a sequence of elements $e_n \in N_n, n \geq 2$ which converges in strong star topology to e_1. Now choose $x_n = \lambda^{-1}v^*e_1e_nv$. Then $x_n \to 1$ in strong star topology, and so $\omega(x_n), \omega(x_nx_n^*)$ converges to 1. On the other hand by definition

$$E_n(x_n^*x_n) = v^*e_n^*e_nv$$

converges to $v^*e_1v = \lambda$ strongly. Hence given any $\epsilon > 0$, we can choose n sufficiently large such that if we set $e = x_n^*$, then $e \in M_n$, and

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, \ |\omega(e^*e) - 1| < \epsilon, \ |\phi_n(ee^*) - \lambda| < \epsilon \ .$$

□

Theorem 2.38. *Assume that M_n is an increasing sequence of factors act on a fixed Hilbert space, $N_n \subset M_n$ are subfactors, and ω is a vector state associated with a vector Ω. Suppose that $E_n : M_n \to N_n, n \geq 1$ is a sequence of conditional expectations such that when restricting to M_n, $E_{n+1} = E_n, n \geq 1$, $\mathrm{Ind}E_n = \lambda^{-1}$ is a positive real number independent of n. If strong operator closure of $\bigcup_n N_n$ contains M_1, then*

$$\lim_{n\to\infty} S(\omega, \omega E_n) = -\ln\lambda.$$

Proof. Set $\phi_n := \omega E_n$. By Pimsner-Popa inequality, $E_n(x) \geq \lambda x$ for any positive $x \in M_n$, it follows that $\phi_n \geq \lambda\omega$, and hence by Theorem 2.1

$$S(\omega, \omega \cdot E_n) \leq -\ln\lambda.$$

Note that by monotonicity of relative entropy $S(\omega, \omega \cdot E_n)$ increases with n, hence $\lim_{n\to\infty} S(\omega, \omega \cdot E_n)$ exists and is less than or equal to $-\ln\lambda$.

Hence we need to show that $S(\omega, \omega E_n)$ can get arbitrarily close to $-\ln\lambda$ when n is sufficiently large. The key ideas are the same as explained in Sections 4.3.1 and 4.3.2 of [21]. We provide full details in the following for the reader's convenience.

Denote $\phi_n = \omega \cdot E_n$. By Kosaki's formula (cf. [13])

$$S(\omega, \omega \cdot E_n) = \sup_{m\in\mathbb{N}} \sup_{x_t+y_t=1} \left(\ln k - \int_{k^{-1}}^{\infty} \left(\omega(x_t^*x_t)\frac{1}{t} + \phi_n(y_ty_t^*)\frac{1}{t^2} \right) dt \right) \ ,$$

where x_t is a step function which is equal to 0 when t is sufficiently large. To motivate the proof, it is instructive to see how we can get $S(\omega, \lambda\omega) = -\ln\lambda, 0 < \lambda < 1$ from Kosaki's formula. By tracing the proof in [13], one can see that the path which gives approximation to $-\ln\lambda$ is given by the following continuous path:

$$x(t) = \frac{\lambda}{\lambda + t}, y(t) = \frac{t}{\lambda + t}, t \geq k^{-1}$$

and with such a choice we have

$$\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big)dt = -\ln(\lambda + 1/k)$$

which tends to $-\ln\lambda$ as k goes to ∞. This suggests that for the proof we need to choose path x_t, y_t such that $\omega(x_t^* x_t)$ and $\phi_n(y_t y_t^*)$ are close to $\big(\frac{\lambda}{\lambda+t}\big)^2$ and $\lambda\big(\frac{t}{\lambda+t}\big)^2$, respectively.

By Kosaki's formula

$$S(\omega, \omega\cdot E_n) = \sup_{m\in\mathbb{N}} \sup_{x_t+y_t=1} \left(\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big)dt\right),$$

where x_t is a step function which is equal to 0 when t is sufficiently large. Since we can approximate any continuous function with step functions in the strong topology and vice versa, we can assume that x_t is continuous and is equal to 0 when t is sufficiently large. Given $\epsilon > 0$, for fixed $k, m \in \mathbb{N}$ choose e as in Proposition 2.37 and

$$x_t = 1 - \frac{t}{\lambda+t}e, k^{-1} \le t \le m\ .$$

We have

$$\omega(x_t^* x_t) = 1 - \frac{t}{\lambda+t}\omega(e) - \frac{t}{\lambda+t}\omega(e^*) + \left(\frac{t}{\lambda+t}\right)^2 \omega(e^* e)$$

and

$$\phi_n(y_t y_t^*) = \left(\frac{t}{\lambda+t}\right)^2 \phi_n(ee^*)\ .$$

By Proposition 2.37 we can choose n large enough such that

$$\int_{k^{-1}}^{m} \left|\omega(x_t x_t^*) - \Big(\frac{\lambda}{\lambda+t}\Big)^2\right| \frac{dt}{t} \le \epsilon\ ,$$

$$\int_{k^{-1}}^{m} \left|\phi_n(y_t y_t^*) - \lambda\Big(\frac{t}{\lambda+t}\Big)^2\right| \frac{dt}{t^2} \le \epsilon\ ,$$

and with such a choice of n we have:

$$\begin{aligned}
&\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big)dt \\
&\qquad \ge \ln k - \int_{k^{-1}}^{m} \left(\Big(\frac{\lambda}{\lambda+t}\Big)^2\frac{1}{t} + \Big(\frac{t}{\lambda+t}\Big)^2\frac{\lambda}{t^2}\right)dt + 1/m - 2\epsilon \\
&\qquad\qquad = \ln\left(\frac{k}{k\lambda+1}\right) - \ln\left(\frac{m}{\lambda+m}\right) + 1/m - 2\epsilon\ .
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} S(\omega, \omega\cdot E_n) \ge \ln\left(\frac{k}{k\lambda+1}\right) - \ln\left(\frac{m}{\lambda+m}\right) + 1/m - 2\epsilon\ .$$

Letting k, m go to ∞ and ϵ go to 0, we have proved theorem. □

Let $I = I_1 \cup I_2 \cup \ldots \cup I_n \in \mathcal{PI}$ and $I' = \hat{I}_1 \cup \hat{I}_2 \cup \ldots \cup \hat{I}_n$. Let us arrange indices such that $\hat{I}_i$ share end points with $I_i, I_{i+1}, 1 \le i \le n-1$. We are interested in shrinking I'. Let us first introduce some terminology. By *a contraction* of I along $\hat{I}_1$ we mean keep $I_1 \cup \overline{\hat{I}_1} \cup I_2 := I_{12}$ fixed and let the length of $\hat{I}_1$ go to 0. We will use a sequence $I_1(k), \hat{I}_1(k), I_2(k)$ such that $\hat{I}_1(k)$ is decreasing to describe such a process. Such a sequence is called a **contraction sequence** along $\hat{I}_1$. An element of such a sequence will be denoted by I^k. We will use $(I^k)'$ to denote the interior of the complement of I^k.

Let $C_1(I) = I_{12} \cup I_3 \ldots \cup I_n \in \mathcal{PI}$.

Theorem 2.39. *Choose a contracting $I_1(k), \hat{I}_1(k), I_2(k)$ sequence along $\hat{I}_1$. Then*

$$\lim_{k\to\infty} S(\omega, \omega F_{I^k}) = S(\omega, \omega F_{C_1(I)}) + \ln \mu_{\mathcal{A}}.$$

Proof. Observe that when restricting $F_{C_1(I)}$ to $\mathcal{A}((I^k)')'$, we get a conditional expectation simply denoted only in the proof by $F_k : \mathcal{A}((I^k)')' \to \mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1(k))'$. Let $E_k : \mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1(k))' \to \mathcal{A}(I^k)$ be the conditional expectation such that E_k restricts to identity on $\mathcal{A}(I_3 \cup \ldots \cup I_n)$, and on $\mathcal{A}(I_{12}) \cap \mathcal{A}(\hat{I}_1(k))'$ is the unique conditional expectation onto $\mathcal{A}(I_1(k) \cup I_2(k))$. Note that the index of E_k is $\mu_{\mathcal{A}}$. Notice that Ω is cyclic and separating for $\mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1(k))'$. By Corollary 2.5 we have

$$S(\omega, \omega F_{I^k}) = S(\omega, \omega E_k F_k) = S(\omega, \omega E_k) + S(\omega, \omega F_k).$$

By (2) of Theorem 2.1 we have $\lim_k S(\omega, \omega F_k) = S(\omega, \omega F_{C_1(I)})$. To finish the proof it is sufficient to show that

$$\lim_k S(\omega, \omega E_k) = \ln \mu_{\mathcal{A}}.$$

Since $\bigcup_k I_1(k) \cup I_2(k)$ is equal to I_{12} minus a point, it follows that $\bigcup_k \mathcal{A}(I_1(k) \cup I_2(k))$ is strongly dense in $\mathcal{A}(I_{12})$, and

$$\lim_k S(\omega, \omega E_k) = \ln \mu_{\mathcal{A}}$$

follows from Theorem 2.38. □

Remark 2.40. We note that we can apply Theorem 2.39 a few times to shrink intervals $\hat{I}_2, \ldots, \hat{I}_{n-1}$ successively. This way we see that

$$\lim_k S(\omega, \omega F_{I_k}) = (n-1) \ln \mu_{\mathcal{A}},$$

where one takes an increasing sequence of n disjoint intervals I_k, such that $\bigcup_k I_k$ is equal to S^1 minus finitely many points. This can of course be proved directly using Theorem 2.38.

Now consider the case of $\mathcal{A} \subset \mathcal{B}$ with finite index. By Lemma 2.15 the index of this inclusion is given by $(\frac{\mu_{\mathcal{A}}}{\mu_{\mathcal{B}}})^{\frac{1}{2}}$.

Lemma 2.41. *Choose a contracting $I_1(k), \hat{I}_1(k), I_2(k)$ sequence along $\hat{I}_1$. Then*

$$\lim_{k\to\infty} S(\omega, \omega E_{I^k}) = 1/2(\ln \mu_{\mathcal{A}} - \ln \mu_{\mathcal{B}}) + S(\omega, \omega E_{C_1(I)}).$$

Proof. For the ease of notation we set $\omega_2 := \omega_{I_3} \otimes \ldots \otimes \omega_{I_n}$. By (5) of Theorem 2.1

$$S(\omega, \omega_{I_1(k)} \otimes \omega_{I_2(k)} \otimes \omega_2) = S(\omega, \omega_{I_1(k)} \otimes \omega_{I_2(k)}) + S(\omega, \omega_2) + S(\omega, \omega_{I_1(k) \cup I_2(k)} \otimes \omega_2).$$

We note that as k goes to infinity, $I_1(k) \cup I_2(k)$ increase to I_{12}, hence

$$\lim_k S(\omega, \omega_{I_1(k) \cup I_2(k)} \otimes \omega_2) = S(\omega, \omega_{I_{12}} \otimes \omega_2).$$

Hence

$$\lim_k S(\omega, \omega E_{I^k}) = \lim_k S(\omega, \omega E_{I_1(k) \cup I_2(k)}) + S(\omega, \omega E_{C_1(I)}).$$

The lemma now follows from Theorem 2.20. □

Proposition 2.42. *Let $\mathcal{A} \subset \mathcal{B}$ be as in Theorem 2.29. Choose a contracting $I_1(k)$, $\hat{I}_1(k)$, $I_2(k)$ sequence along $\hat{I}_1$. Then*

$$\lim_{k\to\infty} S(\omega, \omega E_{(I^k)'}) = S(\omega, \omega E_{C_1(I')}).$$

This follows from Theorems 2.29 and 2.39 and Lemma 2.41. □

The above corollary can be phrased as follows: Let $I_k = I_{1k} \cup I_2 \cup \ldots \cup I_n \in \mathcal{PI}$ be such that I_{1k} is a decreasing sequence such that the length of I_{1k} tends to 0 as n goes to infinity. Then

$$\lim_{k\to\infty} S(\omega, \omega E_{I_k}) = S(\omega, \omega E_{I_2\cup\ldots\cup I_n}).$$

It follows that if either $\mathcal{A}$ or $\mathcal{B}$ has the property that

$$\lim_{k\to\infty} S(\omega, \omega^{\otimes}_{I_k}) = S(\omega, \omega^{\otimes}_{I_2\cup\ldots\cup I_n}),$$

then the other net also has this property. In particular all conformal nets that are chain related to free fermion nets have this property since free fermion nets verify such property. It will be interesting to see if this can be proved under more general conditions.

3. CFT in two dimensions

Since the results in this section are very similar to the previous section, we will be brief and discuss only the necessary modifications.

For a formulation of CFT in two dimensions we refer to §2 of [10] or §2 of [29] for more details.

A two dimensional local conformal quantum field theory is given by a net $\mathcal{B}$ defined on a covering manifold $\hat{M}$ of Minkowski space-time $M = R^{(1,1)}$. This manifold is obtained as follows. One first considers Minkowski space-time as the Cartesian product $R \times R$ of its two chiral light-cone directions. On each light-cone, the Möb group $PSL(2, R)$ acts by the rational transformations $x \to \frac{ax+b}{cx+d}$, and we need to consider the compactification of R to S^1 by addition of the point at ∞. In the quantum field theory, the chiral Möb are only projectively represented, leading to a covering of S^1 (in which R will be identified with the interval $(0, 2\pi)$). The covering manifold $\hat{M}$ is the Cartesian product of the coverings of the two chiral S^1, quotiented by the identification $(x_L, x_R) \equiv (x_L + 2\pi, x_R - 2\pi)$. Each subset $(a, a+2\pi) \times (b, b+2\pi)$ represents one copy of Minkowski space-time M within $\hat{M}$. The covering manifold $\hat{M}$ possesses a global causal structure such that the causal complement of a double cone $O = (a, b) \times (c, d)$ (we always assume that $0 < b - a < 2\pi, 0 < d - c < 2\pi$) is the double cone $O' = (b, a + 2\pi) \times (d - 2\pi, c) \equiv (b - 2\pi, a) \times (d, c + 2\pi)$, and $(O')' = O$. Let $e^{ix} \times e^{iy} : R \times R \to S^1 \times S^1$ be the covering map. Let O_S be the image of double cone $O = (a, b) \times (c, d)$ under this covering map. **For the ease of notation we shall drop the subscript S and simply write O_S as O, and simply call them double cones on $S^1 \times S^1$.**

So we will consider double cone C on $S^1 \times S^1$ which is $I \times J$ where I, J are intervals on the circle S^1. A finite set of double cones $C_1, C_2, \ldots C_n$ are **disjoint** if the closure of C_i is in the casual complement of C_j for all $i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$. To make contact with the previous section, suppose a finite set of disjoint double cones $C_1, C_2, \ldots, C_n$ are given. By shuffling indices if necessary we can assume that $C_i = I_i \times J_i, 1 \leq i \leq n$, and $I_1, \ldots, I_n$ are arranged in counterclockwise order on one circle. Then $J_1, \ldots, J_n$ are arranged in counterclockwise order on another circle.

Let $\mathbf{I} = I_1 \cup I_2 \cup \ldots \cup I_n \in \mathcal{PI}$ and $\mathbf{I}' = \hat{I}_1 \cup \hat{I}_2 \cup \ldots \cup \hat{I}_n \in \mathcal{PI}$, such that $\hat{I}_i$ share end points with $I_i, I_{i+1}, 1 \leq i \leq n-1$. Similarly $\mathbf{J} = J_1 \cup J_2 \cup \ldots \cup J_n \in \mathcal{PI}$ and $\mathbf{J}' = \hat{J}_1 \cup \hat{J}_2 \cup \ldots \cup \hat{J}_n \in \mathcal{PI}$, such that $\hat{J}_i$ share end points with $J_i, J_{i+1}, 1 \leq i \leq n-1$.

Definition 3.1. We will use notation $C(\mathbf{I} \times \mathbf{J})$ to denote the union $C_1 \cup C_2 \cup \ldots \cup C_n$.

Denote by $\mathcal{PC}$ the set which consists of finite union of disjoint double cones. We shall use C' to denote the casual complement of $C \in \mathcal{PC}$. Our Definition 3.1 is designed such that

$$C(\mathbf{I} \times \mathbf{J})' = C(\mathbf{I}' \times \mathbf{J}').$$

We note that the concept of **contraction** as introduced in section 2.5 can now be defined similarly for $C(\mathbf{I} \times \mathbf{J})$, we just define contractions along $\hat{I}_1 \times \hat{J}_1$ to be contractions of $\mathbf{I}$ along $\hat{I}_1$ together with contractions of $\mathbf{J}$ along $\hat{J}_1$.

As in §2 of [10] (also cf. [29]) there is a canonical tensor product net $\mathcal{A} = \mathcal{A}_L \times \mathcal{A}_R \subset \mathcal{B}$.

In this section we will consider the case $\mathcal{A} \subset \mathcal{B}$ where $A(I \times J) = \mathcal{A}_L(I) \times \mathcal{A}_R(J)$, both $\mathcal{A}_L$ and $\mathcal{A}_R$ are rational local conformal net with finite mutual information, and $\mathcal{A} \subset \mathcal{B}$ has finite index, and $\mathcal{B}$ is rational as defined in §2.1 of [10]**.** Denote by c_L, c_R the central charges of $\mathcal{A}_L$ and $\mathcal{A}_R$, and as usual ω is the vacuum state.

Definition 3.2. For a double cone $C = I \times J$ we let $G(C) := c_L/6 \ln r_I + c_R/6 \ln r_J$. If a finite set of double cones $C_1, C_2, \ldots, C_n$ are disjoint, we define

$$G(C_1 \cup C_2 \cup \ldots \cup C_n) = G(C_1) + \ldots + G(C_n) - S(\omega, \omega_{C_1} \otimes \omega_{C_2} \otimes \ldots \otimes \omega_{C_n}).$$

Definition 3.3. We define the deficit for $\mathcal{B}(C), C \in \mathcal{PC}$ to be $D_{\mathcal{B}}(C) := G_{\mathcal{B}}(C) - G_{\mathcal{B}}(C')$.

Note that in the case when the two dimensional net is tensor product $\mathcal{A}_L \otimes \mathcal{A}_R$, and $C = C_1 \cup C_2 \cup \ldots \cup C_n, C_i = I_i \times J_i, 1 \leq i \leq n$, are disjoint, we have

$$G_{\mathcal{A}_L \otimes \mathcal{A}_R}(C) = G_{\mathcal{A}_L}(I_1 \cup I_2 \cup \ldots \cup I_n) + G_{\mathcal{A}_L}(J_1 \cup J_2 \cup \ldots \cup J_n).$$

Let $F_C : \mathcal{B}(C')' \to \mathcal{B}(C)$ be the condition expectation of index $\mu_{\mathcal{B}}^{n-1}$.

Definition 3.4. When C is a union of n disjoint double cones, define

$$\hat{D}_{\mathcal{B}}(C) := S(\omega, \omega F_C) - \frac{n-1}{2} \ln \mu_{\mathcal{B}}.$$

The analogue of Conjecture 2.27 for $\mathcal{B}$ is now the following.

Conjecture 3.5. *For a rational two dimensional conformal net $\mathcal{B}$*

$$D_{\mathcal{B}}(C) = \hat{D}_{\mathcal{B}}(C)\ \forall C \in \mathcal{PC}.$$

The analogue of Conjecture 2.24 for $\mathcal{B}$ is now the following.

Conjecture 3.6. *For a rational two dimensional conformal net $\mathcal{B}$*

$$D_{\mathcal{B}}(C) = 0\ \forall C \in \mathcal{PC}$$

if and only if $\mu_{\mathcal{B}} = 1$.

The proof of Proposition 2.28 apples verbatim to the case of two dimensional conformal net and we have the following.

Proposition 3.7. *Conjecture* 3.5 *implies Conjecture* 3.6.

Similarly the proof of Theorem 2.29 applies verbatim to the case of two dimensional conformal nets $\mathcal{A} \subset \mathcal{B}$, with intervals replaced by double cones, and we have the following.

Theorem 3.8. (1) *Let $\mathcal{A} \subset \mathcal{B}$ be rational two dimensional conformal nets with finite index; then*

$$S(\omega, \omega E_C) - S(\omega, \omega E_{C'}) = \hat{D}_{\mathcal{A}}(C) - \hat{D}_{\mathcal{B}}(C) \ \forall C \in \mathcal{PC}.$$

By Theorem 3.8 and Corollary 2.33 we have the following.

Corollary 3.9. *Suppose $\mathcal{B}$ is chain related to $\mathcal{A}_L \otimes \mathcal{A}_R$, where both $\mathcal{A}_L$ and $\mathcal{A}_R$ are chain related to $\mathcal{A}_r$ or a conformal net associated with an even positive lattice. Then Conjecture* 3.5 *is true for $\mathcal{B}$.*

We also have the following.

Corollary 3.10. (1) *Suppose $\mathcal{B}$ is chain related to $\mathcal{A}_L \otimes \mathcal{A}_R$, where both $\mathcal{A}_L$ and $\mathcal{A}_R$ are chain related to $\mathcal{A}_r$; then $D_{\mathcal{B}} = 0$ if and only if $\mu_{\mathcal{B}} = 1$.*

(2) *Suppose that $\mathcal{A}_L \otimes \mathcal{A}_R \subset \mathcal{B}$, and both $\mathcal{A}_L$ and $\mathcal{A}_R$ are chain related to $\mathcal{A}_r$; then $D_{\mathcal{B}} = 0$ if and only if $\mathcal{B}$ is modular invariant.*

Proof. (1) follows from Corollary 3.9 and Proposition 3.7. For the proof of (2), we note that by our assumptions the representations of $\mathcal{A}_L$ and $\mathcal{A}_R$ give rise to two modular tensor categories $\mathcal{C}_L, \mathcal{C}_R$, respectively. By the comments in the second paragraph of page 4 of [23](also cf. Th. 3.1 of [10]), $\mu_{\mathcal{B}} = 1$ if and only if $\mathcal{B}$ is modular invariant. □

A large class of examples with $\mu_{\mathcal{B}} = 1$ can be obtained as follows: take any conformal net $\mathcal{A}$ which is chain related to free fermion net and take the Longo-Rehren two dimensional net (which corresponds to identity modular invariant). It follows by the above corollary that such net verifies $D_{\mathcal{B}} = 0$.

Remark 3.11. The computation of entropies in physics literature is usually done (cf. [4]) with the replica trick using path integrals, and when the underlying CFT can be described by a Lagrangian it is usually assumed that the CFT is modular invariant. In cases where such computations are done, one finds that the deficit vanishes. Hence (2) of the above corollary is a rigorous formulation of such intuitions.

We note that Theorem 2.39 holds in the case of two dimensional conformal net, where contraction is defined in the paragraph after Definition 3.1. In fact all other results of sections 2.4 and 2.5 apply to two dimensional conformal nets as well, with essentially the same proof.

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