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LONG-TERM ANALYSIS OF A STOCHASTIC SIRS MODEL WITH GENERAL INCIDENCE RATES*

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Abstract. This paper investigates a stochastic SIRS epidemic model with an incidence rate that is sufficiently general and that covers many incidence rate models considered to date in the literature. We classify the extinction and permanence by introducing λ, a real-valued threshold. We show that if $\lambda < 0$, then the disease will eventually disappear (i.e., the disease-free state is globally asymptotically stable); if the threshold value $\lambda > 0$, the epidemic becomes strongly stochastically permanent. This result substantially generalizes and improves the related results in the literature. Moreover, the mathematical development in this paper is interesting in its own right. The essential difficulties lie in that the dynamics of the susceptible class depend explicitly on the removed class resulting in a three-dimensional system rather than a two-dimensional system. Consequently, the methodologies developed in the literature are not applicable here. One of the main ingredients in the analyses is this: Though it is not possible to compare solutions in the interior and on the boundary for all $t \in [0, \infty)$, approximation in a long but finite interval $[0, T]$ can be carried out. Then, using the ergodicity of the solution on the boundary and exploiting the mutual interplay between the distance of solutions in the interior and solutions on the boundary and the exponential decay or growth (depending on the sign of the Lyapunov exponent), one can classify the behavior of the system. The convergence to the invariant measure is established under the total variation norm together with the corresponding rate of convergence. To demonstrate, some numerical examples are provided to illustrate our results.

Key words. SIRS model, extinction, permanence, stationary distribution, ergodicity

AMS subject classifications. 34C60, 34F05, 60H10, 92D25, 92D30

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1. Introduction.

The mathematical theory of infectious diseases has witnessed substantial progress in recent years. Starting from the work of Kermack and McKendrick [28, 29], a vast literature on mathematical epidemiology has been developed. We refer the reader to the excellent books [3, 27, 34], among others, for further reading.

Building on the so-called compartmental models, one of the key ideas in Kermack and McKendrick's work is this: Age of infection affects the transmission and removal rates. Because of the seminal importance to the field of theoretical epidemiology, the set of their papers (originally published in the 1920s and the 1930s) was republished in the *Bulletin of Mathematical Biology* in 1991. Their theory is in fact the source of the classical SIR and SIRS epidemic models as well as their variants. These models subdivide a homogeneous host population into three epidemiologically distinct types of individuals (compartments), the susceptible class (S), the infective class (I), and the

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recovered or removed class (R). The epidemic dynamics of some infectious diseases with permanent immunity such as smallpox, measles, and chickenpox are usually modeled by the well-known SIR model:

$$
\begin{cases} dS(t) = (b - \mu S(t)I(t) - \beta_1 S(t))\,dt, \\ dI(t) = (\mu S(t)I(t) - (\beta_1 + \beta_2 + \beta_3)I(t))\,dt, \\ dR(t) = (\beta_2 I(t) - \beta_1 R(t))dt, \end{cases} \tag{1.1}
$$

where b is the birth rate, μ is the transmission coefficient, β_1 and β_3 are the per capita death rates of healthy individuals and the additional per capita death rate due to disease, respectively, and β_2 denotes the recover rate. In this model, $R(t)$ does not affect the dynamics of $S(t)$, indicating that the recovered individuals are immune to the diseases. However, for some other diseases, for example, syphilis and influenza, the recovered individuals can become susceptible again. Thus the dynamics of the diseased population should be described by SIRS models. The classical SIRS (see, for example, [15, Chapter 6] and [32] among others) is a system of differential equations

$$
\begin{cases} dS(t) = (b - \mu S(t)I(t) - \beta_1 S(t) + \gamma R(t))\,dt, \\ dI(t) = (\mu S(t)I(t) - (\beta_1 + \beta_2 + \beta_3)I(t))\,dt, \\ dR(t) = (\beta_2 I(t) - (\beta_1 + \gamma)R(t))dt, \end{cases} \tag{1.2}
$$

where γ is the rate of loss of immunity.

The so-called (bilinear) incidence rate $\mu S(t)I(t)$ describes the number of new cases per unit time in (1.2). However, it was noted in [24] that there are a number of biological mechanisms that may result in nonlinearities in the transmission rates and several theoretical studies have explored the implications of nonlinear structure in transmission rate for host-parasite interactions (see [24] and references therein). In addition, [10] introduced an SIR epidemic model with a saturated incidence rate motivated by a study of the cholera epidemic spread in Bari, Italy, 1973. The saturation of incidence rate is explained by "psychological" effects: for a very large number of infected individuals the incidence rate might be sublinear because in the presence of a very large number of infected individuals, the population may tend to reduce the number of contacts per unit time. Some other nonlinear forms of incidence rates such as $f(s,i) = \frac{si}{s^h + i^h}$ (Holling-type) or $f(s,i) = \frac{si}{1+m_1 s + m_2 i}$ (Beddington–DeÅngelis type) were also considered (see, e.g., [2, 5, 21, 22, 25, 30, 35, 37, 40]. In this paper, we work with a more general incidence rate where $F(s,i)$ is a locally Lipschitz continuous function in both variables. Thus, our model includes many incidence rates considered to date in the literature. Furthermore, we suppose that the model is perturbed by white noise to accommodate the well-recognized effect of environmental fluctuations due to the random environment. To be specific, we consider the following model:

$$
\begin{cases} dS(t) = \left(b - \dfrac{S(t)I(t)}{F(S(t),I(t))} - \beta_1 S(t) + \gamma R(t)\right)dt + \sigma_1 S(t)dW_1(t), \\ dI(t) = \left(\dfrac{S(t)I(t)}{F(S(t),I(t))} - (\beta_1 + \beta_2 + \beta_3)I(t)\right)dt + \sigma_2 I(t)dW_2(t), \\ dR(t) = (\beta_2 I(t) - (\beta_1 + \gamma)R(t))dt + \sigma_3 R(t)dW_3(t), \end{cases} \tag{1.3}
$$

where (W_1, W_2, W_3) are three correlated Brownian motions with the positive definite covariance matrix $(a_{ij})_{3\times 3}$ satisfying $\langle W_i, W_j \rangle_t = a_{ij} t$ for $i, j = 1, 2, 3$.

The choice of linear diffusion terms is to keep the model sufficiently simple so that we can focus on elaborating and illustrating the new ideas and methods. Combined with [6, 20], the methods introduced in this paper can be applied to a more general model with nonlinear diffusion parts under some additional mild conditions. Despite being seemingly oversimplified as commented on in [1], in which several alternative models were proposed together with an interesting simulation study, linear functions of Gaussian white noise models have been used extensively by biologists for projecting future population sizes and estimating extinction risks in [12, 18, 31] and exploring what stochastic mechanisms facilitate or inhibit the persistence of populations, coexistence of interacting species or genotypes, or maintenance of ecosystem services; see, e.g., [17, 38] and the references therein.

Letting $c_1 = \beta_1$, $c_2 = \beta_1 + \beta_2 + \beta_3$, $c_3 = \beta_1 + \gamma$, and $c_4 = \beta_2$, we can then rewrite (1.3) as

$$
\begin{cases}
dS(t) = \left(b - \dfrac{S(t)I(t)}{F(S(t),I(t))} - c_1 S(t) + \gamma R(t)\right) dt + \sigma_1 S(t) dW_1(t), \\
dI(t) = \left(\dfrac{S(t)I(t)}{F(S(t),I(t))} - c_2 I(t)\right) dt + \sigma_2 I(t) dW_2(t), \\
dR(t) = (c_4 I(t) - c_3 R(t))dt + \sigma_3 R(t) dW_3(t).
\end{cases}
\tag{1.4}
$$

To help the reader, we put the parameters of the models in the following table.

Parameters in Models (1.3) and (1.4)

b	birth rate
β_1	per capita death rate of healthy individual
β_2	per capita recover rate
β_3	additional per capita death rate due to disease
γ	per capita rate of loss of immunity
μ	transmission coefficient
c_1	$= \beta_1$
c_2	$= \beta_1 + \beta_2 + \beta_3$
c_3	$= \beta_1 + \gamma$
c_4	$= \beta_2$
σ_1	intensity of the fluctuation of $S(t)$ due to random environment
σ_2	intensity of the fluctuation of $I(t)$ due to random environment
σ_3	intensity of the fluctuation of $R(t)$ due to random environment

Working with epidemic models, one of the main goals is to find the basic reproduction number R_0 to classify the long-term behavior of the system. In general, it is shown in many deterministic models that if $R_0 < 1$, then the disease becomes extinct while if $R_0 > 1$, the disease persists. Similar results hold for some stochastic SIR models (see, e.g., [13, 14]) and some stochastic SIRS models in which the total population is forced to be bounded (see [19, 23]). In contrast, although stochastic SIRS models with linear diffusion terms have been considered in many papers (see [8, 9, 41, 42] and the references therein), the basic reproduction number R_0 was either undiscovered or extra conditions were needed to obtain criteria for persistence and extinction of the diseases (besides comparing R_0 with 1).

In our recent work [13], we provided sharp conditions for a stochastic SIR model, which essentially almost completely classified the persistence and extinction (of the three classes). One of the main ingredients is the use of Lyapunov exponents. Nevertheless, the approach in [13] is not directly applicable here. The main difficulty stems from the fact that we have to work with a three-dimensional system of the SIRS models rather than a two-dimensional system of the SIR models in [13] and that the comparison arguments do not work well for SIRS models because the dynamics of $S(t)$ depend explicitly on $R(t)$; see, for example, the first equation of (1.3). Recently, based on the ideas of Lyapunov exponents and occupation measures, some beautiful results on stochastic persistence and extinction were obtained in [6]. Unfortunately, the results there are not applicable to our model because some conditions for "H-persistent (strong version)" in [6, Theorem 4.12 or Proposition 4.13] are not satisfied for our Lyapunov function.

It becomes clear that to analyze SIRS models requires new techniques and delicate treatment to overcome the difficulties. In this paper, we rely on the idea that although we cannot compare solutions in the interior and on the boundary in the whole interval $t \in [0, \infty)$, we can carry out approximation in a long but finite interval $[0, T]$. Then, using the ergodicity of the solution on the boundary and exploiting the mutual interplay between the distance of solutions in the interior and solutions on the boundary and the exponential decay or growth (depending on the sign of the Lyapunov exponent), we can classify the behavior of the system.

The rest of the paper is organized as follows. Section 2 presents the main results of the paper. Section 3 provides a discussion on how the noise affects the reproduction number. In addition, some numerical examples are also provided. The detailed proofs of the main results for extinction and persistence are given in sections 4 and 5, respectively. Finally, an appendix containing the proof of Theorem 2.1 is placed at the end of the paper.

2. Main results. Throughout the paper, we denote $\mathbb{R}^3_+ = \{(s,i,r) : s \geq 0, i \geq 0, r \geq 0\}$, $\mathbb{R}^{3,\circ}_+ = \{(s,i,r) : s > 0, i > 0, r > 0\}$, and $\mathbb{R}^{3,*}_+ = \{(s,i,r) : s \geq 0, i > 0, r \geq 0\}$, respectively. Let $W_i(t), i = 1,2,3$ be Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, we assume the following standing assumption throughout the paper.

Assumption 2.1. Assume that on $[0,\infty)^2$, the function $F(s,i)$ is bounded below by $c_0 > 0$ and that the function $i \mapsto \frac{s}{F(s,i)}$ is continuous at $i = 0$ uniformly in $s \in [0, \infty)$. Assume further that $\frac{s}{F(s,0)}$ is nondecreasing in s.

Remark 2.1. This assumption is satisfied by many incidence rates introduced in the literature such as Holling type (with $h \leq 1$), Beddington–DeÅngelis type, and some nonmonotone incidence rates in [8, 11] as well as the saturated incidence rate in [10, 22] when $\frac{i}{F(s,i)} = g(i)$ satisfying $g(x) \geq 0, g(0) = 0$ with continuous derivative $g'(x)$ satisfying $g'(0) > 0$ (for instance, $g(i) = \frac{m_1 i^q}{1+m_2 i^p}$ with $q \geq 1, p > 0, m_1, m_2 > 0$). In fact, the case $g(i) = \frac{m_1 i^q}{1+m_2 i^p}$ with $q < 1$ as well as the Holling type with $h < 1$ can be treated but it would require methods that are substantially different from those introduced in this paper.

We start with a theorem about the well-posedness of the model.

THEOREM 2.1. *The following assertions hold.*

(i) *For any initial value* $(s,i,r) \in \mathbb{R}^3_+$, *there exists a global solution* $(S(\cdot), I(\cdot), R(\cdot))$ *to* (1.3) (*or* (1.4)) *such that*

$$\mathbb{P}_{s,i,r}\{(S(t), I(t), R(t)) \in \mathbb{R}^3_+ \ \forall t \geq 0\} = 1.$$

Moreover,

$$\mathbb{P}_{s,0,r}\{I(t) = 0 \ \forall t \geq 0\} = 1 \tag{2.1}$$

and

$$\mathbb{P}_{s,i,r}\{(S(t),I(t),R(t))\in\mathbb{R}^{3,\circ}_{+}\ \forall t>0\}=1 \qquad \text{for any } (s,i,r)\in\mathbb{R}^{3,*}_{+}. \tag{2.2}$$

(ii) *The following moment bound of the solution holds. Namely, for $p>0$ sufficiently small, there exists an $M_p>0$ such that*

$$\mathbb{E}_{s,i,r}\left(S(t)+I(t)+R(t)\right)^{1+p}\le\frac{(1+s+i+r)^{1+p}}{e^{(1+p)\frac{\beta_1 t}{4}}}+M_p \quad \forall t\ge 0. \tag{2.3}$$

Moreover, for any $H,\varepsilon,T>0$, there exists an $M_{H,\varepsilon,T}>0$ such that

$$\mathbb{P}_{s,i,r}\left\{\sup_{t\in[0,T]}\{S(t)+I(t)+R(t)\}\le M_{H,\varepsilon,T}\right\}\ge 1-\varepsilon \quad \forall (s,i,r)\in[0,H]^3. \tag{2.4}$$

To proceed, consider the dynamics of the population without disease by setting $I(t)=0$ and $R(t)=0$ in the equation for $S(t)$ and denote such a function by $\widetilde{S}(t)$. Then

$$d\widetilde{S}(t)=\left(b-c_1\widetilde{S}(t)\right)dt+\sigma_1\widetilde{S}(t)dW_1(t). \tag{2.5}$$

This equation has an ergodic invariant measure, denoted by μ^0 (see [13]). Since μ^0 is an inverse Gamma distribution with parameters $\alpha:=\frac{2c_1+\sigma_1^2}{\sigma_1^2}>1$ and $\beta:=\frac{2b}{\sigma_1^2}$, the function $x\mapsto x^p$ is μ^0-integrable whenever $0<p<\alpha$. This, together with the fact that $F(s,i)$ is bounded below by c_0, implies that

$$\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^0(ds)<\infty. \tag{2.6}$$

Similar to the intuitive explanation in [13, Remark 2.1], when the density of the disease $I(t)$ is small, the long-term growth rate of $I(t)$ is determined by

$$\lambda:=\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^0(ds)-c_2-0.5a_{22}\sigma_2^2. \tag{2.7}$$

The main results of the paper classifying the extinction and permanence of the disease are stated in the following theorems. Their proofs as well as the proof of Theorem 2.1 are arranged in sections 4, 5 and the appendix, respectively.

THEOREM 2.2. *Suppose $\lambda<0$. For any $(s,i,r)\in\mathbb{R}^{3,\circ}_{+}$,*

$$\mathbb{P}_{s,i,r}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t}=\lambda<0,\lim_{t\to\infty}\frac{\ln R(t)}{t}=\max\{-c_3-0.5a_{33}\sigma_3^3,\lambda\}<0\right\}=1. \tag{2.8}$$

THEOREM 2.3. *Suppose $\lambda>0$. There exists a unique invariant probability measure π^* such that for any $n\in\mathbb{N}$,*

$$\lim_{t\to\infty}t^n\|P(t,(s,i,r),\cdot)-\pi^*(\cdot)\|=0 \quad \forall (s,i,r)\in\mathbb{R}^{3,*}_{+}, \tag{2.9}$$

where $\|\cdot\|$ is the total variation norm. The support of π^* is $\mathbb{R}^{3,\circ}_{+}$. Moreover, for any initial value $(s,i,r)\in\mathbb{R}^{3,*}_{+}$ and a π^*-integrable function f we have

$$\mathbb{P}_{s,i,r}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(S(t),I(t),R(t)\big)dt=\int_{\mathbb{R}^{3,\circ}_{+}}f(s',i',r')\pi^*(ds',di',dr')\right\}=1. \tag{2.10}$$

Concerning these results, Theorem 2.1 establishes the existence and uniqueness of positive solutions of (1.3) or (1.4) with nonnegative initial data. Also presented are moment boundedness and probability boundedness. The probability boundedness is essentially a tightness result, which indicates that no probability is lost. Theorem 2.2 shows the condition for the eventually disappearance of the disease and finds the Lyapunov exponents (exponential convergence rate) associated with $I(t)$ and $R(t)$. Theorem 2.3 demonstrates the convergence to the invariance measure and a law of large number type result. Furthermore, it derives the convergence rate under the total variation norm. It shows that the rate of convergence is in fact polynomial of any degree but short of an exponential function. It indicates that the disease will persist and the probability distribution of the classes can be approximated by an invariant probability. The convergence rate suggests a good approximation can be archived after a modest period of time.

The two theorems leave the critical case $\lambda = 0$ untreated. The contradiction argument in [36] is not applicable because some comparison techniques do not work. [4] suggests the process $(S(t), I(t), R(t))$ in $\mathbb{R}^{3,\circ}_+$ is null recurrent, that is, there exists a sigma-finite invariant measure on $\mathbb{R}^{3,\circ}$. However, [4] is applicable only to the case when the invariant probability measure on the boundary is a Dirac measure. It would be interesting if the results in [4] can be generalized to the case when the boundary has a more general invariant probability measure.

3. Discussion and numerical examples. Before proceeding to the proofs of the results, this section provides some discussions and numerical demonstration.

3.1. Discussion. We first give some comments on the impact of randomness to the basic reproduction number R_0 of the stochastic system (1.4) which is defined by the ratio of the birth rate to the death rate of the diseased class $I(t)$ when its density is small and so

$$R_0 := \frac{\int_{(0,\infty)} \frac{s}{F(s,0)} \mu^0(ds)}{c_2 + 0.5a_{22}\sigma_2^2} = \frac{\lambda}{c_2 + 0.5a_{22}\sigma_2^2} + 1.$$

Let us also consider the deterministic counterpart to (1.4):

$$\begin{cases} dS(t) = \left(b - \dfrac{S(t)I(t)}{F(S(t), I(t))} - c_1 S(t) + \gamma R(t)\right) dt, \\ dI(t) = \left(\dfrac{S(t)I(t)}{F(S(t), I(t))} - c_2 I(t)\right) dt, \\ dR(t) = (c_4 I(t) - c_3 R(t))dt. \end{cases} \tag{3.1}$$

Denote by R_0^d the basic reproduction number of (3.1) and also let λ_d be the growth rate of $I(t)$ at rare density. Because in the disease-free state $S(t)$ converges to $s^* := \frac{b}{c_1}$, it is easy to show that

$$R_0^d = \frac{f(s^*)}{c_2} = \frac{\frac{b}{c_1}}{c_2 F\left(\frac{b}{c_1}, 0\right)} \text{ and } \lambda_d = f(s^*) - c_2 = \frac{\frac{b}{c_1}}{F\left(\frac{b}{c_1}, 0\right)} - c_2,$$

where $f(s) = \frac{s}{F(s,0)}$.

In the disease-free state, the solution to the stochastic system $\widetilde{S}^0(t)$ in (2.5) has a stationary distribution μ^0. Let S be a random variable with distribution μ^0. Then we have

$$\mathbb{E}_{\mu^0} S = \int_0^\infty s\mu^0(ds) = s^*,$$

which does not depend on σ_1. As a result, if $f(s)$ is a linear function, one can see that the basic reproduction number R_0 of the stochastic system (1.4) does not depend on σ_1. However, the fluctuating effect of σ_1 does have impacts on the dynamics of the disease if the incidence rate is nonlinear. This can be explained because the noise makes $\widetilde{S}$ fluctuates around s^*, that is, $\widetilde{S}$ can move between the disease-favored region and the disease-unfavored region. Thus, depending on the nature of the nonlinear function f, this fluctuation can facilitate or obstruct the development of the disease.

Remark 3.1. We consider the following cases.

(i) If $f(s)$ is a linear function, then $\lambda = \lambda_d - 0.5a_{22}\sigma_2^2$ and $\frac{R_0^d}{R_0} = \frac{c_2+0.5a_{22}\sigma_2^2}{c_2} \geq 1$. If $\sigma_2 = 0$, then $\lambda = \lambda_d$, $R_0^d = R_0$.

(ii) If $f(s)$ is a nonlinear concave function on $(0,\infty)$, then $\lambda_d > \lambda$ and $R_0^d > R_0$. As a result, if the deterministic system (3.1) tends to a disease free state, so does the stochastic counterpart (1.4).

(iii) If $f(s)$ is nonlinear and convex, then there are sets of parameters such that $\lambda_d < 0 < \lambda$ or equivalently $R_0^d < 1 < R_0$. In fact, if $\sigma_2 = 0, \sigma_1 \neq 0$, then we always have that $R_0 > R_0^d$ or equivalently $\lambda > \lambda_d$.

To further elaborate on the above remark, we argue as follows. Since we always have that $\mathbb{E}_{\mu^0} S = \int_0^\infty s\mu^0(ds) = s^*$, and μ^0 is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)$, it follows from Jensen's inequality that

$$\mathbb{E}_{\mu^0} f(S) \begin{cases} = f(s^*) & \text{if } f \text{ is linear,} \\ > f(s^*) & \text{if } f \text{ is nonlinear convex,} \\ < f(s^*) & \text{if } f \text{ is nonlinear concave.} \end{cases}$$

Since

$$\lambda = \mathbb{E}_{\mu^0} f(S) - c_2 - 0.5a_{22}\sigma_2^2, \qquad \lambda_d = f(s^*) - c_2,$$

we can easily derive claims (i) and (ii). For claim (iii), if $0.5a_{22}\sigma_2^2 < \mathbb{E}_{\mu^0} f(S) - f(s^*)$, we get that $\lambda > \lambda_d$.

Now we wish to examine how λ (or equivalently R_0) depends on the parameters. It is obvious that λ decreases as c_2 and σ_2^2 increase. Applying a comparison argument, we can easily show that λ increases (decreases reps.) as b (β_1 reps.) increases. The main focus is on the dependence of λ on σ_1^2. For population models perturbed by white noise, it is well-known that the noise has a detrimental effect on the growth rate of the species (see, e.g., [12, 17, 18]). This can be explained by the fact that the fluctuation of the environments makes species more difficult to adapt. Mathematically, in a stochastic logistic differential equation

$$dX = X(a - bX)dt + \sigma X dW(t),$$

the growth rate at rare density is $a - 0.5\sigma^2$, which decreases as σ increases. Moreover, given that $a - 0.5\sigma^2 > 0$, the average population is $\frac{a-0.5\sigma^2}{b}$, which is also a decreasing function of σ^2. In our epidemic model, σ_1 does not change the average disease-free population. But simulations of the density function of μ^0 suggest that σ_1 also has a detrimental effect on the disease-free population (we call it the Type 1 effect) in the sense that as σ_1^2 increases, the disease-free population $\widetilde{S}^0$ is much more likely to stay at

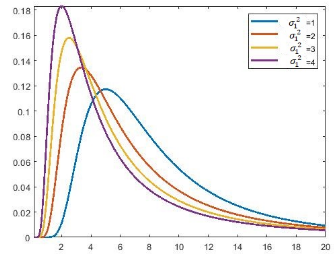

FIG. 1. *The density of the stationary distribution μ^0 with $b = 10, c_1 = 1$ in four cases: $\sigma_1^2 = 1, \ldots, 4$. The average of the free-disease population is $s^* = 10$.*

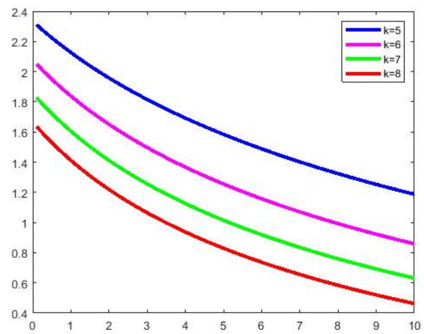

FIG. 2. *Graph of λ as a function of σ_1^2 when $b = 10, c_1 = 1, c_2 + \sigma_2 = 50$. We have four graphs for different incidence rates where $f(s) = \frac{s}{F(s,0)} = \frac{4s}{k+s}$ with $k = 5, \ldots, 8$.*

a lower level than the average s^* (see Figure 1). There is another effect to the growth rate λ of the disease at rare density that comes through the resonance of the fluctuation intensity σ_1^2 with the nonlinearity of the incidence rate (we call it the Type 2 effect).

If f is concave, the combination of concavity of f and the fluctuation of $\widetilde{S}^0(t)$ through σ_1^2 makes the disease harder to develop (as in Remake 3.1). Consequently, both Type 1 and Type 2 effects have detrimental impacts on the development of the disease. It is therefore not surprising that the simulations indicate λ is a decreasing function in σ_1^2; see Figure 2.

In contrast, if f is convex, the fluctuation of $\widetilde{S}^0(t)$ because of σ_1^2 resonates with the convexity of f in favor of the disease to develop. In this case, the development of the disease is subject to two different effects from σ_1^2, one (Type 2) is beneficial, and the other (Type 1) is unfavorable. Figure 3 suggests that λ as a function of σ_1^2 first increases and then decreases. It shows that with small fluctuations, the Type 1 effect is not strong compared to the Type 2 effect (the resonance of the fluctuation with the convexity of f). However, as σ_1^2 is large enough, the Type 1 effect gets stronger and diminishes the Type 2 effect, but it cannot outweigh the Type 2 effect because we show in Remark 3.1 that λ attains its maximum at $\sigma_1 = 0$.

3.2. Numerical simulations. In this section, we provide further numerical experimental results with different sets of parameters. One of the main efforts is to visualize the persistence and extinction of the disease. Euler's numerical algorithm is run with 10^6 iterations and step size $\Delta t = 10^{-4}$. To visualize the behavior, we plot

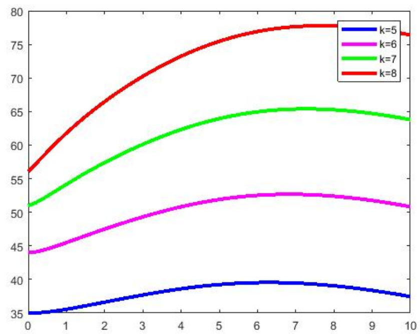

FIG. 3. *Graph of λ as a function of σ_1^2 when $b = 10, c_1 = 1, c_2 + \sigma_2 = 50$. We have four graphs for different incidence rates where $f(s) = \frac{s}{F(s,0)} = \min\{s(1+s), (2k+1)s - k^2\}$ with $k = 5, \ldots, 8$.*

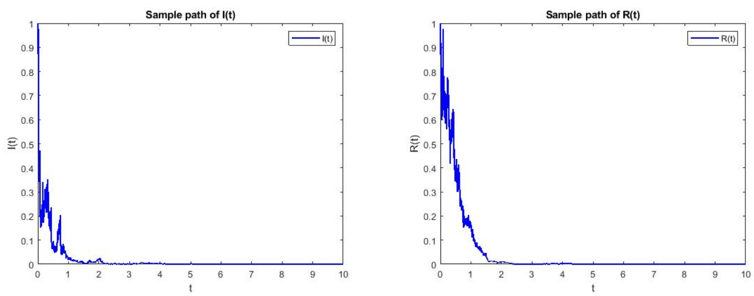

FIG. 4. *Sample paths of $I(t)$ and $R(t)$ with parameters $b = 6; \gamma = 1, \beta_1 = 1.5, \beta_2 = 1.6, \beta_3 = 1, \sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 1$, and $F(s,i) = 1$. $\lambda = -1.1$.*

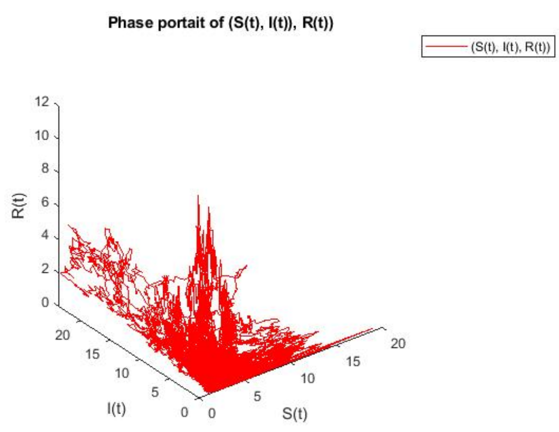

FIG. 5. *Sample paths of $I(t)$ and $R(t)$ with parameters $b = 6; \gamma = 1, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 2$, and $F(s,i) = 1$. $\lambda = 2.5$.*

the corresponding curves. Figure 4 delineates the extinction of the disease when λ is negative. The asymptotic behavior is delineated by the decaying graphs. In contrast, Figure 5 shows the persistence of the disease when λ is positive. In fact, we draw a three-dimensional surface similar to the two-dimensional "phase portrait."

4. Extinction: The case $\lambda < 0$. Focusing on the case of $\lambda < 0$, this section analyzes the extinction. We first consider the perturbed equation of (2.5)

$$d\widetilde{S}^\theta(t) = \left(b + \gamma\theta - c_1\widetilde{S}^\theta(t)\right) dt + \sigma_1\widetilde{S}^\theta(t)dW_1(t). \tag{4.1}$$

Similar to (2.5), (4.1) has an ergodic invariant measure, denoted by μ^θ, which is an inverse Gamma distribution. Similar to the observation in (2.6), we have

$$\int_{(0,\infty)} \frac{s}{F(s,0)}\mu^\theta(ds) < \infty.$$

LEMMA 4.1. *We have*

$$\lim_{\theta\to 0}\left|\int_{(0,\infty)} \frac{s}{F(s,0)}\mu^\theta(ds) - \int_{(0,\infty)} \frac{s}{F(s,0)}\mu^0(ds)\right| = 0.$$

Proof. The density of μ^θ on $(0,\infty)$ is

$$g_\theta(s) := \frac{\widetilde{\beta}_\theta^\alpha}{\Gamma(\alpha)} s^{-1-\alpha}e^{-\frac{\widetilde{\beta}_\theta}{s}}, \quad s > 0, \quad \text{with } \alpha := 1 + \frac{2c_1}{\sigma_1^2} \text{ and } \widetilde{\beta}_\theta := \frac{2(b+\gamma\theta)}{\sigma_1^2}. \tag{4.2}$$

Note that $\widetilde{\beta}_\theta \downarrow \widetilde{\beta}_0 := \frac{2b}{\sigma_1^2}$ as $\theta \downarrow 0$ and hence $g_\theta(s) \to g_0(s)$ pointwise as $\theta \downarrow 0$. Moreover, straightforward computations reveal that for all $\theta \in [0,1]$, we have

$$g_\theta(s) \le g(s) := \frac{\widetilde{\beta}_1^\alpha}{\Gamma(\alpha)} s^{-1-\alpha}e^{-\widetilde{\beta}_0 s^{-1}}.$$

It is easy to see that $\int_0^\infty sg(s)ds < \infty$. On the other hand, by Assumption 2.1, we have $\frac{s}{F(s,\theta)} \le \frac{s}{c_0}$. Thus, an application of the dominated convergence theorem yields

$$\begin{aligned}&\lim_{\theta\to 0}\left|\int_{(0,\infty)} \frac{s}{F(s,0)}\mu^\theta(ds) - \int_{(0,\infty)} \frac{s}{F(s,0)}\mu^0(ds)\right| \\ &\quad\le \lim_{\theta\to 0}\int_{(0,\infty)} \left|\frac{sg_\theta(s)}{F(s,0)} - \frac{sg_0(s)}{F(s,0)}\right| ds = 0.\end{aligned}$$

This completes the proof. ∎

LEMMA 4.2. *For any $\varepsilon > 0$ and $H > 0$, there exists a $\delta_0 > 0$ such that for all $(s,i,r) \in [0,H] \times [0,\delta_0] \times [0,\delta_0]$, we have*

$$\mathbb{P}_{s,i,r}\left\{\lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = 0\right\} \ge 1 - \varepsilon, \tag{4.3}$$

$$\mathbb{P}_{s,i,r}\left\{\limsup_{t\to\infty} \frac{1}{t}\int_0^t S(u)du \le \frac{b+\gamma\theta_1}{c_1}\right\} \ge 1 - \varepsilon, \tag{4.4}$$

where $\theta_1 > 0$ satisfies

$$\left|\int_{(0,\infty)} \frac{s}{F(s,0)}\mu^{\theta_1}(ds) - \int_{(0,\infty)} \frac{s}{F(s,0)}\mu^0(ds)\right| \le 0.1\widetilde{\lambda}, \tag{4.5}$$

and $\widetilde{\lambda} := \min\{-\lambda, c_3 + 0.5a_{33}\sigma_3^2\}$.

Proof. By the ergodicity of $\widetilde{S}_s^{\theta_1}(t)$, with probability 1,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\widetilde{S}_s^{\theta_1}(u)}{F(\widetilde{S}_s^{\theta_1}(u),0)}du=\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^{\theta_1}(ds), \tag{4.6}$$

where the subscript s in $\widetilde{S}_s^{\theta_1}(\cdot)$ indicates the initial value of $\widetilde{S}^{\theta_1}(\cdot)$. Consequently, for any $\varepsilon>0$, there exists a $T=T(H,\varepsilon)>0$ such that $\mathbb{P}_H(\Omega_1)\geq 1-\frac{\varepsilon}{4}$, where

$$\Omega_1=\left\{\omega:\frac{1}{t}\int_0^t\frac{\widetilde{S}^{\theta_1}(u)}{F(\widetilde{S}^{\theta_1}(u),0)}du\leq\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^{\theta_1}(du)+\frac{\widetilde{\lambda}}{10}\quad\forall t\geq T\right\},$$

And the subscript in $\mathbb{P}_H$ shows the initial value of $\widetilde{S}^{\theta_1}(u)$. Since $\widetilde{S}_s^{\theta_1}(u)\leq\widetilde{S}_H^{\theta_1}(u), u\geq 0$ almost surely for $s\leq H$ and $\frac{s}{F(s,0)}$ is increasing, we have that $\mathbb{P}_s(\Omega_1)\geq 1-\frac{\varepsilon}{4}$ for $s\in[0,H]$. Likewise, the strong law of large numbers for Brownian motions

$$\lim_{t\to\infty}\frac{W_k(t)}{t}=0,\ \text{almost surely (a.s.) for } k=1,2,3, \tag{4.7}$$

implies that $\mathbb{P}(\Omega_2)\geq 1-\frac{\varepsilon}{4}$, where

$$\Omega_2=\left\{\omega:\frac{|\sigma_k W_k(t)|}{t}\leq\frac{\widetilde{\lambda}}{10}\quad\forall t\geq T, k=1,2,3\right\}.$$

Furthermore, we can choose $T>1$ such that

$$\frac{\exp\{-0.5\widetilde{\lambda}T\}}{c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}}\leq 1. \tag{4.8}$$

Note that by the definition of $\widetilde{\lambda}$, $c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}>0$.

By virtue of (2.4) in Theorem 2.1 and the fact that $\frac{s}{F(s,0)}\leq\frac{s}{c_0}$, there exists an $M=M(\varepsilon,T,H)>0$ such that

$$\mathbb{P}_{s,i,r}(\Omega_3)\geq 1-\frac{\varepsilon}{4}\ \forall (s,i,r)\in\mathbb{R}^3_+ \text{ with } s+i+r\leq H,$$

where

$$\Omega_3=\left\{\omega:\int_0^T\frac{S(u)}{F(S(u),0)}du\leq M-0.1\widetilde{\lambda}T\right\}. \tag{4.9}$$

Moreover, we can choose M sufficiently large so that

$$\begin{aligned}&\mathbb{P}(\Omega_4)\geq 1-\frac{\varepsilon}{4},\\ &\text{where } \Omega_4=\Big\{\omega:|\sigma_k W_k(t)|\leq M-0.1\widetilde{\lambda}T\ \forall t\in[0,T] \text{ and } k=1,2,3\Big\}.\end{aligned} \tag{4.10}$$

Let $\delta_0>0$ be such that

$$\delta_0 e^M\left(1+c_4\frac{\exp\left\{(c_3+0.5a_{33}\sigma_3^2)T+3M\right\}}{c_3+0.5a_{33}\sigma_3^2}\right)\leq\frac{\theta_1}{2}\ \text{and}\ \delta_0 e^{2M}<\frac{\widetilde{\lambda}}{10\kappa_0}. \tag{4.11}$$

Define

$$\widetilde{\tau} := \inf\left\{t \geq 0 : R(t) \geq \theta_1 \text{ or } I(t) \geq \frac{\widetilde{\lambda}}{10\kappa_0}\right\}. \tag{4.12}$$

Using the second equation of (1.3), we have

$$I(t) = I(0)\exp\left\{\int_0^t \frac{S(u)}{F(S(u), I(u))}du - \left(c_2 + 0.5a_{22}\sigma_2^2\right)t + \sigma_2 W_2(t)\right\}. \tag{4.13}$$

Likewise, the third equation of (1.3) implies that

$$R(t) = \frac{1}{\Phi(t)}\left(R(0) + \int_0^t c_4\Phi(u)I(u)du\right), \tag{4.14}$$

where

$$\Phi(t) = \exp\left(\left(c_3 + 0.5a_{33}\sigma_3^2\right)t - \sigma_3 W_3(t)\right).$$

Thanks to Assumption 2.1,

$$\left|\frac{s}{F(s,i)} - \frac{s}{F(s,0)}\right| < 0.1\widetilde{\lambda} \text{ for any } s \geq 0, \text{ and } 0 \leq i \leq \frac{\widetilde{\lambda}}{10\kappa_0}. \tag{4.15}$$

Thus, in view of (4.9), (4.10), (4.13), and (4.15), we have

$$\begin{aligned} &ie^{-\left(c_2+0.5a_{22}\sigma_2^2\right)t-M} \\ &\quad \leq I(t) \leq ie^{2M} < \frac{\widetilde{\lambda}}{10\kappa_0} \quad \forall t \in [0, T\wedge\widetilde{\tau}], i \leq \delta_0 \text{ and } \omega \in \Omega_3 \cap \Omega_4. \end{aligned} \tag{4.16}$$

We next derive from (4.10) and (4.16) that

$$\begin{aligned} \int_0^t \Phi(u)I(u)du &\leq i\int_0^t \exp\{(c_3 + 0.5a_{33}\sigma_3^2)u + 3M\}du \\ &\leq i\frac{\exp\left\{(c_3 + 0.5a_{33}\sigma_3^2)T + 3M\right\}}{c_3 + 0.5a_{33}\sigma_3^2} \end{aligned} \tag{4.17}$$

for all $t \in [0, T\wedge\widetilde{\tau}]$ and $\omega \in \Omega_3 \cap \Omega_4$. On the other hand

$$\frac{1}{\Phi(t)} \leq \exp(\sigma_3 W_3(t)) \leq e^M \quad \forall t \in [0, T], \omega \in \Omega_4.$$

As a result, if $r, i \in [0, \delta_0)$, then we have from (4.14) that

$$\begin{aligned} R(t) &\leq e^M\left(r + ic_4\frac{\exp\left\{(c_3 + 0.5a_{33}\sigma_3^2)T + 3M\right\}}{c_3 + 0.5a_{33}\sigma_3^2}\right) \\ &< e^M\left(\delta_0 + \delta_0 c_4\frac{\exp\left\{(c_3 + 0.5a_{33}\sigma_3^2)T + 3M\right\}}{c_3 + 0.5a_{33}\sigma_3^2}\right) \\ &\leq \frac{\theta_1}{2} \quad \forall t \in [0, T\wedge\widetilde{\tau}] \text{ and } \omega \in \Omega_3 \cap \Omega_4. \end{aligned} \tag{4.18}$$

Thus, it follows from (4.16) and (4.18) that for any initial value $(s, i, r) \in [0, H] \times [0, \delta_0) \times [0, \delta_0)$, we have

$$\widetilde{\tau} > T \text{ if } \omega \in \Omega_3 \cap \Omega_4. \tag{4.19}$$

Now, we show that for any initial value $(s,i,r)\in[0,H]\times[0,\delta_0)\times[0,\delta_0)$, we have

$$\widetilde{\tau}=\infty \text{ for almost every } \omega\in\bigcap_{k=1}^{4}\Omega_k. \tag{4.20}$$

We have from the comparison principle that $S(u)\le\widetilde{S}^{\theta_1}(u)$, $u\in[0,\widetilde{\tau}]$ with probability 1. Then we can compute using (4.15)

$$\begin{aligned} I(t)&=i\exp\left\{\int_0^t\frac{S(u)}{F(S(u),I(u))}du-\Big(c_2+0.5a_{22}\sigma_2^2\Big)t+\sigma_2W_2(t)\right\}\\ &\le i\exp\left\{\int_0^t\frac{S(u)}{F(S(u),0)}du+0.1\widetilde{\lambda}t-\Big(c_2+0.5a_{22}\sigma_2^2\Big)t+\sigma_2W_2(t)\right\}\\ &\le i\exp\left\{\int_0^t\frac{\widetilde{S}^{\theta_1}(u)}{F(\widetilde{S}^{\theta_1}(u),0)}du+0.2\widetilde{\lambda}t-\Big(c_2+0.5a_{22}\sigma_2^2\Big)t+\sigma_2W_2(t)\right\}\\ &\le i\exp\left\{t\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^{\theta_1}(ds)+0.3\widetilde{\lambda}t-\Big(c_2+0.5a_{22}\sigma_2^2\Big)t+0.1\widetilde{\lambda}t\right\}\\ &\le i\exp\left\{\lambda t+0.4\widetilde{\lambda}t\right\}\quad\text{(due to (4.5))}\\ &\le i\exp\left\{-0.6\widetilde{\lambda}t\right\},\quad t\in[T,\widetilde{\tau}),(s,i,r)\in[0,H]\times[0,\delta_0)\times[0,\delta_0) \end{aligned}\tag{4.21}$$

for almost every $\omega\in\cap_{k=1}^4\Omega_k$.

Using (4.14), we can write for $t\ge T$ that

$$R(t)=\frac{1}{\Phi(t)}\left(R(0)+\int_0^T c_4\Phi(u)I(u)du\right)+\frac{1}{\Phi(t)}\int_T^t c_4\Phi(u)I(u)du. \tag{4.22}$$

When $\omega\in\Omega_2$, we have

$$\exp\left\{\Big(c_3+0.5a_{33}\sigma_3^2-0.1\widetilde{\lambda}\Big)t\right\}\le\Phi(t)\le\exp\left\{\Big(c_3+0.5a_{33}\sigma_3^2+0.1\widetilde{\lambda}\Big)t\right\}\quad\forall t\ge T. \tag{4.23}$$

By virtue of (4.21) and (4.23), for $(s,i,r)\in[0,H]\times[0,\delta_0)\times[0,\delta_0)$, we have the following estimate: For almost every $\omega\in\cap_{k=1}^4\Omega_k$ and $t\in[T,\widetilde{\tau})$,

$$\begin{aligned}\int_T^t\Phi(u)I(u)du&\le i\int_T^t\exp\left\{\big(c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}\big)u\right\}du\\ &\le i\frac{\exp\left\{\big(c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}\big)t\right\}}{c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}},\end{aligned}$$

which together with the first inequality of (4.23) leads to

$$\begin{aligned}&\frac{1}{\Phi(t)}\int_T^t\Phi(u)I(u)du\\ &\le i\exp\left\{-\big(c_3+0.5a_{33}\sigma_3^2-0.1\widetilde{\lambda}\big)t\right\}\frac{\exp\left\{\big(c_3+0.5a_{33}\sigma_3^2-0.6\widetilde{\lambda}\big)t\right\}}{c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}}\\ &=i\frac{\exp\{-0.5\widetilde{\lambda}t\}}{c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}}\le i,\end{aligned}\tag{4.24}$$

where the last inequality follows from (4.8). Let n be any integer greater than T. Applying (4.17), (4.23), and (4.24) to (4.22), we have for any initial value $(s,i,r) \in [0,H] \times [0,\delta_0) \times [0,\delta_0)$ and for almost every $\omega \in \cap_{k=1}^4 \Omega_k$ and $t \in [T, n \wedge \widetilde{\tau})$,

(4.25)
$$\begin{aligned} R(t) &\le e^{-\left(c_3+0.5a_{33}\sigma_3^2-0.1\widetilde{\lambda}\right)t}\left(r + ic_4\frac{e^{(c_3+0.5a_{33}\sigma_3^2)T+3M}}{c_3+0.5a_{33}\sigma_3^2}\right) + ic_4\frac{e^{-0.5\widetilde{\lambda}t}}{c_3+0.5a_{33}\sigma_3^2-0.5\widetilde{\lambda}} \\ &\le \delta_0\left(1 + c_4\frac{e^{(c_3+0.5a_{33}\sigma_3^2)T+3M}}{c_3+0.5a_{33}\sigma_3^2}\right) + \delta_0 c_4 \\ &< \frac{\theta_1}{2} + \frac{\theta_1}{2} = \theta_1, \end{aligned}$$

where the last line follows from (4.11). This and (4.21) imply that $\widetilde{\tau} \ge n$ for almost every $\omega \in \cap_{k=1}^4 \Omega_k$. Since $n \in \mathbb{N}$ is arbitrary, we obtain (4.20).

With (4.20) at our hands, we can let $t \to \infty$ in (4.21) and the first inequality of (4.25) to see that

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = 0 \tag{4.26}$$

for any initial value $(s,i,r) \in [0,H] \times [0,\delta_0) \times [0,\delta_0)$ and almost every $\omega \in \cap_{k=1}^4 \Omega_k$. Note that $\mathbb{P}\{\cap_{k=1}^4 \Omega_k\} \ge 1-\varepsilon$ and thus (4.3) follows.

Moreover, it follows from (4.20) and the comparison principle that $S(u) \le \widetilde{S}^{\theta_1}(u)$ for all $u \in [0,\infty)$ and $\omega \in \cap_{k=1}^4 \Omega_k$. Consequently we have

(4.27)
$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t S(u)du \le \lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{S}^{\theta_1}(u)du = \frac{b+\gamma\theta_1}{c_1} \text{ for almost every } \omega \in \bigcap_{k=1}^4 \Omega_k.$$

This gives (4.4) and hence completes the proof of the lemma. ∎

PROPOSITION 4.1. *For any $\varepsilon > 0$ and $H > 0$, there exists a $\delta > 0$ such that*

(4.28)
$$\mathbb{P}_{s,i,r}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda < 0, \lim_{t\to\infty}\frac{\ln R(t)}{t} = \max\{-c_3 - 0.5a_{33}\sigma_3^3, \lambda\} < 0\right\} \ge 1-\varepsilon$$

for $(s,i,r) \in [0,H] \times (0,\delta) \times [0,H]$.

Proof. Having Lemma 4.2, the proof of this proposition is based on considering the tightness and weak-limits of the family of randomized occupation measures defined as

$$\widetilde{\Pi}^t(\cdot) := \frac{1}{t}\int_0^t \mathbf{1}_{\{(S(u),I(u),R(u))\in\cdot\}}du, \quad t > 0.$$

Note that the tightness of a family of probability measures means the relative compactness in the space of probability measures equipped with the topology of weak convergence; see, e.g. [7]. It is proven in [16, Theorem 4.2] or [20, 39] that with probability 1, any weak-limit of $\widetilde{\Pi}^t$ as $t \to \infty$ is an invariant probability measure of the process $(S(t), I(t), R(t))$ on $\mathbb{R}_+^3$. In view of Lemma 4.2, the collection of measures $\{\widetilde{\Pi}^t(\cdot;\omega), t > 0, \omega \in \cap_{k=1}^4 \Omega_4\}$ is tight in $\mathbb{R}_+^3$ and any weak limit of $\widetilde{\Pi}^t(\cdot)$ as $t \to \infty$ must have support on $[0,\infty) \times \{0\} \times \{0\}$. Clearly, $\mu^0 \times \boldsymbol{\delta}^* \times \boldsymbol{\delta}^*$ is the unique invariant probability measure on $[0,\infty) \times \{0\} \times \{0\}$ for the process $(S(t), I(t), R(t))$, where $\boldsymbol{\delta}^*$ is the Dirac measure with mass at 0. As a result, $\widetilde{\Pi}^t(\cdot)$ converges weakly to $\mu^0\times\boldsymbol{\delta}^*\times\boldsymbol{\delta}^*$ for almost every $\omega\in\cap_{k=1}^4\Omega_k$ as t tends to ∞. By the weak convergence, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{S(u)}{F(S(u),I(u))}du=\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^0(ds) \tag{4.29}$$

for almost every $\omega\in\cap_{k=1}^4\Omega_k$. Note that although $\frac{s}{F(s,0)}$ may be unbounded, by Assumption 2.1, we have $\frac{s}{F(s,i)}\le\frac{s}{c_0}$. Also recall that $S(u)\le\widetilde{S}^{\theta_1}(u)$ for all $u\ge 0$ and $\omega\in\cap_{k=1}^4\Omega_k$. Thus, the limit (4.29) is valid due to the weak convergence and the uniform integrability

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t(S(u))^{1+\widetilde{p}}du\le\lim_{t\to\infty}\frac{1}{t}\int_0^t(\widetilde{S}^{\theta_1}(u))^{1+\widetilde{p}}du=\int_{(0,\infty)}s^{1+\widetilde{p}}\mu^{\theta_1}(ds)<\infty$$

for $\widetilde{p}\in(0,\alpha)$, which follows (4.2) and the ergodicity of $\widetilde{S}^{\theta_1}(t)$. By (4.13), we obtain

$$\frac{\ln I(t)}{t}=\frac{\ln I(0)}{t}+\frac{1}{t}\int_0^t\frac{S(u)}{F(S(u),I(u))}du-\left(c_2+0.5a_{22}\sigma_2^2\right)+\sigma_2\frac{W_2(t)}{t}$$

and hence

$$\lim_{t\to\infty}\frac{\ln I(t)}{t}=\lambda \tag{4.30}$$

for almost every $\omega\in\cap_{k=1}^4\Omega_k$. In view of (4.30) and the limit

$$\lim_{t\to\infty}\frac{\ln\Phi(t)}{t}=c_3+0.5a_{33}\sigma_3^2\quad\text{ a.s.,} \tag{4.31}$$

we can write

$$\begin{aligned}&\frac{1}{\Phi(t)}\int_0^t\Phi(u)I(u)du\\&\quad=\exp\{-(c_3+0.5a_{33}\sigma_3^2)t+o_1(t)\}\int_0^t\exp\{(\lambda+c_3+0.5a_{33}\sigma_3^2)u+o_2(u)\}du,\end{aligned}$$

where $\lim_{t\to\infty}\frac{|o_1(t)|+|o_2(t)|}{t}=0$ for almost every $\omega\in\cap_{k=1}^4\Omega_k$. Then, we can easily show that

$$\lim_{t\to\infty}\frac{1}{t}\ln\frac{1}{\Phi(t)}\int_0^t\Phi(u)I(u)du=\max\{\lambda,-(c_3+0.5a_{33}\sigma_3^2)\} \tag{4.32}$$

for almost every $\omega\in\cap_{k=1}^4\Omega_k$. In light of (4.14), (4.31), and (4.32), we have

$$\lim_{t\to\infty}\frac{\ln R(t)}{t}=\max\{\lambda,-(c_3+0.5a_{33}\sigma_3^2)\}$$

for almost every $\omega\in\cap_{k=1}^4\Omega_k$. Recall that $\mathbb{P}_{s,i,r}(\cap_{k=1}^4\Omega_k)\ge 1-\varepsilon$ for all $(s,i,r)\in[0,H]\times[0,\delta_0)\times[0,\delta_0)$. The proof is therefore complete. ∎

Proof of Theorem 2.2. The process $(S(t),I(t),R(t))$ is transient in $\mathbb{R}_+^{3,\circ}$ by virtue of Proposition 4.1. Thus, the process has no invariant probability measure in $\mathbb{R}_+^{3,\circ}$. It is easy to show that $\mu^0\times\boldsymbol{\delta}^*\times\boldsymbol{\delta}^*$ is the unique invariant probability measure on the boundary and therefore the only invariant probability measure in $\mathbb{R}^3_+$. Let H be sufficiently large that $\mu^0((0,H)) > 1-\varepsilon$. Thanks to (2.3), the process $(S(t), I(t), R(t))$ is tight. Consequently the occupation measure

$$\Pi^t_{s,r,i}(\cdot) := \frac{1}{t}\int_0^t \mathbb{P}_{s,r,i}\left\{(S(u), I(u), R(u)) \in \cdot\right\} du$$

is tight in $\mathbb{R}^3_+$. Since any weak limit of $\Pi^t_{s,r,i}$ as $t \to \infty$ must be an invariant probability measure of $(S(t), I(t), R(t))$, we have that $\Pi^t_{s,r,i}$ converges weakly to $\mu^0 \times \boldsymbol{\delta}^* \times \boldsymbol{\delta}^*$ as $t \to \infty$. As a result, for any $\delta > 0$, there exists a $\widehat{T} > 0$ such that

$$\Pi^{\widehat{T}}_{s,r,i}((0,H) \times (0,\delta) \times (0,\delta)) > 1 - \varepsilon$$

or equivalently

$$\frac{1}{\widehat{T}}\int_0^{\widehat{T}} \mathbb{P}_{s,r,i}\{(S(t), I(t), R(t)) \in (0,H) \times (0,\delta) \times (0,\delta)\}dt > 1 - \varepsilon.$$

Thus,

$$\mathbb{P}_{s,r,i}\{\widehat{\tau} \le \widehat{T}\} > 1 - \varepsilon,$$

where $\widehat{\tau} = \inf\{t \ge 0 : (S(t), I(t), R(t)) \in (0,H) \times (0,\delta) \times (0,\delta)\}$. Using the strong Markov property and Proposition 4.1, we have that

$$\mathbb{P}_{s,i,r}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda < 0, \lim_{t\to\infty}\frac{\ln R(t)}{t} = \max\{-c_3 - 0.5\sigma_2^3, \lambda\} < 0\right\} \ge 1 - \varepsilon$$

for any $(s,i,r) \in \mathbb{R}^{3,*}_+$. Since $\varepsilon > 0$ is arbitrary, (2.8) follows. This completes the proof of Theorem 2.2. ∎

5. Persistence: The case $\boldsymbol{\lambda > 0}$. The proof in this section is a generalization of [13, subsection 2.2]. Throughout this section, we denote $[\ln x]_- = \max\{0, -\ln x\}$.

LEMMA 5.1. *For any $A \in \mathcal{F}$ and positive integer n, there exists a positive constant $\widehat{K}_n$ such that*

(5.1)
$$\mathbb{E}\left([\ln I(t)]_-^{n+1}\mathbf{1}_A\right) \le [\ln i]_-^{n+1}\mathbb{P}(A) + \widehat{K}_n\sqrt{\mathbb{P}(A)}(t+1)[\ln i]_-^n + \widehat{K}_n(t^{n+1}+1)\sqrt{\mathbb{P}(A)},$$

where $I(0) = i$. As a result, for any $H > 0$

$$\mathbb{P}_{s,i,r}\left\{[\ln I(t)]_- > H\right\} \le \frac{[\ln i]_-^{n+1} + \widehat{K}_n(t+1)[\ln i]_-^n + \widehat{K}_n(t^{n+1}+1)}{H^{n+1}}. \tag{5.2}$$

Proof. We have

$$\begin{aligned}-\ln I(t) &= -\ln I(0) - \int_0^t \frac{S(u)}{F(S(u), I(u))}du + (c_2 + 0.5a_{22}\sigma_2^2)t - \sigma_2 W_2(t)\\ &\le -\ln I(0) + (c_2 + 0.5a_{22}\sigma_2^2)t + |\sigma_2 W_2(t)|.\end{aligned}$$

Therefore,

$$[\ln I(t)]_- \le [\ln I(0)]_- + (c_2 + 0.5a_{22}\sigma_2^2)t + |\sigma_2 W_2(t)|.$$

Raising both sides to power $n+1$ and then using the inequalities $(a+b)^{n+1} \leq a^{n+1} + K_n a^n b + K_n b^{n+1}$ for some $K_n > 0$ and $(a+b)^{n+1} \leq 2^{n+1}(a^{n+1} + b^{n+1})$ for nonnegative a and b, we have

$$\begin{aligned}[\ln I(t)]_-^{n+1} &\leq [\ln i]_-^{n+1} + K_n \left((c_2 + 0.5a_{22}\sigma_2^2)t + |\sigma_2 W_2(t)|\right)[\ln i]_-^n \\ &\quad + K_n \left((c_2 + 0.5a_{22}\sigma_2^2)t + |\sigma_2 W_2(t)|\right)^{n+1} \\ &\leq [\ln i]_-^{n+1} + K_n \left((c_2 + 0.5a_{22}\sigma_2^2)t + |\sigma_2 W_2(t)|\right)[\ln i]_-^n \\ &\quad + 2^{n+1} K_n \left(c_2 + 0.5a_{22}\sigma_2^2)t\right)^{n+1} + 2^{n+1} K_n |\sigma_2 W_2(t)|^{n+1}\end{aligned} \tag{5.3}$$

given that $I(0) = i$. By the Hölder inequality, we obtain for $t > 0$ that

$$\mathbb{E}|W_2(t)|\mathbf{1}_A \leq \sqrt{\mathbb{E}W_2^2(t)\mathbb{P}(A)} \leq \sqrt{t\mathbb{P}(A)} \leq (t+1)\sqrt{\mathbb{P}(A)}$$

and

$$\mathbb{E}|W_2^{n+1}(t)|\mathbf{1}_A \leq \sqrt{\mathbb{E}W_2^{2n+2}(t)\mathbb{P}(A)} \leq c_n\sqrt{t^{n+1}\mathbb{P}(A)} \leq c_n(t^{n+1}+1)\sqrt{\mathbb{P}(A)},$$

where c_n is some positive constant. Multiplying both sides of (5.3) by $\mathbf{1}_A$ and then taking expectation on both sides and using the two above estimates as well as the fact that $\mathbb{P}(A) \leq \sqrt{\mathbb{P}(A)}$, we obtain (5.1) for some positive constant $\widehat{K}_n$. Letting $A = \Omega$ and applying the Markov inequality to (5.1) yields (5.2). ∎

LEMMA 5.2. *For any $\varepsilon > 0$, there exist a $\delta \in (0,1)$ and a $T^* > 1$ such that*

$$\mathbb{P}_{s,i,r}\left\{\ln i + \frac{3\lambda T^*}{4} \leq 0 \wedge \ln I(T^*)\right\} \geq 1 - \varepsilon \tag{5.4}$$

for all $(s,i,r) \in [0,\infty) \times (0,\delta] \times [0,\infty)$.

Proof. Let $\theta \geq 0$ and $\widehat{S}^\theta(t)$ be the solution to

$$d\widehat{S}^\theta(t) = \left[b - \left(\frac{\theta}{c_0} + c_1\right)\widehat{S}^\theta(t)\right]dt + \sigma_1 \widehat{S}^\theta(t)dB(t). \tag{5.5}$$

Similar to Lemma 4.1, (5.5) has an ergodic invariant measure, denoted by ν^θ on $(0,\infty)$, satisfying

$$\lim_{\theta\to 0}\left|\int_{(0,\infty)}\left(\frac{s}{F(s,0)}\nu^\theta(ds) - \frac{s}{F(s,0)}\mu^0(ds)\right)\right| = 0.$$

There is therefore a $\theta_0 \in (0,1)$ satisfying

$$\left|\int_{(0,\infty)}\left(\frac{s}{F(s,0)}\nu^\theta(ds) - \frac{s}{F(s,0)}\mu^0(ds)\right)\right| < \frac{\lambda}{12}, \theta \in [0,\theta_0]. \tag{5.6}$$

Let $0 < \theta_2 < \theta_0 < 1$ such that $|\frac{s}{F(s,i)} - \frac{s}{F(s,0)}| \leq \frac{\lambda}{24}$ when $i \leq \theta_2$, which can be found because of the continuity of $\frac{s}{F(s,i)}$ at $i = 0$ uniformly in s. Consider $\widehat{S}_s^{\theta_2}(t)$ of

(5.5), in which the initial condition $s \geq 0$. We have from (5.6) and the ergodicity of $\widehat{S}^{\theta_2}(t)$ that

$$\mathbb{P}_s\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\widehat{S}^{\theta_2}(u)du}{F(\widehat{S}^{\theta_2}(u),0)} = \int_{(0,\infty)} \frac{s}{F(s,0)}\nu^\theta(ds) \geq \int_{(0,\infty)} \frac{u}{F(u,0)}\mu^0(du) - \frac{\lambda}{12}\right\} = 1 \ \forall s \in [0,\infty).$$

Consequently it follows that there exists a $T^* > 1$ such that

$$\begin{aligned} &\mathbb{P}_0(\Omega_5) \geq 1 - \frac{\varepsilon}{3}, \\ &\text{where } \Omega_5 := \left\{\frac{1}{t}\int_0^t \frac{\widehat{S}^{\theta_2}(u)du}{F(\widehat{S}^{\theta_2}(u),0)} \geq \int_{(0,\infty)} \frac{u}{F(u,0)}\mu^0(du) - \frac{\lambda}{6} \ \ \forall\, t \geq T^*\right\}. \end{aligned} \tag{5.7}$$

Because of the uniqueness of solutions, for any $s \in [0,\infty)$, we have $\widehat{S}^{\theta_2}_s(u) \geq \widehat{S}^{\theta_2}_0(u)$ for any $u > 0$ with a probability 1. Thus,

$$\mathbb{P}_s(\Omega_5) \geq 1 - \frac{\varepsilon}{3},\ s \in [0,\infty). \tag{5.8}$$

On the other hand, the strong law of large numbers for Brownian motion

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{W_2(t)}{t} = 0\right\} = 1$$

implies that there exists a $T^* > 1$ such that

$$\mathbb{P}(\Omega_6) \geq 1 - \frac{\varepsilon}{3},\ \text{ where } \Omega_6 := \left\{\frac{\sigma_2 W_2(t)}{t} \geq -\frac{\lambda}{24} \ \ \forall\, t \geq T^*\right\}. \tag{5.9}$$

Without loss of generality, we may assume that the T^* in (5.7) and (5.9) are the same.

Define the stopping time

$$\zeta = \inf\{t \geq 0 : I(t) \geq \theta_2\}. \tag{5.10}$$

Since $\frac{I(t)}{F(S(t),I(t))} \leq \frac{\theta_2}{c_0}$ if $I(t) \leq \theta_2$, we have from the comparison theorem that

$$\mathbb{P}_{s,i,r}\left\{S(t) \geq \widehat{S}^{\theta_2}(t) \ \ \forall\, t \leq \zeta\right\} = 1,$$

which, together with the fact that $|\frac{s}{F(s,i)} - \frac{s}{F(s,0)}| \leq \frac{\lambda}{24}$ when $i \leq \theta_2$ and $\frac{s}{F(s,0)}$ is nondecreasing in s, implies

$$\mathbb{P}_{s,i,r}\left\{\frac{S(t)}{F(S(t),I(t))} \geq \frac{S(t)}{F(S(t),0)} - \frac{\lambda}{24} \geq \frac{\widehat{S}^{\theta_2}(t)}{F(\widehat{S}^{\theta_2}(t),0)} - \frac{\lambda}{24} \ \ \forall\, t \leq \zeta\right\} = 1. \tag{5.11}$$

From (5.9), (5.8), (5.10), and (5.11), we can show for any $(s,i,r)\in[0,\infty)\times(0,\infty)\times[0,\infty)$ that

$$\begin{aligned}
0 &> \ln\theta_2 \geq \ln(I(T^*))\\
&= \ln i + \int_0^{T^*}\frac{S(u)}{F(S(u),I(u))}du - (c_2+0.5a_{22}\sigma_2^2)T^* + \sigma_2 W_2(T^*)\\
&\geq \ln i + \int_0^{T^*}\frac{\widehat S^{\theta_2}(u)}{F(\widehat S^{\theta_2}(u),0)}du - \frac{\lambda T^*}{24} - (c_2+0.5a_{22}\sigma_2^2)T^* + \sigma_2 W_2(T^*)\\
&\geq \ln i + T^*\int_{(0,\infty)}\frac{s}{F(s,0)}\mu^0(ds) - \frac{\lambda T^*}{6} - \frac{\lambda T^*}{24} - (c_2+0.5a_{22}\sigma_2^2)T^* - \frac{\lambda T^*}{24}\\
&\geq \ln i + \frac{3\lambda}{4}T^*, \ \text{ for almost every } \omega\in\Omega_5\cap\Omega_6\cap\{\zeta\geq T^*\}.
\end{aligned} \tag{5.12}$$

In view of (5.2), there exists a $\delta=\delta(\varepsilon,\theta_2)\in(0,1)$ sufficiently small such that

$$\mathbb{P}_{s,i,r}\left\{\ln I(t)\geq \ln\delta + \frac{3\lambda T^*}{4}\right\}\geq 1-\frac{\varepsilon}{3}, \text{ for } (s,i,r)\in[0,\infty)\times[\theta_2,\infty)\times[0,\infty), t\leq T^*.$$

Furthermore, we can choose δ such that $\ln\delta+\frac{3\lambda T^*}{4}\leq 0$. This and the strong Markov property of $(S(t),I(t),R(t))$ implies that

$$\mathbb{P}_{s,i,r}\left\{\zeta\leq T^*, \text{ and } \ln I(T^*)\geq \ln\delta+\frac{3\lambda T^*}{4}\right\}\geq\left(1-\frac{\varepsilon}{3}\right)\mathbb{P}_{s,i,r}\{\zeta\leq T^*\} \tag{5.13}$$

for all $(s,i,r)\in[0,\infty)\times(0,\infty)\times[0,\infty)$ In view of (5.12) and (5.13) we have for $(s,i,r)\in[0,\infty)\times(0,\delta]\times[0,\infty)$ that

$$\begin{aligned}
&\mathbb{P}_{s,i,r}\left\{\ln i+\frac{3\lambda T^*}{4}\leq \ln I(T^*)\right\}\\
&\quad\geq \mathbb{P}_{s,i,r}\Big(\Omega_5\cap\Omega_6\cap\{\zeta\geq T^*\}\Big)+\left(1-\frac{\varepsilon}{3}\right)\mathbb{P}_{s,i,r}\{\zeta\leq T^*\}\geq 1-\varepsilon.
\end{aligned}$$

This and the inequality $\ln\delta+\frac{3\lambda T^*}{4}\leq 0$ yield the desire result. ∎

PROPOSITION 5.1. *There exists a constant* $T^*>1$ *such that for any* $n\in\mathbb{Z}_+$, *we have*

$$\mathbb{E}_{s,i,r}[\ln I(T^*)]_-^{n+1}\leq[\ln i]_-^{n+1}-\frac{\lambda T^*}{4}[\ln i]_-^n + L_n, (s,i,r)\in\mathbb{R}_+^{3,*},$$

where $L_n>0$ *is a constant.*

Proof. Let $n\in\mathbb{N}$ and $\widehat K_n$ be the constants as in the proof of Lemma 5.1. We can choose $\varepsilon>0$ such that

$$\frac{3n\lambda}{8}(1-\varepsilon)-2\widehat K_n\sqrt{\varepsilon}\geq\frac{\lambda}{4}. \tag{5.14}$$

By Lemma 5.2, there exist $T^*>1$, and $\delta\in(0,1)$ such that

$$\mathbb{P}_{s,i,r}(\widehat\Omega)\geq 1-\varepsilon \quad \forall (s,i,r)\in[0,\infty)\times(0,\delta]\times[0,\infty), \tag{5.15}$$

where

$$\widehat\Omega := \left\{\ln i+\frac{3\lambda T^*}{4}\leq 0\wedge\ln I(T^*)\right\}.$$

For $(s,i,r)\in[0,\infty)\times(0,\delta]\times[0,\infty)$ and $\omega\in\widehat\Omega$,

$$[\ln I(T^*)]_-\le-\left(\ln i+\frac{3\lambda T^*}{4}\right)=[\ln i]_--\frac{3\lambda T^*}{4}.$$

Raising both sides to power $n+1$ and using the inequality for positive a,b: $(a-b)^{n+1}\le a^{n+1}-\dfrac{n}{2}a^nb+\check K_nb^{n+1}$ for some $c>0$, we have

$$[\ln I(T^*)]_-^{n+1}\le[\ln i]_-^{n+1}-\frac{3n\lambda T^*}{8}[\ln i]_-^n+\check K_n\frac{(3\lambda T^*)^{n+1}}{4^{n+1}}\tag{5.16}$$

for $(s,i,r)\in[0,\infty)\times(0,\delta]\times[0,\infty)$ and $\omega\in\widehat\Omega$. On the other hand, it follows from Lemma 5.1 that

$$\mathbb E\left([\ln I(t)]_-^{n+1}\mathbf 1_{\widehat\Omega^c}\right)\le[\ln i]_-^{n+1}\mathbb P(\widehat\Omega^c)+\widehat K_n\sqrt{\mathbb P(\widehat\Omega^c)}(t+1)[\ln i]_-^n+\widehat K_n(t^{n+1}+1)\sqrt{\mathbb P(\widehat\Omega^c)}.\tag{5.17}$$

In view of (5.16) and (5.17), for $(s,i,r)\in[0,\infty)\times(0,\delta]\times[0,\infty)$,

$$\begin{aligned}\mathbb E[\ln I(T^*)]_-^{n+1}&=\mathbb E\left([\ln I(T^*)]_-^{n+1}\mathbf 1_{\widehat\Omega}\right)+\mathbb E\left([\ln I(T^*)]_-^{n+1}\mathbf 1_{\widehat\Omega^c}\right)\\&\le[\ln i]_-^{n+1}+\left(\widehat K_n\sqrt{\mathbb P(\widehat\Omega^c)}(T^*+1)-\frac{3n\lambda T^*}{8}\mathbb P(\widehat\Omega)\right)[\ln i]_-^n\\&\quad+\widehat K_n({T^*}^{n+1}+1)+\check K_n\frac{(3\lambda T^*)^{n+1}}{4^{n+1}}.\\&\le[\ln i]_-^{n+1}-\frac{\lambda T^*}{4}[\ln i]_-^n+\widetilde L_1,\end{aligned}\tag{5.18}$$

where we used (5.14) to derive the last inequality, and $\widetilde L_1:=\widehat K_n({T^*}^{n+1}+1)+\check K_n\frac{(3\lambda T^*)^{n+1}}{4^{n+1}}$.

If $i\ge\delta$, we have from Lemma 5.1 that

$$\begin{aligned}&\mathbb E\left([\ln I(T^*)]_-^{n+1}\right)\\&\quad\le[\ln i]_-^{n+1}+\widehat K_n(T^*+1)[\ln i]_-^n+\widehat K_n((T^*)^{n+1}+1)\\&\quad\le[\ln i]_-^{n+1}-\frac{\lambda T^*}{4}[\ln i]_-^n+\left(\widehat K_n(T^*+1)+\frac{\lambda T^*}{4}\right)[\ln i]_-^n+\widehat K_n((T^*)^{n+1}+1)\\&\quad\le[\ln i]_-^{n+1}-\frac{\lambda T^*}{4}[\ln i]_-^n+\widetilde L_2,\end{aligned}\tag{5.19}$$

where $\widetilde L_2:=(\widehat K_n(T^*+1)+\frac{\lambda T^*}{4})[\ln\delta]_-^n+\widehat K_n((T^*)^{n+1}+1)$. By combining (5.18) and (5.19), the proof is complete. ∎

LEMMA 5.3. *For any $0<h<H<\infty$, $T\ge1$, and $\varepsilon>0$, there exists a $\widehat\delta=\widehat\delta(H,h,T,\varepsilon)>0$ such that*

$$\mathbb P_{s,i,r}\{\min\{S(T),I(T),R(T)\}\ge\widehat\delta\}\ge1-\varepsilon\quad\text{for all }(s,i,r)\in[0,H]\times[h,H]\times[0,H].\tag{5.20}$$

Proof. Recall from the proof of Lemma 4.2 that there exists an $\widetilde M>0$ such that $\mathbb P_{s,i,r}(\widetilde\Omega)>1-\varepsilon$ for all $(s,i,r)\in[0,H]^3$, where

$$\widetilde\Omega=\left\{\sup_{t\in[0,T],k=1,2,3}|\sigma_kW_k(t)|\vee\int_0^T\frac{S(u)}{F(S(u),I(u))}du\le\widetilde M\right\}.$$

Similar to (4.16) we have

$$ie^{-\left(c_2+0.5a_{22}\sigma_2^2\right)t-\widetilde{M}} \leq I(t) \leq ie^{2\widetilde{M}} \text{ for } t \in [0,T] \text{ and } \omega \in \widetilde{\Omega}. \tag{5.21}$$

By the variation of constant method (see [33, Chapter 3]), we can write

$$S(t) = \Psi(t)\left(S(0) + \int_0^t (b+\gamma R(u))\Psi^{-1}(u)du\right),$$

where

$$\Psi(t) = \exp\left\{-\int_0^t \frac{I(u)}{F(S(u),I(u))}du - (c_1+0.5a_{11}\sigma_1^2)t+\sigma_1 W_1(t)\right\}.$$

In view of (5.21) and the inequality $-\frac{i}{F(s,i)} \geq -\frac{i}{c_0}$, for any $\omega \in \widetilde{\Omega}$ and $(s,i,r) \in [0,H]^3$, we have

$$\begin{aligned} S(T) &\geq \exp\left\{-T\left(\frac{He^{2\widetilde{M}}}{c_0} + c_1 + 0.5a_{11}\sigma_1^2\right) - \widetilde{M}\right\}\left(S(0) + b\int_0^T \exp(-\widetilde{M})ds\right) \\ &\geq bT\exp\left\{-T\left(\frac{He^{2\widetilde{M}}}{c_0} + c_1 + 0.5a_{11}\sigma_1^2\right) - 2\widetilde{M}\right\}. \end{aligned} \tag{5.22}$$

Recall that

$$R(t) = \frac{1}{\Phi(t)}\left(R(0) + c_4\int_0^t \Phi(u)I(u)du\right) \geq \frac{c_4}{\Phi(t)}\int_0^t \Phi(u)I(u)du, \tag{5.23}$$

where $\Phi(t) = \exp((c_3+0.5a_{33}\sigma_3^2)t - \sigma_3 W_3(t))$. When $(s,i,r) \in [0,H]^3$ and $\omega \in \widetilde{\Omega}$, we have

$$\exp\{-\widetilde{M}\} \leq \Phi(t) \leq \exp\left(\left(c_3+0.5a_{33}\sigma_3^2\right)t + \widetilde{M}\right), t \in [0,T],$$

which in combination with (5.21) and (5.23) yields

$$\begin{aligned} R(T) &\geq \exp\{-(c_3+0.5a_{33}\sigma_3^2)T - \widetilde{M}\}\int_0^T e^{-\widetilde{M}} i\exp\{-(c_2+0.5a_{22}\sigma_2^2)u - \widetilde{M}\}du \\ &\geq im_{T,\widetilde{M}} \end{aligned} \tag{5.24}$$

for some positive constant $m_{T,\widetilde{M}}$ depending on T and $\widetilde{M}$. Combining (5.21), (5.22), and (5.24) we can easily find $\widehat{\delta}$ satisfying (5.20). ∎

LEMMA 5.4. *Let T^* be as in Proposition* 5.1. *Any set of the form $[0,H]\times[h,H]\times[0,H]$ for $0 < h < H < \infty$ is petite with respect to the Markov chain $\{(S(kT^*), I(kT^*), R(kT^*)), k \in \mathbb{Z}_+\}$. That is, there exists a nontrivial measure ν on $\mathbb{R}^{3,*}$ and a nonnegative sequence $\{a_n\}_{n=1}^\infty$ such that*

$$\sum_{n=1}^\infty a_n = 1 \textit{ and } \sum_{n=1}^\infty a_n \mathbb{P}_{s,i,r}\left\{(S(nT^*), I(nT^*), R(nT^*)) \in A\right\} \geq \nu(A)$$

for any Borel set $A \subset \mathbb{R}_+^{3,*}$ and $(s,i,r) \in [0,H]\times[h,H]\times[0,H]$.

Proof. Since the diffusion process $(S(t), I(t), R(t))$ is nondegenerate in $\mathbb{R}^{3,\circ}_+$, by [20, Lemma 3.6], the Markov chain $\{(S(kT^*), I(kT^*), R(kT^*)), k \in \mathbb{Z}_+\}$ is irreducible and aperiodic and any compact set of $\mathbb{R}^{3,\circ}_+$ is petite. By (2.4) and Lemma 5.3, there exist $\widehat{\delta} > 0$ and $\widehat{M} > 0$ such that

$$\mathbb{P}_{s,i,r}\left\{(S(T^*), I(T^*), R(T^*)) \in \mathcal{K}\right\} \geq \frac{1}{2},\ (s,i,r) \in [0,H]\times[h,H]\times[0,H], \tag{5.25}$$

where $\mathcal{K} = \{(s,i,r) : s\wedge i\wedge r \geq \widehat{\delta}, s\vee i\vee r \leq \widehat{M}\}$. Since $\mathcal{K}$ is a compact set in $\mathbb{R}^{3,\circ}_+$, it is petite, that is, there exists a nontrivial positive measure ν on $\mathbb{R}^{3,*}_+$ and a nonnegative sequence (a_n) with $\sum_{i=1}^\infty a_n = 1$ such that

$$\sum_{k=1}^\infty a_k \mathbb{P}_{s,i,r}\left\{(S(kT^*), I(kT^*), R(kT^*)) \in A\right\} \geq \nu(A) \tag{5.26}$$

for any Borel set $A \subset \mathbb{R}^{3,*}_+$ and $(s,i,r) \in \mathcal{K}$.

In view of (5.25), (5.26), and the Markov property of $\{(S(kT^*), I(kT^*), R(kT^*)), k \in \mathbb{Z}_+\}$, we have for any $(s,i,r) \in [0,H]\times[h,H]\times[0,H]$ that

$$\sum_{k=2}^\infty a_{k-1}\mathbb{P}_{s,i,r}\left\{(S(kT^*), I(kT^*), R(kT^*)) \in A\right\} \geq \frac{\nu(A)}{2} \text{ for any Borel set } A \subset \mathbb{R}^{3,*}_+, \tag{5.27}$$

which shows that $[0,H]\times[h,H]\times[0,H]$ is also a petite set. ∎

Finally we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let $n \in \mathbb{Z}$. By virtue of (2.3), there are $h_1 \in (0,1)$ and $\widehat{L} > 0$ satisfying

$$\begin{aligned} &\mathbb{E}_{s,i,r}(S(T^*) + I(T^*) + R(T^*)) \\ &\quad \leq (1-h_1)(s+i+r) + \widehat{L} \\ &\quad \leq (s+i+r) - h_1(s+i+r)^{\frac{n}{n+1}} + \widehat{L} + 1 \ \forall (s,i,r) \in \mathbb{R}^{3,*}_+. \end{aligned} \tag{5.28}$$

Let $V(s,i,r) = s+i+r+[\ln i]_-^{n+1}, n \geq 1$. In view of (5.28) and Proposition 5.1,

$$\mathbb{E}V(S(T^*), I(T^*), R(T^*)) \leq V(s,i,r) - 2h_2(V(s,i,r))^{\frac{n}{n+1}} + \widetilde{L} + \widehat{L} + 1 \ \forall (s,i,r) \in \mathbb{R}^{3,*}_+ \tag{5.29}$$

for some $\widehat{L} > 0$, $0 < 2h_2 < h_1$, which depend on n but are independent of (s,i,r). Let h be sufficiently small and H sufficiently large such that $h_2(V(s,i,r))^{\frac{n}{n+1}} \geq \widetilde{L} + \widehat{L} + 1$ for all $(s,i,r) \in \mathbb{R}^{3,*}_+ \setminus [0,H]\times[h,H]\times[0,H]$. Putting

$$H_2 := \sup_{(s,i,r)\in[0,H]\times[h,H]\times[0,H]} \left\{V(s,i,r) - 2h_2(V(s,i,r))^{\frac{n}{n+1}} + \widetilde{L} + \widehat{L} + 1\right\} < \infty,$$

we have from (5.29)

$$\begin{aligned} &\mathbb{E}_{s,i,r}V(S(T^*), I(T^*), R(T^*)) \\ &\quad \leq V(s,i,r) - h_2(V(s,i,r))^{\frac{n}{n+1}} + H_2\mathbf{1}_{\{(s,i,r)\in[0,H]\times[h,H]\times[0,H]\}}. \end{aligned} \tag{5.30}$$

Applying (5.30) and Lemma 5.4 to [26, Theorem 3.6], we obtain that

$$k^n\|P(kT^*, (s,i,r), \cdot) - \pi^*\| \to 0 \text{ as } k \to \infty \text{ for any } (s,i,r) \in \mathbb{R}^{3,*}_+ \tag{5.31}$$

for some invariant probability measure π^* of the Markov chain $\{S(kT^*),\ I(kT^*)),\ R(kT^*),\ k \in \mathbb{Z}_+\}$. Using the argument in the proof of [13, Theorem 2.2], we can show that π^* is also an invariant probability measure of the Markov process $\{(S(t), I(t), R(t)), t \geq 0\}$. Since $\|P(t,(s,i,r),\cdot) - \pi^*\|$ is decreasing in t, we easily deduce from (5.30) that $t^n\|P(t,(s,i,r),\cdot) - \pi^*\| \to 0$ as $t \to \infty$. Since the diffusion is nondegenerate in $\mathbb{R}^{3,\circ}_+$, supp $(\pi^*) = \mathbb{R}^{3,\circ}_+$ and the strong law of large numbers (2.10) is well-known. ∎

Appendix A. Proof of Theorem 2.1.

Proof of Theorem 2.1. The proof for the existence and uniqueness of global nonnegative solutions is standard, which is similar to [36, Theorem 2.1] or [42, Theorem 3.1]. The properties (2.1) and (2.2) can also be proved in the same manner as in [36, Lemma 2.1]. We now prove (2.3). Define $V(s,i,r) = (1+s+i+r)^{p+1}$ for some $p > 0$ to be chosen later. Then

$$\begin{aligned}\mathcal{L}V(s,i,r) &= (1+p)(1+s+i+r)^p\,(b+\beta_1 - \beta_1(1+s+i+r) - \beta_3 i)\\ &\quad + \frac{p(1+p)}{2}(1+s+i+r)^{p-1}(\sigma_1 s, \sigma_2 i, \sigma_3 r)\Sigma(\sigma_1 s, \sigma_2 i, \sigma_3 r)^\top\\ &\leq (1+p)(1+s+i+r)^p(b+\beta_1 - \beta_1(1+s+i+r))\\ &\quad + \kappa_\Sigma\frac{p(1+p)}{2}(1+s+i+r)^p\\ &\leq (1+p)(1+s+i+r)^p\left(b+\beta_1 - \left(\beta_1 - \kappa_\Sigma\frac{p}{2}\right)(1+s+i+r)\right)\\ &\leq C - (1+p)\frac{\beta_1}{4}V(s,i,r) \quad \forall\,(s,i,r) \in \mathbb{R}^3_+,\end{aligned}$$

where $\kappa_\Sigma = \sup_{(s,i,r)}\{\frac{(\sigma_1 s,\sigma_2 i,\sigma_3 r)\Sigma(\sigma_1 s,\sigma_2 i,\sigma_3 r)^\top}{(1+s+i+r)^2}\} < \infty$, $p > 0$ is sufficiently small so that $\kappa_\Sigma\frac{p}{2} < \frac{\beta_1}{2}$, and C is a sufficiently large positive number.

Let $\eta_k = \inf\{t \geq 0 : 1+S(t)+I(t)+R(t) \geq k\}$. Then $\eta_k \to \infty$ a.s. as $k \to \infty$. Applying Itô's formula for $V(s,i,r) = (1+s+i+r)^{p+1}$, we have

$$\begin{aligned}&\mathbb{E}_{s,i,r}\left[e^{\frac{\beta_1(1+p)}{4}(t\wedge\eta_k)}V(S(t\wedge\eta_k), I(t\wedge\eta_k), R(t\wedge\eta_k))\right]\\ &= V(s,i,r) + \mathbb{E}_{s,i,r}\int_0^{t\wedge\eta_k} e^{\frac{\beta_1(1+p)}{4}u}\left(\mathcal{L} + \frac{(1+p)\beta_1}{4}\right)V(S(u),I(u),R(u))du\\ &\leq V(s,i,r) + \mathbb{E}_{s,i,r}\int_0^{t\wedge\eta_k} e^{\frac{\beta_1(1+p)}{4}u}C du\\ &\leq V(s,i,r) + \frac{4C}{(1+p)\beta_1}e^{\frac{\beta_1(1+p)}{4}t}.\end{aligned} \tag{A.1}$$

Letting $k \to \infty$ in (A.1) and applying Fatou's lemma, we obtain

$$\mathbb{E}_{s,i,r}\left[e^{\frac{\beta_1(1+p)t}{4}}V(S(t),I(t),R(t))\right] \leq V(s,i,r) + \frac{4C}{(1+p)\beta_1}e^{\frac{\beta_1(1+p)}{4}t}.$$

As a result,

$$\mathbb{E}_{s,i,r}(1+S(t)+I(t)+R(t))^{1+p} \leq \frac{(1+s+i+r)^{1+p}}{e^{\frac{\beta_1(1+p)}{4}t}} + \frac{4C}{(1+p)\beta_1}.$$

This obviously implies (2.3).

On the other hand, for any $T > 0$, (A.1) implies that

$$\begin{aligned} k^{p+1}\mathbb{P}_{s.i.r}\{\eta_k \le T\} &= \mathbb{E}_{s,i,r}\Big[V(S(\eta_k), I(\eta_k), R(\eta_k))1_{\{\eta_k\le T\}}\Big] \\ &\le \mathbb{E}_{s,i,r}\Big[e^{\frac{\beta_1(1+p)}{4}\eta_k}V(S(\eta_k), I(\eta_k), R(\eta_k))1_{\{\eta_k\le T\}}\Big] \\ &\le V(s,i,r) + \frac{4C}{(1+p)\beta_1}e^{\frac{\beta_1(1+p)}{4}}T. \end{aligned}$$

Then (2.4) follows. The proof of this theorem is complete. ∎

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