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General Section

Twisted arithmetic Siegel Weil formula on $X_0(N)$ ☆

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In this paper, we study twisted arithmetic divisors on the modular curve $\mathcal{X}_0(N)$ with N square-free. For each pair (Δ, r), where $\Delta \equiv r^2 \mod 4N$ and Δ is a fundamental discriminant, we construct a twisted arithmetic theta function $\widehat{\phi}_{\Delta,r}(\tau)$ which is a generating function of arithmetic twisted Heegner divisors. We prove that the arithmetic pairing $\langle \widehat{\phi}_{\Delta,r}(\tau), \widehat{\omega}_N \rangle$ is equal to the special value, rather than the derivative, of some Eisenstein series, thanks to some cancellation, where $\widehat{\omega}_N$ is a normalized metric Hodge line bundle. We also prove the modularity of $\widehat{\phi}_{\Delta,r}(\tau)$.

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1. Introduction

Eisenstein series play an important role in arithmetic geometry and number theory. Kudla conjectured that the derivatives of Eisenstein series are closely related to arithmetic intersection numbers on Shimura varieties via a conjectural arithmetic version of the Siegel-Weil formula. In [KRY1], Kudla, Rapoport and Yang proved such a formula on a division Shimura curve. Kudla and Yang worked out a result on the modular curve $X_0(1)$ [Ya]. The associated Eisenstein series is Zagier's famous Eisenstein series [HZ] of weight 3/2. In [BF2], Bruinier and Funke gave another proof of the main formula in [Ya] via theta lifting. We extended the arithmetic Siegel-Weil formula to the modular curve $X_0(N)$ with N square free in [DY].

This paper is a sequel to [DY]. We replace the Heegner divisors by twisted Heegner divisors, which were studied by Bruinier and Ono in [BO]. Interestingly, the derivative aspect of the Eisenstein series disappears in such a case, thanks to some cancellation.

Let $N > 0$ be a positive integer, and let

$$V = \{w = \left(\begin{smallmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{smallmatrix}\right) \in M_2(\mathbb{Q}) : \mathrm{tr}(w) = 0\}, \tag{1.1}$$

with the quadratic form $Q(w) = N \det w = -N w_2 w_3 - N w_1^2$. Let

$$L = \left\{w = \begin{pmatrix} b & \frac{-a}{N} \\ c & -b \end{pmatrix} \in M_2(\mathbb{Z}) : \ a, b, c \in \mathbb{Z}\right\} \subset V, \tag{1.2}$$

and let $L^\sharp$ be its dual lattice. Then $\mathrm{SL}_2 \cong \mathrm{Spin}(V)$ acts on V by conjugation, i.e., $g \cdot w = gwg^{-1}$, and $\Gamma_0(N)$ acts on $L^\sharp/L$ trivially. Let $\mathbb{D}$ be the associated Hermitian domain of positive lines in $V_{\mathbb{R}}$, then $\mathbb{D}$ is isomorphic to upper half plane $\mathbb{H}$ via (2.2) which preserves the action of SL_2. We identify $X_0(N)$ with the standard compactification of $\Gamma_0(N)\backslash\mathbb{H}$.

For each $\mu \in L^\sharp/L$, let $L_\mu = \mu + L$ and

$$L_\mu[n] = \{w \in L_\mu : \ Q(w) = n\}.$$

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant which is a square modulo $4N$. Let $L^\Delta = \Delta L$ with quadratic form $Q_\Delta(x) = \frac{Q(x)}{|\Delta|}$, then its dual lattice is $L^{\Delta,\sharp} = L^\sharp$. Associated to Δ is a generalized 'genus character' $\chi_\Delta : L^\sharp/L^\Delta \to \{\pm 1\}$, (see Section 2.2 for details). For an integer r with $\Delta = r^2 \mod (4N)$, $\mu \in L^\sharp/L$, and a positive rational number $n \in \mathrm{sgn}(\Delta)Q(\mu) + \mathbb{Z}$, we define a twisted Heegner divisor

$$Z_{\Delta,r}(n, \mu) = \sum_{w \in \Gamma_0(N)\backslash L_{r\mu}[|\Delta| n]} \chi_\Delta(w) Z(w) \in \mathrm{Div}(X_0(N))_{\mathbb{Q}}, \tag{1.3}$$

which is defined over $\mathbb{Q}(\sqrt{\Delta})$. Here $Z(w) = \mathbb{R}w$ is the point on $X_0(N)$ given by the positive line $\mathbb{R}w$. This divisor depends only on the congruence class ($r \mod 2N$).

We will construct twisted Kudla's Green function $\Xi_{\Delta,r}(n,\mu,v)$ for $Z_{\Delta,r}(n,\mu)$ in Section 4. All these functions are smooth at the cusps and are different from the Green functions $\Xi(n,\mu,v)$ in [DY]. These functions are well-defined and smooth when $n \le 0$.

Now assume that N is square free. Let $\mathcal{X}_0(N)$ be the canonical integral model over $\mathbb{Z}$ of $X_0(N)$ as defined in [KM] (see Section 5). It is regular over $\mathbb{Z}$ and smooth over $\mathbb{Z}[\frac{1}{N}]$. We define twisted arithmetic divisors in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$ by

$$\widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v) = \begin{cases} (\mathcal{Z}_{\Delta,r}(n,\mu), \Xi_{\Delta,r}(n,\mu,v)) & \text{if } n>0, \\ (0, \Xi_{\Delta,r}(n,\mu,v)) & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\mathcal{Z}_{\Delta,r}(n,\mu)$ is the Zariski closure of $Z_{\Delta,r}(n,\mu)$ in $\mathcal{X}_0(N)$.

Definition 1.1. Define the formal generating series (twisted arithmetic theta function)

$$\widehat{\phi}_{\Delta,r}(\tau) = \sum_{\substack{n \equiv \mathrm{sgn}(\Delta) Q(\mu) \pmod{\mathbb{Z}} \\ \mu \in L^\sharp/L}} \widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v) q^n e_\mu, \tag{1.5}$$

which belongs to $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N)) \otimes \mathbb{C}[L^\sharp/L][[q,q^{-1}]]$. Here $q = e^{2\pi i \tau}$ and $v = \Im(\tau)$.

Let Γ' be the metaplectic cover of $\mathrm{SL}_2(\mathbb{Z})$ which acts on $\mathbb{C}[L^\sharp/L]$ via the Weil representation ρ_L via (2.3), and let $\{e_\mu : \mu \in L^\sharp/L\}$ be the standard basis of $\mathbb{C}[L^\sharp/L]$. Then

$$E_L(\tau,s) = \sum_{\gamma' \in \Gamma'_\infty \backslash \Gamma'} \left(v^{\frac{s-1}{2}} e_0\right) |_{3/2,\tilde{\rho}_L} \gamma'$$

is a vector valued Eisenstein series of weight 3/2. Here the Petersson slash operator is defined on functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$ by

$$\left(f\, |_{3/2,\tilde{\rho}_L} \gamma'\right)(\tau) = \phi(\tau)^{-3} \tilde{\rho}_L^{-1}(\gamma') f(\gamma\tau),$$

and $\gamma' = (\gamma, \phi) \in \Gamma'$. Define the normalized Eisenstein series [DY, equation (1.5)]

$$\mathcal{E}_L(\tau,s) = -\frac{s}{4}\pi^{-s-1}\Gamma(s)\zeta^{(N)}(2s) N^{\frac{1}{2}+\frac{3}{2}s} E_L(\tau,s), \tag{1.6}$$

where

$$\zeta^{(N)}(s) = \zeta(s) \prod_{p|N} (1-p^{-s}).$$

Let ω_N be the Hodge bundle on $\mathcal{X}_0(N)$, which we identify with the line bundle $\mathcal{M}_1(\Gamma_0(N))$ of modular forms of weight 1. For a modular form f of weight k, we define its normalized Petersson norm

$$\|f(z)\| = |f(z)(4\pi e^{-C}y)^{\frac{k}{2}}| \tag{1.7}$$

where $C = \frac{\log 4\pi+\gamma}{2}$. This induces a metrized Hodge bundle $\widehat{\omega}_N$ with $\widehat{\omega}_N^k \cong \widehat{\mathcal{M}}_k(\Gamma_0(N))$. These line bundles have log singularities as explained in Section 5.1, where we also explain the arithmetic intersection and degree used in the following theorem.

Theorem 1.2. *Let $\Delta \neq 1$ be a fundamental discriminant, then*

$$\deg \widehat{\phi}_{\Delta,r}(\tau) = 0 \tag{1.8}$$

and

$$\langle \widehat{\phi}_{\Delta,r}(\tau), \widehat{\omega}_N \rangle = \begin{cases} \frac{1}{\varphi(N)} \log(u_\Delta) h(\Delta) \mathcal{E}_L(\tau, 1) & \text{if } \Delta > 1, \\ 0, & \text{if } \Delta < 0. \end{cases} \tag{1.9}$$

Here deg *is defined by (5.6) and $\langle\ ,\ \rangle$ is the arithmetic intersection in the sense of Gillet-Soulé.*

It is interesting to compare it with the main result in [DY], which we state here for convenience.

Theorem 1.3. *[DY, Theorem 1.3] When $\Delta = 1$,*

$$\deg \widehat{\phi}_{\Delta,r}(\tau) = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1),$$

and

$$\langle \widehat{\phi}_{\Delta,r}(\tau), \widehat{\omega}_N \rangle = \frac{1}{\varphi(N)} \Big(\mathcal{E}'_L(\tau, 1) - \sum_{p|N} \frac{p}{p-1} \mathcal{E}_L(\tau, 1) \log p \Big).$$

Here is the basic idea in the proof of Theorem 1.2. For any $\Gamma_0(N)$ invariant function f, we define the twisted theta lift

$$I_{\Delta,r}(\tau, f) = \int_{X_0(N)} f(z) \Theta_{\Delta,r}(\tau, z), \tag{1.10}$$

when the integral is convergent. Here $\Theta_{\Delta,r}(\tau, z)$ is the twisted Kudla-Millson theta kernel defined by (2.13), following [AE, Section 4].

Define the normalized Eisenstein series of weight 0 as in [DY] by

$$\mathcal{E}(N, z, s) = N^{2s} \pi^{-s} \Gamma(s) \zeta^{(N)}(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\Im(\gamma z))^s.$$

We prove the following theorem in Section 3.

Theorem 1.4. *When Δ is a fundamental discriminant,*

$$I_{\Delta,r}(\tau, \mathcal{E}(N,z,s)) = \begin{cases} \Delta^{\frac{s}{2}}\Lambda(\varepsilon_\Delta, s)\mathcal{E}_L(\tau,s), & \text{if } \Delta > 0, \\ 0, & \text{if } \Delta < 0, \end{cases} \tag{1.11}$$

where $\Lambda(\varepsilon_\Delta, s) = L(\varepsilon_\Delta, s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}$ is the completed L-series associated to the character $\varepsilon_\Delta(n) = \left(\frac{\Delta}{n}\right)$.

Alfes and Ehlen studied the case when $N = 1$ in [AE, Theorem 6.1].

Taking the residues of both sides, we obtain the first identity in Theorem 1.2.

Combining the above theorem with the Kronecker limit formula for $\Gamma_0(N)$ [DY, Theorem 1.5], we obtain the following result, which is also the generalization of [AE, Theorem 6.2].

Theorem 1.5.

$$-\frac{1}{12}\mathrm{I}_{\Delta,r}(\tau, \log\|\Delta_N\|) = \begin{cases} \mathcal{E}_L'(\tau,1) & \text{if } \Delta = 1, \\ \log(u_\Delta)h(\Delta)\mathcal{E}_L(\tau,1) & \text{if } \Delta > 1, \\ 0, & \text{if } \Delta < 0, \end{cases} \tag{1.12}$$

where $u_\Delta > 1$ is the fundamental unit and $h(\Delta)$ is the class number of the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$.

The final step to prove Theorem 1.2 is to identify the metrized line bundle $\widehat{\omega_N}^k$ (with log singularity) with the explicit arithmetic divisor $\widehat{\mathrm{Div}}(\Delta_N)$ (with log-log singularity). That way, we can compute the arithmetic intersection, and relate it with the twisted theta lifting $I_{\Delta,r}(\log\|\Delta_N\|)$. This, combined with Theorem 1.5, proves Theorem 1.2. Finally, the same argument as in [DY, Section 8] using Theorem 1.2, gives the following modularity result.

Theorem 1.6. *$\widehat{\phi}_{\Delta,r}(\tau)$ is a vector valued modular form for Γ' of weight $\frac{3}{2}$, valued in $\mathbb{C}[L^\sharp/L] \otimes \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$.*

In [BO, Section 6], Bruinier and Ono proved that

$$A_{\Delta,r}(\tau) = \sum_{\mu\in L^\sharp/L}\sum_{n>0} Z_{\Delta,r}(n,\mu)q^n e_\mu$$

is a cusp form valued in $S_{\frac{3}{2},\tilde{\rho}_L} \bigotimes J(\mathbb{Q}(\sqrt{\Delta}))$, where J is the Jacobian of $X_0(N)$ and $q = e^{2\pi i\tau}$. Notice $A_{\Delta,r}(\tau)$ is the generic component of $\widehat{\phi}_{\Delta,r}(\tau)$. So modularity of $\widehat{\phi}_{\Delta,r}(\tau)$ (Theorem 1.6) is an integral version of their result.

This paper is organized as follows. We will introduce some notation and introduce the twisted Kudla-Millson theta function in Section 2. We will introduce the twisted theta lifting and use it to prove the Theorems 1.4 and 1.5 in Section 3. In Section 4, we define twisted Kudla's Green functions, and show these functions are smooth at cusps. In Section 5, we will define the arithmetic theta functions and prove Theorem 1.2 and Theorem 1.6.

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2. Basic set-up and theta lifting

Let V (resp. L) be the quadratic space (resp. even integral lattice) defined in the introduction. Then $\mathrm{SL}_2 \cong \mathrm{Spin}(V)$ acts on V by conjugation, i.e., $g \cdot w = gwg^{-1}$. Notice that $\Gamma_0(N)$ acts on $L^\sharp/L$ trivially. Let $\mathbb{D}$ be the Hermitian domain of positive real lines in $V_\mathbb{R}$:

$$\mathbb{D} = \{z \subset V_\mathbb{R} : \dim z = 1 \text{ and } (\ ,\ )\,|_z > 0\}.$$

The isomorphism between $\mathbb{H}$ and $\mathbb{D}$ is given by the map

$$z = x + iy \mapsto w(z) = \frac{1}{\sqrt{N}y}\begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix}. \tag{2.1}$$

The inverse is

$$w = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mapsto z(w) = \frac{2aN + \sqrt{D}}{2cN}, \quad D = -4NQ(w). \tag{2.2}$$

This isomorphism is $\mathrm{SL}_2(\mathbb{R})$-compatible and induces an isomorphism between $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ and $\Gamma_0(N)\backslash\mathbb{D}$. Let $X_0(N)$ be the standard compactification of $Y_0(N)$.

For $w \in V(\mathbb{Q})$ with $Q(w) > 0$, the Heegner point $Z(w)$ is the image of $z(w)$ in $X_0(N)$. When $Q(w) \le 0$, set $z(w) = 0$.

2.1. The Weil representation

Let $\mathrm{Mp}_{2,\mathbb{R}}$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$, which can be realized as pairs $(g, \phi(g,\tau))$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $\phi(g,\tau)$ is a holomorphic function of $\tau \in \mathbb{H}$ such that $\phi(g,\tau)^2 = j(g,\tau) = c\tau + d$. Let Γ' be the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_{2,\mathbb{R}}$ with two generators

$$S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \quad T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right).$$

We denote the standard basis of $\mathbb{C}[L^\sharp/L]$ by $\{e_\mu = L_\mu : \mu \in L^\sharp/L\}$. Then there is a Weil representation ρ_L of Γ' on $\mathbb{C}[L^\sharp/L]$ given by [Bor]

$$\begin{aligned} \rho_L(T)e_\mu &= e(Q(\mu))e_\mu, \\ \rho_L(S)e_\mu &= \frac{e(\frac{1}{8})}{\sqrt{|L^\sharp/L|}} \sum_{\mu' \in L^\sharp/L} e(-(\mu,\mu'))e_{\mu'}. \end{aligned} \tag{2.3}$$

This Weil representation ρ_L is closely related to the Weil representation ω of $\mathrm{Mp}_{2,\mathbb{A}}$ on $S(V_\mathbb{A})$ (see [BHY]).

2.2. Twisted Heegner divisors

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant which is a square modulo $4N$, and let $L^\Delta = \Delta L$ with renormalized quadratic form $Q_\Delta(w) = \frac{Q(w)}{|\Delta|}$. Then it is easy to check $L^{\Delta,\sharp} = L^\sharp$. Let Γ_Δ be the subgroup of $\Gamma_0(N)$ which acts on $L^{\Delta,\sharp}/L^\Delta$ trivially. It is not hard to check that the map

$$\chi_\Delta(\begin{pmatrix} \frac{b}{2N} & \frac{-a}{N} \\ c & -\frac{b}{2N} \end{pmatrix}) = \begin{cases} (\frac{\Delta}{n}), & \text{if } \Delta \mid b^2-4Nac \text{ and } \frac{b^2-4Nac}{\Delta} \text{ is a} \\ & \text{square modulo } 4N \text{ and } (a,b,c,\Delta)=1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

gives a well-defined map

$$\chi_\Delta : L^{\Delta,\sharp}/L^\Delta \to \{\pm 1\},$$

where n is any integer prime to Δ represented by one of the quadratic forms $[N_1a, b, N_2c]$ with $N_1N_2 = N$, N_1, $N_2 > 0$. Here we denote $[a,b,Nc] = ax^2 + bxy + Ncy^2$. Indeed, $\chi_\Delta(w) = \chi_\Delta([a,b,Nc])$ is the generalized genus character defined in [GKZ, Section 1] (see also [BO, Section 4]). We leave it to the reader to check that $\chi(w + L^\Delta) = \chi(w)$ and so χ_Δ induces a map on $L^{\Delta,\sharp}/L^\Delta$. It is known [GKZ] that the map is invariant under the action of $\Gamma_0(N)$ and the action of all Atkin-Lehner involutions, i.e.,

$$\chi_\Delta(\gamma w \gamma^{-1}) = \chi_\Delta(w) \text{ and } \chi_\Delta(w_M w w_M^{-1}) = \chi_\Delta(w), \tag{2.5}$$

where $\gamma \in \Gamma_0(N)$ and w_M is the Atkin-Lehner involution with $M \| N$.

Choose and fix $r \in \mathbb{Z}$ with $r^2 \equiv \Delta \mod 4N$. For any $\mu \in L^\sharp/L$ and a positive rational number $n \in \mathrm{sgn}(\Delta)Q(\mu) + \mathbb{Z}$, we define the twisted Heegner divisor by

$$Z_{\Delta,r}(n,\mu) := \sum_{w \in \Gamma_0(N)\backslash L_{r\mu}[n|\Delta|]} \chi_\Delta(w)Z(w) \in \mathrm{Div}(X_0(N))_\mathbb{Q}, \tag{2.6}$$

which is defined over $\mathbb{Q}(\sqrt{\Delta})$. Notice that we count each point $Z(w) = \mathbb{R}w$ with multiplicity $\frac{2}{|\Gamma_w|}$ in the orbifold $X_0(N)$, where Γ_w is the stabilizer of w in $\Gamma_0(N)$. So our definition is the same as that in [AE, Section 5] and [BO, Section 5].

Now define for $\delta \in L^{\Delta,\sharp}/L^{\Delta}$

$$Z_\Delta(n,\delta) := \sum_{w \in \Gamma_\Delta \backslash L_\delta^\Delta[n]} Z(w) \in \mathrm{Div}(X_{\Gamma_\Delta}). \tag{2.7}$$

Recall the natural map

$$\pi_{\Gamma_\Delta} : X_{\Gamma_\Delta} \longrightarrow X_0(N)$$

is a covering map with the degree $[\overline{\Gamma_0(N)} : \overline{\Gamma_\Delta}]$, where X_{Γ_Δ} is the modular curve $\Gamma_\Delta \backslash \mathbb{H}^*$, and $\bar{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$.

Lemma 2.1. *Let $n \equiv \mathrm{sgn}(\Delta) Q(\mu) (\mod \mathbb{Z})$ be a positive number, then*

$$\sum_{\substack{\delta \in L^{\Delta,\sharp}/L^\Delta \\ \delta \equiv r\mu (L)}} \chi_\Delta(\delta) Z_\Delta(n,\delta) = \pi_{\Gamma_\Delta}^*(Z_{\Delta,r}(n,\mu)), \tag{2.8}$$

where $\pi_{\Gamma_\Delta}^*$ is the pullback

$$\pi_{\Gamma_\Delta}^* : Z^1(X_0(N)) \longrightarrow Z^1(X_{\Gamma_\Delta}).$$

Proof. Write $\Gamma = \Gamma_0(N)$. For $w \in \Gamma \setminus L_{r\mu}[|\Delta| n]$, we have

$$\pi_{\Gamma_\Delta}^{-1}(Z(w)) = \{Z(w_1), ..., Z(w_g)\},$$

and then

$$\pi_{\Gamma_\Delta}^*(Z(w)) = Z(w_1) + ... + Z(w_g).$$

So we have

$$\begin{aligned}
\pi_{\Gamma_\Delta}^*(Z_{\Delta,r}(n,\mu)) &= \sum_{w \in \Gamma \backslash L_{r\mu}[|\Delta| n]} \chi_\Delta(w) \pi_{\Gamma_\Delta}^*(Z(w)) \\
&= \sum_{\substack{\delta \in L^\sharp/L^\Delta \\ \delta \equiv r\mu (L)}} \chi_\Delta(\delta) \sum_{w \in \Gamma_\Delta \backslash L_\delta^\Delta[n]} Z(w) \\
&= \sum_{\substack{\delta \in L^\sharp/L^\Delta \\ \delta \equiv r\mu (L)}} \chi_\Delta(\delta) Z_\Delta(n,\delta).
\end{aligned}$$

This proves the lemma. $\square$

2.3. Twisted Kudla-Millson theta function

For $z = x + iy \in \mathbb{H}$, recall $w(z) \in V_{\mathbb{R}}$ via (2.1). Let $w(z)^{\perp}$ be the orthogonal complement of $\mathbb{R}w(z)$ with the following orthogonal decomposition

$$\begin{aligned} V_{\mathbb{R}} &= \mathbb{R}w(z) \oplus w(z)^{\perp}, \\ w &= w_z + w_{z^{\perp}}. \end{aligned}$$

Following Kudla and Millson ([KMi], [BF2, Section 3]), define for $w \in V_{\mathbb{R}}$

$$R(w, z)_{\Delta} = -(w_{z^{\perp}}, w_{z^{\perp}})_{\Delta},$$

and the majorant

$$(w, w)_{\Delta, z} = (w_z, w_z)_{\Delta} + R(w, z)_{\Delta},$$

where $(,)_{\Delta} = \frac{(\ ,\ )}{|\Delta|}$ is the bilinear form associated to the quadratic form Q_{Δ}. One has

$$R(w, z)_{\Delta} = \frac{1}{2}(w, w(z))_{\Delta}^2 - (w, w)_{\Delta}. \tag{2.9}$$

For $w = \left(\begin{smallmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{smallmatrix}\right) \in V_{\mathbb{R}}$, we have

$$(w, w(z))_{\Delta} = -\frac{\sqrt{N}}{y\sqrt{|\Delta|}}(w_3 z\bar{z} - w_1(z + \bar{z}) - w_2). \tag{2.10}$$

Let

$$\varphi_{\Delta}^0(w, z) = \left((w, w(z))_{\Delta}^2 - \frac{1}{2\pi}\right) e^{-2\pi R(w,z)_{\Delta}} \mu(z)$$

and

$$\varphi_{\Delta}(w, \tau, z) = e(Q_{\Delta}(w)\tau)\varphi_{\Delta}^0(\sqrt{v}w, z),$$

be the differential forms on $V_{\mathbb{R}}$ valued in $\Omega^{1,1}(\mathbb{D})$, where $\mu(z) = \frac{dx\, dy}{y^2}$.

For any $\delta \in L^{\Delta,\sharp}/L^{\Delta}$, define

$$\Theta_{\delta}(\tau, z) = \sum_{w \in L_{\delta}^{\Delta}} \varphi_{\Delta}(w, \tau, z), \tag{2.11}$$

where $L_{\delta}^{\Delta} = \delta + L^{\Delta}$. Then

$$\Theta_{L^{\Delta}}(\tau, z) = \sum_{\delta \in L^{\Delta,\sharp}/L^{\Delta}} \Theta_{\delta}(\tau, z) e_{\delta} \tag{2.12}$$

is a vector valued Kudla-Millson theta function, which is a non-holomorphic modular form of weight $3/2$ of $(\Gamma', \rho_{L^\Delta})$ with respect to the variable τ with values in $\Omega^{1,1}(X_{\Gamma_\Delta})$.

Following Bruinier and Ono's work [BO], Alfes and Ehlen constructed a $\mathbb{C}[L^\sharp/L]-$valued twisted theta function [AE, Section 4]

$$\Theta_{\Delta,r}(\tau, z) := \sum_{\mu} \Theta_{\Delta,r,\mu}(\tau, z) e_\mu, \tag{2.13}$$

where $(L^\sharp = L^{\Delta,\sharp})$

$$\Theta_{\Delta,r,\mu}(\tau, z) = \sum_{\substack{\delta \in L^\sharp/L^\Delta \\ \delta \equiv r\mu(L) \\ Q_\Delta(\delta) \equiv \mathrm{sgn}(\Delta) Q(\mu) \pmod{\mathbb{Z}}}} \chi_\Delta(\delta) \Theta_\delta(\tau, z).$$

This twisted theta function has good transformation properties just like the classical Kudla-Millson theta functions.

Proposition 2.2. *[AE, Proposition 4.1] The theta function $\Theta_{\Delta,r}(\tau, z)$ is a non-holomorphic $\mathbb{C}[L^\sharp/L]$-valued modular form of weight $3/2$ for the representation $\tilde{\rho}_L$ in the variable τ. Furthermore, it is a non-holomorphic automorphic form of weight 0 for $\Gamma_0(N)$ in the variable $z \in \mathbb{D}$. Here*

$$\tilde{\rho}_L = \begin{cases} \rho_L & \text{if } \Delta > 0, \\ \bar{\rho}_L & \text{if } \Delta < 0. \end{cases} \tag{2.14}$$

3. Twisted theta lift

Following Alfes and Ehlen [AE], we consider the twisted theta lifting: for any $\Gamma_0(N)$-invariant function $f(z)$, the lifting is given by

$$I_{\Delta,r}(\tau, f) := \int_{X_0(N)} f(z) \Theta_{\Delta,r}(\tau, z) = \sum_{\mu \in L^\sharp/L} \int_{X_0(N)} f(z) \Theta_{\Delta,r,\mu}(\tau, z) e_\mu, \tag{3.1}$$

if the integral is convergent. Recall from [BF2, Proposition 4.1] that the theta series $\Theta_{\Delta,r}(\tau, z)$ decays like e^{-Cy^2} as y goes to the infinity (for some $C > 0$), and behaves similarly at other cusps. In particular, the twisted theta lifts of Eisenstein series $\mathcal{E}(N, z, s)$ and Petersson norm $\log \|\Delta_N\|$ are well-defined. We first recall a result of Alfes and Ehlen.

Proposition 3.1. *([Al, Proposition 3.1], [Eh]) Let $K = \mathbb{Z}$ with the quadratic form $Q(x) = -Nx^2$. Then $K^\sharp/K \cong L^\sharp/L$ as quadratic modules, and*

$$\Theta_{\Delta,r}(\tau,z) = -\bar{\epsilon}y\frac{N^{3/2}}{|\Delta|}\sum_{n=1}^{\infty} n^2(\frac{\Delta}{n})\sum_{\gamma\in\Gamma'_\infty\diagdown\Gamma'}$$
$$\times\Big[exp(-\pi\frac{y^2Nn^2}{v|\Delta|})v^{-3/2}\sum_{\lambda\in K^\sharp} e(|\Delta|Q(\lambda)\overline{\tau}-2N\lambda nx)e_{r\lambda}\Big]\big|_{3/2,\tilde{\rho}_K}\gamma dxdy.$$

Here $\epsilon = 1$ *is* $\Delta > 0$ *and* $\epsilon = i$ *if* $\Delta < 0$.

Now we are ready to prove Theorem 1.4 which we restate here for convenience. We follow the idea in the proof of [AE, Theorem 6.1], where the case $N = 1$ is considered.

Theorem 3.2. *When* Δ *is a fundamental discriminant,*

$$I_{\Delta,r}(\tau,\mathcal{E}(N,z,s)) = \begin{cases}\Delta^{\frac{s}{2}}\Lambda(\varepsilon_\Delta,s)\mathcal{E}_L(\tau,s), & \text{if } \Delta>0,\\ 0, & \text{if } \Delta<0,\end{cases} \tag{3.2}$$

where $\Lambda(\varepsilon_\Delta,s) = L(\varepsilon_\Delta,s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}$ *is the completed L-series associated to the character* $\varepsilon_\Delta(n) = \left(\frac{\Delta}{n}\right)$.

Proof. One has by Proposition 3.1,

$$\begin{aligned}&\Theta_{\Delta,r}(\tau,z)\\ &= -\bar{\epsilon}y\frac{N^{3/2}}{|\Delta|}\sum_{n=1}^{\infty}\sum_{\gamma\in\Gamma'_\infty\diagdown\Gamma'} n^2(\frac{\Delta}{n})\exp(-\pi\frac{y^2Nn^2\mid c\tau+d\mid^2}{v|\Delta|})v^{-3/2}\mid c\tau+d\mid^3\\ &\times(c\tau+d)^{-3/2}\sum_{\lambda\in K^\sharp} e(|\Delta|Q(\lambda)\overline{\gamma\tau}-2N\lambda nx)\tilde{\rho}_K^{-1}(\gamma)e_{r\lambda}dxdy\\ &= -\bar{\epsilon}\frac{yN^{3/2}}{v^{3/2}|\Delta|}\sum_{n=1}^{\infty}\sum_{\gamma\in\Gamma'_\infty\diagdown\Gamma'} n^2(\frac{\Delta}{n})\exp(-\frac{\pi y^2Nn^2\mid c\tau+d\mid^2}{v|\Delta|})\\ &\times(c\overline{\tau}+d)^{3/2}\sum_{\lambda\in K^\sharp} e(|\Delta|Q(\lambda)\overline{\gamma\tau}-2N\lambda nx)\tilde{\rho}_K^{-1}(\gamma)e_{r\lambda}dxdy.\end{aligned}$$

Here we used that fact that for every coprime pair (c,d), there is unique $\gamma = \left(\begin{smallmatrix}* & *\\ c & d\end{smallmatrix}\right) \in \Gamma'_\infty\diagdown\Gamma'$ associated to it.

Unfolding the twisted theta lifting integral, for $\Re(s) > 1$, we have

$$\begin{aligned}&I_{\Delta,r}(\tau,E(N,z,s))\\ &= \int\limits_{\Gamma_\infty\diagdown\mathbb{H}} \Theta_{\Delta,r}(\tau,z)y^s\\ &= -\bar{\epsilon}\frac{N^{\frac{3}{2}}}{v^{\frac{3}{2}}|\Delta|}\sum_{n=1}^{\infty} n^2(\frac{\Delta}{n})\sum_{\gamma\in\Gamma'_\infty\diagdown\Gamma'}(c\overline{\tau}+d)^{3/2}\int_0^\infty e^{(-\frac{\pi Ny^2n^2}{|\Delta|v}|c\tau+d|^2)}y^{s+1}dy\end{aligned}$$

$$\times\tilde{\rho}_K^{-1}(\gamma)\int_0^1 \sum_{\lambda\in K^\sharp} e(|\Delta|Q(\lambda)\overline{\gamma\tau}-2N\lambda nx)e_{r\lambda}dx.$$

Notice that

$$\int_0^1 \sum_{\lambda\in K^\sharp} e(|\Delta|Q(\lambda)\overline{\gamma\tau}-2N\lambda nx)e_{r\lambda}dx=e_0.$$

So we have

$$\begin{aligned}
&I_{\Delta,r}(\tau,E(N,z,s))\\
&=-\frac{N^{\frac{3}{2}}\bar{\epsilon}}{2v^{\frac{3}{2}}|\Delta|}\sum_{n=1}^{\infty}n^2\big(\frac{\Delta}{n}\big)\sum_{\gamma\in\Gamma'_\infty\setminus\Gamma'}\frac{v^{\frac{s+2}{2}}|\Delta|^{\frac{s+2}{2}}(c\overline{\tau}+d)^{3/2}\Gamma\big(\frac{s}{2}+1\big)}{\pi^{\frac{s+2}{2}}\mid c\tau+d\mid^{s+2}N^{\frac{s+2}{2}}n^{s+2}}\tilde{\rho}_K^{-1}(\gamma)e_0\\
&=-\frac{\bar{\epsilon}|\Delta|^{\frac{s}{2}}L(\varepsilon_\Delta,s)\Gamma\big(\frac{s}{2}+1\big)}{2N^{\frac{s-1}{2}}\pi^{\frac{s+2}{2}}}\sum_{\gamma\in\Gamma'_\infty\setminus\Gamma'}\frac{v^{\frac{s-1}{2}}(c\overline{\tau}+d)^{3/2}}{\mid c\tau+d\mid^{s+2}}\tilde{\rho}_K^{-1}(\gamma)e_0\\
&=-\frac{\bar{\epsilon}|\Delta|^{\frac{s}{2}}L(\varepsilon_\Delta,s)\Gamma\big(\frac{s}{2}+1\big)}{2N^{\frac{s-1}{2}}\pi^{\frac{s+2}{2}}}E_L(\tau,s).
\end{aligned}$$

Here

$$E_L(\tau,s)=\sum_{\gamma'\in\Gamma'_\infty\setminus\Gamma'}\big(v^{\frac{s-1}{2}}e_0\big)\mid_{3/2,\tilde{\rho}_L}\gamma',$$

is the Eisenstein series defined in the introduction. When $\Delta>0$, we obtain

$$I_{\Delta,r}(\tau,\mathcal{E}(N,z,s))=\Delta^{\frac{s}{2}}\Lambda(\varepsilon_\Delta,s)\mathcal{E}_L(\tau,s).$$

When $\Delta<0$, a simple calculation gives

$$E_L(\tau,s)\mid_{\frac{3}{2},\bar{\rho}_L}Z=-E_L(\tau,s),$$

for $Z=\left(\begin{pmatrix}-1&\\&-1\end{pmatrix},i\right)$. On the other hand, the modularity of Eisenstein series implies

$$E_L(\tau,s)\mid_{\frac{3}{2},\bar{\rho}_L}Z=E_L(\tau,s).$$

So $E_L(\tau,s)=0$. $\square$

Taking residue of both sides of the equation (3.2) at $s=1$, we have the following result.

Corollary 3.3.

$$\mathrm{I}_{\Delta,r}(\tau,1)=\begin{cases}\frac{2}{\varphi(N)}\mathcal{E}_L(\tau,1) & \textit{if } \Delta=1,\\ 0 & \textit{if } \Delta\neq 1.\end{cases}\tag{3.3}$$

Recall the modular form $\Delta_N(z)$ of weight $k=12\varphi(N)$ for $\Gamma_0(N)$ defined in [DY, (1.6)]:

$$\Delta_N(z)=\prod_{t|N}\Delta(tz)^{a(t)}\tag{3.4}$$

with

$$a(t)=\sum_{r|t}\mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(\frac{N}{r})},$$

where $\mu(n)$ is the Möbius function and $\varphi(N)$ is the Euler function.

For a modular form f of weight k and level N, recall its normalized Petersson metric

$$\|f(z)\|=|f(z)(4\pi e^{-C}y)^{\frac{k}{2}}|\tag{3.5}$$

where $C=\frac{\log 4\pi+\gamma}{2}$ with Euler constant γ.

Theorem 3.4. *Let the notation be as above, one has*

$$-\frac{1}{12}\mathrm{I}_{\Delta,r}(\tau,\log\|\Delta_N\|)=\begin{cases}\mathcal{E}'_L(\tau,1) & \textit{if } \Delta=1,\\ \log(u_\Delta)h(\Delta)\mathcal{E}_L(\tau,1) & \textit{if } \Delta>1,\\ 0, & \textit{if } \Delta<0,\end{cases}\tag{3.6}$$

where u_Δ is the fundamental unit and $h(\Delta)$ is the class number of the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$.

Proof. By the Kronecker limit formula for $\Gamma_0(N)$ [DY, Theorem 1.5]

$$\lim_{s\to 1}\Big(\mathcal{E}(N,z,s)-\varphi(N)\zeta^*(2s-1)\Big)=-\frac{1}{12}\log\big(y^{6\varphi(N)}\mid\Delta_N(z)\mid\big),$$

we have

$$\begin{aligned}&-\frac{1}{12}I_{\Delta,r}(\tau,\log\mid\Delta_N(z)y^{6\varphi(N)}\mid)\\ &=\lim_{s\to 1}\big(I_{\Delta,r}(\tau,\mathcal{E}(N,z,s))-\varphi(N)\zeta^*(2s-1)I_{\Delta,r}(\tau,1)\big).\end{aligned}$$

Here $\zeta^*(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Now the theorem follows from Theorem 3.2, Corollary 3.3, and the class number formula. □

4. Twisted Kudla's Green function

Following Kudla's methods in [Ku1], we construct a twisted Kudla's Green function for $Z_{\Delta,r}(n,\mu)$ and study its properties in this section.

For $r>0$ and $s\in\mathbb{R}$, let

$$\beta_s(r)=\int_1^\infty e^{-rt}t^{-s}dt, \tag{4.1}$$

and

$$\xi_\Delta(w,z)=\beta_1(2\pi R(w,z)_\Delta). \tag{4.2}$$

Definition 4.1 *(Twisted Kudla's Green functions).* For $n\in\operatorname{sgn}(\Delta)Q(\mu)+\mathbb{Z}$, define

$$\Xi_{\Delta,r}(n,\mu,v)(z)=\sum_{\substack{\delta\in L^\sharp/L^\Delta\\ \delta\equiv r\mu(L)\\ Q_\Delta(\delta)=n}}\chi_\Delta(\delta)\Xi_{L^\Delta}\big(n,\delta,v\big)(z), \tag{4.3}$$

where $\Xi_{L^\Delta}\big(n,\delta,v\big)(z)$ is Kudla's Green function associated to the lattice L^Δ with quadratic form Q_Δ given by

$$\Xi_{L^\Delta}\big(n,\delta,v\big)(z)=\sum_{0\neq w\in L^\Delta_\delta[n]}\xi_\Delta(\sqrt{v}w,z). \tag{4.4}$$

So one has

$$\Xi_{\Delta,r}(n,\mu,v)(z)=\sum_{0\neq w\in L_{r\mu}[n|\Delta|]}\chi_\Delta(w)\xi_\Delta(\sqrt{v}w,z), \tag{4.5}$$

which is clearly invariant under $\Gamma_0(N)$. Recall from Lemma 2.1

$$\sum_{\substack{\delta\in L^\sharp/L^\Delta\\ \delta\equiv r\mu(L)}}\chi_\Delta(\delta)Z_\Delta(n,\delta)=\pi^*_{\Gamma_\Delta}(Z_{\Delta,r}(n,\mu)).$$

Since $\Xi_{L^\Delta}(n,\delta,v)$ is a Green function for $Z_\Delta(n,\delta)$ on $X_{\Gamma_\Delta}=\Gamma_\Delta\backslash\mathbb{H}$ by [Ku1, Section 12], we have the following lemma.

Lemma 4.2. *When $n>0$, $\Xi_{\Delta,r}(n,\mu,v)(z)$ is a Green function for $Z_{\Delta,r}(n,\mu)$ on $Y_0(N)=\Gamma_0(N)\backslash\mathbb{H}$, and satisfies the following current equation,*

$$dd^c[\Xi_{\Delta,r}(n,\mu,v)(z)]+\delta_{Z_{\Delta,r}(n,\mu)}=[\omega_{\Delta,r}(n,\mu,v)], \tag{4.6}$$

where $\omega_{\Delta,r}(n,\mu,v)$ is the differential form

$$\omega_{\Delta,r}(n,\mu,v) = \sum_{w\in L_\mu[n|\Delta|]} \chi_\Delta(w)\varphi^0_\Delta(w,z). \tag{4.7}$$

In particular, when $n \le 0$, $\Xi_{\Delta,r}(n,\mu,v)(z)$ is smooth on $Y_0(N)$.

Now we consider the behavior of these Green functions at the cusps. When $D = -4N|\Delta|n$ is not a square, the Green function $\Xi_{\Delta,r}(n,\mu,v)(z)$ is smooth at the cusps by [DY, Theorem 5.1]. For the rest of this section, we assume that $D = -4N|\Delta|n$ is a square.

Let $\mathrm{Iso}(V)$ be the set of isotropic lines $\ell = \mathbb{Q}w$ of V. Two isotropic lines ℓ_1 and ℓ_2 are equivalent if there is some $\gamma \in \Gamma_0(N)$ with $\gamma\ell_1 = \ell_2$. Given $\ell = \mathbb{Q}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{Iso}(V)$, let $P_\ell = \frac{a}{c}$ be the associated cusp, which depends only on the equivalence class of isotropic line ℓ.

Let $\ell_\infty = \mathbb{Q}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right) \in \mathrm{Iso}(V)$, and $P_\infty = \infty$ be the associated cusp. In general, for an isotropic line ℓ, there exists $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\ell_\infty = \sigma_\ell \cdot \ell$. Then

$$\sigma_\ell \Gamma_\ell \sigma_\ell^{-1} = \{\pm \left(\begin{smallmatrix} 1 & m\kappa_\ell \\ 0 & 1\end{smallmatrix}\right); m \in \mathbb{Z}\},$$

where $\Gamma_\ell \subseteq \Gamma_0(N)$ is the stabilizer of ℓ and $\kappa_\ell > 0$ is the classical width of the associated cusp P_ℓ. On the other hand, there is another positive number $\beta_\ell > 0$, depending on L and the cusp P_ℓ, such that $\left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0\end{smallmatrix}\right)$ is a primitive element in $\ell_\infty \bigcap \sigma_\ell \cdot L$. We call $\varepsilon_\ell = \frac{\kappa_\ell}{\beta_\ell}$ the Funke constant at cusp P_ℓ, which is defined in [Fu, Section 3]. We will denote $\kappa = \kappa_\infty$.

For $w \in L^\sharp$, $w^\perp$ is a split space if and only if $-4N|\Delta|Q_\Delta(w)$ is square. For any $\ell \in \mathrm{Iso}(V)$, let $\delta_{w,\ell}$ be the number of isotropic lines $\ell_w \in \mathrm{Iso}(V)$ which are perpendicular to w and equivalent to ℓ. We often drop the index ℓ when $\ell = \ell_\infty$.

Lemma 4.3. *Let $D = -4Nn|\Delta| > 0$ be a square integer, then one has*

$$\sum_{w\in\Gamma_0(N)\backslash L_\mu[n|\Delta|]} \delta_w \chi_\Delta(w) = 0. \tag{4.8}$$

Proof. For any $w \in L_\mu[n|\Delta|]$ with $(w, \ell_\infty) = 0$, one has

$$w = w(a,b) = \begin{pmatrix} \frac{a}{2N} & \frac{b}{N} \\ & -\frac{a}{2N}\end{pmatrix}, a^2 = D \text{ and } a \equiv r_0 \mod (2N), \tag{4.9}$$

where $\mu = \begin{pmatrix} \frac{r_0}{2N} & \\ & -\frac{r_0}{2N}\end{pmatrix}$. It is known that $w(a,b)$ is $\Gamma_0(N)$ equivalent to $w(a,b')$ if and only if $b \equiv b' \mod a$.

When $2\mu \notin L$, let a be the unique square root of D satisfying $a \equiv r_0 \mod (2N)$. By [DY, Lemma 6.2], the representatives for $\Gamma_0(N) \setminus L_\mu[n|\Delta|]$ are given by

$$\left\{ \begin{pmatrix} \frac{a}{2N} & \frac{b}{N} \\ & -\frac{a}{2N}\end{pmatrix} \mid 0 \le b < \sqrt{D}\right\},$$

and $\delta_{w(a,b)} = 1$. Since $-b$ is represented by the quadratic form $-bx^2 + axy$, we have

$$\chi_\Delta\left(\begin{pmatrix} \frac{a}{2N} & \frac{b}{N} \\ & -\frac{a}{2N} \end{pmatrix}\right) = (\frac{\Delta}{-b}).$$

So

$$\sum_{w\in\Gamma_0(N)\backslash L_\mu[n|\Delta|]} \delta_w \chi_\Delta(w) = \sum_{b=0}^{\sqrt{D}-1} \left(\frac{\Delta}{-b}\right) = 0.$$

When $2\mu \in L$, the equation (4.9) has two solutions $a = \pm\sqrt{D}$. The set of representatives is a subset of

$$S := \left\{ \begin{pmatrix} \pm\frac{\sqrt{D}}{2N} & \frac{b}{N} \\ & \mp\frac{\sqrt{D}}{2N} \end{pmatrix} \mid 0 \le b < \sqrt{D} \right\}.$$

[DY, Lemma 6.2] asserts that

$$w(a,b) \text{ is } \Gamma_0(N) \text{ equivalent to } w(-a,b') \Longleftrightarrow \delta_{w(a,b)} = \delta_{w(-a,b')} = 2.$$

Moreover, there is the unique one $w(-a,b') \in S$ equivalent to $w(a,b)$ when $\delta_{w(a,b)} = 2$. So

$$\begin{aligned} R := &\left\{w(\sqrt{D},b) \mid 0 \le b < \sqrt{D}\right\} \\ &\cup \left\{w(-\sqrt{D},b') \mid \delta_{w(-\sqrt{D},b')} = 1,\ 0 \le b' < \sqrt{D}\right\}, \end{aligned} \tag{4.10}$$

is a set of representatives for $\Gamma_0(N) \setminus L_\mu[n|\Delta|]$. Clearly, one has $S = R \cup B$ with

$$B = \left\{w(-\sqrt{D},b') \mid \delta_{w(-\sqrt{D},b')} = 2,\ 0 \le b' < \sqrt{D}\right\}.$$

So we have

$$\begin{aligned} \sum_{w\in\Gamma_0(N)\backslash L_\mu[n|\Delta|]} \delta_w \chi_\Delta(w) &= \sum_{w\in R} \delta_w \chi_\Delta(w) \\ &= \sum_{\substack{w\in R \\ \delta_w=2}} 2\chi_\Delta(w) + \sum_{\substack{w\in R \\ \delta_w=1}} \chi_\Delta(w) \\ &= \sum_{\substack{w\in R \\ \delta_w=2}} \chi_\Delta(w) + \sum_{\substack{w\in B \\ \delta_w=2}} \chi_\Delta(w) + \sum_{\substack{w\in R \\ \delta_w=1}} \chi_\Delta(w) \\ &= \sum_{w\in S} \chi_\Delta(w) = 2\sum_{b=0}^{\sqrt{D}-1} \left(\frac{\Delta}{-b}\right) = 0. \end{aligned}$$

This proves the lemma. □

The main purpose of this section is to prove the following result.

Theorem 4.4. *Let the notation be as above and* $n \in \operatorname{sgn}(\Delta)Q(\mu)+\mathbb{Z}$, *and assume that* N *is square free. Then* $\Xi_{\Delta,r}(n,\mu,v)(z)$ *is smooth and vanishes at the cusps:*

$$\lim_{q_\ell \to 0} \Xi_{\Delta,r}(n,\mu,v)(z) = 0,$$

where q_ℓ *is a local parameter at the cusp* P_ℓ *associated to* $\ell \in \operatorname{Iso}(V)$.

Proof. Let $D = -4N|\Delta|n$ and we split the proof into three cases: D is not a square, $D > 0$ is a square and $D = 0$.

Case 1: We first assume that D is not a square. This case follows directly from the result [DY, Theorem 5.1].

Case 2: Next we assume that $D > 0$ is a square. We first work on X_{Γ_Δ} as $\Xi_{L^\Delta}(n,\delta,v)$ is defined over X_{Γ_Δ}. Let P_ℓ be the cusp associated to an isotropic line $\ell \in \operatorname{Iso}(V)$. We have, near the cusp P_ℓ, by [DY, Theorem 5.1] and equation (4.3)

$$\Xi_{\Delta,r}(n,\mu,v)(z) = -g_{\Delta,r}(n,\mu,v,P_\ell)\log \mid q_\ell \mid^2 -2\psi_\Delta(n,\mu,v;q_\ell), \tag{4.11}$$

where $\psi_\Delta(n,\mu,v;q_\ell)$ is a smooth function of q_ℓ (as a function of two real variables q_ℓ and $\bar{q}_\ell$) and

$$\lim_{q_\ell \to 0} \psi_\Delta(n,\mu,v;q_\ell) = 0.$$

Here

$$g_{\Delta,r}(n,\mu,v,P_\ell) = \frac{1}{8\pi\sqrt{-nv}}\beta_{3/2}(-4\pi nv)\alpha_{\Delta,r}(n,\mu,P_\ell),$$
$$\alpha_{\Delta,r}(n,\mu,P_\ell) = \sum_{w \in \Gamma_0(N)\backslash L_{r\mu}[n|\Delta|]} \chi_\Delta(w)\delta_{w,\ell}$$

and $0 \le \delta_{w,\ell} \le 2$ is the number of isotropic lines $\ell_w \in \operatorname{Iso}(V)$ which are perpendicular to w and belong to the same cusp as ℓ. Notice that all terms in (4.11) can be descended to the modular curve $X_0(N)$.

Since N is square-free, the Atkin-Lehner involutions act on the cusps of $X_0(N)$ transitively. It is known that χ_Δ is invariant under the Atkin-Lehner involutions. So

$$\begin{aligned}\alpha_{\Delta,r}(n,\mu,P_\ell) &= \sum_{w \in \Gamma_0(N)\backslash L_{r\mu}[n|\Delta|]} \chi_\Delta(w)\delta_{w,\ell} \\ &= \sum_{w \in \Gamma_0(N)\backslash L_{r\sigma_\ell\cdot\mu}[n|\Delta|]} \chi_\Delta(w)\delta_{w,\ell_\infty},\end{aligned}$$

which is zero by Lemma 4.3. Here $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ is an Atkin-Lehner operator such that $\sigma_\ell \cdot \ell = \ell_\infty$ and $L_{r\sigma_\ell\cdot\mu} = L + r\sigma_\ell \cdot \mu$.

Case 3: We finally assume $D = 0$, i.e., $n = 0$. In this case, only the terms with $r\mu = 0$ contribute. We have around the cusp P_ℓ, by [DY, Theorem 5.1],

$$\begin{aligned}
&\Xi_{\Delta,r}(0,\mu,v) \\
=& -\sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta)\left(\frac{\varepsilon_\ell}{2\pi\sqrt{vN}}(\log|q_\ell|^2) + 2\log(-\log|q_\ell|^2)\right) \\
&-2\sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta)\psi_\ell(0,\delta,v;q_\ell).
\end{aligned} \tag{4.12}$$

Here ε_ℓ is the Funke constant at ℓ and $\psi_\ell(0,\delta,v;q_\ell)$ is a smooth function of q_ℓ, and

$$\lim_{q_\ell\to 0}\psi_\ell(0,\delta,v;q_\ell) = \begin{cases} a_\ell & \text{if } \delta\in L^\Delta \\ b_\ell & \text{if } \delta\notin L^\Delta \end{cases} \tag{4.13}$$

for some constant a_ℓ and b_ℓ. When $\delta\in L^\Delta$, $\chi_\Delta(\delta)=0$, so we have

$$\lim_{q_\ell\to 0}\sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta)\psi_\ell(0,\delta,v;q_\ell) = \sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta)b_\ell. \tag{4.14}$$

Combining it with equation (4.12), it suffices to show

$$\sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta) = 0. \tag{4.15}$$

We assume that $\ell\cap L = \mathbb{Z}\lambda_\ell$. Then the representatives for all $\delta\in L/L^\Delta$ with $\ell\cap L_\delta^\Delta\neq\phi$ are given by

$$\{m\lambda_\ell \mid m = 0,1,...,\Delta-1\}.$$

One has

$$\sum_{\substack{\delta\in L/L^\Delta \\ \ell\cap L_\delta^\Delta\neq\phi}} \chi_\Delta(\delta) = \sum_{m=0}^{\Delta-1}\chi_\Delta(m\lambda_\ell) = 0.$$

This proves that $\Xi_{\Delta,r}(0,\mu,v)$ is smooth around all cusps P_ℓ and goes to zero when $q_\ell\to 0$. □

Let

$$Z_{\Delta,r}(n,\mu)=\begin{cases}Z_{\Delta,r}(n,\mu) & \text{if } n>0,\\ 0 & \text{if otherwise.}\end{cases}$$

Corollary 4.5. *Let the notation and assumption be as in Theorem 4.4. Then $\Xi_{\Delta,r}(n,\mu,v)$ is a Green function for $Z_{\Delta,r}(n,\mu)$ on $X_0(N)$ in the usual Gillet-Soulé sense, i.e.,*

$$dd^c[\Xi_{\Delta,r}(n,\mu,v)]+\delta_{Z_{\Delta,r}(n,\mu)}=[\omega_{\Delta,r}(n,\mu,v)].$$

5. Twisted arithmetic theta function

In this section, we assume that N is square free. Following [KM], let $\mathcal{Y}_0(N)$ ($\mathcal{X}_0(N)$) be the moduli stack over $\mathbb{Z}$ of cyclic isogenies of degree N of elliptic curves (generalized elliptic curves) $\pi: E\to E'$, such that $\ker\pi$ meets every irreducible component of each geometric fiber. The stack $\mathcal{X}_0(N)$ is regular, proper, and flat over $\mathbb{Z}$ and $\mathcal{X}_0(N)(\mathbb{C})=X_0(N)$. It is a DM-stack. It is smooth over $\mathbb{Z}[\frac{1}{N}]$.

When $p|N$, the special fiber $\mathcal{X}_0(N) \pmod p$ has two irreducible components $\mathcal{X}_p^\infty$ and $\mathcal{X}_p^0$. Let $\mathcal{X}_p^\infty(\mathcal{X}_p^0)$ be the component which contains the cusp $\mathcal{P}_\infty \pmod p(\mathcal{P}_0 \pmod p)$. Here $\mathcal{P}_\infty$ and $\mathcal{P}_0$ are the Zariski closure of the cusps infinity and zero. Let $\mathcal{Z}_{\Delta,r}(n,\mu)$ be the Zariski closure of $Z_{\Delta,r}(n,\mu)$.

We define an arithmetic divisor in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$ in the sense of Gillet-Soulé by

$$\widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v)=(\mathcal{Z}_{\Delta,r}(n,\mu),\Xi_{\Delta,r}(n,\mu,v)). \tag{5.1}$$

The twisted arithmetic theta function ($q=e^{2\pi i\tau}$) is defined to be

$$\widehat{\phi}_{\Delta,r}(\tau)=\sum_{\substack{n\equiv \mathrm{sgn}(\Delta)Q(\mu) \pmod{\mathbb{Z}}\\ \mu\in L^\sharp/L}}\widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v)q^n e_\mu. \tag{5.2}$$

5.1. The metrized Hodge bundle

Let ω_N be the Hodge bundle (see [KM]) and $\mathcal{M}_k(\Gamma_0(N))$ be the line bundle of weight k modular form on $\mathcal{X}_0(N)$. It is known that $\omega_N^2\cong\Omega_{\mathcal{X}_0(N)/\mathbb{Z}}(-S)\cong M_2(\Gamma_0(N))$, where S is the set of the cusps. The normalized Petersson metric for modular forms given in (1.7) induces a metrized line bundle $\widehat{\omega}_N^k\cong\widehat{\mathcal{M}_k}(\Gamma_0(N))$ with 'log singularity' along cusps. Indeed, let $k=12\varphi(N)$, the modular form $\Delta_N(z)$ of weight k, given by (3.4), is a section of $\mathcal{M}_k(\Gamma_0(N))$, and has log singularity at all cusps with index $\alpha=k/2$ in the following sense. For example at cusp P_∞, $q=e(z)$ is a local parameter, and we have

$$\|\Delta_N(z)\|=(-\log|q_z|^2)^{\frac{k}{2}}|q_z|^{\frac{r}{12}k}\varphi(q_z),$$

with (here C is the normalization constant in (1.7))

$$\varphi(q_z) = e^{-\frac{kC}{2}} \prod_{n=1}^{\infty} |(1-q_z)^{24C_N(n)}|.$$

In the notation of [Kü1] (see also [DY, Part II]), one has only $\widehat{\omega}_N \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N), S)$. We recall the following fact from [DY, Section 6].

Lemma 5.1. *Let $k = 12\varphi(N)$. Under the isomorphism $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathcal{X}_0(N),S) \cong \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N),S)$, one has $\widehat{\omega_N}^k \cong \widehat{\mathrm{Div}}(\Delta_N) = (\mathrm{Div}\,\Delta_N, -\log\|\Delta_N\|^2)$ with*

$$\mathrm{Div}\,\Delta_N = \frac{rk}{12}\mathcal{P}_\infty - k\sum_{p|N} \frac{p}{p-1}\mathcal{X}_p^0, \tag{5.3}$$

and

$$r = N\prod_{p|N}(1+p^{-1}) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)].$$

Here $\mathcal{P}_\infty$ is the Zariski closure of the cusp ∞.

5.2. Proof of Theorems 1.2 and 1.6

The arithmetic intersection theory of Gillet-Soulé has been extended to the case of arithmetic divisors with 'log-log' singularities and metrized line bundles with 'log'-singularities, see ([BKK], [Kü1], [Kü2]), in particular [Kü1, Proposition 1.4]). In particular, if $\widehat{\mathcal{Z}_1} = (\mathcal{Z}_1, g_1) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$ and $\widehat{\mathcal{Z}_2} = (\mathcal{Z}_2, g_2) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N), S)$ intersect properly, we have

$$\langle \widehat{\mathcal{Z}_1}, \widehat{\mathcal{Z}_2} \rangle = (\mathcal{Z}_1.\mathcal{Z}_2)_{fin} + \frac{1}{2} g_1 * g_2, \tag{5.4}$$

with

$$g_1 * g_2 = g_1(Z_2) + \int_{X_0(N)} g_2\omega_1. \tag{5.5}$$

Here ω_1 is a smooth $(1,1)$ form satisfying the current equation

$$dd^c g_1 + \delta_{Z_1} = [\omega_1].$$

We define

$$\deg \widehat{\mathcal{Z}} = \langle \widehat{\mathcal{Z}}, (0,2) \rangle = \int_{X_0(N)} \omega \tag{5.6}$$

to be the degree of $\widehat{\mathcal{Z}}$. When $\widehat{\mathcal{Z}} \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$, the degree of $\widehat{\mathcal{Z}}$ is just the degree of its generic fiber $\mathcal{Z}(\mathbb{C})$.

Firstly, we prove the following proposition which is an analogue of [DY, Proposition 6.7], although the proof is more involved.

Proposition 5.2. *Let $\Delta \neq 1$ be a fundamental discriminant, for every prime $p|N$, one has*

$$\langle \widehat{\phi}_{\Delta,r}(\tau), \mathcal{X}_p^0 \rangle = \langle \widehat{\phi}_{\Delta,r}(\tau), \mathcal{X}_p^\infty \rangle = 0.$$

Proof. The proof for $\Delta > 0$ is similar to that of [DY, Proposition 6.8] and is left to the reader.

We now assume $\Delta < 0$. First notice that

$$\chi_\Delta(-w) = \mathrm{sgn}(\Delta)\chi_\Delta(w) \text{ and } w_N^*(\widehat{\phi}_{\Delta,r}(\tau)) = \mathrm{sgn}(\Delta)\widehat{\phi}_{\Delta,r}(\tau).$$

It is known that $w_N^* \mathcal{X}_p^0 = \mathcal{X}_p^\infty$ with $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Since w_N is an isomorphism, we have

$$\langle \widehat{\phi}_{\Delta,r}(\tau), \mathcal{X}_p^0 \rangle = -\langle \widehat{\phi}_{\Delta,r}(\tau), \mathcal{X}_p^\infty \rangle.$$

So

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0 \rangle_p = -\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^\infty \rangle_p. \tag{5.7}$$

Let $\mu = \mathrm{diag}(r_0/2N, -r_0/2N)$ and $D = -4Nn|\Delta|$, then $D \equiv (rr_0)^2 \mod (4N)$. Let $\mathfrak{n} = [N, \frac{rr_0+\sqrt{D}}{2}]$, which is a proper ideal of the order $\mathcal{O}_D$ and has norm N.

For $w = \begin{pmatrix} \frac{b}{2N} & -\frac{a}{N} \\ c & -\frac{b}{2N} \end{pmatrix} \in L_{r\mu}(|\Delta|n)$, the associated CM point $\mathbb{R}w$ corresponds to $[w] = (E, G)$ in the moduli interpretation of $\mathcal{X}_0(N)$, where $E = \mathbb{C}/\mathfrak{a}$ over $\mathbb{C}$ with $\mathfrak{a} = [a, \frac{b+\sqrt{D}}{2}]$ and $G \cong \mathfrak{n}^{-1}\mathfrak{a}/\mathfrak{a} \cong \mathcal{O}_D/\mathfrak{n}$ being a cyclic subgroup scheme of E of order N. By the theory of complex multiplication, $[w]$ is actually defined over $\mathcal{O}_H$, where H is the ring class field of $\mathcal{O}_D$. It is the Zariski closure of $\mathbb{R}w$ in $\mathcal{X}_0(N)$. We refer to [GZ] for detail on modular curves and CM points.

First assume $p \nmid D$, then p is split in $\mathbb{Q}(\sqrt{D})$ (as $D \equiv (rr_0)^2 \mod (4N)$). Let $\mathfrak{q}$ be a prime of H above p, and $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}_D$. Look at the reduction at $\mathfrak{q}$, and denote $\mathbb{F}_\mathfrak{q} = \mathcal{O}_H/\mathfrak{q}$. Then $(E, G)(\bar{\mathbb{F}}_\mathfrak{q}) \in \mathcal{X}_p^0(N)(\bar{\mathbb{F}}_\mathfrak{q})$ if and only if $G(\bar{\mathbb{F}}_\mathfrak{q}) \cong \mu_p \times (\mathbb{Z}/(N/p))$, which is equivalent to $\mathfrak{p} \nmid \mathfrak{n}$ (as $G \cong \mathcal{O}_D/\mathfrak{n}$). Therefore, when $\mathfrak{p}|\mathfrak{n}$, $\mathcal{Z}_{\Delta,r}(n,\mu)(\bar{\mathbb{F}}_\mathfrak{q})$ lies in $\mathcal{X}_p^\infty$, and we have

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0 \rangle_\mathfrak{q} = 0.$$

When $\mathfrak{p} \nmid \mathfrak{n}$, we have

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0 \rangle_\mathfrak{q} = -\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^\infty \rangle_\mathfrak{q} = 0.$$

Adding all $\mathfrak{q}$ over p together, we see

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0\rangle_p = 0$$

Next we assume $p|D$. Since N is square-free, we have $p\|D$, and that p is ramified in $\mathbb{Q}(\sqrt{D})$. In this case, all the CM points in the support of $\mathcal{Z}_{\Delta,r}(n,\mu)$ in $\bar{\mathbb{F}}_p$ are supersingular and are all in the intersection $\mathcal{X}_p^0 \cap \mathcal{X}_p^\infty$. This implies

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0\rangle_p = \langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^\infty\rangle_p.$$

Combining this with equation (5.7), we see again

$$\langle \mathcal{Z}_{\Delta,r}(n,\mu), \mathcal{X}_p^0\rangle_p = 0.$$

This proves the proposition. □

Proof of Theorem 1.2. First, we have

$$\deg \widehat{\phi}_{\Delta,r}(\tau) = \int_{X_0(N)} \Theta_{\Delta,r}(\tau,z) = I_{\Delta,r}(1) = 0$$

by Corollary 3.3. Next, denote

$$\widehat{\Delta}_N = (\frac{rk}{12}\mathcal{P}_\infty, -\log\|\Delta_N(z)\|^2). \tag{5.8}$$

Then we have by Proposition 5.2

$$\widehat{\mathrm{Div}}(\Delta_N) = \widehat{\Delta}_N - k\sum_{p|N}\frac{p}{p-1}\mathcal{X}_p^0,$$

and

$$\langle \widehat{\phi}_{\Delta,r}(\tau), \widehat{\mathrm{Div}}(\Delta_N)\rangle = \langle \widehat{\phi}_{\Delta,r}(\tau), \widehat{\Delta}_N\rangle.$$

Since CM points never become cusps in the specialization, we have by (5.4)

$$\begin{aligned}
&\langle \widehat{\phi}_{\Delta,r}, \widehat{\Delta}_N\rangle \\
&= \frac{rk}{12}\sum_{n>0}(Z_{\Delta,r}(n,\mu), \mathcal{P}_\infty)_{fin} q^n e_\mu - \int_{X_0(N)} \log\|\Delta_N\|\Theta_{\Delta,r}(\tau,z) \\
&= -I_{\Delta,r}(\log\|\Delta_N\|).
\end{aligned}$$

From Theorem 1.5, we prove the second formula of Theorem 1.2.

Proof of Theorem 1.6. Now the proof of Theorem 1.6 is exactly the same as proof of [DY, Theorem 8.4], and we refer to [DY] for details.

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