VON NEUMANN ALGEBRA INDEX AND MAXIMAL RELATIVE ENTROPY

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Abstract. We revisit the connection between von Neumann algebra index and relative entropy. We observe that the Pimsner-Popa index in [22] connects to maximal sandwiched p-Rényi relative entropy for all $1/2 \leq p \leq \infty$, including the Umegaki's relative entropy at $p = 1$. Based on that, we introduce a new notation of maximal relative entropy for a inclusion of finite von Neumann algebras. These maximal relative entropy generalizes subfactors index and has application in estimating decoherence time of quantum Markov semigroup.

1. Introduction

The index $[\mathcal{M} : \mathcal{N}]$ for a II_1 subfactor $\mathcal{N} \subset \mathcal{M}$ was first constructed by Jones [14] as the coupling constant of the representation of $\mathcal{N}$ on $L_2(\mathcal{M})$. Motivated from classical egordic theory, Connes and Störmer [7] introduced the relative entropy $H(\mathcal{M}|\mathcal{N})$ for an inclusion of finite (dimensional) $\mathcal{N} \subset \mathcal{M}$. The connection between these two quantities was first studied by Pimsner and Popa [22] and they proved the general relation

$$\log[\mathcal{M} : \mathcal{N}] \geq H(\mathcal{M}|\mathcal{N}) \,. \tag{1}$$

A key concept in their discussion is the following index for an inclusion $\mathcal{N} \subset \mathcal{M}$ of finite von Neumann algebras,

$$\lambda(\mathcal{M} : \mathcal{N}) = \max\{\lambda \mid \lambda\rho \leq E(\rho) \,, \text{ for all } \rho \in \mathcal{M}_+\} \tag{2}$$

where $E : \mathcal{M} \to \mathcal{N}$ is the trace preserving conditional expectation onto $\mathcal{N}$. It was proved in [22] that $[\mathcal{M} : \mathcal{N}] = \lambda(\mathcal{M} : \mathcal{N})^{-1}$ for II_1 subfactors and $\log \lambda(\mathcal{M} : \mathcal{N})^{-1} \geq H(\mathcal{M}|\mathcal{N})$ in general, from which (24) follows. In this paper, we revisit these concepts and connect them to sandwiched Rényi relative entropies D_p recently introduced in quantum information theory (see Section 2 for definitions). The starting point is the observation that the quantity $\lambda(\mathcal{M} : \mathcal{N})$ is closely related to the sandwiched Rényi relative entropy D_p at $p = \infty$. Based on that, we obtain the following connection between index and p-Rényi relative entropy for all $1/2 \leq p \leq \infty$, including Umegaki's relative entropy at $p = 1$.

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Theorem 1.1. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of II_1 factor or hyperfinite von Neumann algebras. For $1/2 \le p \le \infty$,*

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = \sup_{\rho \in S(\mathcal{M})} D_p(\rho || \mathcal{E}(\rho)) = \sup_{\rho \in S(\mathcal{M})} \inf_{\sigma \in S(\mathcal{N})} D_p(\rho || \sigma) \ , \tag{3}$$

where the supremum takes all density operators ρ in $\mathcal{M}$ and the infimum takes all density operators σ in $\mathcal{N}$.

For a density operator $\rho \in \mathcal{M}$, we define $D_p(\rho||\mathcal{N}) = \inf_\sigma D(\rho||\sigma)$ where the infimum takes all density $\sigma \in \mathcal{N}$. This notation measures the distance of the state ρ to the states of subalgebra $\mathcal{N}$. It unifies several information measure studied in quantum information theory, such as (Rényi) conditional entropy [20], relative entropy of decoherence [31] and relative entropy asymmetry [19]. Theorem 1.1 says that the von Neumann algebra index can be viewed as the maximal relative entropy to the subalgebra. Motivated from that, we introduce new notations of relative entropy for an inclusion $\mathcal{M} \subset \mathcal{N}$

$$D_p(\mathcal{M}||\mathcal{N}) := \sup_{\rho \in \mathcal{M}} D_p(\rho||\mathcal{N}) \ , D_{p,cb}(\mathcal{M}||\mathcal{N}) := \sup_n D_{p,cb}(M_n(\mathcal{M})||M_n(\mathcal{N})) \ ,$$

Such relative entropies differ with Connes-Störmer $H(\mathcal{M}|\mathcal{N})$ but are more related to the index $\lambda(M : N)$ and $[\mathcal{M} : \mathcal{N}]$. In particular, for $p = 1, \infty$, $D_{1,cb}$ and $D_{\infty,cb}$ satisfies additivity under tensor product.

One application of $D_{p,cb}$ is to estimate the decoherence time of quantum Markov semigroup. A quantum Markov semigroup $(T_t)_t : \mathcal{M} \to \mathcal{M}$ is an ultra-weak continuous family of normal unital completely positive maps. When $\mathcal{M} = B(H)$, quantum Markov semigroups are also called GLKS equation in physics literature (see [6]). It models the evolution of open quantum system that potentially interacts with environment.

Theorem 1.2. *Let $T_t = e^{-At} : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $\mathcal{N}$ be incoherent subalgebra of T_t. Suppose $D_{2,cb}(\mathcal{M}||\mathcal{N}) < \infty$ and T_t has λ-spectral gap that $\lambda \parallel x - E(x) \parallel_2^2 \le tr(x^*Ax)$. Then for any density $\rho \in M_n(\mathcal{M})$, we have $\parallel id \otimes T_t(\rho) - id \otimes E(\rho) \parallel_1 \le \epsilon$ if*

$$t \ge \frac{1}{\lambda}\big(2 \log \frac{2}{\epsilon} + D_{2,cb}(\mathcal{M}||\mathcal{N})/2\big)$$

The incoherent subalgebra is common multiplicative domain of T_t for all $t \ge 0$. A semigroup T_t is non-primitive if $\mathcal{N}$ is nontrivial. A non-primitive semigroup describes the a general decoherence process that a quantum state ρ lose its coherence and converges to the incoherent state $E(\rho)$, where $\mathcal{E}$ is the conditional expectation onto $\mathcal{N}$. Theorem 1.2 gives an estimate of the decoherence time independent of the dimension of auxiliary system M_n. In particular, when $\mathcal{N}$ is a commutative algebra (classical system), $id \otimes E(\rho)$ is always a separable state. Then the above estimates also bounds the entanglement remained in $T_t(\rho)$, which gives the entanglement-breaking time of the semigroup.

The rest of paper is organized as follows. In Section 2, we review definition and basic properties about sandwiched Rényi relative. The connection between $D_p(\rho||\mathcal{N})$ and amalgamated L_p-spaces is also mentioned. Section 3 proves Theorem 1.1 and discuss some further properties about maximal relative entropy $D_p(\mathcal{M}||\mathcal{N})$ and $D_{p,cb}(\mathcal{M}||\mathcal{N})$. Section 4 is devoted to application of $D_{p,cb}(\mathcal{M}||\mathcal{N})$ in the decoherence time and proves Theorem 1.2

2. Relative entropy

2.1. **Sandwiched Rényi relative entropy.** Let $\mathcal{M}$ be a finite von Neumann algebra equipped with normal faithful trace state tr. For $1 \le p < \infty$, the space $L_p(\mathcal{M})$ is defined as the norm completion with respect to L_p-norm $\| x \|_p = tr(|x|^p)^{\frac{1}{p}}$. In particular, $L_\infty(\mathcal{M}) := \mathcal{M}$ and the predual space $\mathcal{M}_* \cong L_1(\mathcal{M})$ via the duality

$$a \in L_1(\mathcal{M}) \longleftrightarrow \phi_a \in \mathcal{M}_*,\ \phi_a(x) = tr(ax)\,.$$

We say an element $\rho \in L_1(\mathcal{M})$ a density operator if $\rho \ge 0$ and $tr(\rho) = 1$. We denote $S(\mathcal{M})$ for all density operator of $\mathcal{M}$, which correspond to the normal states of $\mathcal{M}$. Let $p \in [\frac{1}{2}, 1) \cup (1, \infty]$ and $\frac{1}{p'} + \frac{1}{p} = 1$. For two density ρ and σ, the sandwiched Rényi relative entropy is defined as

$$D_p(\rho||\sigma) = \begin{cases} p' \log \| \sigma^{-\frac{1}{2p'}} \rho \sigma^{-\frac{1}{2p'}} \|_p, & \text{if } \rho << \sigma \\ +\infty, & \text{otherwise.} \end{cases}$$

Here $\rho << \sigma$ means that the support projection satisfies $supp(\rho) \le supp(\sigma)$. The negative power $\sigma^{-\frac{1}{2p'}}$ can be interpreted as generalized inverse on the support and in most discussion we can assume σ is faithful. This definition was originally introduced in [29, 20] for matrix algebras and recently generalized to general von Neumann algebra via different methods [4, ?, ?, 10]. When $p \to 1$, D_p recovers the relative entropy

$$D(\rho||\sigma) = tr(\rho \log \rho - \rho \log \sigma) \tag{4}$$

which was first introduced by Umegaki [26] and later extended to general von Neumann algebra by Araki [1]. Umegaki's definition is the noncommutative generalization of Kullback-Leibler divergence in probablity theory. It is an fundamental quantity that have been intensive studied and widely used in quantum information theory (see [27] for a survey). As relative entropy usually has operational meaning in the asymptotic i.i.d setting (e.g. [21]), the sandwiched Rényi relative entropy D_p has been found useful in proving strong converse theorem and one shot rate [29, 11, 17]. For all $\frac{1}{2} \le p \le \infty$, $D_p(\rho||\sigma)$ is a measure of difference between ρ and σ. In particular the case $p = \infty$,

$$D_\infty(\rho||\sigma) = \log \| \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \|_\infty = \log \inf\{\lambda | \rho \le \lambda\sigma\}$$

is also called D_{max} and $D_{\frac{1}{2}}$ is essentially the fidelity. We summerise here some important properties of D_p. Let ρ, σ be two densities operator

i) $D_p(\rho||\sigma) \ge 0$. Moreover, $D_p(\rho||\sigma) = 0$ if and only if $\rho = \sigma$
ii) $D_p(\rho||\sigma)$ is non-decreasing over $p \in [\frac{1}{2}, \infty]$ and $\lim_{p\to 1} D_p(\rho||\sigma) = D(\rho||\sigma)$.
iii) For a complete positive trace preserving map (CPTP) $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$, $D_p(\rho||\sigma) \ge D_p(\Phi(\rho)||\Phi(\sigma))$. In particular, $D_p(\rho||\sigma)$ is joint convex for ρ and σ.

i), ii) and iii) was proved in [20, 29] for matrix algebras. The discussion for the case of general von Nuemann algebra can be found in [4, 12, 13, 10].

2.2. **Relative entropy with respect to a subalgebra.** Let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra. Motivated from the asymmetry measure of group in [18], we introduced the following definition of relative entropy with respect to a subalgebra: for a density $\rho \in L_1(\mathcal{M})$,

$$D_p(\rho||\mathcal{N}) = \inf_{\sigma \in S(\mathcal{N})} D_p(\rho||\sigma) \ .$$

where the infimum takes over all densities $\sigma \in S(\mathcal{N})$. This definition connects several concepts in the literature:

a) Let $\alpha : G \to Aut(\mathcal{M})$ be an action of a group G as $*$-automorphism of $\mathcal{M}$. Let $\mathcal{N} = \mathcal{M}^G := \{x \in \mathcal{M} | \alpha_g(x) = x \,\forall\, g \in G\}$ be the invariant subalgebra. Then $D_p(\rho||\mathcal{M}^G)$ is a G-asymmetry measure introduced in [18].
b) For $\mathcal{M} = B(H_A) \otimes B(H_B)$ and $\mathcal{N} = \mathbb{C}1 \otimes B(H_B) \subset B(H_A) \otimes B(H_B)$, $D_p(\rho||\mathcal{N})$ gives the sandwiched Rényi relative entropy $H_p(A|B)$ in [20] up to a constant $D_p(\rho||\mathcal{N}) = H_p(A|B)_\rho + \log |A|$. The constant comes from that the matrix trace on $B(H_A) \otimes B(H_B)$ differs with $B(H_B)$ by a factor of $|A|$.
c) Let $\mathcal{N} = l_\infty^n \subset M_n = \mathcal{M}$ be the diagonal matrices inside the matrix algebra M_n. $D_p(\rho||\mathcal{N})$ gives the sandwiched Rényi relative entropy of coherence.

We have the basic properties of $D_p(\rho||\mathcal{N})$ parallel to $D(\rho||\sigma)$.

Proposition 2.1. *For $1/2 \le p \le \infty$ and density $\rho \in S(\mathcal{M})$,*

i) $D_p(\rho||\mathcal{N}) \ge 0$. *Moreover* $D_p(\rho||\mathcal{N}) = 0$ *if and only if* $\rho \in S(\mathcal{N})$
ii) $D_p(\rho||\mathcal{N})$ *is non-decreasing over* $p \in [\frac{1}{2}, \infty]$ *and* $\lim_{p\to 1} D_p(\rho||\mathcal{N}) = D(\rho||\mathcal{N})$.
iii) *Let* $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ *be a CPTP such that* $\Phi(L_1(\mathcal{N})) \subset L_1(\mathcal{N})$. *Then* $D_p(\rho||\mathcal{N}) \ge D_p(\Phi(\rho)||\mathcal{N})$. *In particular,* $D_p(\rho||\mathcal{N})$ *is convex for* ρ.
iv) *For* $p = 1$,

$$D(\rho||\mathcal{N}) = D(\rho||\mathcal{E}(\rho)) = H(E(\rho)) - H(\rho)$$

where $H(\rho) = -tr(\rho \log \rho)$ *is the von Neumann entropy.*

Proof. i)-iii) follows from the corresponding properties of $D_p(\rho||\sigma)$ by taking the infimum. When $p=1$, for any density $\sigma \in S_1(\mathcal{N})$,

$$\begin{aligned} D(\rho||\sigma) &= tr(\rho\log\rho - \rho\log\sigma) = tr(\rho\log\rho) - \tau(E(\rho)\log\sigma) \\ &= tr(\rho\log\rho - E(\rho)\log E(\rho)) - tr(E(\rho)\log\sigma - E(\rho)\log E(\rho)) \\ &= D(\rho||E(\rho)) + D(\sigma||E(\rho))\,. \end{aligned} \tag{5}$$ p

Because $D(\sigma||E(\rho)) \geq 0$ and $D(\sigma||E(\rho)) = 0$ implies $\sigma = \mathcal{E}(\rho)$, so the infimum attains uniquely at $E(\rho)$. Moreover, by the condition expectation property,

$$D(\rho\|E(\rho)) = tr(\rho\log\rho - \rho\log E(\rho)) = tr(\rho\log\rho - E_(\rho)\log E(\rho)) = H(E(\rho)) - H(\rho)\,.$$

this verifies iv). ■

Form above properties, we see that $D_p(\rho||\mathcal{N})$ are natural measures of the difference ρ is from a density of $\mathcal{N}$. Viewing $E(\rho)$ as the projection of ρ, $D_p(\rho||\mathcal{E}(\rho))$ is also a measure with respect to the subalgebra $\mathcal{N}$ and coincides with $D_p(\rho||\mathcal{N})$ at $p=1$. We note that for general p, $D_p(\rho||\mathcal{N}) \neq D_p(\rho||\mathcal{E}(\rho))$.

Example 2.2. Let $\mathcal{N} \cong l_\infty^2$ be the diagonal matrix in $\mathcal{M} = M_2$. For $0 \leq a \leq 1$, consider the pure state $\rho = \begin{bmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{bmatrix}$. One can calculate that for $1 < p \leq \infty$ and $q = \frac{p}{2p-1}$,

$$\begin{aligned} &D_p(\rho||\mathcal{N}) = D_p(\rho||\sigma_p) = p'\log(1 + a^q(1-a)^{1-q} + (1-a)^q a^{1-q})\,, \\ &\sigma_p = \begin{bmatrix} \frac{a^q}{a^q+(1-a)^q} & 0 \\ 0 & \frac{(1-a)^q}{a^q+(1-a)^q} \end{bmatrix} \\ &D_p(\rho||E(\rho)) = p'\log(a^{\frac{1}{p}} + (1-a)^{\frac{1}{p}})\,,\ E(\rho) = \begin{bmatrix} a & 0 \\ 0 & 1-a \end{bmatrix} \end{aligned}$$

2.3. **Connection to amalgamated L_p-spaces.** The Rényi relative entropy $D_p(\rho||\mathcal{N})$ are closely related to the amalgamated L_p-spaces and conditional L_p-spaces introduced in [15] (JPmemo). Here we briefly recall the basic definitions and refer to the appendix and [15] (JPmemo) for more information.

Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{1}$. The amalgamated L_p-space $L_1^p(\mathcal{N} \subset \mathcal{M})$ is the completion of $\mathcal{M}$ with respect the the norm

$$\|\,x\,\|_{L_1^p(\mathcal{N}\subset\mathcal{M})} = \inf_{x=ayb} \|\,\alpha\,\|_{L_{2p'}(\mathcal{N})} \|\,y\,\|_{L_p(\mathcal{M})} \|\,\beta\,\|_{L_{2p'}(\mathcal{N})} \tag{6}$$ augmented

where the infimum runs over all factorization $x = ayb$ with $a, b \in \mathcal{N}$ and $y \in \mathcal{M}$. For positive $x \geq 0$, it suffices to consider positive $a = b \geq 0$ in the infimum and

$$\|\,x\,\|_{L_1^p(\mathcal{N}\subset\mathcal{M})} = \inf_{\sigma\in S(\mathcal{N})} \|\,\sigma^{-\frac{1}{2p'}}\rho\sigma^{-\frac{1}{2p'}}\,\|_p, \tag{7}$$ positive

where the negative power are inverse on the support. Therefore, for $1 < p \le \infty$,

$$D_p(\rho||\mathcal{N}) = p' \log \| \rho \|_{L_1^p(\mathcal{N}\subset\mathcal{M})} \ .$$

It follows from Hölder inequality that $\| \rho \|_{L_1^p(\mathcal{N}\subset\mathcal{M})} \ge \| \rho \|_1$ and $\| \rho \|_{L_1^p(\mathcal{N}\subset\mathcal{M})} = \| \rho \|_1$ if and only if $\rho \in L_1(\mathcal{N})$. This corresponds to the positivity $D_p(\rho||\mathcal{N}) \ge 0$ and $D_p(\rho||\mathcal{N}) = 0$ if and only if $\rho \in S(\mathcal{N})$. For $1 \le p \le q \le \infty$ and $\frac{1}{q} + \frac{1}{2r} = \frac{1}{p}$, we define $L_q^p(\mathcal{N} \subset \mathcal{M})$ as the completion of $\mathcal{M}$ with respect the norm

$$\| x \|_{L_q^p(\mathcal{N}\subset\mathcal{M})} = \sup_{\|a\|_{L_{2r}(\mathcal{N})}=\|b\|_{L_{2r}(\mathcal{N})}=1} \| axb \|_{L_p(\mathcal{M})} \tag{8}$$ conditional

where the supremum runs over all a, b in the unit ball of $L_{2r}(\mathcal{N})$. The connection of $D_p(\rho||\mathcal{N})$ for $\frac{1}{2} \le p < 1$ goes with conditional L_p-norm via $\rho^{\frac{1}{2}}$. Let $1 \le q = 2p \le 2$ and $\frac{1}{q} = \frac{1}{r} + \frac{1}{2}$. We define the norm

$$\| x \|_{L^2_{(r,\infty)}(\mathcal{N}\subset\mathcal{M})} = \sup_{\|a\|_{L_r(\mathcal{N})}=1} \| ax \|_{L_q(\mathcal{M})} \ .$$

where the supreme runs over all $a \in \mathcal{N}$ with $\| a \|_{L_r(\mathcal{N})} = 1$. For $1 \le q = 2p < 2$

$$D_p(\rho||\mathcal{N}) = -r \log \| \rho^{\frac{1}{2}} \|_{L^2_{(r,\infty)}(\mathcal{N}\subset\mathcal{M})} \ .$$

We show that the infimum in $D_p(\rho||\mathcal{N})$ is always attained. The proof uses uniform convexity of L_p-spaces and is included in the appendix.

unique **Proposition 2.3.** *For $1/2 \le p \le \infty$, the infimum $D_p(\rho||\mathcal{N}) = \inf_{\sigma\in\mathcal{S}(\mathcal{N})} D_p(\rho||\sigma)$ is attained at some σ. For $1/2 < p < \infty$, such σ is unique.*

3. Maximal relative entropy

Recall the Popa-Pimsner index for a finite von Neumann algebra is defined as

$$\lambda(\mathcal{M} : \mathcal{N}) = \max\{\lambda | \lambda x \le \mathcal{E}(x) \ \forall \ x \in M_+\}$$

This definition can be written by D_∞ as follows

$$\begin{aligned}
\log \lambda(\mathcal{M} : \mathcal{N}) &= \log \sup\{\lambda | \lambda x \le \mathcal{E}(x) \text{ for all } x \in \mathcal{M}_+\} \\
&= \log \inf_{x\in\mathcal{M}_+} \sup\{\lambda | \lambda x \le \mathcal{E}(x)\} \\
&= \inf_{x\in\mathcal{M}_+} (\log \inf\{\mu | x \le \mu\mathcal{E}(x)\})^{-1} \\
&= \Big(\sup_{x\in\mathcal{M}_+} \log \inf\{\mu | x \le \mu\mathcal{E}(x)\} \Big)^{-1} \\
&= \Big(\sup_{x\in S(\mathcal{M})} D_\infty(x||E(x)) \Big)^{-1}
\end{aligned}$$

where the last equality follows from the fact $\mathcal{M}_+$ is norm-dense in $L_1(\mathcal{M})_+$. Thus we have

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = \sup_{\rho \in S(\mathcal{M})} D_\infty(\rho || E(\rho)). \tag{9}$$

in

We now prove the main theorem.

index **Theorem 3.1.** *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of II_1 subfactors or hyperfinite finite von Neumann algebras. Then for $1/2 \leq p \leq \infty$,*

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = \sup_{\rho \in S(\mathcal{M})} D_p(\rho || \mathcal{E}(\rho)) = \sup_{\rho \in S(\mathcal{M})} D_p(\rho || \mathcal{N})$$

Proof. By monotonicity,

$$D_{\frac{1}{2}}(\rho||\mathcal{N}) \leq D_p(\rho||\mathcal{N}) \leq D_\infty(\rho||\mathcal{N}) \leq D_\infty(\rho||\mathcal{E}(\rho))$$
$$D_{\frac{1}{2}}(\rho||\mathcal{N}) \leq D_{\frac{1}{2}}(\rho||\mathcal{E}(\rho)) \leq D_p(\rho||\mathcal{E}(\rho)) \leq D_\infty(\rho||\mathcal{E}(\rho)) \ ,$$

it suffices to prove that

$$\sup_{\rho \in S(\mathcal{M})} D_{\frac{1}{2}}(\rho||\mathcal{N}) \geq -\log \lambda(\mathcal{M} : \mathcal{N}) \ .$$

Not that

$$\begin{aligned} D_{\frac{1}{2}}(\rho||\mathcal{N}) &= \inf_{\sigma \in S(\mathcal{N})} D_{\frac{1}{2}}(\rho||\sigma) = \inf -2\log \| \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \|_1 \\ &= -2 \log \sup \| \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \|_1 \ . \end{aligned}$$

Let $e = \mathrm{supp}(\rho)$ be the support projection of ρ. By Hölder inequality, for any $\sigma \in S(\mathcal{N})$,

$$\begin{aligned} \| \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \|_1 &\leq \| \sigma^{\frac{1}{2}} e \|_2 \| \rho^{\frac{1}{2}} \|_2 \\ &= tr(\sigma e)^{\frac{1}{2}} = tr(\sigma \mathcal{E}(e))^{\frac{1}{2}} \leq \| E(e) \|_\infty^{\frac{1}{2}} \ . \end{aligned}$$

Therefore, $D_{\frac{1}{2}}(\rho||\mathcal{N}) \geq -\log \| E(e) \|_\infty$ and

$$\sup_\rho D_{\frac{1}{2}}(\rho||\mathcal{N}) \geq -\log \inf\{ \| \mathcal{E}(e) \| \ | \ e \text{ projection in } \mathcal{M} \} \ .$$

It has been proved in [22, Theorem 2.2, Proposition 2.6 and Corollary 5.6] (pipo) that the infimum at the right hand side equals $\lambda(\mathcal{M} : \mathcal{N})$ when $\mathcal{M}, \mathcal{N}$ are II_1 factors or hyperfinite. That completes the proof. ■

The above theorem basically used the monotonicity of D_p over p and the following key equality

$$\max\{\lambda | \lambda x \leq \mathcal{E}(x) \ \forall \ x \in M_+\} = \inf\{ \| \mathcal{E}(e) \| \ | \ e \text{ projection in } \mathcal{M}\} \ . \tag{10}$$

key

proved for II_1 factors and hyperfinite von Neumann algebras. The "$\leq$" direction always holds form convexity. The converse inequality is open in general. In both finite dimensional or subfactor cases, it follows from the fact that there exists a projection $e \in \mathcal{M}$ such that $\mathcal{E}(e)$ is $\lambda(\mathcal{M}:\mathcal{N})$ times a projection. Let $\rho_0 = tr(e)^{-1}e$ be the normalized density of e. As a consequence of monotonicity, $D_p(\rho_0||\mathcal{N})$ attains the index for all $1/2 \le p \le \infty$,

$$\sup_{\rho\in S(\mathcal{M})} D_p(\rho||\mathcal{N}) = D_p(\rho_0||\mathcal{N}) \ = D_p(\rho_0||E(\rho_0)) \ . \tag{11}$$

Let us briefly review the value of $\lambda(\mathcal{M}:\mathcal{N})$ and the optimal density ρ from [22].

For II_1 subfactor $\mathcal{N}\subset\mathcal{M}$, there is a projection $e\in\mathcal{M}$ such that $E(e) = [\mathcal{M}:\mathcal{N}]^{-1}$. This implies

$$\lambda(\mathcal{M}:\mathcal{N})^{-1} = [\mathcal{M}:\mathcal{N}] \ ,$$

For finite dimensional cases, let $\mathcal{N}\cong \oplus_k M_{n_k}, \mathcal{M}\cong\oplus_l M_{m_l}$ and assume that the unital inclusion $\iota:\mathcal{N}\hookrightarrow\mathcal{M}$ is given by

$$\iota(\oplus_k x_k) = \oplus_l(\oplus_k x_k\otimes 1_{a_{kl}}) \ .$$

Here 1_n denotes the identity matrix in M_n and a_{kl} is called the inclusion matrix, which means that each block M_{m_l} of $\mathcal{M}$ contains a_{kl} copy of M_{n_k} blocks from $\mathcal{N}$. Let t_l be the trace of minimal projection in M_{m_l} block of $\mathcal{M}$ and s_k be the trace of minimal projection in M_{n_k} block of $\mathcal{N}$. Then $s=(s_k), t=(t_l), n=(n_k), m=(m_l)$ as column vectors satisfy $s=At$ and $m=A^Tn$, where $A=(a_{kl})$ and A^T is the transpose of A. Based on (11), it is equivalent to consider the optimal element for D_p of any $1/2\le p\le\infty$.

Without losing generosity, we assume the trace of $\mathcal{M}$ is an induced matrix trace by a further inclusion $\mathcal{M}\cong\oplus_l M_{m_l}\otimes 1_{t_l}\subset M_d$. Based on Theorem 3.1, an equivalent approach is to maximize $D(\rho||E(\rho)) = H(E(\rho)) - H(\rho)$. By convexity of D, it suffices to consider a minimal projection $e = |\psi\rangle\langle\psi|\otimes 1_{t_l}$ in one of the block M_{m_l}. Then $\rho = |\psi\rangle\langle\psi|\otimes\frac{1_{t_l}}{t_l}$ is the normalized density and $H(\rho)=\log t_l$. Denote $P_{k,i}$ be the projection in M_{m_l} corresponding to the ith copy of M_{n_k} and write $|\psi_{k,i}\rangle = P_{k,i}|\psi\rangle$. The conditional expectation of ρ is given by

$$E_{\mathcal{N}}(\rho) = \oplus_k(\sum_{i=1}^{a_{kl}}|\psi_{k,i}\rangle\langle\psi_{k,i}|)\otimes\frac{1}{s_k}1_{s_k} \ .$$

The largest possible rank of $\mathcal{E}_{\mathcal{N}}(\rho)$ is $\sum_k\min(a_{kl},n_k)s_k$ because the part in the M_{n_k} block of $\mathcal{N}$

$$\sum_{i=1} P_{i,k}|\psi\rangle\langle\psi|P_{i,k} = \sum_{i=1}^{a_{kl}}|\psi_{k,i}\rangle\langle\psi_{k,i}|$$

is of rank at most $\min(a_{kl},n_k)$. Then the maximal entropy $H(\mathcal{E}(\rho))$ is attained by choosing $|\psi_{k,i}\rangle\langle\psi_{k,i}|$ mutually orthogonal and $\parallel\psi_{k,i}\parallel^2=\frac{s_k}{\sum_k\min(a_{kl},n_k)s_k}$. In this case,

$$E_{\mathcal{N}}(\rho) = \oplus_k(\sum_{i=1}^{a_{kl}}|\psi_{k,i}\rangle\langle\psi_{k,i}|)\otimes\frac{1}{s_k}1_{s_k} = \frac{1}{\sum_k\min(a_{kl},n_k)s_k}\oplus_k(\sum_{i=1}^{a_{kl}}|\tilde{\psi}_{k,i}\rangle\langle\tilde{\psi}_{k,i}|)\otimes\frac{1}{s_k}1_{s_k}$$

where $|\tilde{\psi}_{k,i}\rangle = |\psi_{k,i}\rangle / \parallel \psi_{k,i} \parallel_2$ are normalized vector. Then

$$\begin{aligned} D(\rho||E(\rho)) =& H(E(\rho)) - H(\rho) = \log \sum_k \min(a_{kl}, n_k) s_k - \log t_l \\ =& \log \sum_k \min(a_{kl}, n_k) s_k / t_l \; . \end{aligned}$$

This leads to the formula in pipo [22, Theorem 6.1]

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = \max_\rho D(\rho||\mathcal{N}) = \log \max_l \sum_k \min(a_{kl}, n_k) s_k / t_l \; . \tag{12}$$

formula

Motivated from above we introduce for finite von Neumann algebras $\mathcal{N} \subset \mathcal{M}$, the maximal relative entropy $D(\mathcal{M}||\mathcal{N})$ and its Rényi version $D_p(\mathcal{M}||\mathcal{N})$

$$\begin{aligned} D(\mathcal{M}||\mathcal{N}) :=& \sup_{\rho \in S(\mathcal{M})} D(\rho||\mathcal{N}) \; , \\ D_p(\mathcal{M}||\mathcal{N}) :=& \sup_{\rho \in S(\mathcal{M})} D_p(\rho||\mathcal{N}) \end{aligned}$$

As a consequence of Theorem index 3.1, for II_1 subfactors or hyperfinite $\mathcal{N} \subset \mathcal{M}$, $D_p(\mathcal{M}||\mathcal{N}) = D(\mathcal{M} : \mathcal{N})$ is independent of p, while in general such equality is open. These definition are different with the Connes-Stormer relative entropy

$$H(\mathcal{M}|\mathcal{N}) = \sup_{\sum_i x_i = 1} \sum_i tr(x_i \log x_i - x_i \log E(x_i))$$

where the supreme runs over all partition of unity $\sum_i x_i = 1, x_i \geq 0$. We now discuss the relation between $\lambda(\mathcal{M} : \mathcal{N})$, $D_p(\mathcal{M}||\mathcal{N})$ and $H(\mathcal{M}|\mathcal{N})$.

compare **Proposition 3.2.** *Let $\mathcal{N} \subset \mathcal{M}$ be finite von Neumann algebras.*

i) *$D_p(\mathcal{M}||\mathcal{N})$ is monotone for $1/2 \leq p \leq \infty$.*

ii) *For $1 \leq p \leq \infty$,*

$$-\log \lambda(\mathcal{M} : \mathcal{N}) \geq D_p(\mathcal{M}||\mathcal{N}) \geq H(\mathcal{M}|\mathcal{N}) \; .$$

iii) *If $\mathcal{N} \subset \mathcal{M}$ are II_1 subfactors or hyperfinite, then for $\frac{1}{2} \leq p \leq \infty$,*

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = D_p(\mathcal{M}||\mathcal{N}) \; .$$

Proof. i) follows from the monotonicity of D_p. For ii), we have by in (9) that

$$-\log \lambda(\mathcal{M} : \mathcal{N}) = \sup_\rho D_\infty(\rho||\mathcal{E}(\rho)) \geq D_\infty(\mathcal{M}||\mathcal{N}) \geq D_p(\mathcal{M}||\mathcal{N}) \; .$$

Let $x_i \in \mathcal{M}$ such that $\sum_{i=1}^n x_i = 1$ and $x_i \geq 0$. Write $\tilde{x_i} = \frac{x_i}{tr(x_i)}$ as the normalized density. Then

$$H(\mathcal{M}|\mathcal{N}) = \sup_{\{p_i\}, \tilde{x_i}} \sum_i p_i D(\tilde{x_i}||\mathcal{E}(\tilde{x_i})) - \sum_i p_i \log p_i = \sup_{\{p_i\}, \tilde{x_i}} D(\rho||id \otimes E_\mathcal{N}(\rho))$$

where $\rho = \sum_i p_i |i\rangle\langle i| \otimes \tilde{x}_i$ is a density operator in $l_\infty(\mathcal{M})$. It follows from convexity that for any finite n, $D(l_\infty^n(\mathcal{M})||l_\infty^n(\mathcal{N})) = D(\mathcal{M}||\mathcal{N})$. Then for $1 \le p \le \infty$,

$$H(\mathcal{M}|\mathcal{N}) \le \sup_n D(l_\infty^n(\mathcal{M})||l_\infty^n(\mathcal{N})) = D(\mathcal{M}||\mathcal{N}) \le D_p(\mathcal{M}||\mathcal{N}) \le -\log \lambda(\mathcal{M}:\mathcal{N}) \ .$$

iii) is a direct consequence of Theorem 3.1. ■

Remark 3.3. Recall that Petz's Rényi relative entropy for two density ρ and σ is defined as

$$\tilde{D}_p(\rho||\sigma) = p' \log tr(\rho^p \sigma^{1-p})^{\frac{1}{p}} \ .$$

For $p = \frac{1}{2}$, $D_{\frac{1}{2}}(\rho||\sigma) \le \tilde{D}_{\frac{1}{2}}(\rho||\sigma)$ and for $1 < p$, it was proved in [12, Corollary 3.3] that $\tilde{D}_{2-\frac{1}{p}}(\rho||\sigma) \le D(\rho||\sigma) \le \tilde{D}_p(\rho||\sigma)$. Therefore, for $\mathcal{N} \subset \mathcal{M}$ subfactor or hyperfinite, the maximal relative entropy expression also holds for $\tilde{D}_p$ with $\frac{1}{2} \le p \le 2$,

$$-\log \lambda(\mathcal{M}:\mathcal{N}) = \tilde{D}_p(\mathcal{M}||\mathcal{N}) := \sup_{\rho \in S(\mathcal{M})} \inf_{\sigma \in S(\mathcal{N})} \tilde{D}_p(\rho||\sigma) \ .$$

As was observed in [22], $-\log \lambda(\mathcal{M}:\mathcal{N})$ does not always equal to $[\mathcal{M},\mathcal{N}]$ for finite dimensional subfactors. Indeed, for $n < m$,

$$D(M_{mn}||M_n) = \log \min(n,m) m \ne \log m^2 = \log[M_{mn}:M_n] \ .$$

Moreover, the subfactors index satisfies the multiplicative properties

i) for $\mathcal{N} \subset \mathcal{M} \subset \mathcal{L}$, $[\mathcal{L}:\mathcal{N}] = [\mathcal{L}:\mathcal{M}][\mathcal{M}:\mathcal{N}]$
ii) for $\mathcal{N}_1 \subset \mathcal{M}_1, \mathcal{N}_2 \subset \mathcal{M}_2$, $[\mathcal{M}_1 \otimes \mathcal{M}_2 : \mathcal{N}_1 \otimes \mathcal{N}_2] = [\mathcal{M}_1:\mathcal{N}_1][\mathcal{M}_2:\mathcal{N}_2]$

The follow proposition shows that this also differs with $D(\mathcal{M}||\mathcal{N})$.

property **Proposition 3.4.** *Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ be finite von Neumann algebras.*

i) *for $\mathcal{N} \subset \mathcal{M} \subset \mathcal{L}$, $D(\mathcal{L}||\mathcal{N}) \le D(\mathcal{L}||\mathcal{M}) + D(\mathcal{M}||\mathcal{N})$;*
ii) *for $\mathcal{N}_1 \subset \mathcal{M}_1, \mathcal{N}_2 \subset \mathcal{M}_2$, $D(\mathcal{M}_1 \otimes \mathcal{M}_2||\mathcal{N}_1 \otimes \mathcal{N}_2) \ge D(\mathcal{M}_1||\mathcal{N}_1) + D(\mathcal{M}_2||\mathcal{N}_2)$.*

In general both inequalities can be strict.

Proof. i) Let $E_\mathcal{M}$ (resp. $E_\mathcal{N}$) be the conditional expectation from $\mathcal{L}$ onto $\mathcal{M}$ (resp. $\mathcal{N}$). Because $E_\mathcal{N} \circ E_\mathcal{M} = E_\mathcal{N}$, for $\rho \in S(\mathcal{L})$,

$$\begin{aligned} D(\rho||\mathcal{N}) &= H(E_\mathcal{N}(\rho)) - H(\rho) = H(E_\mathcal{N}(\rho)) - H(E_\mathcal{M}(\rho)) + H(E_\mathcal{N}(\rho)) - H(\rho) \\ &= D(E_\mathcal{M}(\rho)||\mathcal{N}) + D(\rho||\mathcal{M}) \le D(\mathcal{M}||\mathcal{N}) + D(\mathcal{L}||\mathcal{M}) \end{aligned}$$

which proves i). For the strict inequality case, we have

$$D(M_4||M_2) = \log 4 \ , \ D(M_2||\mathbb{C}) = \log 2 \ , \ D(M_4||\mathbb{C}) = \log 4 \ne D(M_4||M_2) + D(M_2||\mathbb{C}) \ .$$

For ii), let $E_i, i = 1, 2$ be the conditional expectation from M_i to N_i. The inequality follows from that

$$D(\rho||\mathcal{E}_1(\rho)) + D(\sigma||\mathcal{E}_2(\sigma)) = D(\rho \otimes \sigma||\mathcal{E}_1(\rho) \otimes \mathcal{E}_2(\sigma)) \leq D(\mathcal{M}_1 \otimes \mathcal{M}_2||\mathcal{N}_1 \otimes \mathcal{N}_2) \ .$$

This inequality is strict for the case

$$D(M_6||M_2) = \log 6 \ , \ D(M_6||M_3) = \log 4 \ ,$$
$$D(M_{36}||M_6) = \log 36 \neq D(M_6||M_2) + D(M_6||M_3)$$

Another example is $\mathcal{N} = (M_2 \otimes \mathbb{C}1_3) \oplus (M_3 \otimes \mathbb{C}1_2) \subset M_{12} = \mathcal{M}$. Then

$$D(M_{12}||\mathcal{N}) = \log(4 + 6) = \log 10 \ ,$$
$$D(M_{12} \otimes M_{12}||\mathcal{N} \otimes \mathcal{N}) = \log(4 \times 9 + 6 \times 6 + 6 \times 6 + 4 \times 4) = \log 126 \ . \qquad \blacksquare$$

The following is an example of left regular representation of finite groups.

Remark 3.5. Form the above example, we know that there exists a bipartite state $\rho \in M_{12} \otimes M_{12}$ such that

$$D(\rho_1||\mathcal{N}) + D(\rho_2||\mathcal{N}) < D(\rho||\mathcal{N} \otimes \mathcal{N}) \ ,$$

where ρ_1 and ρ_2 are the reduced densities of ρ on each component. Hence the relative entropy with respect to subalgebra is **super-additive**. The super-additivity implies that ρ is an entangled state, which means ρ is not a convex combination of tensor product densities. This phenomenon for the coherent information is of particular interest in quantum information superact [25].

Example 3.6. Let G be a finite group and $\mathcal{L}(G) = \text{span}\lambda(G) \subset B(l_2(G))$ be the gourp von Neumann algebra of left regular representation λ. For a subgroup $H \subset G$, denote $\mathcal{L}(H)$ as the subalgebra generated by $\lambda(H)$. Then for inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$,

$$D(\mathcal{L}(G)||\mathcal{L}(H)) = \log[G : H] \ .$$

First, by Peter-Weyl formula (cf. bump [5]) that $\mathcal{L}(G) \cong \oplus_k M_{n_k} \otimes \mathbb{C}1_{n_k}$ and $|G| = \sum_k n_k^2$. Thus by the formula (formula 12),

$$D(\mathcal{L}(G)||\mathbb{C}) = \log|G| \ , \ D(B(l_2(G))||\mathcal{L}(G)) = \log(\sum_k n_k^2) = \log|G|.$$

Consider $G = H \cup Hg_1 \cup \cdots Hg_{n-1}$ decomposed as a disjoint union of cosets and $n = [G : H]$. Let P_i be the projection onto $l_2(Hg_i)$ as a subspace of $l_2(G)$. So $\mathcal{L}(H)$ is a left regular representation of H of multiplicity n on $\oplus_i P_i l_2(G) = l_2(G)$. Thus

$$D(\mathcal{L}(H)||\mathbb{C}) = \log|H| \ , D(\mathcal{L}(G)||\mathcal{L}(H)) \geq [G : H]$$

by Proposition 3.4 i) for the inclusion $\mathbb{C} \subset \mathcal{L}(H) \subset \mathcal{L}(G)$. On the other hand, the conditional expectation $E_H : \mathcal{L}(G) \to \mathcal{L}(H)$ is given by

$$\mathcal{E}_H(\sum_{g\in G} \alpha_g \lambda(g)) = \sum_{g\in H} \alpha_g \lambda(g) = \sum_i P_i(\sum_{g\in G} \alpha_g \lambda(g)) P_i \ ,$$

where $\lambda(g)$ is the unitary of left shifting by g. For $g \in H$, $P_i\lambda(g)P_i = 0$ because for any $h_1, h_2 \in H$, $gh_1g_i = h_2g_i$ implies $g = h_2h_1^{-1} \in H$. Note that the trace on $\mathcal{L}(G)$ coincides with the induced normalized matrix trace of $B(l_2(G))$. Consider $\mathcal{N} = \oplus B(l_2(Hg_i)) \subset B(l_2(G))$. We have $D(\mathcal{M}||\mathcal{N}) = \log n$ adn $E_{\mathcal{N}}(\rho) = \sum_i P_i\rho P_i$ is the conditional expectation. Thus

$$\begin{aligned} D(\mathcal{L}(G)||\mathcal{L}(H)) &= \sup_{\rho\in\mathcal{L}(G)} D(\rho||\mathcal{E}_H(\rho)) = \sup_{\rho\in\mathcal{L}(G)} D(\rho||\mathcal{E}_{\mathcal{N}}(\rho)) \\ &\le \sup_{\rho\in B(l_2(G))} D(\rho||\mathcal{E}_{\mathcal{N}}(\rho)) = D(\mathcal{M}||\mathcal{N}) = \log n \end{aligned}$$

Therefore we obtain $D(\mathcal{M}||\mathcal{N}) = [G : H]$.

The continuity of $D(\cdot||\mathcal{N})$ follows from [30, Lemma 7]

Proposition 3.7. *Let $\rho, \sigma \in S(\mathcal{M})$ be two densities with $\| \rho - \sigma \|_1 = \epsilon$. Then*

$$|D(\rho||\mathcal{N}) - D(\sigma||\mathcal{N})| \le 2\epsilon D(\mathcal{M}||\mathcal{N}) + (1 + 2\epsilon)h(\frac{\epsilon}{1 + 2\epsilon}),$$

where $h(\lambda) = -\lambda \log \lambda - (1 - \lambda) \log(1 - \lambda)$ is the binary entropy function.

We know by convexity that adding an auxiliary classical (commutative) system l_∞^n does not change the maximal relative entropy,

$$D_p(l_\infty^n(\mathcal{M})||l_\infty^n(\mathcal{N})) = D_p(\mathcal{M}||\mathcal{N}) \ .$$

However this is not the case if we replace l_∞ by a quantum system M_n. For finite von Neumann algebras $\mathcal{N} \subset \mathcal{M}$, we define the *cb*-maximal relative entropy

$$D_{cb,p}(\mathcal{M}||\mathcal{N}) := \sup_n D_p(M_n(\mathcal{M}))||M_n(\mathcal{N}))$$

In general, $D_{cb,p}(\mathcal{M}||\mathcal{N}) \ge D_p(\mathcal{M}||\mathcal{N})$ and the inequality can be strict. In particular, for all $\frac{1}{2} \le p \le \infty$

$$\begin{aligned} &D_p(M_n \otimes M_m||M_n) = mn = -\log \lambda(M_n \otimes M_m : M_n) \ , \\ &D_{p,cb}(M_n \otimes M_m||M_n) = m^2 = \log[M_n \otimes M_m : M_n] \ . \end{aligned} \quad (13)$$

which are different when $n < m$. Using the properties of $D(\mathcal{M}||\mathcal{N})$, we immediately obtain

Corollary 3.8. i) *$D_{p,cb}(\mathcal{M}||\mathcal{N})$ is monotone for $p \in [1/2, \infty]$.*

ii) *If $\mathcal{N} \subset \mathcal{M}$ are II_1 subfactors or hyperfinite, $D_{p,cb}(\mathcal{M}||\mathcal{N})$ is independent of p.*

iii) *For $\mathcal{N} \subset \mathcal{M}$ finite subfactors,*

$$\log[\mathcal{M} : \mathcal{N}] = D_{p,cb}(\mathcal{M}||\mathcal{N}) \,. \tag{14}$$ cb

Proof. For iii), the finite dimensional case is (14). For II_1 subfactors, $D_{cb}(\mathcal{M}||\mathcal{N}) = D(\mathcal{M}||\mathcal{N}) = \log[\mathcal{M} : \mathcal{N}]$ because subfactor index $[\mathcal{M} : \mathcal{N}]$ is multiplicative [14]. ■

The above proposition suggests that (the exponential of) $D_{p,cb}$ are extensions of subfactor index $[\mathcal{M} : \mathcal{N}]$ to finite von Neumann algebras. Using the connection between $D_p(\rho||\mathcal{N})$ and $\| \rho \|_{L_1^p(\mathcal{N}\subset\mathcal{M})}$ for $1 < p \leq \infty$, we see that $D_p(\mathcal{M}||\mathcal{N})$ is basically the norm of identity map from $L_1(\mathcal{M})$ to $L_1^p(\mathcal{N} \subset \mathcal{M})$. Indeed, it suffices to consider positive elements because for $x = yz$,

$$\| x \|_{L_1^p(\mathcal{N}\subset\mathcal{M})} = \| y \|_{L_{2p'}(\mathcal{N})L_{2p}(\mathcal{M})} \| z \|_{L_{2p}(\mathcal{M})L_{2p'}(\mathcal{N})} \leq \| yy^* \|_{L_1^p(\mathcal{N}\subset\mathcal{M})} \| z^*z \|_{L_1^p(\mathcal{N}\subset\mathcal{M})}$$

Thus, for $1 < p \leq \infty$,

$$D_p(\mathcal{M}||\mathcal{N}) = p' \log \| id : L_1(\mathcal{M}) \to L_1^p(\mathcal{N} \subset \mathcal{M}) \|$$

For $\frac{1}{2} < p < 1$ and $\frac{1}{2p} = \frac{1}{r} + \frac{1}{2}$, the maximal relative entropy is

$$D_p(\mathcal{M}||\mathcal{N}) = 2p' \log \| id : L_2(\mathcal{M}) \to L^2_{(r,\infty)}(\mathcal{N} \subset \mathcal{M}) \|$$

We shall show that for $1 < p \leq \infty$, $D_{p,cb}$ are indeed given by the completely bounded norms. We discuss in the appendix that the natural operator space of $L_1^p(\mathcal{N} \subset \mathcal{M})$ is given by

$$S_1^n \widehat{\otimes} L_1^p(\mathcal{N} \subset \mathcal{M}) \cong L_1^p(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \,. \tag{15}$$ os2

where $S_1^n = (M_n)^*$ is n operator space of trace class operators.

Proposition 3.9. *Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

i) *for $1 \leq p \leq \infty$*

$$D_{p,cb}(\mathcal{M}||\mathcal{N}) = \sup_{\mathcal{R}} D_p(\mathcal{R}\overline{\otimes}\mathcal{M}||\mathcal{R}\overline{\otimes}\mathcal{N}) \,. \tag{16}$$ equal

where the supremum runs over all finite von Neumann algebra $\mathcal{R}$.

ii) *for $1 < p \leq \infty$, $D_{p,cb}(\mathcal{M}||\mathcal{N}) = p' \log \| id : L_1(\mathcal{M}) \to L_1^p(\mathcal{N} \subset \mathcal{M}) \|_{cb}$.*

iii) *For $\mathcal{N}_i \subset \mathcal{M}_i, i = 1, 2$ finite von Neumann algebras*

$$D_{cb}(\mathcal{M}_1 \otimes \mathcal{M}_2||\mathcal{N}_1 \otimes \mathcal{N}_2) = D_{cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{cb}(\mathcal{M}_2||\mathcal{N}_2) \,. \tag{17}$$

$$D_{p,cb}(\mathcal{M}_1 \otimes \mathcal{M}_2||\mathcal{N}_1 \otimes \mathcal{N}_2) \leq D_{p,cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{\infty,cb}(\mathcal{M}_2||\mathcal{N}_2) \,. \tag{18}$$

In particular, for $p = \infty$,

$$D_{\infty,cb}(\mathcal{M}_1 \otimes \mathcal{M}_2||\mathcal{N}_1 \otimes \mathcal{N}_2) = D_{\infty,cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{\infty,cb}(\mathcal{M}_2||\mathcal{N}_2) \,.$$

Proof. Let $\mathcal{R}\subset B(H)$ and $\rho\in S(\mathcal{R}\overline{\otimes}\mathcal{M}),\sigma\in S(\mathcal{R}\overline{\otimes}\mathcal{N})$. Let $\tilde{\rho}$ (resp. $\tilde{\rho}$) be a normal state on $B(H)\overline{\otimes}\mathcal{M}$ (resp. $B(H)\overline{\otimes}\mathcal{N}$) extending ρ (resp. σ). Let $\iota:\mathcal{R}\hookrightarrow B(H)$ be the inclusion. ι is a normal unital completely positive map. Its adjoint on the predual $\iota^\dagger: B(H)_*\to R_*$ is the restriction

$$\iota^\dagger(\phi)=\phi|\mathcal{R}$$

In particular, using the identification $B(H)_*\cong S_1(H)$ and $L_1(\mathcal{R}_*)$, $\iota^\dagger$ is a completely positive trace preserving map. We have

$$\rho=\iota^\dagger\otimes id_{\mathcal{M}_*}(\tilde{\rho})\,,\sigma=\iota^\dagger\otimes id_{\mathcal{M}_*}(\tilde{\sigma})\,.$$

Then by data processing inequality,

$$D_p(\rho||\sigma)=D_p(\iota^\dagger\otimes id(\tilde{\rho})||\iota^\dagger\otimes id(\tilde{\sigma}))\le D_p(\tilde{\rho}||\tilde{\sigma})$$

Since $B(H)$ is approximate finite dimensional, we can find $\tilde{\rho}_n\in S(M_n(\mathcal{M})),\tilde{\sigma}_n\in S(M_n(\mathcal{N}))$ such that

$$\|\tilde{\rho}_n-\tilde{\rho}\|_1\to 0\,,\|\tilde{\sigma}_n-\tilde{\sigma}\|_1\ .$$

By the lower-semicontinuity of D_p for $1\le p\le\infty$ [12, Proposition 3.7],

$$D_p(\tilde{\rho}||\tilde{\sigma})\le\liminf_{n\to\infty}D_p(\tilde{\rho}_n||\tilde{\sigma}_n)\le\sup_n D_p(M_n(\mathcal{M})||M_n(\mathcal{N}))=D_{p,cb}(\mathcal{M}||\mathcal{N})\,.$$

ii) follows from (15) and [23, Lemma 1.7]. For iii), let $E_i:\mathcal{M}_i\to\mathcal{N}_i$ be the conditional expectation. For a density $\rho\in\mathcal{R}\overline{\otimes}\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$,

$$\begin{aligned}D(\rho||id\otimes E_1\otimes E_2(\rho))&=D(\rho||id\otimes id\otimes E_2(\rho))+D(id\otimes id\otimes E_1(\rho)||id\otimes E_1\otimes E_2(\rho))\\&\le D(\mathcal{R}\otimes\mathcal{M}_1\otimes\mathcal{M}_2)||\mathcal{R}\otimes\mathcal{M}_1\otimes\mathcal{N}_2)+D(\mathcal{R}\otimes\mathcal{M}_1\otimes\mathcal{N}_2)||\mathcal{R}\otimes\mathcal{N}_1\otimes\mathcal{N}_2)\\&\le D_{cb}(\mathcal{M}_1||\mathcal{N}_1)+D_{cb}(\mathcal{M}_2||\mathcal{N}_2)\,.\end{aligned}$$

This proves the case $p=1$. For $p>1$, let $\sigma\in\mathcal{N}_1\overline{\otimes}\mathcal{N}_2$ be a invertible density

$$\|\sigma^{-\frac{1}{2p'}}\rho\sigma^{-\frac{1}{2p'}}\|_p\le\|\sigma_1^{-\frac{1}{2p'}}\rho\sigma_1^{-\frac{1}{2p'}}\|_p\|\sigma^{-\frac{1}{2p'}}\sigma_1^{\frac{1}{2p'}}\|_\infty^2$$

for some invertible density $\sigma_1\in\mathcal{N}_1\overline{\otimes}\mathcal{M}_2$. Note that

$$\|\sigma^{-\frac{1}{2p'}}\sigma_1^{\frac{1}{2p'}}\|_\infty^2\le\|\sigma^{-\frac{1}{2}}\sigma_1^{\frac{1}{2}}\|_\infty^{\frac{2}{p'}}=\|\sigma^{-\frac{1}{2}}\sigma_1\sigma^{-\frac{1}{2}}\|_\infty^{\frac{1}{p'}}\ .$$

By relative entropy, we have

$$D_p(\rho||\sigma)\le D_p(\rho||\sigma_1)+D_\infty(\sigma_1||\sigma)\,.$$

Taking infimum for both $\sigma_1\in\mathcal{N}_1\overline{\otimes}\mathcal{M}_2$ and $\sigma\in\mathcal{N}_1\overline{\otimes}\mathcal{N}_2$, we have

$$D_p(\rho||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)\le D_p(\rho||\mathcal{N}_1\overline{\otimes}\mathcal{M}_2)+D(\sigma_1||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)\,.$$

Taking supremum over ρ, we have

$$D_p(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)\le D_p(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2||\mathcal{N}_1\overline{\otimes}\mathcal{M}_2)+D_\infty(\mathcal{N}_1\overline{\otimes}\mathcal{M}_2||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)$$

$$\leq D_{p,cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{\infty,cb}(\mathcal{M}_2||\mathcal{N}_12)\ .$$

Replacing $\mathcal{N}_1 \subset \mathcal{M}_1$ by $\mathcal{R}\overline{\otimes}\mathcal{N}_1 \subset \mathcal{R}\overline{\otimes}\mathcal{M}_1$ yeilds the inequality for $D_{p,cb}(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)$. The equality follows from choosing tensor product elements. ■

Up to this writing, we do not know whether $D_{cb,p} = D_{cb}$ independent of p holds for general finite von Neumann algebras. Recall that for subfactor ot hyperfinite case, this follows from the equality (10), which is open for general von Neumann algebras as mentioned in [22].

4. Applications to decoherence time

In this section, we discuss the applications to decoherence time of quantum Markov processes. We discuss the symmetric case and briefly mention the modification for non-symmetric ones in Appendix. We start with the continuous time setting. Let $(\mathcal{M}, tr)$ be a finite von Neumann algebra $\mathcal{M}$ equipped with faithful normal tracail state tr. A quantum Markov semigroup $(T_t)_{t\geq 0} : M \to M$ for $t \geq 0$ is a w^*-continuous family of maps that satisfies

i) T_t is a normal unital completely positive (normal UCP) map for all $t \geq 0$.
ii) $T_t \circ T_s = T_{s+t}$ for any $t, s \geq 0$ and $T_0 = id$.
iii) for each $x \in \mathcal{M}$, $t \to T_t(x)$ is continuous in σ-weak topology.

We denote by A the generator of T_t, that is the densely defined operator on $L_2(\mathcal{M})$ given by

$$Ax = w^* - \lim_{t\to 0^+} \frac{1}{t}(x - T_t(x))$$

for all $x \in \mathcal{M}$ such that the σ-weak limit exists. We denote

$$\mathcal{N} = \{a \in \mathcal{M} | T_t(a^*)T_t(a) = T_t(a^*a) \text{ and } T_t(a)T_t(a^*) = T_t(aa^*)\ , \forall\, t\}$$

as the common multiplicative domain of T_t. We call $\mathcal{N}$ the *incoherent subalgebra.* When $\mathcal{N} = \mathbb{C}1$ is trivial, T_t is called primitive and has a unique invariant state. In general, $(T_t)_{t\geq 0}$ restricted on $\mathcal{N}$ is a semigroup of $*$-homomorphism.

We say a quantum markov semigroup $(T_t)_{t\geq 0}$ is symmetric if for all $x, y \in \mathcal{M}$ and $t \geq 0$, $tr(x^*T_t(y)) = tr(T_t(x)^*y)$. Namely, $T_t = T_t^\dagger$ is self-adjoint with respect to trace. As a consequence T_t is trace preserving $tr(T_t(\rho)) = tr(\rho)$ and invariant on $\mathcal{N}$. Indeed, for $a, b \in \mathcal{N}$,

$$tr(aT_{2t}(b)) = tr(T_t(a)T_t(b)) = tr(T_t(ab)) = tr(ab)\ .$$

Let $E : \mathcal{M} \to \mathcal{N}$ be the trace preserving conditional expectation onto $\mathcal{N}$. By the above discussion, we know

$$A \circ E = 0\ , T_t \circ E = E \circ T_t = E\ .$$

One important functional inequality which relates the convergence property is the modified logarithmic Sobolev inequality. We say $(T_t)_{t\geq 0}$ satisfies λ-modified logarithmic Sobolev inequality (or λ-MSLI) for $\lambda > 0$ if for any density $\rho \in \mathcal{M}$

$$\lambda D(\rho||\mathcal{N}) \leq I_A(\rho) =: tr\big((A\rho)\ln\rho\big)\ .$$

This is equivalent to exponential decay of relative entropy CLSI,bardet [9, 2]

$$D(T_t(\rho)||\mathcal{N}) = D(T_t(\rho)||E(\rho)) \leq e^{-\lambda t} D(\rho||E(\rho))\ . \tag{19}$$

entropydecay

By quantum Pinker inequality (c.f. watrous [28]),

$$D(\rho||\sigma) \geq \frac{1}{2} \parallel \rho - \sigma \parallel_1^2,$$

this gives estimate of *decoherence time*

$$t_{deco}(\epsilon) = \min\{t \geq 0\ |\ \parallel T_t(\rho) - E(\rho) \parallel_1 \leq \epsilon\ \forall \text{ density } \rho \in \mathcal{M}\}.$$

Suppose the maximal relative entropy $D(\mathcal{M}||\mathcal{N}) = \sup_\rho D(\rho||\mathcal{N}) < \infty$ is finite, we have

$$\lambda - \text{LSI} \implies t_{deco}(\epsilon) \leq \frac{1}{\lambda}\big(2\log\frac{1}{\epsilon} + \log 2D(\mathcal{M}||\mathcal{N})\big)\ . \tag{20}$$

deco

Another important functional inequality is the spectral gap (also called Poincaré inequality) For $\lambda > 0$, we say $(T_t)_t$ has λ-spectral gap (or λ-PI) if for any $x \in \mathcal{M}$,

$$\lambda \parallel x - E(x) \parallel_2^2 \leq tr(x^* A x)$$

Write I as the identity map on $L_2(\mathcal{M})$ and $I - E$ is the projection onto the orthgonoal complement $L_2(\mathcal{N})^\perp$. λ-PI is the spectral gap condition (see bardet [2]) that

$$\parallel A^{-1}(I - E) : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \parallel \leq \lambda$$

$$\text{or equivalently} \quad \parallel T_t - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \parallel \leq e^{-\lambda t} \tag{21}$$

2decay

This means for each x, the L_2-distance between $T_t(x)$ and its equilibrium $E(x)$ decays exponentially. In general, λ-MLSI implies λ-PI bardet [2], which means that the entropy decay entropydecay (19) is stronger than L_2-norm decay 2decay (21). The next theorem shows that the spectral gap condition implies a weaker exponential decay of relative entropy.

d2 **Theorem 4.1.** *Let $(T_t)_{t\geq 0} : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $\mathcal{N}$ be the incoherent subalgebra of T_t. Suppose T_t satisfies λ-PI. Then for density $\rho \in \mathcal{M}$,*

$$D(T_t(\rho)||\mathcal{N}) \leq 2e^{-\lambda t + D_2(\rho||\mathcal{N})/2}\ . \tag{22}$$

decay

If in additional, $D_2(\mathcal{M}||\mathcal{N}) = \sup_\rho D_2(\rho||\mathcal{N}) < \infty$, then

$$t_{deco}(\epsilon) \leq \frac{1}{\lambda}\big(2\log\frac{2}{\epsilon} + D_2(\mathcal{M}||\mathcal{N})/2\big)$$

Proof. The λ-spectral gap property is equivalent to

$$\| T_t - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| \leq e^{-\lambda t}$$

Since both T and E are $\mathcal{N}$-bimodule map, it follows from CLSI [9, Lemma 3.12] that

$$\begin{aligned} \| T_t - E : L_1^2(\mathcal{N} \subset \mathcal{M}) \to L_1^2(\mathcal{N} \subset \mathcal{M}) \| &= \| T_t - E : L_2^2(\mathcal{N} \subset \mathcal{M}) \to L_2^2(\mathcal{N} \subset \mathcal{M}) \| \\ &= \| T_t - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| \leq e^{-\lambda t} \end{aligned}$$

(see Appendix for definition of $L_p^q(\mathcal{N} \subset \mathcal{M})$ for general $1 \leq p, q \leq \infty$.) Then for a density $\rho \in \mathcal{M}$,

$$\begin{aligned} D(T_t(\rho)||\mathcal{N}) \leq & D(T_t(\rho)||\mathcal{N}) \\ \leq & 2 \log \| T_t(\rho) \|_{L_1^2(\mathcal{N} \subset \mathcal{M})} \\ \leq & 2 \log \Big(\| E(\rho) \|_{L_1^2(\mathcal{N} \subset \mathcal{M})} + \| T_t - E(\rho) \|_{L_1^2(\mathcal{N} \subset \mathcal{M})} \Big) \\ \leq & 2 \log(1 + e^{-\lambda t} \| \rho \|_{L_1^2(\mathcal{N} \subset \mathcal{M})}) \leq 2 e^{-\lambda t + D_2(\rho||\mathcal{N})/2} . \end{aligned}$$

■

The decoherence time estimate follows from quantum Pinsker inequality.

Let us compare the above theorem with the decay property entropydecay (19) obtained from λ-MLSI. Because the MLSI constant $\geq$ PI constant, the exponent in decay (23) is at least as the MLSI constant but the constant factor in decay (23) is larger.

On the other hand, tensorization is an important property of MLSI for classical Markov semigroup. However, tensorization propetry is not known for MLSI of quantum Markov semigroup. We say $(T_t)_{t\geq 0}$ satisfies λ-complete logarithmic Sobolev inequality (or λ-CLSI) if for any n, $id_{M_n} \otimes T_t : M_n(\mathcal{M}) \to M_n(\mathcal{M})$ satisfies λ-MSLI. It follows from data processing inequality that λ-CLSI is tensor stable. However, it is not clear in the noncommutative case λ-LSI implies λ-CLSI. We refer to CLSI [9] for more discussion about CLSI and related examples.

For $\mathcal{M} = B(H)$, quantum Markov semigroups are also called GLKS equation in quantum physics (see GLKS [6]). It models the evolution of open quantum system which potentially interacts with environment. In this setting, CLSI estimates the *complete decoherence time*

$$t_{c.deco} = \inf\{ t \geq 0 \,|\, \| id \otimes T_t(\rho) - id \otimes E(\rho) \|_1 \leq \epsilon, \, \forall \, n \geq 1 \text{ and density } \rho \in M_n(\mathcal{M}) \}$$

Suppose $D_{cb}(\mathcal{M}||\mathcal{N}) < \infty$, we have as analog of deco (20)

$$\lambda\text{-CLSI} \implies t_{c.deco}(\epsilon) \leq \frac{1}{\lambda} \big(2 \log \frac{1}{\epsilon} + \log 2 D_{cb}(\mathcal{M}||\mathcal{N}) \big)$$

The complete version of decoherence time estimates the convergence rate independent of the dimension of auxiliary system M_n. In particular, when $\mathcal{N}$ is a commutative algebra (classical system), $t_{c.deco}$ also bounds the entanglement breaking time.

In contrast to MLSI, the spectral gap property or PI is stable under tensorization. Indeed, for any n, the generator A has the same spectral as $I_{\otimes}A$, the generator of $id_{M_n} \otimes T_t$. Based on this, Theorem (d2 4.1) also applies to $id_{M_n} \otimes T_t$, which leads to an estimate of complete decoherence time.

Corollary 4.2. *Let $(T_t)_{t\geq 0} : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $\mathcal{N}$ be the incoherent subalgebra of T_t. Suppose T_t satisfies λ-PI. Then for any n and density $\rho \in M_n(\mathcal{M})$,*

$$D(id \otimes T_t(\rho)||M_n(\mathcal{N})) \leq 2e^{-\lambda t + D_2(\rho||M_n(\mathcal{N}))/2} \,. \tag{23}$$ decay

If in additional $D_{2,cb}(\mathcal{M}||\mathcal{N}) < \infty$, then

$$t_{c.deco}(\epsilon) \leq \frac{1}{\lambda}\big(2\log\frac{2}{\epsilon} + D_{2,cb}(\mathcal{M}||\mathcal{N})/2\big)$$

The above theorem also works for tensor product of semigroups. Indeed, for two semigroups $S_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_t : \mathcal{M}_2 \to \mathcal{M}_2$

i) If S_t satisfies λ_1-PI and T_t satisfies λ_2-PI, then $S_t \otimes T_t$ satisfies $\min\{\lambda_1, \lambda_2\}$-PI.
ii) If $D_{2,cb}(\mathcal{M}_1||\mathcal{N}_1) < \infty$ and $D_{\infty,cb}(\mathcal{M}_2||\mathcal{N}_2) < \infty$, then $D_{2,cb}(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2||\mathcal{N}_1\overline{\otimes}\mathcal{N}_2) = D_{2,cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{\infty,cb}(\mathcal{M}_2||\mathcal{N}_2) < \infty$ by Theorem cbp ??.

We now discuss the discrete time setting. A quantum Makrov map $T : M \to M$ is a symmetric normal completely positive unital map. Let $\mathcal{N} = \{a \in \mathcal{M} \mid T(a^*a) = T(a^*)T(a)\}$ be the multiplicative domain of T. T restricted on $\mathcal{N}$ is a normal trace preserving $*$-homomorphism. T^2 is identity on $\mathcal{N}$ because for any $a, b \in \mathcal{N}$

$$tr(aT^2(b)) = tr(T(a)T(b)) = tr(T(ab)) = tr(ab) \,.$$

and T is a isometry on $L_2(\mathcal{N})$. Let $E : \mathcal{M} \to \mathcal{N}$ be the conditional expectation onto $\mathcal{N}$ and I be the identity operator on $L_2(\mathcal{M})$. We have

$$T^2 \circ E = E \circ T^2 = E \,, T \circ E = E \circ T \,. \tag{24}$$ relation

Theorem 4.3. *Let $T : \mathcal{M} \to \mathcal{M}$ be a symmteric quantum Markov map and let $\mathcal{N}$ be multiplicative domain of T. Suppose $\| T(I - E) : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| \leq \mu < 1$. Then for any $n \geq 1$ and density $\rho \in M_n(\mathcal{M})$, we have*

$$D(T^k(\rho)||M_n(\mathcal{N})) \leq 2\mu^k e^{D_2(\rho||M_n(\mathcal{N}))/2} \,.$$

Moreover, for $k \geq (\log\frac{1}{\mu})^{-1}(\log\frac{4}{\epsilon^2} + D_{2,cb}(\mathcal{M}||\mathcal{N})/2)$,

$$\| id \otimes T^k(\rho) - id \otimes E(\rho) \|_1 \leq \epsilon \quad \text{for } k \text{ even,}$$

$$\| id \otimes T^k(\rho) - id \otimes T \circ E(\rho) \|_1 \leq \epsilon \quad \text{for } k \text{ odd.}$$

Proof. Using the relation (24), we have

$$(T(I-E))^2 = (T - T\circ E)^2 = T^2 - 2T^2\circ E + T^2\circ E = T^2 - E\,.$$

Then

$$(T - T\circ E)^{2k} = T^{2n} - E\,,\ (T - T\circ E)^{2k+1} = T^{2k+1} - E\circ T\,.$$

By [9, Lemma 3.12] again, since $(T-E)^k$ are $\mathcal{N}$-bimodule map,

$$\begin{aligned}&\|\,(T - T\circ E)^k : L_1^2(\mathcal{N}\subset\mathcal{M}) \to L_1^2(\mathcal{N}\subset\mathcal{M})\,\| \\ =&\|\,(T - T\circ E)^k : L_2(\mathcal{M}) \to L_2(\mathcal{M})\,\| \le \mu^k\end{aligned}$$

The rest of argument is similar to Theorem 4.1. Here we show the case for k odd,

$$\begin{aligned}D(T^{2m+1}(\rho)||\mathcal{N}) \le& D_2(I\otimes T^k(\rho)||\mathcal{N}) \\ \le& 2\log\|\,T^k(\rho)\,\|_{L_1^2(\mathcal{N}\subset\mathcal{M})} \\ \le& 2\log\Big(\,\|\,E\circ T^k(\rho)\,\|_{L_1^2(M_m(\mathcal{N})\subset M_m(\mathcal{M}))} + \|\,(T - T\circ E)^k(\rho)\,\|_{L_1^2(\mathcal{N}\subset\mathcal{M})}\,\Big) \\ \le& 2\log(1+\mu^k\,\|\,\rho\,\|_{L_1^2(\mathcal{N}\subset\mathcal{M})}) \\ \le& 2\mu^k e^{D_2(\rho||\mathcal{N})/2}\,.\end{aligned}$$

Applying the same argument for $\rho\in M_n(\mathcal{M})$ yields the desired estimate. ■

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Appendix A

A.1. **Amalgamated L_p-space and Conditional L_p-spaces.** In this section, we recall the definition of amalgamated L_p-space and conditional L_p-spaces needed for our discussion. For general cases, we refer to JPmemo [15]. For $1 \leq p, q \leq \infty$ and $|\frac{1}{p} - \frac{1}{q}| = \frac{1}{2r}$, we define $L_q^p(\mathcal{N} \subset \mathcal{M})$ as the completion of $\mathcal{M}$ with respect the norm

$$\| x \|_{L_q^p(\mathcal{N}\subset\mathcal{M})} = \begin{cases} \inf_{x=ayb,\ a,b\in\mathcal{N}} \| a \|_{L_{2r}(\mathcal{N})} \| y \|_{L_q(\mathcal{M})} \| b \|_{L_{2r}(\mathcal{N})} & \text{if } p \leq q \\ \sup_{\| a \|_{L_{2r}(\mathcal{N})} = \| b \|_{L_{2r}(\mathcal{N})} = 1} \| axb \|_{L_p(\mathcal{M})} & \text{if } p \geq q. \end{cases} \quad (25)$$

conditional

For $p \leq q$, $L_q^p(\mathcal{N} \subset \mathcal{M})$ is called amalgamated L_p-space and for $p \geq q$ conditional L_p-space. It follows from Hölder inequality that

i) $L_p^p(\mathcal{N} \subset \mathcal{M}) = L_p(\mathcal{M})$,
ii) for $q_1 \leq p \leq q_2$, $\| x \|_{L_p^{q_1}(\mathcal{N}\subset\mathcal{M})} \leq \| x \|_{L_p(\mathcal{M})} \leq \| x \|_{L_p^{q_2}(\mathcal{N}\subset\mathcal{M})}$,
iii) $L_p(\mathcal{N}) \subset L_p^q(\mathcal{N} \subset \mathcal{M})$ for any $1 \leq q \leq \infty$. Moreover, $\| x \|_{L_p^q(\mathcal{N}\subset\mathcal{M})} = \| x \|_{L_p(\mathcal{N})}$ if and only if $x \in L_p(\mathcal{N})$

For $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, we have the duality $L_q^p(\mathcal{N} \subset \mathcal{M})^* = L_{q'}^{p'}(\mathcal{N} \subset \mathcal{M})$ via

$$\| x \|_{L_q^p(\mathcal{N}\subset\mathcal{M})} = \sup\{ |tr(xy)| \mid \| y \|_{L_{q'}^{p'}(\mathcal{N}\subset\mathcal{M})} \leq 1 \} ,$$

For $q = 1$, $L_1^p(\mathcal{N} \subset \mathcal{M}) \subset L_\infty^{p'}(\mathcal{N} \subset \mathcal{M})^*$ as a w^*-dense subspace. (see JPmemo [15, Propsition 4.5]). In particular, the dual of amalgamated space is conditional space and vice versa. We also have complex interpolation relation

$$L_q^p(\mathcal{N} \subset \mathcal{M}) = [L_{q_0}^{p_0}(\mathcal{N} \subset \mathcal{M}), L_{q_1}^{p_1}(\mathcal{N} \subset \mathcal{M})]_\theta$$

isometrically where $(1-\theta)/p_0 + \theta/p_1 = 1/p, (1-\theta)/q_0 + \theta/q_1 = 1/q$ and $(p_1 - q_1)(p_2 - q_2) \geq 0$.

We will also need some "square root" version of above L_p-spaces. For $2 \leq r \leq \infty, 1 \leq p, q \leq \infty$ and $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, we define the norm

$$\| x \|_{L_{(r,\infty)}^p(\mathcal{N}\subset\mathcal{M})} = \sup_{\| a \|\ L_r(\mathcal{N})=1} \| ax \|_{L_q(\mathcal{M})} \ .$$

where the supreme runs over all $a \in \mathcal{N}$ with $\| a \|_{L_r(\mathcal{N})} = 1$. The dual spaces are the amalgamated space $L_{q'}(\mathcal{M})L_r(\mathcal{N})$ given by

$$\| y \|_{L_{q'}(\mathcal{M})L_r(\mathcal{N})} = \inf_{y=za} \| z \|_{L_{q'}(\mathcal{M})} \| a \|_{L_r(\mathcal{N})} \ .$$

For $1 < q < \infty$, we have the dual relation

$$\begin{aligned} \| x \|_{L^2_{(r,\infty)}(\mathcal{N}\subset\mathcal{M})} &= \sup\{ \| ax \|_{L_q(\mathcal{M})} \ | \ \| a \|_{L_r(\mathcal{N})} = 1\} \\ &= \sup\{ |tr(zax)| \ | \ \| a \|_{L_r(\mathcal{N})} = 1, \| z \|_{L_{q'}(\mathcal{M})} = 1\} \\ &= \sup\{ |tr(yx)| \ | \ \| y \|_{L_{q'}(\mathcal{M})L_r(\mathcal{N})} = 1\} \end{aligned} \tag{26}$$

These spaces also interpolates (see Theorem 4.6 from [15]). Note that the property ii) and iii) in Proposition 2.1 can also be obtained from complex interpolation relation of the space $L^p_q(\mathcal{N} \subset \mathcal{M})$ and $L^p_{(r,\infty)}$ proved in [15]. We now prove Proposition 2.3.

Proposition A.1. *For $1/2 \le p \le \infty$, $D_p(\rho||\mathcal{N}) = \inf_{\sigma\in\mathcal{S}(\mathcal{N})} D_p(\rho||\sigma)$ attains the infimum at an σ. For $1/2 < p < \infty$, such σ is unique.*

Proof. The case for $p = 1$ follows from (5). For $1 < p < \infty$, we use the norm expression

$$D_p(\rho||\mathcal{N}) = p' \log \inf_{\rho=aya} \| a \|^2_{2p'} \| y \|_p = \inf_{\rho^{\frac{1}{2}}=a\eta} \| a \|_{2p'} \| \eta \|_{2p} \ ,$$

where $a \in L_{2p'}(\mathcal{N}), y \in L_p(\mathcal{M}), \eta \in L_{2p}(\mathcal{M})$ and a positive. It suffices to show that the above infimum is attained at unique a. Assume $\| x \|_{L^p_1(\mathcal{N}\subset\mathcal{M})} = 1$. We find sequences $(a_n) \subset L_{2p'}(\mathcal{N})$ and $(\eta_n) \subset L_{2p}(\mathcal{M})$ such that for each n, $\sqrt{x} = a_n\eta_n$, $\| a_n \|_{2p'} = 1$ and

$$\| \eta_n \|_{2p} \ge 1 \ , \ \lim_{n\to\infty} \| \eta_n \|_{2p} \to 1 \ .$$

Write $a_{n,m} = (\frac{1}{2}a_n^2 + \frac{1}{2}a_m^2)^{\frac{1}{2}}$. Consider the factorization

$$\sqrt{x} = \begin{bmatrix} \frac{a_n}{\sqrt{2}} & \frac{a_m}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\eta_n}{\sqrt{2}} \\ \frac{\eta_m}{\sqrt{2}} \end{bmatrix} = a_{n,m}\eta_{n,m} \ ,$$

where $\eta_{n,m} = a_{n,m}^{-1}(\frac{1}{2}a_n\eta_n + \frac{1}{2}a_m\eta_m)$. Note that

$$\begin{aligned} \| a_{n,m} \|_{2p'} &= \left\| \frac{a_n^2 + a_m^2}{2} \right\|_{p'}^{\frac{1}{2}} \ , \\ \| \eta_{n,m} \|_{2p} &= \left\| \frac{\eta_n^*\eta_n + \eta_m^*\eta_m}{2} \right\|_{p}^{\frac{1}{2}} \le (\frac{1}{2} \| \eta_n \|^2_{2p} + \frac{1}{2} \| \eta_m \|^2_{2p})^{\frac{1}{2}} \to 1 \end{aligned}$$

when $n, m \to \infty$. Because $\sqrt{x} = a_{n,m}\eta_{n,m}$, $\| a_{n,m} \|_{2p'} \| \eta_{n,m} \| \ge 1$ for any n, m. Then we have

$$\lim_{N\to\infty} \inf_{n,m\ge N} \| \frac{a_n^2 + a_m^2}{2} \|_{p'} \ge 1 \ .$$

By uniform convexity of noncommutative L_p space (c.f. kosaki84,fack86 [16, 8]), this implies that (a_n^2) converges in $L_{2p'}$. Using the inequality $\| a^2 - b^2 \|_{2p} \geq \| a - b \|_p^{\frac{1}{2}}$ from ricard15 [24, Lemma 2.1], we have that (a_n) converges in $L_{p'}(\mathcal{N})$. On the other hand, because $L_{2p}(\mathcal{M})$ is reflexive, there exists a subsequence $\eta_{n_k} \to \eta$ weakly and $\| \eta \|_{2p} \leq 1$. Thus $\sqrt{x} = a_{n_k}\eta_{n_k} \to a\eta$ weakly in $L_2(\mathcal{M})$. Hence $\sqrt{x} = a\eta$ and $\| a \|_{2p'} = \| \eta \|_{2p} = 1$. Note that we have shown that for any sequence a_n with $\sqrt{x} = a_n\eta_n$ and

$$\| a_n \|_{2p'} = 1, \lim_{n\to\infty} \| \eta_n \|_{2p} \to 1, \tag{27}$$ con

a_n converges to some a in $L_{2p'}$. Let b_n be another such sequence with $x = b_n\eta_n'$ and converges to b. Define $c_{2n-1} = a_n, c_{2n} = b_n, \xi_{2n-1} = \eta_n, \xi_{2n} = \eta_n'$. Then $x = c_n\xi_n$ satisfies same condition of (con 27). Then c_n converges to some c in $L_{2p'}$ which implies that the limit $a = b = c$ is unique. For $p = \infty$, we know

$$D_\infty(\rho||\mathcal{N}) = \log \inf\{\lambda \,|\rho \leq \lambda\sigma, \text{for some } \| \sigma \|_{L_1(\mathcal{N})} = 1\} \,.$$

Let $\lambda = \inf\{\lambda \,|\rho \leq \lambda\sigma, \| \sigma \|_{L_1(\mathcal{N})} = 1\}$ and let σ_n be a sequence of densities in $L_1(\mathcal{N}) \cong \mathcal{N}_*$ such that $\lambda_n := \min\{\lambda \,|\rho \leq \lambda\sigma\} \to \lambda$ monotonically non-increasing. By w^*-compactness of state space in $\mathcal{N}^*$, we have a subsequence σ_{n_k} converges to some state $\sigma \in \mathcal{N}^*$ in the weak* topology. Then for any k, $\lambda_{n_k}\sigma_{n_m} \geq \rho$ in $\mathcal{N}^*$ for $m \geq k$. Passing to the limit, we have $\lambda\sigma \geq \rho$ for some state $\sigma \in \mathcal{N}^*$. We show that $\sigma \in \mathcal{N}_*$. By the decomposition of the double dual space $\mathcal{N}^{**} = \mathcal{N} \oplus e\mathcal{N}^{**}e$ for some projection $e \in \mathcal{N}^{**}$, $\sigma = \sigma_0 \oplus \sigma_s$ decomposed as a normal part $\sigma_0 \in \mathcal{N}_*$ supported on $\mathcal{N}$ and a singular part $\sigma_s \in \mathcal{N}^{**}$ supported on $e\mathcal{N}^{**}e$. Suppose $\sigma_s \neq 0$. Then $\sigma_0(1) = \mu < 1$ and

$$\rho \leq \lambda\sigma \Rightarrow \rho \leq \lambda\sigma_0 \,.$$

Take the normalized density $\tilde{\sigma} = \frac{1}{\mu}\sigma_0 \in \mathcal{N}_*$. We have $\rho \leq \dfrac{\lambda}{\mu}\tilde{\sigma}$ with $\lambda/\mu > \lambda$ which is a contradiction. This proves the existence of σ.

For $1 < q = 2p < 2$ and $\frac{1}{q} = \frac{1}{r} + \frac{1}{2}$., it sufficient to show that the norm

$$\| \rho^{\frac{1}{2}} \|_{L^2_{(r,\infty)}(\mathcal{N}\subset\mathcal{M})} = \sup_{\|a\|_{L_r(\mathcal{N})}=1} \| a\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})}$$

is attained for some $\| a \|_{L_r(\mathcal{N})} = 1$. Let $\| \rho^{\frac{1}{2}} \|_{L^2_{(r,\infty)}(\mathcal{N}\subset\mathcal{M})} = \lambda$ and $a_n \geq 0$ be a positive sequence in $\| a_n \|_{L_r(\mathcal{N})} = 1$ such that $\| a_n\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})} \to \lambda$. Write $a_{n,m} = (\frac{a_n^2 + a_m^2}{2})^{\frac{1}{2}}$. We have

$$\begin{bmatrix} a_n\rho^{\frac{1}{2}} & a_n\rho^{\frac{1}{2}} \\ a_m\rho^{\frac{1}{2}} & a_m\rho^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} a_n a_{n,m}^{-1} & 0 \\ a_m a_{n,m}^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} a_{n,m}\rho^{\frac{1}{2}} & a_{n,m}\rho^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}$$

Suppose $\| a_n\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})}, \| a_m\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})} \geq (1-\epsilon)\lambda$. Then we have

$$\| \begin{bmatrix} a_n\rho^{\frac{1}{2}} & a_n\rho^{\frac{1}{2}} \\ a_m\rho^{\frac{1}{2}} & a_m\rho^{\frac{1}{2}} \end{bmatrix} \|_{L_q(M_2(\mathcal{M}))} \geq \| \begin{bmatrix} a_n\rho^{\frac{1}{2}} & \\ & a_m\rho^{\frac{1}{2}} \end{bmatrix} \|_{L_q(M_2(\mathcal{M}))} \geq 2^{\frac{1}{q}}(1-\epsilon)\lambda\ ,$$

$$\| \begin{bmatrix} a_n a_{n,m}^{-1} & 0 \\ a_m a_{n,m}^{-1} & 0 \end{bmatrix} \|_{L_\infty(M_2(\mathcal{M}))} = 1$$

$$\| \begin{bmatrix} a_{n,m}\rho^{\frac{1}{2}} & a_{n,m}\rho^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} \|_{L_q(M_2(\mathcal{M}))} = \| \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \|_{L_q(M_2)} \| a_{n,m}\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})} = 2^{\frac{1}{q}} \| a_{n,m}\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})}$$

By the definition of λ,

$$(1-\epsilon)\lambda \leq \| a_{n,m}\rho^{\frac{1}{2}} \|_{L_q(\mathcal{M})} \Rightarrow (1-\epsilon) \leq \| a_{n,m} \|_{L_r(\mathcal{N})}\ .$$

Thus we have shown

$$\lim_{N\to\infty} \inf_{n,m\geq N} \| \frac{a_n^2 + a_m^2}{2} \|_{\frac{r}{2}} \geq 1\ .$$

Following the same argument of the case of $1 < p < \infty$, we obtain that a_n converges a in $L_r(\mathcal{N})$ and such limit a is unique for $\rho^{\frac{1}{2}}$. Finally, we discuss the case for $p = 1/2$. It suffices to show the following supremum is attained

$$\begin{aligned} \| z \|_{L^2_{(2,\infty)}(\mathcal{N}\subset\mathcal{M})} &= \sup\{ \| az \|_{L_1(\mathcal{M})} \mid \| a \|_{L_2(\mathcal{N})} = 1\} \\ &= \sup\{ |tr(azy)| | \| a \|_{L_2(\mathcal{N})} = 1, y \in \mathcal{M} \text{ unitary}\} \\ &= \sup\{ \| E(zy) \|_2 \, | y \in \mathcal{M} \text{ unitary}\}\ . \end{aligned} \tag{28}$$

b

Consider the set

$$C = \{ (id - E)(zy) \mid y \in \mathcal{M} \text{ unitary}\}\ .$$

C is a weakly closed set in $L_2(\mathcal{M})$. Indeed, for any y_n such that $(id-E)(zy_n) \to x$ weakly in $L_2(\mathcal{M})$, we can find a subsequence $y_{n_k} \to y$ weakly in $\mathcal{M}$. Then $(id-E)(zy_{n_k}) \to (id-E)(zy)$ weakly in $L_2(\mathcal{M})$. Hence $x = (id-E)(zy)$ which proves the closeness. We show that C admits an element attains the infimum

$$\inf_{x\in C} \| x \|_{L_2(\mathcal{M})} := \lambda$$

Let x_n be a sequence such that $\| x_n \|_2 \to \lambda$. For a weakly converging subsequence $x_{n_k} \to x$, we have $x \in C$ by closeness and

$$\| x \|_2 \leq \liminf_{k\to\infty} \| x_{n_k} \|_2 = \lambda\ .$$

Hence the infimum norm for is attained. Since $\mathcal{E} : L_2(\mathcal{M}) \to L_2(\mathcal{N})$ is a projection,

$$\| E(zy) \|_2^2 + \| (id-E)(zy) \|_2^2 = \| zy \|_2^2 = 1$$

We have the supremum

$$\sup\{\| E(zy) \|_2 \ \mid y \in \mathcal{M} \text{ unitary } \}$$

is attained by some y_0. Therefore the supremum in (28) is attained with $a = |E(zy_0)|$. ■

A.2. **Operator space structures.** We shall now discuss the operator space structures of $L_1^p(\mathcal{N} \subset \mathcal{M})$. Let us introduce the short notation

$$L_\infty^r(\mathcal{N} \subset \mathcal{M}) = L^\infty_{(2,\infty)}(\mathcal{N} \subset \mathcal{M}) \,, L_\infty^c(\mathcal{N} \subset \mathcal{M}) = L^\infty_{(\infty,c)}(\mathcal{N} \subset \mathcal{M})$$

Recall that the norm of these two spaces are given by

$$\| x \|_{L_\infty^r(\mathcal{N} \subset \mathcal{M})} = \| E(xx^*) \|_\infty \,, \| x \|_{L_\infty^c(\mathcal{N} \subset \mathcal{M})} = \| E(x^*x) \|_\infty \ .$$

We define the operator space structure as follows,

$$M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M})) \cong L_\infty^r(M_n(\mathcal{N}) \subset M_n(\mathcal{M}))$$
$$M_n(L_\infty^c(\mathcal{N} \subset \mathcal{M})) \cong L_\infty^c(M_n(\mathcal{N}) \subset M_n(\mathcal{M}))$$

Namely for $a = \sum_j a_j \otimes x_j \in M_n \otimes M$,

$$\| a \|_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} := \| id \otimes E(aa^*) \|^{1/2}_{M_n(\mathcal{N})} = \| a \|_{L_\infty^r(M_n(\mathcal{N}) \subset M_n(\mathcal{M}))} \,,$$
$$\| a \|_{M_n(L_\infty^c(\mathcal{N} \subset \mathcal{M}))} := \| id \otimes E(a^*a) \|^{1/2}_{M_n(\mathcal{N})} = \| a \|_{L_\infty^c(M_n(\mathcal{N}) \subset M_n(\mathcal{M}))} \ .$$

We verify the above norms satisfies Ruan's axioms. For $a = a_1 \oplus a_2 \in M_n(\mathcal{M}) \oplus M_m(\mathcal{M})$,

$$\begin{aligned} \| a \|_{M_{n+m}(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} &= \| id \otimes E(aa^*) \|^{1/2}_{M_{n+m}(\mathcal{N})} \\ &= \| id_n \otimes E(a_1 a_1^*) \oplus id_m \otimes E(a_2 a_2^*) \|^{1/2}_{M_{n+m}(\mathcal{N})} \\ &= \max\{\| a_1 \|_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} \,, \| a_2 \|_{M_m(L_\infty^r(\mathcal{N} \subset \mathcal{M}))}\} \end{aligned}$$

For $a \in M_n(\mathcal{M}), b_1, b_2 \in M_n$, we have

$$(b_1 \otimes 1) a (b_2 \otimes 1) \Big((b_1 \otimes 1) a (b_2 \otimes 1) \Big)^* = (b_1 \otimes 1) a (b_2 b_2^* \otimes 1) a^* (b_1^* \otimes 1) \leq \| b_2 \|^2_\infty \, (b_1 \otimes 1) a a^* (b_1^* \otimes 1)$$

Thus we have

$$\begin{aligned} \| (b_1 \otimes 1) a (b_2 \otimes 1) \|^2_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} &= \| id \otimes E\Big((b_1 \otimes 1) a (b_2 b_2^* \otimes 1) a^* (b_1^* \otimes 1) \Big) \|_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} \\ &\leq \| b_2 \|^2_\infty \| (b_1 \otimes 1) id \otimes E(aa^*) (b_1^* \otimes 1) \|_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} \\ &\leq \| b_2 \|^2_\infty \| b_1 \|^2_\infty \| id \otimes E(aa^*) \|_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} \\ &\leq \| b_2 \|^2_\infty \| b_1 \|^2_\infty \| a \|^2_{M_n(L_\infty^r(\mathcal{N} \subset \mathcal{M}))} \end{aligned}$$

The argument for $M_n(L_\infty^c(\mathcal{N} \subset \mathcal{M}))$ is similar. Using injectivity of minimal tensor product $\otimes_{min}$, we have for a finite von Neumann algebra $\mathcal{R} \subset B(H)$,

$$\mathcal{R} \otimes_{min} L_\infty^1(\mathcal{N} \subset \mathcal{M}) \subset B(H) \otimes_{min} L_\infty^1(\mathcal{N} \subset \mathcal{M}) \,,$$

and for $a \in \mathcal{R} \otimes \mathcal{M}$,

$$\| a \|_{\mathcal{R} \otimes_{min} L^r_\infty(\mathcal{N} \subset \mathcal{M})} = \| a \|_{B(H) \otimes_{min} L^r_\infty(\mathcal{N} \subset \mathcal{M})} = \| id \otimes E(aa^*) \|^{1/2}_\infty = \| a \|_{L^r_\infty(\mathcal{R} \overline{\otimes} \mathcal{N} \subset \mathcal{R} \overline{\otimes} \mathcal{M})}$$

Therefore, $\mathcal{R} \otimes_{min} L^r_\infty(\mathcal{N} \subset \mathcal{M}) \subset L^r_\infty(\mathcal{R} \overline{\otimes} \mathcal{N} \subset \mathcal{R} \overline{\otimes} \mathcal{M})$ as a subspace. It is easy to verify that with above operator space structure $L^r_\infty(\mathcal{N} \subset \mathcal{M})$ (resp. $L^c_\infty(\mathcal{N} \subset \mathcal{M})$) is a right (resp. left) operator $\mathcal{M}$-module. It was proved in [15, Lemma 4.9] that for $z \in \mathcal{M}$,

$$\| z \|_{L^1_\infty(\mathcal{N} \subset \mathcal{M})} = \inf\{ \| x \|_{L^r_\infty(\mathcal{N} \subset \mathcal{M})} \| y \|_{L^c_\infty(\mathcal{N} \subset \mathcal{M})} \mid z = xy, x, y \in \mathcal{M} \} . \tag{29}$$

The lemma was stated for L^p_∞ with $1 < p < \infty$ although the proof works for $p = 1$ as well). It suggests the following decomposition by module Haargerup tensor product (see [] for operator module and Haargerup tensor product).

Lemma A.2. *We have isometric isomorphism*

$$L^r_\infty(\mathcal{N} \subset \mathcal{M}) \otimes_{\mathcal{M},h} L^c_\infty(\mathcal{N} \subset \mathcal{M}) \cong L^1_\infty(\mathcal{N} \subset \mathcal{M}) ,$$

where $\otimes_{\mathcal{M},h}$ is the module Haagerup tensor product. Moreover, this induce the operator space structure

$$M_n(L^1_\infty(\mathcal{N} \subset \mathcal{M})) \cong L^1_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) ,$$
$$S^n_1 \widehat{\otimes} L^\infty_1(\mathcal{N} \subset \mathcal{M}) \cong L^\infty_1(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) .$$

where $S^n_1 = (M_n)^*$ is the n-dimensional trace class.

Proof. Let us consider the map

$$m : L^r_\infty(\mathcal{N} \subset \mathcal{M}) \otimes_h L^c_\infty(\mathcal{N} \subset \mathcal{M}) \to L^1_\infty(\mathcal{N} \subset \mathcal{M}) , \; m(y \otimes z) = yz$$

This is a contraction because for $\sum_{j=1}^n y_j \otimes z_j$,

$$\begin{aligned} \| \sum_j y_j z_j \|_{L_1(\mathcal{N} \subset \mathcal{M})} &= \sup\{ \| \sum_j a y_j z_j b \|_1 \mid \| a \|_{L_2(\mathcal{N})} = \| b \|_{L_2(\mathcal{N})} = 1 \} \\ &\le \sup_{\| a \|_{L_2(\mathcal{N})}=1} \| \sum_j a y_j y_j^* a^* \|_1^{\frac{1}{2}} \sup_{\| b \|_{L_2(\mathcal{N})}=1} \| \sum_j b^* z_j^* z_j b \|_1^{\frac{1}{2}} \\ &\le \sup_{\| a \|_{L_2(\mathcal{N})}=1} \| E(\sum_j y_j y_j^*) \|_1^{\frac{1}{2}} \sup_{\| b \|_{L_2(\mathcal{N})}=1} \| E(\sum_j z_j^* z_j) \|_1^{\frac{1}{2}} \\ &= \| (y_1, \cdots, y_n) \|_{R_n(L_1(\mathcal{N} \subset \mathcal{M}))} \| (z_1, \cdots, z_n) \|_{C_n(L_1(\mathcal{N} \subset \mathcal{M}))} \end{aligned}$$

where R_n (resp. C_n) are row (resp. column). space. Also, m induces a map on the module tensor product $L^r_\infty(\mathcal{N} \subset \mathcal{M}) \otimes_{\mathcal{M},h} L^c_\infty(\mathcal{N} \subset \mathcal{M})$ since $y, z, a \in \mathcal{M}$, the element $ya \otimes z - y \otimes az$ is in the kernel of m. By the inequality (29), m is an isometry. Morover,

m is also surjective, because $\mathcal{M} \subset L_1(\mathcal{N} \subset \mathcal{M})$ is dense. Thus we proves the isometric isomorphism. Based on that, we obtain

$$\begin{aligned} M_n(L^1_\infty(\mathcal{N} \subset \mathcal{M})) &\cong M_n(L^r_\infty(\mathcal{N} \subset \mathcal{M})) \otimes_{M_n(\mathcal{M}),h} M_n(L^c_\infty(\mathcal{N} \subset \mathcal{M})) \\ &\cong L^r_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \otimes_{M_n(\mathcal{M}),h} L^c_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \\ &\cong L^1_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \end{aligned}$$

We then define the operator space structure of L^∞_1 by duality that

$$L^1_\infty(\mathcal{N} \subset \mathcal{M}) \subset \big(L^\infty_1(\mathcal{N} \subset \mathcal{M})\big)^*$$

as w^*-dense subspace. Then other identity follows from that

$$\begin{aligned} L^1_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) &\subset \big(L^\infty_1(M_n(\mathcal{N}) \subset M_n(\mathcal{M}))\big)^* \\ M_n(L^1_\infty(\mathcal{N} \subset \mathcal{M})) &\subset \big(S^n_1 \widehat{\otimes} L^\infty_1(\mathcal{N} \subset \mathcal{M})\big)^* \end{aligned}$$

both as w^*-dense subspace. ■

Recall the complex interpolation relation for $1 \le p \le \infty$,

$$\begin{aligned} L^p_1(\mathcal{N} \subset \mathcal{M}) &= [L^\infty_1(\mathcal{N} \subset \mathcal{M}), L_1(\mathcal{M})]_{1/p} = L^p_1(\mathcal{N} \subset \mathcal{M}) \, . \\ L^p_\infty(\mathcal{N} \subset \mathcal{M}) &= [L_\infty(\mathcal{M}), L^1_\infty(\mathcal{N} \subset \mathcal{M})]_{1/p} = L^p_\infty(\mathcal{N} \subset \mathcal{M}) \end{aligned}$$

Note that $S^n_1 \widehat{\otimes} L_1(\mathcal{M}) = L_1(M_n(\mathcal{M}))$ and $M_n(L_\infty(\mathcal{M})) = L_\infty(M_n(\mathcal{M}))$. Then by interpolation, we obtain the operator space structure for L^p_1 and L^p_∞.

Corollary A.3. *For* $1 \le p \le \infty$,

$$\begin{aligned} M_n(L^p_\infty(\mathcal{N} \subset \mathcal{M})) &\cong L^p_\infty(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \, , \\ S^n_1 \widehat{\otimes} L^p_1(\mathcal{N} \subset \mathcal{M}) &\cong L^p_1(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \, . \end{aligned} \tag{30}$$

operatorspac

Question: 1. Do we have $L_1(\mathcal{R}) \widehat{\otimes} L^p_1(\mathcal{N} \subset \mathcal{M}) \cong L^p_1(\mathcal{R} \overline{\otimes} \mathcal{N} \subset \mathcal{R} \overline{\otimes} \mathcal{M})$? (complete) isometrically? Or we do have the identity map is a contraction

$$id : L_1(\mathcal{R}) \widehat{\otimes} L^p_1(\mathcal{N} \subset \mathcal{M}) \to L^p_1(\mathcal{R} \overline{\otimes} \mathcal{N} \subset \mathcal{R} \overline{\otimes} \mathcal{M}) \, .$$

This gives an alternative way to show

$$D_{p,cb}(\mathcal{M} || \mathcal{N}) = \sup_{\mathcal{R}} D(\mathcal{R} \otimes \mathcal{M} || \mathcal{R} \otimes \mathcal{N})$$

for $\mathcal{R}$ finite von Neumann algebra.
It suffices to show $\mathcal{R} \otimes_{min} L^1_\infty(\mathcal{N} \subset \mathcal{M}) \subset L^1_\infty(\mathcal{R} \overline{\otimes} \mathcal{N} \subset \mathcal{R} \overline{\otimes} \mathcal{M})$ isometrically.

2. Do we have the identity map is a (complete) contraction?

$$L^p_1(\mathcal{N}_1 \subset \mathcal{M}_1) \widehat{\otimes} L^p_1(\mathcal{N}_1 \subset \mathcal{M}_1) \to L^p_1(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2 \subset \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) \, , \; x \otimes y \mapsto x \otimes y \tag{31}$$

This implies the additivity for $1 < p \leq \infty$

$$D_{p,cb}(\mathcal{M}||\mathcal{N}) = D_{p,cb}(\mathcal{M}_1||\mathcal{N}_1) + D_{p,cb}(\mathcal{M}_2||\mathcal{N}_2)$$

It suffices to show that $L_1^p(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2 \subset \mathcal{M}_1\overline{\otimes}\mathcal{M}_2)$ induced a subcross norm on $L_1^p(\mathcal{N}_1 \subset \mathcal{M}_1) \otimes L_1^p(\mathcal{N}_2 \subset \mathcal{M}_2)$. Actually, I can show the dual space $L_\infty^p(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2 \subset \mathcal{M}_1\overline{\otimes}\mathcal{M}_2)$ gives a cross norm on $L_\infty^p(\mathcal{N}_1 \subset \mathcal{M}_1) \otimes L_\infty^p(\mathcal{N}_2 \subset \mathcal{M}_2)$ and for $\rho_1 \in S_1^n\hat{\otimes}L_1^p(\mathcal{N}_1 \subset \mathcal{M}_1), \rho_2 \in S_1^m\hat{\otimes}L_1^p(\mathcal{N}_2 \subset \mathcal{M}_2)$

$$\| \rho_1 \otimes \rho_2 \|_{S_1^{nm}\hat{\otimes}L_1^p(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} = \| \rho_1 \|_{S_1^n\hat{\otimes}L_1^p(\mathcal{N}_1\subset\mathcal{M}_1)} \| \rho_1 \|_{S_1^m\hat{\otimes}L_1^p(\mathcal{N}_1\subset\mathcal{M}_1)} \tag{32}$$

equality

Below is a proof.

Proof. Let $E_i : \mathcal{M}_i \to \mathcal{N}_i$ be the conditional expectation. It is clear from definition that for $x \in \mathcal{M}_1, y \in \mathcal{M}_2$,

$$\| x \otimes y \|_{L_1^p(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} \leq \| x \|_{L_1^p(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_1^p(\mathcal{N}_2\subset\mathcal{M}_2)} ,$$
$$\| x \otimes y \|_{L_\infty^{p'}(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} \geq \| x \|_{L_\infty^{p'}(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_\infty^{p'}(\mathcal{N}_2\subset\mathcal{M}_2)} ,$$

For $p = \infty, p' = 1$,

$$\begin{aligned}
&\| x \otimes y \|_{L_\infty^1(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} \\
= &\inf_{x\otimes y = ab} \| E_1 \otimes E_2(aa^*) \|_1 \| E_1 \otimes E_2(b^*b) \|_1 \\
\leq &\inf_{x\otimes y = a_1b_1\otimes a_2b_2} \| E_1 \otimes E_2(a_1a_1^* \otimes a_2a_2^*) \|_1 \| E_2 \otimes E_2(b_1^*b_1 \otimes b_2^*b_2) \|_1 \\
\leq &\inf_{x = a_1a_2} \| E_1(a_1a_1^*) \|_1 \| E_1(b_1^*b_1) \|_1 \inf_{y = b_1b_2} \| E_2(a_2a_2^*) \|_1 \| E_2(b_2^*b_2) \|_1 \\
= &\| x \|_{L_\infty^1(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_\infty^1(\mathcal{N}_2\subset\mathcal{M}_2)}
\end{aligned}$$

Thus we have

$$\| x \otimes y \|_{L_\infty^1(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} = \| x \|_{L_\infty^1(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_\infty^1(\mathcal{N}_2\subset\mathcal{M}_2)} \tag{33}$$

2

Then by duality,

$$\begin{aligned}
&\| x \otimes y \|_{L_1^\infty(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} \\
= &\sup\{|tr_1 \otimes tr_2((x \otimes y)z)| \; \| z \|_{L_\infty^1(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2)} = q\} \\
\geq &\sup\{|tr_1 \otimes tr_2((x \otimes y)(a \otimes b))| \; \| a \otimes b \|_{L_\infty^1(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} = 1\} \\
= &\sup\{|tr_1(xa)tr_2(yb)| \; \| a \|_{L_\infty^1(\mathcal{N}_1\subset\mathcal{M}_1)} = \| b \|_{L_\infty^1(\mathcal{N}_2\subset\mathcal{M}_2)} = 1\} \\
= &\| x \|_{L_1^\infty(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_1^\infty(\mathcal{N}_2\subset\mathcal{M}_2)}
\end{aligned}$$

Thus,

$$\| x \otimes y \|_{L_1^\infty(\mathcal{N}_1\overline{\otimes}\mathcal{N}_2\subset\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)} = \| x \|_{L_1^\infty(\mathcal{N}_1\subset\mathcal{M}_1)} \| y \|_{L_1^\infty(\mathcal{N}_2\subset\mathcal{M}_2)} \quad . \tag{34}$$

1

By interpolation, we have for all $1 \leq p \leq \infty$,

$$\| x \otimes y \|_{L^p_\infty(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2 \subset \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)} = \| x \|_{L^p_\infty(\mathcal{N}_1 \subset \mathcal{M}_1)} \| y \|_{L^1_\infty(\mathcal{N}_2 \subset \mathcal{M}_2)}$$
$$\| x \otimes y \|_{L^p_1(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2 \subset \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)} = \| x \|_{L^p_1(\mathcal{N}_1 \subset \mathcal{M}_1)} \| y \|_{L^p_1(\mathcal{N}_2 \subset \mathcal{M}_2)} \ .$$

The same argument works for $M_n(\mathcal{N}_1) \subset M_n(\mathcal{M}_1)$ and $M_m(\mathcal{N}_2) \subset M_n(\mathcal{M}_2)$, which implies $L^p_\infty(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2 \subset \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)$ gives a cross operator space norm on $L^p_\infty(\mathcal{N}_1 \subset \mathcal{M}_1) \otimes L^p_\infty(\mathcal{N}_2 \subset \mathcal{M}_2)$ and the equality (equality 32). ■

A.3. **Non-tracial Cases.** In previous discussion, we considered amalgamated L_p space and conditional L_p space with respect to a normal faithful finite trace. These spaces in JPmemo [15] was studied more generally for a normal faithful state. We follow the idea of bardet2 [3] to use the non-tracial cases for non-symmetric quantum Markov semigroups. For simplicity, we consider $\mathcal{M} = M_n$ the matrix algebras equipped with normalized trace $tr(1) = 1$.

Let $(T_t)_{t \geq 0} : \mathcal{M} \to \mathcal{M}$ be a quantum Markov semigroup and

$$\mathcal{N} = \{ a \in \mathcal{M} \mid T_t(a^*a) = T_t(a^*)T_t(a) \ , T_t(aa) = T_t(a)T_t(a^*) \ , \ \forall \, t \geq 0 \}$$

be the incoherent subalgebra of T_t. Denote $(T_t^\dagger)_{t \geq 0} : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ as the adjoint semigroup on the predual. $(T_t^\dagger)_{t \geq 0}$ models the time evolution of states in Schrödinger pciture whereas $(T_t)_{t \geq 0}$ transforms observables in Heisenberg picture. We assume that $(T_t)_{t \geq 0}$ admits an invariant normal faithful state σ satisfying $T_t^\dagger(\sigma) = \sigma$. Let $E : \mathcal{M} \to \mathcal{N}$ be the σ-preserving conditional expectation onto $\mathcal{N}$. The natural reference state is

$$\sigma_0 = E^\dagger(1) \ .$$

Note that σ_0 restricted on $\mathcal{N}$ is the trace

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