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Dimension-Free Properties of Strong Muckenhoupt and Reverse Hölder Weights for Radon Measures

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Abstract In this paper, we prove self-improvement properties of strong Muckenhoupt and Reverse Hölder weights with respect to a general Radon measure on $\mathbb{R}^n$. We derive our result via a Bellman function argument. An important feature of our proof is that it uses only the Bellman function for the one-dimensional problem for Lebesgue measure; with this function in hand, we derive dimension-free results for general measures and dimensions.

Keywords Bellman function · Dimension-free estimates · Muckenhoupt weights · Reverse Hölder weights

Mathematics Subject Classification Primary 42B35 · Secondary 43A85

1 Introduction

It is well known that Muckenhoupt weights on a real line with respect to the Lebesgue measure satisfy self-improvement properties in the following sense: for $p > q$, we always have $A_q \subset A_p$; but also for any function $w \in A_p$, there is an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$ (we refer to Definition 1 for precise definitions). Besides that, there always exists a q such that $w \in RH_q$. These self-improvement properties allow one to prove many important results in harmonic analysis, see, e.g., [4] or a more recent paper [5].

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In [7], the authors considered *strong Muckenhoupt classes*; in particular, it was proven that for a Radon measure μ on $\mathbb{R}^n$ which is absolutely continuous with respect to the Lebesgue measure dx, any weight $w \in A_p^*$ satisfies a Reverse Hölder property with an exponent that does not depend on the dimension n.

For $p > 1$, we say that w belongs to the *strong Muckenhoupt class with respect to* μ, $w \in A_p^*$, if there exists a number $Q > 1$ such that for any rectangular box $R \subset \mathbb{R}^n$ with edges parallel to axis, we have

$$\langle w \rangle_R \langle w^{-1/(p-1)} \rangle_R^{p-1} \leqslant Q,$$

where $\langle \varphi \rangle_R$ denotes the average of the function φ over R:

$$\langle \varphi \rangle_R := \frac{1}{\mu(R)} \int_R \varphi(x) \mathrm{d}\mu(x).$$

For $p > 1$, we say that w belongs to the *strong Reverse Hölder class with respect to* μ, $w \in RH_p^*$, if there exists a constant $Q > 1$ such that for any rectangular box R with edges parallel to axis, we have

$$\langle w^p \rangle_R^{1/p} \leqslant Q \langle w \rangle_R.$$

We proceed with the following definition.

Definition 1 Let μ be a Radon measure on $\mathbb{R}^n$ and w be a function which is positive μ-a.e. For $p > 1$, we denote the A_p^**-characteristic of* w by

$$[w]_p := \sup_R \langle w \rangle_R \langle w^{-1/(p-1)} \rangle_R^{p-1}$$

and the *Reverse Hölder characteristic of* w by

$$[w]_{RH_p} := \sup_R \langle w^p \rangle_R^{1/p} \langle w \rangle_R^{-1},$$

where both suprema are taken over rectangular boxes R with edges parallel to axis. If $[w]_p < \infty$, we have $w \in A_p^*$ and if $[w]_{RH_p} < \infty$, we have $w \in RH_p^*$.

In [7], it was proved that if μ is an absolutely continuous Radon measure on $\mathbb{R}^n$ and $[w]_p < \infty$, then for some $q > 1$ we have $[w]_{RH_q} < C < \infty$ with an explicit dimension-free estimates on q and C. It is of a particular importance that we can take

$$q = 1 + \frac{1}{2^{p+2}[w]_p}.$$

To prove this result, the authors used a clever version of the Calderón–Zygmund decomposition from [6]. The aim of this paper is to derive a sharp result from the one-dimensional case for Lebesgue measure (i.e., for the classical A_p and RH_p classes on $\mathbb{R}$). In this case, the result from [7] can be obtained, for example, by means of the so-called Bellman function; i.e., a function of two variables that satisfies certain boundary and concavity conditions in its domain. In the one-dimensional case, this function is known explicitly, see [10]. It has been understood for some time that, for classes of functions like A_p, RH_p or BMO_p, when we work with their strong multidimensional analogs (e.g., A_p^* and RH_p^*), the one-dimensional Bellman function should prove the higher-dimensional results with dimension-free constants. For the Lebesgue measure and the inclusion $RH_p^* \subset A_q^*$, this was carried out in [1]. The trick of using the Bellman function for one-dimensional problems was also used in [2,3,9] (in a slightly different setting, the same trick was also used in [8]). In this paper, we present a simple version of this trick for general measures; we prove the result from [7] as well as all other results of self-improving type for strong Muckenhoupt and Reverse Hölder weights.

2 Statement of the Main Result

2.1 Properties of Muckenhoupt Weights A_p^*

For $p_1 := -1/(p-1)$ and every $t \in [0, 1]$, define $u_{p_1}^{\pm}(t)$ to be solutions of the equation:

$$(1-u)(1-p_1u)^{-1/p_1} = t.$$

The function $u_{p_1}^{+}$ is decreasing and maps $[0, 1]$ onto $[0, 1]$; the function $u_{p_1}^{-}$ is increasing and maps $[0, 1]$ onto $[1/p_1, 0]$. For a fixed $Q > 1$, define

$$s_{p_1}^{\pm} = s_{p_1}^{\pm}(Q) := u_{p_1}^{\pm}(1/Q). \tag{1}$$

Our first main result is as follows:

Theorem 2.1 *Let μ be a Radon measure on $\mathbb{R}^n$ with $\mu(H) = 0$ for every hyperplane H orthogonal to one of the coordinate axis. Fix numbers $p > 1$ and $Q > 1$ and set $p_1 := -1/(p-1)$. Then for every weight w with $[w]_p = Q$ we have*

$$w \in A_q^*, \qquad 1 - s_{p_1}^{-}(Q) < q < \infty,$$

and

$$w \in RH_q^*, \qquad 1 \leqslant q < 1/s_{p_1}^{+}(Q),$$

where $s_{p_1}^{\pm}(Q)$ are defined in (1). These ranges for q are sharp for $n = 1$ and $\mu = dx$.

2.2 Properties of Reverse Hölder Weights

For $p > 1$ and every $t \in [0, 1]$, we define $v_p^{\pm}(t)$ to be solutions of the equation

$$(1 - pv)^{1/p}(1 - v)^{-1} = t.$$

In this case, v_p^+ is a decreasing function that maps $[0, 1]$ onto $[0, 1/p]$ and v_p^- is an increasing function that maps $[0, 1]$ onto $[-\infty, 0]$. As before for a fixed $Q > 1$, we define

$$s_p^{\pm} = s_p^{\pm}(Q) := v_p^{\pm}(1/Q). \tag{2}$$

Our second main result concerning Reverse Hölder weights is the following.

Theorem 2.2 *Let μ be a Radon measure on $\mathbb{R}^n$ with $\mu(H) = 0$ for every hyperplane H orthogonal to one of the coordinate axis. Fix numbers $p > 1$ and $Q > 1$. Then for every weight w with $[w]_{RH_p} = Q$, we have*

$$w \in A_q^*, \qquad 1 - s_p^-(Q) < q < \infty,$$

and

$$w \in RH_q^*, \qquad 1 \leqslant q < 1/s_p^+(Q),$$

where $s_p^{\pm}(Q)$ are defined in (2). These ranges for q are sharp for $n = 1$ and $\mu = dx$.

3 Proof of the Main Results

We begin with the following Theorem from [10]. This theorem ensures the existence of a certain Bellman function for a one-dimensional problem. In what follows, by letters without sub-indices (e.g., $x, x^{\pm}$), we denote points in $\mathbb{R}^2$ and by letters with sub-indices, we denote the corresponding coordinates (e.g., x_1^+ denotes the first coordinate of x^+).

Theorem 3.1 (Theorem 1 in [10]) *Fix $p > 1$ and set $p_1 := -1/(p-1)$. Also fix an $r \in (1/s_{p_1}^-, p_1] \cup [1, 1/s_{p_1}^+)$ for $s_{p_1}^{\pm}(Q)$ defined in (1). For every $Q > 1$, there exists a non-negative function $B_Q(x)$ defined in the domain $\Omega_Q := \{x = (x_1, x_2) \in \mathbb{R}^2 : 1 \leqslant x_1 x_2^{-1/p_1} \leqslant Q\}$ with the following property: $B_Q(x)$ is continuous in x and Q, and for any line segment $[x^-, x^+] \subset \Omega_Q$ and $x = \lambda x^- + (1-\lambda)x^+$, $\lambda \in [0, 1]$, we have*

$$B_Q(x) \geqslant \lambda B_Q(x^-) + (1-\lambda) B_Q(x^+).$$

Moreover, $B(x_1, x_1^{p_1}) = x_1^r$ and $B_Q(x) \leqslant c(r, Q) x_1^r$ for some positive constant $c(r, Q)$ and every $x \in \Omega_Q$.

To use the concavity property of the function B_Q for our proof, we need the following lemma. Its proof is given in [10, Lemma4] with an interval instead of the rectangle; however, the proof remains the same in our case.

Lemma 3.2 *Let the measure μ be as before. Fix two numbers $Q_1 > Q > 1$ and a rectangular box $R \subset \mathbb{R}^n$ with edges parallel to the axis. For every coordinate vector* **e***, there exists a hyperplane H normal to* **e** *that splits R into two rectangular boxes R^1 and R^2 with the following properties:*

(i) *For $i = 1, 2$, we have $\mu(R^i)/\mu(R) \in (c, 1-c)$ for some constant $c \in (0, 1)$;*
(ii) *For every weight w with $[w]_p \leqslant Q$, we have $[x^1, x^2] \subset \Omega_{Q_1}$ and, therefore,*

$$B_{Q_1}(x_1, x_2) \geqslant \frac{\mu(R^1)}{\mu(R)} B_{Q_1}(x_1^1, x_2^1) + \frac{\mu(R^2)}{\mu(R)} B_{Q_1}(x_2^i, x_2^2),$$

where

$$x_1 = \langle w \rangle_R, \qquad x_2 = \langle w^{p_1} \rangle_R$$
$$x_1^i = \langle w \rangle_{R^i}, \qquad x_2^i = \langle w^{p_1} \rangle_{R^i}.$$

We are ready to prove our main result.

Proof of Theorem 2.1 Fix a rectangular box R with edges parallel to the axis, and take any $Q_1 > Q$. We first explain how we split R into two rectangular boxes. Take one of the $(n-1)$-dimensional faces of R, call it R_{n-1}, which has the largest $(n-1)$-area. Among all $(n-2)$-dimensional faces of R_{n-1}, take one of those (call it R_{n-2}) that have the largest $(n-2)$-area. We proceed like this to get $R_{n-1}, \ldots, R_1$. Now take a vector **e** that is orthogonal to every R_i, $i = 1, \ldots, n-1$.[1] We now split R according to Lemma 3.2. Notice that all the corresponding i-dimensional faces of R^1 and R^2 have smaller i-areas than the corresponding i-dimensional faces of R. We now take the boxes R^1 and R^2 and repeat the same procedure. If we repeat this M times, we get a family of rectangular boxes $\mathscr{R} = \{R^{i,M}\}_{i=1\ldots 2^M}$. Denote

$$x_1 = \langle w \rangle_R, \qquad x_2 = \langle w^{p_1} \rangle_R$$
$$x_1^{i,M} = \langle w \rangle_{R^{i,M}}, \qquad x_2^{i,M} = \langle w^{p_1} \rangle_{R^{i,M}}.$$

Abusing the notation, we also define step-functions

$$x_1^M(t) := \sum_{i=1}^{2^M} x_1^{i,M} \mathbb{1}_{R^{i,M}}(t), \qquad x_2^M(t) := \sum_{i=1}^{2^M} x_2^{i,M} \mathbb{1}_{R^{i,M}}(t).$$

From the construction of rectangular boxes, we notice that $x_1^M(t) \to w(t)$ and $x_2^M(t) \to w^{p_1}(t)$ as $M \to \infty$ for μ-a.e. $t \in R$. Indeed, our splitting procedure (and the fact that we have $\mu(R^i)/\mu(R) \in (1-c, c)$ at every step) guarantees that

$$\max_{i=1,\ldots,2^M} \operatorname{diam}(R^{i,M}) \to 0, \quad M \to \infty,$$

[1] In the case that $n = 2$, we just take **e** orthogonal to the longest side of R.

and we obtain the convergence of $x_1^M(t)$ and $x_2^M(t)$ from the Lebegue differentiation theorem for Radon measures. Therefore,

$$B_{Q_1}(x_1, x_2) \geqslant \sum_{i=1}^{2^M} \frac{\mu(R^{i,M})}{\mu(R)} B_{Q_1}(x_1^{i,M}, x_2^{i,M}) = \frac{1}{\mu(R)} \int_R B_{Q_1}(x_1^M(t), x_2^M(t)) \mathrm{d}\mu(t).$$

By the Fatou lemma,

$$\begin{aligned} B_{Q_1}(x_1, x_2) &\geqslant \frac{1}{\mu(R)} \int_R \lim_{M\to\infty} B_{Q_1}\left(x_1^M(t), x_2^M(t)\right) \mathrm{d}\mu(t) \\ &= \frac{1}{\mu(R)} \int_R B_{Q_1}(w(t), w^{p_1}(t)) \mathrm{d}\mu(t) = \frac{1}{\mu(R)} \int_R w^r(t) \mathrm{d}t = \langle w^r \rangle_R. \end{aligned} \tag{3}$$

Since $B_{Q_1}(x_1, x_2)$ is continuous in Q_1 and the above estimate holds for any $Q_1 > Q$, we get

$$c(r, Q)\langle w\rangle_R^r = c(r, Q)x_1^r \leqslant \langle w^r \rangle_R.$$

If we use this estimate for $q = r \in [1, 1/s_{p_1}^+)$, we obtain $w \in RH_q^*$. If we use this estimate for $-1/(q-1) = r \in (1/s_{p_1}^-, p_1]$, we obtain $w \in A_q^*$ for $1 - s_{p_1}^-(Q) < q < \infty$. □

To prove Theorem 2.2, we need to use a different Bellman function B_Q. Namely, the following result holds.

Theorem 3.3 (Theorem 1 in [10]) *Fix* $p > 1$ *and* $r \in (1/s_p^-, 1] \cup [p, 1/s_p^+)$ *for* $s_p^\pm$ *defined in* (2). *For every* $Q > 1$, *there exists a non-negative function* $B_Q(x)$ *defined in the domain* $\Omega_Q := \{x = (x_1, x_2) \in \mathbb{R}^2 \colon 1 \leqslant x_1 x_2^{-1/p} \leqslant Q\}$ *with the following property:* $B_Q(x)$ *is continuous in* x *and* Q, *and for any line segment* $[x^-, x^+] \subset \Omega_Q$ *and* $x = \lambda x^- + (1-\lambda)x^+$, $\lambda \in [0, 1]$, *we have*

$$B_Q(x) \geqslant \lambda B_Q(x^-) + (1-\lambda) B_Q(x^+).$$

Moreover, $B(x_1, x_1^p) = x_1^r$ *and* $B_Q(x) \leqslant c(r, Q)x_1^r$ *for some positive constant* $c(r, Q)$ *and every* $x \in \Omega_Q$.

We also notice that the analog of Lemma 3.2 reads the same, and with this in hand, the proof of Theorem 2.1 is analogous to the proof of Theorem 2.2; we leave the details to the reader.

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References

1. Dindoš, M., Wall, T.: The sharp A_p constant for weights in a reverse-Hölder class. Rev. Mat. Iberoam. **25**(2), 559–594 (2009)
2. Domelevo, K., Petermichl, S., Wittwer, J.: A linear dimensionless bound for the weighted Riesz vector. Bull. Sci. Math. **141**(5), 385–407 (2017)
3. Dragičević, O., Volberg, A.: Bellman functions and dimensionless estimates of Littlewood-Paley type. J. Oper. Theory **56**(1), 167–198 (2006)
4. García-Cuerva, J., de Francia, J.L.R.: Weighted norm inequalities and related topics. North-Holland Mathematics Studies, vol. 116. North-Holland Publishing Co., Amsterdam. Notas de Matemática [Mathematical Notes], 104 (1985)
5. Hytönen, T., Pérez, C.: Sharp weighted bounds involving A_∞. Anal. PDE **6**(4), 777–818 (2013)
6. Korenovskyy, A.A., Lerner, A.K., Stokolos, A.M.: On a multidimensional form of F. Riesz's, "rising sun" lemma. Proc. Am. Math. Soc. **133**(5), 1437–1440 (2005)
7. Luque, T., Pérez, C., Rela, E.: Reverse Hölder property for strong weights and general measures. J. Geom. Anal. **27**(1), 162–182 (2017)
8. Nazarov, F., Reznikov, A., Treil, S., Volberg, A.: A Bellman function proof of the L^2 bump conjecture. J. Anal. Math. **121**, 255–277 (2013)
9. Treil, S.: Sharp A_2 estimates of Haar shifts via Bellman function. In: Recent Trends in Analysis, Theta Ser. Adv. Math., vol. 16, pp. 187–208. Theta, Bucharest (2013)
10. Vasyunin, V.I.: Mutual estimates for L^p-norms and the Bellman function. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), (Issledovaniya po Lineınym Operatoram i Teorii Funktsiı. 36) **355**, 81–138, 237–238 (2008)