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SYLOW SUBGROUPS, EXPONENTS AND CHARACTER VALUES

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ABSTRACT. If G is a finite group, p is a prime, and P is a Sylow p-subgroup of G, we study how the exponent of the abelian group P/P' is affected and affects the values of the complex characters of G. This is related to Brauer's Problem 12. How this is exactly done is one of the last unsolved consequences of the McKay–Galois conjecture.

1. INTRODUCTION

In Problem 12 of the celebrated list [Br], Richard Brauer asks: *Given the character table of a group G and a prime p dividing $n = |G|$, how much information about the structure of the p-Sylow group P can be obtained?*

In [IN], it was proved that if G is a finite p-solvable group and p is a prime, then a single Galois automorphism and the character table of G determined the exponent of P/P', and it was hinted that this could hold true for every finite group. (Here in this paper $P' = [P, P]$ is the commutator subgroup of P, and recall that its exponent is the smallest prime power p^e such that $x^{p^e} = 1$ for all $x \in P/P'$.) This was the origin of a wider generalization of the McKay conjecture: the so-called *McKay–Galois* conjecture in [N2] which implies the following.

Conjecture A. *Let $e \geq 1$ be an integer. Let σ_e be the Galois automorphism of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}})$ that fixes roots of unity of order not divisible by p, and sends p-power roots of unity ξ to ξ^{1+p^e}. Let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$. Then the exponent of P/P' is less than or equal to p^e if and only if all the irreducible characters of p'-degree of G are σ_e-fixed.*

In the first main result of this paper, we prove the *if* direction of Conjecture A. In fact, a stronger result is obtained.

Theorem B. *Let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$. If all the irreducible characters of p'-degree of the principal p-block of G are σ_e-fixed, then the exponent of P/P' is less than or equal to p^e.*

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If the proof of Theorem B already depends on the classification of finite simple groups and on delicate properties of their characters, the *only if* direction of Conjecture A seems to lie even deeper. It is quite exciting too since it proposes that a *small* abelian p-group P/P' affects the character values of every finite group G that happens to have P as a Sylow p-subgroup.

Our second main result reduces the *only if* direction of Conjecture A to decorated simple groups.

Theorem C. *Conjecture A is true for every finite group, if it is true for almost quasi-simple groups.*

In the last section of this paper, we prove that certain almost quasi-simple groups satisfy Conjecture A, giving further evidence of its truth. (G. Malle has informed us that using Theorem C he has proved Conjecture A for $p = 2$ very recently in [M].) As pointed out in several places, the present knowledge of the actions of $\mathrm{Aut}(S)$ and $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on the set $\mathrm{Irr}(S)$ of the irreducible characters of a simple group of Lie type S is not enough to fully answer questions as Conjecture A, for the time being. This appears to be one of the main problems of the representation theory of finite groups today.

We have mentioned that Conjecture A is implied by the McKay-Galois conjecture. (For a proof, see Theorem 9.12 of [N3].) At the time of this writing, there is a draft of a reduction of the McKay–Galois conjecture to a problem on simple groups in [NSV]. As happens with the reduction of the McKay conjecture, and unlike our Theorem C, this is not a straight reduction to decorated simple groups, but something far more complex. In any case, there are no shortcuts: Conjecture A (as well as the McKay–Galois conjecture) will need to be proved for decorated simple groups eventually, and this, as we have said, will require a much better understanding of the values of the characters of simple groups and of their extensions than is currently available. We consider our Theorem B as a contribution to this problem.

For $p = 2$, Theorem B bears a similarity with a conjecture of R. Gow that we proved in [NT1]: *If G is a finite real group, then P/P' is elementary abelian, where $P \in \mathrm{Syl}_2(G)$.* In fact, the main result of [NT1] is that P/P' is elementary abelian if all the odd-degree irreducible characters in the principal 2-block of G are real-valued. Unlike Conjecture A, the converse of this result is not true (outside solvable groups). This shows, again, that in the global/local questions only the Galois automorphisms described in [N2] seem to behave perfectly.

As a corollary of Theorem B, we do have, however, a new result on a classical family of groups: the rational groups. In fact, our result holds more generally for *quadratic-rational* groups. Recall that if G is a finite group and $\chi \in \mathrm{Irr}(G)$, then $\mathbb{Q}(\chi)$ is the subfield of $\mathbb{C}$ generated by the values of χ.

Corollary D. *Let G be a finite group, let p be an odd prime, and let $P \in \mathrm{Syl}_p(G)$. Assume that $|\mathbb{Q}(\chi) : \mathbb{Q}| \leq 2$ for all $\chi \in \mathrm{Irr}(G)$ of p'-degree in the principal p-block of G. Then P/P' is elementary abelian.*

Proof. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ be fixing p'-roots of unity and sending every p-power root of unity ξ to ξ^{1+p}. Let $m = |G|_{p'}$, and let $n = |G|$. Now, for any $\chi \in \mathrm{Irr}(G)$ of p'-degree in the principal p-block of G, we have that the extension $\mathbb{Q}_m(\chi)/\mathbb{Q}_m$ has degree 1 or 2. Also $\mathbb{Q}_m(\chi)/\mathbb{Q}_m$ is contained in $\mathbb{Q}_n/\mathbb{Q}_m$, which has cyclic Galois group. Using Gauss sums, the only sub-extension of degree 2 of $\mathbb{Q}_n/\mathbb{Q}_m$ is $\mathbb{Q}_m(i\sqrt{p})$ or $\mathbb{Q}_m(\sqrt{p})$ depending on the congruence of p mod 4. In any case, $i\sqrt{p}$ or $\sqrt{p}$ are sums of roots of unity of order dividing p, so they are fixed by σ. So χ is fixed by σ, and Theorem B applies for $\sigma = \sigma_1$. $\square$

Note that the $p = 2$ analogue of Corollary D is not true. For instance, all the irreducible odd-degree characters of $G = \mathrm{SU}_3(3)$ are in the principal 2-block of G, are rational-valued or have quadratic field of values. However, if P is a Sylow 2-subgroup of G, then $P/P' = C_2 \times C_4$. Of course, this cannot happen if G is a rational group, by the main result of [NT1] mentioned above.

Finally, let us mention that after the recent result in [SF], the *if* direction of Conjecture A is one of the last unproven consequences of the McKay-Galois conjecture.

2. Theorem B

In this section, we prove Theorem B assuming that Theorem 2.3 below on almost-simple groups is true.

Our notation for characters follows [Is], while the notation for blocks follows [N1]. If G is a finite group, then $\mathrm{Irr}(G)$ is the set of the irreducible complex characters of G. If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of irreducible characters $\chi \in \mathrm{Irr}(G)$ such that $[\chi_N, \theta] \neq 0$. If p is a prime, then $\mathrm{Irr}_{p'}(G)$ is the set of $\chi \in \mathrm{Irr}(G)$ of degree $\chi(1)$ not divisible by p, and $\mathrm{Irr}_{p'}(G|\theta) = \mathrm{Irr}(G|\theta) \cap \mathrm{Irr}_{p'}(G)$. We denote by $B_0(G)$ the principal p-block of G, and by $\mathrm{Irr}_{p'}(B_0(G))$ the complex irreducible characters in it of p'-degree.

About our Galois automorphism $\sigma = \sigma_e \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}})$, we notice the following. If G is a finite group of order dividing some integer n, then, by elementary number theory, we see that the restriction τ of σ to the n^{th} cyclotomic field $\mathbb{Q}_n$ has order a power of p, and τ acts like σ on the ordinary characters of every group of order a divisor of n.

We will use several times the following easy result.

Lemma 2.1. *Let $H \leq G$ be a finite group and let A be a p-group for some prime p. Suppose that A acts on the characters of G and H such that $[\chi^a, \psi^a] = [\chi, \psi]$ for all the characters χ, ψ of G (and of H) and such that $(\chi_H)^a = (\chi^a)_H$ for every character χ of G and every $a \in A$.*

(i) *Let ψ be an A-invariant character of G. If there exists an A-invariant $\xi \in \mathrm{Irr}(H)$ with $[\psi_H, \xi]$ not divisible by p, then there exists some A-invariant $\tau \in \mathrm{Irr}(G)$ such that $[\psi, \tau][\tau_H, \xi]$ is not divisible by p.*

(ii) *If ψ is an A-invariant p'-degree character of G, then ψ has an A-invariant p'-degree irreducible constituent with p'-multiplicity.*

Proof. Notice that A permutes the irreducible characters of G and of H, since $[\chi^a, \chi^a] = [\chi, \chi]$ for characters χ of G (and of H).

(i) We have that

$$[\psi_H, \xi] = \sum_{\tau \in \mathrm{Irr}(G)} [\psi, \tau][\tau_H, \xi].$$

Since we have that $[\psi, \tau^a][(\tau^a)_H, \xi] = [\psi, \tau][\tau_H, \xi]$ using our hypotheses, we deduce that

$$[\psi_H, \xi] \equiv \sum_{\tau \in \mathrm{Irr}_A(G)} [\psi, \tau][\tau_H, \xi] \bmod p$$

where $\mathrm{Irr}_A(G)$ is the set of irreducible A-invariant characters of G. Since $[\psi_H, \xi]$ is not divisible by p, the first part easily follows.

Part (ii) follows from part (a) by setting $H = 1$. □

The first two parts of the following lemma are trivial, while the third (due to M. Murai) lies deeper.

Lemma 2.2. *Let G be a finite group, and let $N \triangleleft G$.*

(i) *We have that $B_0(G/N) \subseteq B_0(G)$.*

(ii) *If H_i are finite groups and $\gamma_i \in \mathrm{Irr}(B_0(H_i))$, then*
$$\gamma_1 \times \cdots \times \gamma_t \in \mathrm{Irr}(B_0(H_1 \times \cdots \times H_t)).$$
(iii) *Suppose that $\theta \in \mathrm{Irr}(B_0(N))$ has p'-degree and extends to NP, where $P \in \mathrm{Syl}_p(G)$. Then there exists $\chi \in \mathrm{Irr}(B_0(G))$ of p'-degree over θ.*

Proof. See Lemma 2.6 of [NT1]. □

Next is the exact result that we need from almost simple groups in order to prove Theorem B, and whose proof we defer until the next section.

Theorem 2.3. *Let p be a prime, and $e \geq 1$. Suppose that $S \lhd G$, where S is a non-abelian finite simple group, G/S is a p-group, and $\mathbf{C}_G(S) = 1$. Let $P \in \mathrm{Syl}_p(G)$ and $Q = P \cap S$. If all the P-invariant $\chi \in \mathrm{Irr}_{p'}(S)$ in the principal p-block of S are σ_e-fixed, then every linear P-invariant character of Q is σ_e-fixed.*

Now, we are ready to prove Theorem B.

Theorem 2.4. *Let G be a finite group. Let $\sigma = \sigma_e$, where $e \geq 1$ is some integer. If all the characters in $\mathrm{Irr}_{p'}(B_0(G))$ are σ-fixed, then the exponent of P/P' is at most p^e, where $P \in \mathrm{Syl}_p(G)$.*

Proof. We may assume that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, so that σ has p-power order. We assume that all characters in $\mathrm{Irr}_{p'}(G)$ in the principal p-block of G are σ-fixed, and by induction on $|G|$, we prove that the exponent of P/P' is at most p^e. If λ is a linear character of P, notice that
$$\lambda^\sigma = \lambda^{1+p^e}.$$
Therefore, using that P/P' is isomorphic to the group of linear characters of P, it is enough to prove that all the linear characters of P are σ-fixed. Let $1 < K$ be a minimal normal subgroup of G. Then we know by induction that all the characters of $PK/P'K$ are σ-fixed (using that $B_0(G/K)$ is contained in $B_0(G)$ by Lemma 2.2(ii)). We may assume then that $\mathbf{O}_{p'}(G) = 1$.

Suppose now that K is a p-group. Let $\lambda \in \mathrm{Irr}(P)$ be linear. Write $\theta = \lambda_K \in \mathrm{Irr}(K)$ and let T be the stabilizer of θ in G. Then θ is σ-fixed, because K is elementary abelian. Now, by using the Schur-Zassenhaus theorem, we can write $P\mathbf{C}_G(P) = P \times U$, and consider $\hat{\lambda} = \lambda \times 1_U \in \mathrm{Irr}(P\mathbf{C}_G(P))$. Now, by Lemma (6.4) of [N1] and the Third Main Theorem (6.7) of [N1], there exists some irreducible constituent $\psi \in \mathrm{Irr}(T)$ of p'-degree, over $\hat{\lambda}$, in the principal p-block of T. Since ψ lies over θ, then ψ^G is irreducible by the Clifford correspondence. Also, ψ^G has p'-degree and lies in the principal p-block of G (by Corollary (6.2) of [N1] and the Third Main Theorem). We deduce that ψ^G is σ-fixed by hypothesis. By the uniqueness of the Clifford correspondence (using that θ is σ-fixed), then we have that ψ is σ-fixed. Now, by Lemma 2.1(ii) with $A = \langle\sigma\rangle$, we have that ψ_P has some σ-invariant linear constituent τ. Since $\tau_K = \theta = \lambda_K$, we have that $\lambda = \tau\rho$, for some linear $\rho \in \mathrm{Irr}(P/K)$ (by Gallagher's Corollary (6.17) of [Is]). By induction, recall that ρ is σ-fixed. Therefore λ is σ-fixed.

Hence, we may assume that K is non-abelian of order divisible by p. Next, we claim that $KP = G$. Let $\psi \in \mathrm{Irr}(KP)$ be of p'-degree in the principal block of KP. We claim that ψ is σ-invariant. We have that $\psi_K = \theta$ is irreducible and P-invariant by Corollary (11.29) of [Is]. By Corollary (9.2) of [N1], we deduce that θ is in the principal block of K. By Lemma 2.2(iii), there exists some $\chi \in \mathrm{Irr}(G)$ in the principal p-block of G over θ of p'-degree. By hypothesis, χ is σ-invariant. By Lemma 2.1(ii), χ_{KP} has some irreducible constituent τ which is σ-invariant of p'-degree. Now, $\tau_K \in \mathrm{Irr}(K)$ is σ-invariant and G-conjugate to θ, so we deduce that θ is σ-invariant. Now, since θ has p'-degree and has trivial determinant (because K is perfect), we have by Corollary (6.28) of [Is] that θ has a canonical extension $\hat{\theta}$ to KP. By uniqueness, $\hat{\theta} \in \mathrm{Irr}(KP)$ is also σ-invariant. Then, by Gallagher, $\psi = \hat{\theta}\lambda$, for some linear $\lambda \in \mathrm{Irr}(KP/K)$, which we know is σ-invariant by induction. Thus ψ is σ-invariant. Hence, we may assume that $G = KP$.

Let $S \triangleleft K$ be non-abelian simple. Let $H = \mathbf{N}_G(S)$. Thus $G = HP$ and $Q = P \cap H \in \mathrm{Syl}_p(H)$. Let $R = K \cap P = K \cap Q \in \mathrm{Syl}_p(K)$, and let $R_1 = R \cap S = P \cap S = Q \cap S \in \mathrm{Syl}_p(S)$. We can write $K = S^{x_1} \times \cdots \times S^{x_t}$, where $P = \bigcup_{j=1}^t Qx_j$ is a disjoint union, with $x_1 = 1$. (We use here that $G = \bigcup_{j=1}^t Hx_j$ is also a disjoint union.) Notice that

$$R = R_1^{x_1} \times \cdots \times R_1^{x_t}\,.$$

Furthermore, we claim that $Q = \mathbf{N}_P(R_1)$. Since $R_1 = P \cap S$ and $Q = \mathbf{N}_P(S)$, it follows that $Q \leq \mathbf{N}_P(R_1)$. Conversely, suppose that $z \in \mathbf{N}_P(R_1)$. Let $1 \neq v \in R_1$. Then $v^z \in R_1 \leq S$. On the other hand $v^z \in S^z = S^{x_j}$ for some j and $v^z \in S \cap S^{x_j}$. Necessarily $S^{x_j} = S = S^z$ and $z \in \mathbf{N}_P(S) = Q$.

Now, let $C = \mathbf{C}_G(S)$. Thus $S \leq K \leq SC \leq H$, and H/C is almost simple with H/SC a p-group. Also $QC/C \in \mathrm{Syl}_p(H/C)$. Using that $C \cap S = 1$, we have that $R_1C/C \in \mathrm{Syl}_p(SC/C)$, and therefore $(QC/C) \cap (SC/C) = R_1C/C = (R_1 \times C)/C$. We wish to apply Theorem 2.3 to H/C. Let $\gamma \in \mathrm{Irr}_{p'}(SC/C)$ be QC/C-invariant in the principal p-block. Since $C \cap S = 1$, we have that $\gamma_S = \tau \in \mathrm{Irr}(S)$ is H-invariant of p'-degree in the principal p-block. By Lemma 4.1 of [NTT1], we have that $\rho = \tau^{x_1} \times \cdots \times \tau^{x_t} \in \mathrm{Irr}(K)$ is G-invariant of p'-degree (and in the principal p-block, using Lemma 2.2(ii)). Now, ρ has a canonical extension $\hat{\rho}$ to G (by Corollary (6.28) of [Is]), which lies in the principal p-block (using Corollary (9.6) of [N1]) and therefore is σ-invariant by hypothesis. In particular, ρ is σ-invariant. Since ρ_S is a multiple of τ, we conclude that τ (and therefore γ) are σ-invariant. By Theorem 2.3, we have that all the Q-invariant linear characters of R_1 are σ-fixed.

Finally, let $\lambda \in \mathrm{Irr}(P)$ be linear. Then $\nu_1 = \lambda_{R_1}$ is Q-invariant, and therefore is σ-fixed. Also, since $x_i \in P$, we have that $\lambda_{R_1^{x_i}} = (\nu_1)^{x_i}$, and we conclude that $\nu = \lambda_R = \nu_1^{x_1} \times \cdots \times \nu_1^{x_t}$ is σ-fixed. Now, by applying Lemma 2.1(ii) to the group $A = P \times \langle \sigma \rangle$, we have that ν^K contains an irreducible constituent $\eta \in \mathrm{Irr}(K)$ which is P-invariant and σ-invariant, of p'-degree with $[\nu^K, \eta] = [\eta_R, \nu]$ not divisible by p. Let $\psi = \hat{\eta}$ be the canonical extension of η to G. By uniqueness, we have that ψ is σ-invariant. Also, $[\psi_R, \nu] = [\eta_R, \nu]$ is not divisible by p. We have that $\rho = \psi_P$ is σ-invariant with $[\rho_R, \nu] \not\equiv 0 \bmod p$. Since ν is σ-invariant, by Lemma 2.1(i) it follows that ρ has a σ-invariant constituent $\mu \in \mathrm{Irr}(P)$ such that $[\mu_R, \nu]$ is not divisible by p. Since ν is R-invariant and P is a p-group, it follows that $\mu_R = \nu$. Now, $\lambda = \epsilon\mu$, by Gallagher, for some linear $\epsilon \in \mathrm{Irr}(P/R)$. Since the linear characters of P/R are the linear characters of G/K, and they are σ-invariant by induction, then the proof is complete. $\square$

3. Almost Simple Groups. I

In this and the next section we prove Theorem 2.3, thus completing the proof of Theorem B.

Definition 3.1. Let p be a prime and a be any positive integer. A finite p-group P is called *p^a-good*, if $\exp(P/P') \leq p^a$. A finite group G is called *p^a-good* if $P \in \mathrm{Syl}_p(G)$ is p^a-good.

In what follows, N_p denotes the p-part of any positive integer N. We begin with some elementary observations.

Lemma 3.2. *Let p be a prime and a any positive integer, and let P, Q, R be finite p-groups.*

(i) *The set $P[a] := \{x \in P \mid x^{p^a} \in P'\}$ is a normal subgroup of P. Furthermore, P is p^a-good if and only $P[a] = P$. Also, P is p-good if and only if $P' = \Phi(P)$.*

(ii) *Suppose that Q and R are p^a-good subgroups of P. Then $\langle Q, R\rangle$ as well as any quotient of Q are p^a-good.*

(iii) *Suppose that* $P = \langle P', x_1, \dots, x_m\rangle$ *with* $x_i^{p^a} \in P'$. *Then* P *is* p^a*-good.*

Proof. See [NT1, Lemma 3.2]. $\square$

Lemma 3.3. *Let* p *be a prime and let* G *be a finite group.*

(i) *Suppose that* $\mathbf{N}_G(P) = P \times A$ *for* $P \in \mathrm{Syl}_p(G)$ *and some subgroup* A. *Then the number of* p*-blocks of* G *of maximal defect is* $|\operatorname{Irr}(A)|$.

(ii) *Suppose that* $S = G/N$ *for some* p'*-subgroup* $N \lhd G$. *Let* $\alpha, \beta \in \operatorname{Irr}(S)$ *and view* α, β *as characters of* G. *Then* α, β *belong to the same* p*-block of* S *if and only if* α, β *belong to the same* p*-block of* G. *Furthermore,* $\operatorname{Irr}(B_0(G)) = \operatorname{Irr}(B_0(S))$.

(iii) *Suppose that* G *has only one conjugacy class of* p*-central subgroups of order* p, $P \in \mathrm{Syl}_p(G)$ *is non-abelian, and* $C \leq \mathbf{Z}(P)$ *has order* p. *Then* $C \leq P'$.

(iv) *Suppose that* $A \leq B$ *are normal subgroups of* G *such that* A *and* G/B *are* p*-groups. If* B/A *has a self-normalizing Sylow* p*-subgroup, then so does* G.

Proof. See [NT1, Lemma 2.7]. $\square$

We will use the following consequences of Lemma 2.1.

Lemma 3.4. *Let* p *be any prime,* $e \geq 1$*, and let* $R \leq G$ *be finite groups.*

(i) *Let* $R \leq H \leq G$ *and let* $\chi \in \operatorname{Irr}(G)$ *be* σ_e*-fixed. Suppose there exists a* σ_e*-fixed* $\nu \in \operatorname{Irr}(R)$ *such that* $p \nmid [\chi_R, \nu]$. *Then there exists a* σ_e*-fixed* $\gamma \in \operatorname{Irr}(H)$ *such that*
$$p \nmid [\chi_H, \gamma] \cdot [\gamma_R, \nu].$$

(ii) *Let* $R \leq K \leq G$ *be such that* $p \nmid [K : R]$, *and let* P *be a* p*-subgroup of* G *normalizing both* K *and* R. *If there exists a* σ_e*-fixed* P*-invariant* $\nu \in \operatorname{Irr}(R)$ *of* p'*-degree, then there exists a* σ_e*-fixed* P*-invariant* $\gamma \in \operatorname{Irr}(K)$ *of* p'*-degree such that* $p \nmid [\gamma_R, \nu]$.

Proof. (i) Apply Lemma 2.1(i) to $\psi = \chi_H$ and $A = \langle \sigma_e \rangle$. (ii) Apply Lemma 2.1(ii) to $\psi = \nu^R$ and $A = \langle \sigma_e \rangle \times P$. $\square$

The following result generalizes [NT1, Lemma 2.8]:

Lemma 3.5. *Let* p *be a prime,* $e \geq 1$*, and let* $N \leq G$ *be normal subgroups of a finite group* $H = RN$ *with* $p \nmid |N/N'|$ *and* $R \in \mathrm{Syl}_p(H)$. *Let* $P := R \cap G$, $Q := R \cap N$, *and suppose that*

(a) *every* R*-invariant linear character of* P/Q *is* σ_e*-fixed, and*

(b) *every* R*-invariant linear character of* Q *is* σ_e*-fixed.*

Then every R*-invariant linear character of* P *is* σ_e*-fixed.*

Proof. Let $\lambda \in \operatorname{Irr}(P/P')$ be R-invariant. Then $\nu = \lambda_Q$ is R-invariant, and so σ_e-fixed according to (b). Note that R normalizes both N and Q. By Lemma 3.4(ii) applied to $Q \leq N \leq G$, there exists $\eta \in \operatorname{Irr}(N)$ which is R-invariant (and so H-invariant), of p'-degree, σ_e-fixed, with $p \nmid [\eta_Q, \nu]$. As $p \nmid |N/N'|$, the determinantal order $o(\eta)$ is coprime to p. On the other hand, $H/N \cong R/(R \cap Q)$, so by [Is, Corollary (8.16)], η has a unique extension χ to H with $o(\chi) = o(\eta)$. In particular, χ is σ_e-fixed, and $[\chi_Q, \nu] = [\eta_Q, \nu]$ is coprime to p. By Lemma 3.4(i) applied to $Q \leq R \leq H$, χ_R contains a σ_e-fixed character $\xi \in \operatorname{Irr}(R)$ such that $p \nmid [\xi_Q, \nu]$. As R/Q is a p-group and ν is R-invariant, it follows that $\xi_Q = \nu$. Clearly, ξ_P is R-invariant and $(\xi_P)_Q = \nu = \lambda_Q$. Hence $\lambda = \epsilon(\xi_P)$ for some R-invariant linear $\epsilon \in \operatorname{Irr}(P/Q)$ by Gallagher's Corollary (6.17) of [Is]. According to (a), ϵ is σ_e-fixed, and so λ is σ_e-fixed, as stated. $\square$

Corollary 3.6. *Let* p *be a prime,* $e \geq 1$*, and let* N *be a normal subgroup of a finite group* G *with* $p \nmid |N/N'|$. *Suppose that both* N *and* G/N *are* p^e*-good. Then* G *is* p^e*-good.*

Proof. We may take $P \in \mathrm{Syl}_p(G)$ and assume $G = PN$. Now take $R = P$, $H = G$, and apply Lemma 3.5. $\square$

Lemma 3.7. *Let p be any prime and let n be any positive integer. Then A_n, S_n, and all 26 sporadic finite simple groups are p-good.*

Proof. See [NT1, Lemmas 3.3, 3.4]. $\square$

Proposition 3.8. *Let p be any prime and let $S \not\cong {}^2F_4(2)'$ be a simple group of Lie type in characteristic p. Then S is p-good.*

Proof. The case $p = 2$ was already treated in [NT1, Proposition 4.5]. So let's assume that $p > 2$ and let $P \in \mathrm{Syl}_p(S)$. If $(S, p) \neq (G_2(q), 3)$, then P/P' is elementary abelian by [GLS, Theorem 3.3.1(b)], and so we are done. But even in the case $(S, p) = (G_2(q), 3)$, P can be chosen to be generated by root subgroups X_α, α a positive root, and all X_α are elementary abelian. It follows by Lemma 3.2(iii) that P is p-good. $\square$

In what follows, we use the notation SL^ϵ to denote SL if $\epsilon = +$ and SU if $\epsilon = -$, and similarly for GL^ϵ. We also use the notation E_6^ϵ to denote E_6 when $\epsilon = +$ and ${}^2E_6$ when $\epsilon = -$. Slightly abusing the notation, we will treat ϵ with $\epsilon = \pm$ as $\epsilon 1$ in expressions like $q - \epsilon$, etc.

Proposition 3.9. *Let S be a simple group of Lie type in odd characteristic and let $p = 2$. If S is one of the following groups*

(a) $\mathrm{PSL}^\epsilon_{2m}(q)$,*where either $4|(q-\epsilon)$ and m is a 2-power, or if $4|(q+\epsilon)$,*
(b) $\mathrm{PSp}_{2m}(q)$, $P\Omega_{2m+1}(q)$, $P\Omega^{\pm}_{2m}(q)$, ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_7(q)$, $E_8(q)$,
(c) $E_6^\epsilon(q)$ *with* $q \equiv -\epsilon (\mathrm{mod}\, 4)$,

then S is p-good.

Proof. This statement follows immediately from Propositions 3.5, 3.7, 3.8, Corollary 3.9, and Proposition 4.1 of [NT1]. $\square$

Corollary 3.10. *Theorem 2.3 holds if (S, p) is one of the cases listed in Lemma 3.7, Proposition 3.8, or Proposition 3.9.*

Proof. In all of the listed cases, S is p-good, hence the conclusion of Theorem 2.3 holds for all $e \geq 1$. $\square$

Next we observe that the proof of [NT1, Proposition 2.9] also yields the following result, which will be useful in constructing irreducible characters belonging to the principal p-block of finite simple groups of Lie type. We refer the reader to [C], [DM] for basics of the Deligne-Lusztig theory.

Proposition 3.11. *Let p be a prime, $e \geq 1$, and let $\mathcal{G}$ be a simple algebraic group over a field of characteristic $\ell \neq p$ of adjoint type. Let $F : \mathcal{G} \to \mathcal{G}$ be a Frobenius endomorphism, $G := \mathcal{G}^F$, $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and let $G^* := (\mathcal{G}^*)^{F^*}$, $S := [G, G]$. Let $s \in G^*$ be a semisimple element. Then the following statements hold.*

(i) *Suppose that s is a p-element. Then the semisimple character χ_s corresponding to s belongs to the principal p-block $B_0(G)$ of G. Furthermore, every irreducible constituent of $(\chi_s)_S$ belongs to the principal p-block of S.*
(ii) *Suppose that s is not G^*-conjugate to sz whenever $1 \neq z \in \mathbf{Z}(G^*)$. Then $\theta := (\chi_s)_S \in \mathrm{Irr}(S)$. More generally, any $\varphi \in \mathcal{E}(G, (s))$ is irreducible over S. Moreover, if $k \in \mathbb{Z}$ is such that s^k is not G^*-conjugate to sz whenever $1 \neq z \in \mathbf{Z}(G^*)$, then $(\chi_s)_S = (\chi_{s^k})_S$ if and only if s and s^k are conjugate in G^*. In particular, if s^{1+p^e} is not G^*-conjugate to any sz with $1 \neq z \in \mathbf{Z}(G^*)$, then θ is σ_e-fixed if and only if s and s^{1+p^e} are conjugate in G^*.*

(iii) *Suppose that* $\gcd(|s|, |\mathbf{Z}(G^*)|) = 1$. *Then s is not G^*-conjugate to sz whenever $1 \neq z \in \mathbf{Z}(G^*)$.*

Lemma 3.12. *Let p be a prime and let G be a finite group. Suppose that $s \in G$ is a p-central p-element and there exists an integer k such that*
(a) *s^k and s are conjugate in G; and*
(b) *either $k \equiv 1 (\bmod\, p)$, or $|s|$ divides $k^{p^c} - 1$ for some $c \in \mathbb{Z}_{\geq 0}$.*
Then in fact $s^k = s$.

Proof. The statement obviously holds if $s = 1$. So we will assume that $|s| = p^a$ for some $a \in \mathbb{Z}_{\geq 1}$. In this case, we note that the two conditions in (b) are actually equivalent. (Indeed, if $p|(k-1)$, then $p^a|(k^{p^{a-1}} - 1)$. Conversely, if $p^a|(k^{p^c} - 1)$, then $k \equiv k^{p^c} \equiv 1 (\bmod\, p)$.)

Since s is p-central, $\mathbf{C}_G(s)$ contains a Sylow p-subgroup P of G, and so $s \in \mathbf{Z}(P)$. Now (a) implies by Burnside's fusion control lemma that $gsg^{-1} = s^k$ for some $g \in \mathbf{N}_G(P)$. Write $|g| = mp^b$ with $p \nmid m$ and $b \geq 0$, and let $h := g^m$. Then $hsh^{-1} = s^{k^m}$ and $|h| = p^b$. As $\mathbf{N}_G(P)/P$ is a p'-group, the latter implies that $h \in P$, and so $hsh^{-1} = s$. Thus $|s| = p^a$ divides $k^m - 1$. As noted above, $p|(k-1)$, whence $p \nmid (k^m - 1)/(k-1)$. It follows that p^a divides $k - 1$, i.e. $s^k = s$. □

Lemma 3.13. *Suppose that, for the group G in Theorem 2.3, there exists a a simple algebraic group $\mathcal{G}$ of adjoint type in characteristic $\ell \neq p$ and a Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$ such that $S = \mathcal{G}^F$ and the following conditions hold.*
(a) *$\mathcal{G} \cong \mathcal{G}^*$, where $(\mathcal{G}^*, F^*)$ is dual to $(\mathcal{G}, F)$;*
(b) *$G = S \rtimes \langle h \rangle$, $P \in \mathrm{Syl}_p(G)$, and the conjugation by $h \in P$ induced an automorphism of S, which is obtained by restricting to S a Steinberg endomorphism $\gamma : \mathcal{G} \to \mathcal{G}$ with $F = \gamma^m$ for some $m \in \mathbb{Z}_{\geq 1}$; and*
(c) *$Q = P \cap S = Q_1 \times Q_2 \times \ldots \times Q_k$ is a direct product of k γ-stable cyclic subgroups of the same order.*

Then Theorem 2.3 holds for G.

Proof. Using (a), we will identify $\mathcal{G}^*$ with $\mathcal{G}$, γ^* with γ, F^* with F, and $G^* = (\mathcal{G}^*)^{F^*}$ with S. By (c) we can write $Q_i = \langle t_i \rangle \cong C_{p^a}$ and

$$\gamma(t_i) = t^{n_i} \tag{3.1}$$

for $1 \leq i \leq k$ and for some $a, n_i \in \mathbb{Z}_{\geq 1}$. Let $p^{b_i} := (n_i - 1)_p$.

Consider any h-invariant linear character $\lambda \in \mathrm{Irr}(Q)$. If $a \leq e$, then λ is σ_e-fixed. So we will assume $a > e$. Next, $\mathrm{Ker}(\lambda)$ contains $t_i^{n_i - 1}$ by (3.1). It follows that λ is σ_e-fixed if $\max_i b_i \leq e$. So we will assume that $b_1 > e$. Now, we can choose an element $s \in Q_1$ of order p^{e+1}. By Proposition 3.11, we see that $\theta := (\chi_s)_S \in \mathrm{Irr}(S)$ belongs to $B_0(S)$, and it has p'-degree as $s \in \mathbf{Z}(Q)$. As $b_1 > e$, (3.1) implies that $\gamma(s) = s$. It follows by [NTT2, Corollary 2.5] that θ is γ-invariant, and so P-invariant. By hypothesis, θ is then σ_e-fixed, whence s and s^{1+p^e} are conjugate in S by Proposition 3.11(iii). This in turn implies by Lemma 3.12 that $s = s^{1+p^e}$. But this contradicts the choice of s to be of order p^{e+1}. □

Lemma 3.14. *Suppose that $2 < p|(q - \epsilon)$ and $n = p^k$ for some $k \geq \mathbb{Z}_{\geq 1}$. Then $\mathrm{SL}_n^\epsilon(q)$ is p-good.*

Proof. We proceed by induction on $k \geq 1$ and fix $\alpha \in \overline{\mathbb{F}}_q^\times$ of order $p^a = (q - \epsilon)_p$. We also fix a basis $(e_1, \ldots, e_n)$ of the natural module $V = \mathbb{F}_q^n$ for $\mathrm{SL}_n(q)$, and an orthonormal basis $(e_1, \ldots, e_n)$ of the natural module $V = \mathbb{F}_{q^2}^n$ for $\mathrm{SU}_n(q)$.

(i) For the induction base $k = 1$, we can choose a Sylow p-subgroup of $\mathrm{SL}_p^\epsilon(q)$ to be

$$R = \langle s, x_1 x_2^{-1}, x_2 x_3^{-1}, \ldots, x_{p-1} x_p^{-1} \rangle \tag{3.2}$$

in the chosen basis of V, where

$$s : e_1 \mapsto e_2 \mapsto e_3 \mapsto \ldots \mapsto e_p \mapsto e_1,\ x_1 = \mathrm{diag}(\alpha, 1, 1, \ldots, 1),\ x_i = s^{i-1}x_1s^{1-i},\ 2 \le i \le p.$$

Then we have

$$x_1x_2^{-1} \equiv x_2x_3^{-1} \equiv \ldots \equiv x_{p-1}x_p^{-1} \equiv x_px_1^{-1} (\mathrm{mod}\, [R, R]).$$

But certainly

$$x_1x_2^{-1} \cdot x_2x_3^{-1} \cdot \ldots \cdot x_{p-1}x_p^{-1} \cdot x_px_1^{-1} = 1.$$

It follows that all the generators of R in (3.2) have their p^{th}-powers belonging to $[R, R]$. Hence R is p-good by Lemma 3.2(iii).

(ii) For the induction step, let $n = p^k = mp \ge p^2$, and take $V = \oplus_{i=1}^{p} V_i$ with

$$V_i := \langle e_{(i-1)m+1}, \ldots, e_{im}\rangle.$$

Let

$$y_1 := \mathrm{diag}(\alpha, 1, 1, \ldots, 1),\ t : e_j \mapsto e_{m+j} \mapsto e_{2m+j} \mapsto \ldots \mapsto e_{(p-1)m+j} \mapsto e_j,\ 1 \le j \le m.$$

By the induction hypothesis, a Sylow p-subgroup R_1 of $\mathrm{SL}^{\epsilon}(V_1) \cong \mathrm{SL}^{\epsilon}_m(q)$ that acts diagonally in the basis $(e_1, \ldots, e_m)$ is p-good. Define

$$y_i := t^{i-1}y_1t^{1-i},\ R_i := t^{i-1}R_1t^{1-i}$$

for $2 \le i \le p$. Then note that

$$T := \langle t, y_1y_2^{-1}, y_2y_3^{-1}, \ldots, y_{p-1}y_p^{-1}\rangle$$

is isomorphic to the subgroup R defined in (3.2), and so it is p-good. Now

$$\langle R_1, R_2, \ldots, R_p, T\rangle$$

is a Sylow p-subgroup of $\mathrm{SL}^{\epsilon}_n(q)$, and it is p-good by Lemma 3.2(ii). □

Proposition 3.15. *Theorem 2.3 holds in the case S is a simple Suzuki or Ree group.*

Proof. By Corollary 3.10, we may assume that S is a simple Suzuki or Ree group over a field of characteristic $\ell \ne p$, $p \ne 2$ if $S = {}^2G_2(q^2)$, and that $S \not\cong {}^2F_4(2)'$. Then we can write $q^2 = \ell^{2a+1}$ for some $a \in \mathbb{Z}_{\ge 1}$, and find a Steinberg endomorphism δ of $\mathcal{G} \cong \mathcal{G}^*$ (of type B_2, G_2, or F_4) such that δ^2 is the standard Frobenius endomorphism of $\mathcal{G}$, induced by the field automorphism $x \mapsto x^{\ell}$, and $F = \delta^{2a+1}$. Now if $|G/S| =: m$, then we can take $\gamma := \delta^{(2a+1)/m}$ to fulfill condition (b) of Lemma 3.13.

Suppose first that $S = {}^2B_2(q^2)$ with $q^2 > 2$. Then $p \ne 2$, and one can check that G also fulfills condition (c) of Lemma 3.13, with Q being cyclic. Hence we are done in this case. The same arguments apply in the case $S = {}^2G_2(q^2)$ with $q^2 > 3$.

Next suppose that $S = {}^2F_4(q^2)$ with $q^2 > 2$. If furthermore $p \nmid (q^8 - 1)$, then Q is cyclic, and we are done as above. Suppose $3 \ne p|(q^4 - 1)$. Then by the main result of [LSS] (see Table 5.1 therein), there is a δ-stable connected reductive subgroup $\mathcal{D}$ of $\mathcal{G}$ such that $\mathbf{N}_S(\mathcal{D}^F) \cong \mathrm{Sp}_4(q^2){\cdot}2$ and $\mathbf{N}_G(\mathcal{D}^F)$ is a maximal subgroup of S. As $F = \delta^{2a+1}$, $\mathcal{D}^F$ is δ-stable, and so $\mathbf{N}_G(\mathcal{D}^F) \cong (\mathrm{Sp}_4(q^2) \cdot 2) \rtimes \langle h\rangle$. The assumptions on p ensure that we can take

$$P < M := \mathrm{Sp}_4(q^2) \rtimes \langle h\rangle = \mathrm{Sp}_4(q^2) \rtimes \langle h^2\rangle,$$

where h^2 is induced by $\gamma^2 = (\delta^2)^{(2a+1)/m}$, with δ^2 acting on $\mathcal{D}$ as the standard Frobenius endomorphism. Working in M, we can represent Q as a direct product of $k = 2$ h^2-stable cyclic subgroups of the same order, fulfilling condition (c) of Lemma 3.13. The same arguments apply in the case $p|(q^4 + 1)$, where $\mathcal{D}^F \cong {}^2B_2(q^2) \times {}^2B_2(q^2)$ and the odd-order element h stabilizes each of the two factors ${}^2B_2(q^2)$. Finally, if $p = 3$, then we can put Q in a subgroup $\mathrm{SU}_3(q^2)$ of S, which is 3-good by Lemma 3.14. □

4. Almost Simple Groups. II

Throughout this section, we will assume that

$$S \text{ is a simple group of Lie type in characteristic } \ell \neq p \text{ but not a Suzuki or Ree group, and } (S,p) \text{ is not listed in Corollary 3.10.} \tag{4.1}$$

We will then view $S = [\mathcal{G}^F, \mathcal{G}^F]$ for some simple algebraic group of adjoint type over a field of characteristic ℓ and a Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$. Let $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, $G^* = (\mathcal{G}^*)^{F^*}$, and let q denote the common absolute value of eigenvalues of F acting on the character group $X(\mathcal{T})$ of an F-stable maximal torus $\mathcal{T}$ of $\mathcal{G}$. Recall the notion of *d-tori* in algebraic groups as defined in [MTe, Definition 25.6]. Also, let $\Phi_d(\cdot)$ denote the d^{th} cyclotomic polynomial, of degree $\varphi(d)$. Given p and q, let d be the order of q modulo p, so that $p|\Phi_d(q)$ but $p \nmid \Phi_e(q)$ for all $1 \leq e < d$. Note that $1 \leq d \leq p-1$.

Lemma 4.1. *Let $\mathcal{G}$ be a simple algebraic group with a Steinberg endomorphism γ and let $F = \gamma^m$ for some $m \in \mathbb{Z}_{\geq 1}$. Suppose that*

(a) *the common absolute value of eigenvalues of F acting on the character group $X(\mathcal{T})$ of some F-stable maximal torus $\mathcal{T}$ of $\mathcal{G}$ is q;*
(b) *$\mathcal{S}$ is a γ-stable d-torus of $\mathcal{G}$ for some $d \in \mathbb{Z}_{\geq 1}$; and*
(c) *$j \in \mathbb{Z}_{\geq 1}$ is such that $\mathcal{G}^{\gamma^{jd}}$ is not a Suzuki or Ree group.*

Then γ^{jd} acts on $\mathcal{S}$ via $x \mapsto x^{q^{jd/m}}$.

Proof. We may choose $\mathcal{T}$ to be a γ-stable maximal torus of $\mathcal{G}$. Now, if λ is any eigenvalue for γ acting on $X(\mathcal{T})$, then λ^m is an eigenvalue for $F = \gamma^m$ acting on $X(\mathcal{T})$, whence $|\lambda^m| = q$ by (a), and so $|\lambda| = q^{1/m}$. It now follows from the proof of [MTe, Proposition 25.7] that γ acts on $X := X(\mathcal{S})$ as $q^{1/m}\phi$ for some linear transformation ϕ with $\phi^d = 1_X$. Hence γ^{jd} acts on $X(\mathcal{S})$ as $q^{jd/m} \cdot 1_X$. Now $q^{jd/m}$ is an integer because of (c), and so the statement follows from [NT1, Lemma 4.2(ii)]. □

In the next generalization of Lemma 3.13, by *a natural permutation action* of a finite group Y on a direct product $X = X_1 \times X_2 \times \ldots \times X_k$ with $X_1 \cong X_2 \cong \ldots \cong X_k$ we mean an identification of X with $\{(x_1, x_2, \ldots, x_k) \mid x_i \in X_1\}$ (with component-wise product) and an embedding $\pi : Y \hookrightarrow \mathsf{S}_k$ such that $y \in Y$ sends $(x_1, x_2, \ldots, x_k)$ to $(x_{\pi(y)(1)}, x_{\pi(y)(2)}, \ldots, x_{\pi(y)(k)})$.

Proposition 4.2. *Let S be as in* (4.1), *$G = S \rtimes \langle h \rangle$, and suppose that all the following conditions hold.*

(a) *The conjugation by $h \in P$ induced an automorphism of S, which is obtained by restricting to S a Steinberg endomorphism $\gamma : \mathcal{G} \to \mathcal{G}$ with $F = \gamma^m$ for some p-power $m \in \mathbb{Z}_{\geq 1}$.*
(b) *There is a γ-stable d-torus $\mathcal{S}$ of $\mathcal{G}$ such that $Q = \mathbf{O}_p(\mathcal{S}^F) \rtimes R$ for some p^e-good subgroup R.*
(c) *Let $(\mathcal{G}^*, \gamma^*)$ be dual to $(\mathcal{G}, \gamma)$. There exists a quotient $\mathcal{H} = \mathcal{G}^*/Z$, where $Z \leq \mathbf{Z}(\mathcal{G}^*)$ and γ^*-stable d-tori $\mathcal{R}_j$ of $\mathcal{H}$, $1 \leq j \leq l$, all of rank $\varphi(d)$, such that*

$$Q^* = (\mathbf{O}_p(\mathcal{R}_1^{F^*}) \times \mathbf{O}_p(\mathcal{R}_2^{F^*}) \times \ldots \times \mathbf{O}_p(\mathcal{R}_l^{F^*})) \rtimes R^*$$

is a Sylow p-subgroup of $H := \mathcal{H}^{F^*}$, where the p-subgroup R^* naturally permutes the subgroups $\mathbf{O}(\mathcal{R}_j^{F^*})$, $1 \leq j \leq l$.
(d) *$p \nmid |\mathbf{Z}((\mathcal{G}^*)^{F^*})|$.*

Then Theorem 2.3 holds in this case.

Proof. (i) Let $(\mathcal{G}^*,\gamma^*)$ be dual to $(\mathcal{G},\gamma)$. Then we may take $F^*=(\gamma^*)^m$. By [MTe, Proposition 25.7], $\mathcal{S}=\mathcal{S}_1\times\mathcal{S}_2\times\ldots\times\mathcal{S}_k$ is a direct product of γ-stable d-tori $\mathcal{S}_i$, each of rank $\varphi(d)$. It follows that

$$\mathcal{S}^F=\mathcal{S}_1^F\times\mathcal{S}_2^F\times\ldots\times\mathcal{S}_k^F.$$

As mentioned in the proof of [MTe, Proposition 25.7], each $\mathcal{S}_i^F$ is cyclic of order $\Phi_d(q)$. Also, γ acts on $\mathcal{S}_i^F$. Hence, if $\mathcal{S}_i^F=\langle t_i\rangle$, then

$$\gamma(t_i)=t_i^{n_i} \tag{4.2}$$

for some $n_i\in\mathbb{Z}_{\geq 1}$. On the other hand, $\gamma^d(t_i)=t_i^{q^{d/m}}$ by Lemma 4.1. Observe that $\mathcal{G}^\gamma$ is not a Suzuki or Ree group. (Indeed, otherwise either $\mathcal{G}^F$ itself would be a Suzuki or Ree group for $p>2$, or $(p,\ell,\mathcal{G}^F)=(2,3,G_2(q))$, both contradicting (4.1).) Hence, it follows from the proof of Lemma 4.1 that $q^{1/m}\in\mathbb{Z}$, and so

$$n_i^d\equiv q^{d/m}(\bmod\,\Phi_d(q)) \tag{4.3}$$

for all i. Let

$$(\Phi_d(q))_p=p^a,\ (q^{d/m}-1)_p=p^b,\ (n_i^d-1)_p=p^{b_i},\ \mathbf{O}_p(\mathcal{S}_i^F)=\langle s_i\rangle.$$

The choice of d implies that $(q^d-1)_p=p^a>1$. As m is a p-power, we have

$$q^{d/m}-1\equiv q^d-1\equiv 0(\bmod\,p)$$

and furthermore $p|(q^d-1)/(q^{d/m}-1)$ if $m>1$. It follows from (4.3) that

$$b_1=b_2=\ldots=b_k=b<a \tag{4.4}$$

if $m>1$.

(ii) Note that q is also the common absolute value of eigenvalues of F^* acting on the character group $X(\mathcal{T}^*)$ of an F^*-stable maximal torus $\mathcal{T}^*$ of $\mathcal{G}^*$ or of $\mathcal{H}$. Hence, the same arguments as in (i) show that

$$c_1=c_2=\ldots=c_l=b<a \tag{4.5}$$

if $m>1$. Here, we write $\mathcal{R}_j^{F^*}=\langle t_j^*\rangle$, $\gamma^*(t_j^*)=(t_j^*)^{n_j^*}$, and $((n_j^*)^d-1)_p=p^{c_j}$ for $1\leq j\leq l$.

Furthermore, condition (d) implies that Q^* is contained in $[H,H]$, which is a quotient of $(\mathcal{G}^*)^{F^*}$. Let $\hat{Q}^*$ denote the full inverse image of Q^* in $\mathcal{G}^*$. Then (d) implies that $\mathbf{O}(\hat{Q}^*)\in\mathrm{Syl}_p(G^*)$ and there is a γ^*-equivariant isomorphism $\mathbf{O}_p(\hat{Q}^*)\cong Q^*$. So, without any loss, we may view Q^* as a Sylow p-subgroup of $(\mathcal{G}^*)^{F^*}$, with prescribed action of γ^*.

(iii) Consider any h-invariant linear character $\lambda\in\mathrm{Irr}(Q)$. If $a\leq e$, then Q is p^e-good by Lemma 3.2(ii), and so λ is σ_e-fixed. So we will assume $a>e$. Next, $\mathrm{Ker}(\lambda)$ contains $t_i^{n_i-1}$ by (4.2). It follows by Lemma 3.2(iii) that λ is σ_e-fixed if $b_i\leq e$ for all i. So we will assume that $\max_i b_i>e$. If $m>1$, then this implies by (4.4) that $b>e$. If $m=1$, then we also have $b=a>e$.

Note that $|\mathbf{O}_p(\mathcal{R}_i^{F^*})|=(\Phi_d(q))_p=p^a$. Since $a>e$, hypothesis (c) allows us to choose an element

$$s\in\mathbf{Z}(Q^*)\cap(\mathbf{O}_p(\mathcal{R}_1^{F^*})\times\mathbf{O}_p(\mathcal{R}_2^{F^*})\times\ldots\times\mathbf{O}_p(\mathcal{R}_l^{F^*}))$$

of order p^{e+1}. Using (d) and Proposition 3.11, we see that $\theta:=(\chi_s)_S\in\mathrm{Irr}(S)$ belongs to $B_0(S)$, and it has p'-degree as $s\in\mathbf{Z}(Q^*)$.

If $m=1$, then s is γ^*-stable as $\gamma^*=F^*$ in this case. If $m>1$, then (4.5) implies that $p^{e+1}|((n_j^*)^d-1)$ for all j, and so $(\gamma^*)^d(s)=s$, i.e. s is $(\gamma^*)^d$-stable. As s is stable under $F=\gamma^m$ and $\gcd(m,d)=1$, we see that s is γ^*-stable in this case as well. It follows by [NTT2, Corollary 2.5] that θ is γ-invariant, and so P-invariant. By hypothesis, θ is then σ_e-fixed, whence s and s^{1+p^e} are conjugate in $(\mathcal{G}^*)^{F^*}$ by Proposition 3.11(iii). This in turn implies by Lemma 3.12 that $s = s^{1+p^e}$. But this contradicts the choice of s to be of order p^{e+1}. □

Proposition 4.3. *Let $S = [\mathcal{G}^F, \mathcal{G}^F]$ be as in (4.1) and $p > 2$. Suppose in addition that $\mathcal{G}^F$ is a classical group, $p \nmid \gcd(n, q - \epsilon)$ if $S = \mathrm{PSL}_n^\epsilon(q)$, and that $p \neq 3$ if $S \cong P\Omega_8^+(q)$. Then Theorem 2.3 holds in this case.*

Proof. (i) By the assumptions, $p \nmid |\mathbf{Z}((\mathcal{G}^*)^{F^*})|$. Recall by [GLS, Theorem 2.5.12] that $\mathrm{Aut}(S) = J \rtimes \Phi_S\Gamma_S$ is split over $J = \mathrm{Inndiag}(S)$. Let $m := |G/S|$, a p-power. Then $|GJ| = m|J|$, and so $GJ = J \rtimes C_m$ for some subgroup $C_m \leq \Phi_S\Gamma_S$. Let δ denote the standard Frobenius endomorphism of $\mathcal{G}$ induced by the field automorphism $x \mapsto x^\ell$. Then we have $F = \tau\delta^f$, where τ is either trivial or a graph automorphism of order 2 of $\mathcal{G}$ commuting with δ, and $C_m = \langle \gamma|_S \rangle$ with $\gamma = \tau\delta^{f/m}$. As m is a p-power and $p \nmid |J/S| = |\mathbf{Z}((\mathcal{G}^*)^{F^*})|$, $P \in \mathrm{Syl}_p(GJ)$, and so $h := \gamma|_S \in P^x$ for some $x \in GJ \leq \mathrm{Aut}(S)$. Replacing G by G^x, we then have $G = S \rtimes \langle h \rangle$ with h fulfilling condition (a) of Proposition 4.2.

(ii) Assume in addition that $p \nmid (q - \epsilon)$ if $S = \mathrm{PSL}_n^\epsilon(q)$. In the notation of Proposition 4.2(c), we can choose $\mathcal{H}$ such that

$$H = \mathrm{SL}_n^\epsilon(q),\ \mathrm{SO}_{2n+1}(q),\ \mathrm{Sp}_{2n}(q),\ \mathrm{SO}_{2n}^\epsilon(q),$$

according as $S = \mathrm{PSL}_n^\epsilon(q)$, $\mathrm{PSp}_{2n}(q)$, $\Omega_{2n+1}(q)$, $\mathrm{P}\Omega_{2n}^\epsilon(q)$. Now, the conditions on p and the construction of Sylow p-subgroups of H [GL, Chapter 3, §8], displayed in [BFMNST, §2.2], shows that condition (c) of Proposition 4.2 holds, with R^* being a Sylow p-subgroup of S_l. By Lemma 3.7, R^* is p-good. Arguing as in p. (ii) of the proof of Proposition 4.2, we can again replace $\mathcal{G}$ by an isogenous simple algebraic group and then replace $\mathcal{G}^F$ by

$$\mathrm{SL}_n^\epsilon(q),\ \mathrm{Sp}_{2n}(q),\ \mathrm{SO}_{2n+1}(q),\ \mathrm{SO}_{2n}^\epsilon(q),$$

according as $S = \mathrm{PSL}_n^\epsilon(q)$, $\mathrm{PSp}_{2n}(q)$, $\Omega_{2n+1}(q)$, $\mathrm{P}\Omega_{2n}^\epsilon(q)$. The above description of Sylow p-subgroups then implies that condition (b) of Proposition 4.2 holds. Now we are done by applying Proposition 4.2.

(iii) Next we consider the case $S = \mathrm{PSL}_n^\epsilon(q)$ but $p|(q-\epsilon)$. Then $p \nmid n$, so we write $n = ap+r$ with $a, r \in \mathbb{Z}_{\geq 1}$ and $1 \leq r \leq p-1$. Now $(\mathcal{G}^*)^{F^*} = \mathrm{SL}_n^\epsilon(q)$, and we can find γ^*-stable one-dimensional d-tori $\mathcal{R}_j$, $1 \leq j \leq n-1$, such that

$$\mathcal{R} := \left\{ \mathrm{diag}(x_1, x_2, \ldots, x_{n-1}, \prod_{j=1}^{n-1} x_j^{-1}) \mid x_j \in \mathcal{R}_j \right\}$$

is a maximal torus of $\mathcal{G}^*$ (where $d = 1$ if $\epsilon = +$ and $d = 2$ if $\epsilon = -$). It is easy to see that condition (c) of Proposition 4.2 holds with $l = n-1$ and $R^* \in \mathrm{Syl}_p(\mathsf{S}_{n-r})$ fixing each of $\mathcal{R}_j$ for $ap+1 \leq j \leq n-1$. We can then take $\mathcal{S} = \mathcal{R}/\mathbf{Z}(\mathcal{G}^*)$ to see that condition (b) of Proposition 4.2 holds with $k = n-1$ and $R \in \mathrm{Syl}_p(\mathsf{S}_{n-r})$ being p-good. Now we can again apply Proposition 4.2. □

Proposition 4.4. *Let $q = \ell^f$ be a power of a prime $\ell \neq p$, $\epsilon = \pm$, $S = \mathrm{PSL}_n^\epsilon(q)$, and $2 < p|\gcd(n, q-\epsilon)$. Let $S \lhd G$, where G is a finite group, G/S is a p-group, and $\mathbf{C}_G(S) = 1$. Let $R \in \mathrm{Syl}_p(G)$ and $P = R \cap S$. Suppose that all the R-invariant complex irreducible characters of p'-degree in the principal p-block of S are σ_e-fixed. Then every R-invariant linear character of P is σ_e-fixed.*

Proof. (a) We will view $S = L/\mathbf{Z}(L)$, where $L = \mathrm{SL}_n^\epsilon(q)$, and set $r := (q-\epsilon)_p \geq p$. We will work with the basis $(e_1, \ldots, e_n)$ of the natural L-module V as specified in the proof of Lemma 3.14. Using this basis, we can define the standard Frobenius automorphism $\delta : Y = (y_{ij}) \mapsto (y_{ij}^\ell)$ of the groups

$\mathrm{GL}^\epsilon(V) \cong \mathrm{GL}^\epsilon_n(q)$, L, and S. Write $q = \ell^f$ and $f = p^c f_0$ for some $c \in \mathbb{Z}_{\geq 0}$ and $p \nmid f_0$, and let $\delta_0 := \delta^{f_0}$ if $\epsilon = +$ and $\delta_0 := \delta^{2f_0}$ if $\epsilon = -$. Then the automorphism δ_0 of L has order p^c. Also, we set

$$M := \{Y \in \mathrm{GL}^\epsilon(V) \mid \det(Y)^r = 1\},\ \Gamma := M \rtimes \langle \delta_0 \rangle,\ Z := \mathbf{Z}(M).$$

Then Γ/Z induces a p'-index subgroup of $\mathrm{Aut}(S)$. Since $\mathbf{C}_G(S) = 1$ and G/S is a p-group, after a suitable conjugation in $\mathrm{Aut}(S)$, we may assume that $G \leq \Gamma/Z$.

Fix $\alpha \in \overline{\mathbb{F}}_q^\times$ of order r, and define

$$x_i := \mathrm{diag}(I_{i-1}, \alpha, I_{n-i}),\ 1 \leq i \leq n$$

in the chosen basis of V. We also let $A := \mathsf{A}_n$ act naturally on the basis $(e_1, \ldots, e_n)$ (by permuting the indices of e_i), and fix $T \in \mathrm{Syl}_p(A)$. Now it is easy to check that $\hat{P} := Q \rtimes T \in \mathrm{Syl}_p(L)$, where

$$Q := \langle x_i x_j^{-1} \mid 1 \leq i \neq j \leq n \rangle. \tag{4.6}$$

Moreover, denoting

$$B := \langle x_1 \rangle \rtimes \langle \delta_0 \rangle, \tag{4.7}$$

we see that $R^* := \hat{P} \rtimes B$ is a Sylow p-subgroup of Γ. Let $\hat{G}$ denote the full inverse image of G in $\Gamma = L \rtimes B$. Then

$$\hat{G} = L \rtimes (\hat{G} \cap B),\ \hat{R} = \hat{P} \rtimes (\hat{G} \cap B) \in \mathrm{Syl}_p(\hat{G}). \tag{4.8}$$

We can and will identify P with $\hat{P}/(Z \cap \hat{P})$, and R with $\hat{R}/(Z \cap \hat{R})$.

(b) In view of Lemma 3.14, we may assume that n is not a p-power, and so we can find $d \in \mathbb{Z}_{\geq 1}$ such that $p^d < n < p^{d+1}$. Now write $n = a + b$, where

$$a = p^d \geq p,\ 1 \leq b = n - a = p^{d'} b',\ 0 \leq d' \leq d,\ p \nmid b'.$$

If $d' = d$, set $j := -1$ and

$$s := \mathrm{diag}(\alpha I_a, \alpha^j I_a, I_{n-2a}) \in L. \tag{4.9}$$

If $d' < d$, then, since $\gcd(b', r) = 1$, we can find $x, y \in \mathbb{Z}$ such that $xb' = yr - 1$. Setting

$$j = p^{d-d'} x, \tag{4.10}$$

we then have $a + bj = p^d yr$, and can now consider the p-element

$$s := \mathrm{diag}(\alpha I_a, \alpha^j I_b) \in L. \tag{4.11}$$

In what follows, we will work with some p-powers l, where $r > l \geq 1$. For such an l, $\alpha^l \neq \alpha^{lj}$, and so it is straightforward to check that

$$[L : \mathbf{C}_L(s^l)] = [\mathrm{GL}^\epsilon(V) : \mathbf{C}_{\mathrm{GL}^\epsilon(V)}(s^l)]$$

is coprime to p.

Next we observe that if s^{kl} is L-conjugate to $s^l z$ for some $z \in \mathbf{Z}(L)$ and some $k \in p\mathbb{Z} + 1$, then

$$z = 1,\ \alpha^{l(k-1)} = 1. \tag{4.12}$$

Indeed, let β denote the (unique) eigenvalue of z (acting on V). The choice of j implies that $\alpha^l \neq \alpha^{lj}$ and $\alpha^{kl} \neq \alpha^{klj}$. First suppose that $d' < d$. Then $a > b$, and by comparing eigenvalues and their multiplicities we see that

$$\alpha^{kl} = \alpha^l \beta,\ \alpha^{klj} = \alpha^{lj} \beta.$$

It follows that $\beta = \alpha^{l(1-k)}$; in particular, β is a p-element; and that $\beta^{j-1} = 1$. Since $p|j$ by (4.10), we conclude that $\beta = 1$ and $z = 1$.

Suppose $a = b$. Then
$$\{\alpha^{kl}, \alpha^{-kl}\} = \{\alpha^l\beta, \alpha^{-l}\beta\}.$$
In particular, $\beta^2 = 1$ and $\beta = \alpha^{l(1+\gamma k)}$ is a p-power for some $\gamma = \pm 1$. As $p > 2$, we see that $\beta = 1$ and $z = 1$. Moreover, if $\gamma = 1$, then, as $k \equiv 1 (\mathrm{mod}\, p)$, we see that $\alpha^l = 1$, a contradiction. Hence $\gamma = -1$ and $\alpha^{l(1-k)} = 1$, as stated.

Suppose $d = d'$ and $b > a$. Then
$$\{\alpha^{kl}, \alpha^{-kl}, 1\} = \{\alpha^l\beta, \alpha^{-l}\beta, \beta\}.$$
In particular, $\beta^3 = 1$ and $\beta = \alpha^{il}$ with $i \in \{0, \pm l\}$, whence β is a p-element. It follows that $p = 3$, and so $b \neq 2a$ as $n = a + b$ is not a p-power. In this case, by comparing the eigenvalues with multiplicity $b - a$, we see that $\beta = 1$ and $z = 1$. Now $\beta = \alpha^{l(1+\gamma k)}$ for some $\gamma = \pm 1$. Arguing as above, we conclude that $\gamma = -1$ and $\alpha^{l(1-k)} = 1$.

(c) We can also view L as the dual group H^* and S as $[H, H]$, where $H = \mathcal{G}^F \cong \mathrm{PGL}_n^\epsilon(q)$ is of adjoint type. By virtue of (4.12) with $k = 1$, we can apply Proposition 3.11(i), (ii) to the semisimple character χ_{s^l} of H and conclude that $\theta_l := (\chi_{s^l})_S$ is an irreducible character in $B_0(S)$, of degree
$$\theta_l(1) = [L : \mathbf{C}_L(s^l)]_{\ell'}$$
which is coprime to p.

Let $\lambda \in \mathrm{Irr}(P/P')$ be R-invariant. By inflation we can view λ as an $\hat{R}$-invariant linear character of $\hat{P}$. Now recall that $T \in \mathrm{Syl}_p(\mathsf{A}_n)$ is p-good by Lemma 3.7. Hence by Lemma 3.2(ii) we see that $\hat{P} = Q \rtimes T$ is p^e-good if $r \leq p^e$, and so λ is σ_e-fixed in this case. We may therefore assume that

$$r > p^e. \tag{4.13}$$

Next, recalling (4.6)–(4.8), we note that x_1 centralizes Q, and assume that $\hat{G} \cap B$ induces the subgroup $\langle \delta_1 \rangle$ in the quotient $B/\langle x_1 \rangle$. Then, for a suitable ℓ-power q_1, we have that $\delta_1(x) = x^{\epsilon q_1}$ for all $x \in Q$. Denoting
$$(q_1 - \epsilon)_p = p^{e_1},$$
and using the $\hat{G} \cap B$-invariance of λ, we then see that $\mathrm{Ker}(\lambda) \ni x^{p^{e_1}}$ for all $x \in Q$. As T is p-good, Lemma 3.2(ii) again implies that λ is σ_e-fixed if $e_1 \leq e$. So we will assume that

$$e_1 > e. \tag{4.14}$$

Using (4.13), we now choose $l = r/p^{e+1}$, so that s^l has order p^{e+1}. The construction of s in (4.9), (4.11) shows by (4.14) that s^l is δ_1-invariant and so $(s^l)^L$ is $\hat{R}$-invariant. By hypothesis, θ_l is σ_e-fixed. Applying (4.12) to $(l, k) = (l, 1 + p^e)$, we see by Proposition 3.11(ii) that s^l and $(s^l)^{1+p^e}$ are conjugate in L, and $1 = \alpha^{lp^e} = \alpha^{r/p}$. But this contradicts the choice of α to be of order r. $\square$

The next result is obtained along the lines of the proof of [NT1, Proposition 3.10], but with several modifications.

Proposition 4.5. *Let $q = \ell^f$ be a power of an odd prime ℓ, $\epsilon = \pm$, $4|(q - \epsilon)$, and $S = \mathrm{PSL}_n^\epsilon(q)$, where $n \geq 3$ is not a 2-power. Let $S \lhd G$, where G is a finite group, G/S is a 2-group, and $\mathbf{C}_G(S) = 1$. Let $R \in \mathrm{Syl}_2(G)$ and $P = R \cap S$. Suppose that all the R-invariant complex irreducible characters of odd degree in the principal 2-block of S are σ_e-fixed. Then every linear R-invariant character of P is σ_e-fixed.*

Proof. (a) Write $n = 2m + \kappa$ with $\kappa \in \{0, 1\}$. We will view $S = L/\mathbf{Z}(L)$, where $L = \mathrm{SL}_n^\epsilon(q)$, and set $r := (q - \epsilon)_2 \geq 4$. Let $q^* := q$ if $\epsilon = +$, and $q^* := q^2$ if $\epsilon = -$. We again use the basis $(e_1, \ldots, e_n)$ of the natural L-module $V = \mathbb{F}_{q^*}^n$ as described in the proof of Lemma 3.14. Using this basis, we can define the transpose-inverse automorphism $\tau : Y \mapsto {}^t Y^{-1}$ and the standard Frobenius automorphism $\delta : Y = (y_{ij}) \mapsto (y_{ij}^{\ell})$ of the groups $\mathrm{GL}^{\epsilon}(V) \cong \mathrm{GL}^{\epsilon}_n(q)$, L, and S. Write $f = 2^c f_0$ for some $c \in \mathbb{Z}_{\geq 0}$ and odd f_0, and let $\delta_0 := \delta^{f_0}$. (Note that the automorphism δ_0 of L has order 2^c if $\epsilon = +$ and 2^{c+1} if $\epsilon = -$; in the latter case, $\tau = \delta_0^{2^c}$.) Also, we set

$$M := \{Y \in \mathrm{GL}^{\epsilon}(V) \mid \det(Y)^r = 1\},\ \Gamma := M \rtimes \langle \tau, \sigma_0 \rangle,\ Z := \mathbf{Z}(M).$$

Then Γ/Z induces an odd-index subgroup of $\mathrm{Aut}(S)$. Since $\mathbf{C}_G(S) = 1$ and G/S is a 2-group, after a suitable conjugation in $\mathrm{Aut}(S)$, we may assume that $G \leq \Gamma/Z$.

Fix $\alpha \in \mathbb{F}_{q^*}^{\times}$ of order r, and define

$$x_i = \mathrm{diag}(I_{i-1}, \alpha, I_{n-i}),\ 1 \leq i \leq n;\ t_j = \mathrm{diag}(I_{2j-2}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_{n-2j}),\ 1 \leq j \leq m$$

in the chosen basis of V. We also consider the "flips"

$$\tau_{1i} : e_1 \leftrightarrow e_{2i-1},\ e_2 \leftrightarrow e_{2i},\ e_j \mapsto e_j,\ \forall j \neq 1, 2, 2i-1, 2i$$

for $1 \leq i \leq m$. Then $A := \langle \tau_{1i} \mid 2 \leq i \leq m \rangle \cong \mathsf{S}_m$, and we fix $T \in \mathrm{Syl}_2(A)$. Now it is easy to check that $\hat{P} := Q \rtimes T \in \mathrm{Syl}_2(L)$, where

$$Q := \langle x_i x_j^{-1}, t_k \mid 1 \leq i \neq j \leq n, 1 \leq k \leq m \rangle. \tag{4.15}$$

Moreover, denoting

$$B := \langle x_1 \rangle \rtimes \langle \tau, \delta_0 \rangle,$$

we see that $R^* := \hat{P} \rtimes B$ is a Sylow 2-subgroup of Γ. Let $\hat{G}$ denote the full inverse image of G in $\Gamma = L \rtimes B$. Then

$$\hat{G} = L \rtimes (\hat{G} \cap B),\ \hat{R} = \hat{P} \rtimes (\hat{G} \cap B) \in \mathrm{Syl}_2(\hat{G}). \tag{4.16}$$

We can and will identify P with $\hat{P}/(Z \cap \hat{P})$, and R with $\hat{R}/(Z \cap \hat{R})$.

(b) Since n is not a 2-power, we can write $n = a + b$, where

$$a = 2^d \geq 2,\ 1 \leq b = 2^{d'} b' \leq 2^d - 1,\ 0 \leq d' < d,\ 2 \nmid b'.$$

As $\gcd(b', r) = 1$, we can find $x, y \in \mathbb{Z}$ such that $xb' = yr - 1$. Setting

$$j = 2^{d-d'} x, \tag{4.17}$$

we then have $a + bj = 2^d yr$ and so we can consider the 2-element

$$s = \mathrm{diag}(\alpha I_a, \alpha^j I_b) \in L. \tag{4.18}$$

Note that the choice (4.17) implies that $2|j$. In what follows, we will work with some 2-powers l, where $r > l \geq 1$. For such an l, $\alpha^l \neq \alpha^{lj}$, and so

$$\mathbf{C}_{\mathrm{GL}^{\epsilon}(V)}(s^l) \cong \mathrm{GL}^{\epsilon}_a(q) \times \mathrm{GL}^{\epsilon}_b(q).$$

Next we observe that if k is any odd integer and s^{kl} is L-conjugate to $s^l z$ for some $z \in \mathbf{Z}(L)$, then

$$z = 1,\ \alpha^{l(k-1)} = 1. \tag{4.19}$$

Indeed, if β is the (unique) eigenvalue of z (acting on V), then the conditions $a > b$, $\alpha^l \neq \alpha^{lj}$, and $\alpha^{kl} \neq \alpha^{klj}$ imply by comparing eigenvalues and their multiplicities that

$$\alpha^{kl} = \alpha^l \beta,\ \alpha^{klj} = \alpha^{lj} \beta.$$

It follows that $\beta = \alpha^{l(k-1)}$; in particular, β is a 2-element; and that $\beta^{j-1} = 1$. Since $2|j$, we conclude that $\beta = 1$ and $z = 1$.

(c) We can also view L as the dual group H^* and S as $[H,H]$, where $H = \mathcal{G}^F \cong \mathrm{PGL}_n^\epsilon(q)$ is of adjoint type. By virtue of (4.19) with $k=1$, we can apply Proposition 3.11(i), (ii) to the semisimple character χ_{s^l} of H and conclude that $\theta_l := (\chi_{s^l})_S$ is an irreducible character in $B_0(S)$, of degree

$$\theta_l(1) = [L : \mathbf{C}_L(s^l)]_{\ell'} = \frac{|\,\mathrm{GL}_{a+b}^\epsilon(q)|_{\ell'}}{|\,\mathrm{GL}_a^\epsilon(q)|_{\ell'} \cdot |\,\mathrm{GL}_b^\epsilon(q)|_{\ell'}}$$

which is odd, since $\binom{a+b}{a}$ is odd, see [NT1, Lemma 4.4(i)].

Let $\lambda \in \mathrm{Irr}(P/P')$ be R-invariant. By inflation we can view λ as an $\hat{R}$-invariant linear character of $\hat{P}$. Now recall that $T \in \mathrm{Syl}_2(\mathsf{S}_m)$ is 2-good by Lemma 3.7. Hence by Lemma 3.2(ii) we see that $\hat{P} = Q \rtimes T$ is p^e-good if $r \leq p^e$, and so λ is σ_e-fixed in this case. We may therefore assume that $r > 2^e$. Now, applying (4.19) to $(l,k) = (1, 2^e+1)$, we see by Proposition 3.11(ii) that θ_1 is not σ_e-fixed. The latter implies by the hypothesis that θ_1 cannot be R-invariant. Since $M/Z \leq H$ and θ_1 is clearly H-invariant, we have therefore shown that $R > P$ and $G(M/Z) > M/Z$, whence $\hat{G}M > M$.

(d) Assume now that $\tau \in \hat{G}M$. Since $\hat{G} > L$ and $M = L\langle x_1\rangle$, it follows that $\tau = \tau' x_1^{-i_0}$ for some $\tau' \in \hat{G}$ and $i_0 \in \mathbb{Z}$. Thus $\tau' = \tau x_1^{i_0} \in \hat{G} \cap B$, and so $\tau' \in \hat{R}$ by (4.16). As λ is $\hat{R}$-invariant, $\lambda = \lambda^{\tau'}$ and so $K := \mathrm{Ker}(\lambda)$ contains $\tau'(x)x^{-1}$ for all $x \in \hat{P}$. Certainly, x_1 centralizes all x_i and τ inverts each x_i. Hence

$$\hat{P}' \leq K \ni \tau'(x_i x_j^{-1})(x_i x_j^{-1})^{-1} = (x_i x_j^{-1})^{-2}$$

for all $1 \leq i \neq j \leq n$. Since $4|r$, this implies by [NT1, (3.5)] that $t_k^2 = (x_{2k-1}x_{2k}^{-1})^{r/2} \in K$. As T is 2-good by Lemma 3.7, we also have that $\hat{P}' \ni y^2$ for all $y \in T$. It follows from (4.15) and Lemma 3.2(iii) that $\hat{P}/K$ is 2-good, whence λ is σ_e-fixed.

(e) Note that if $\epsilon = -$, then $c = 0$ and $B = \langle x_1, \tau\rangle$ with $M = L\langle x_1\rangle$. In this case, $\hat{G}M > M$ implies that $\tau \in \hat{G}M$ and so we are done by (d). So it remains to consider the case where $\epsilon = +$, $\tau \notin \hat{G}M$, and $c \geq 1$. Note that $B_1 := \langle \delta_0\rangle \times \langle\tau\rangle \cong C_{2^c} \times C_2$ and $\hat{G}M = M \rtimes (\hat{G}M \cap B_1)$. Also,

$$\hat{R} \leq \tilde{R} := \langle \hat{P}, x_1\rangle \rtimes (\hat{G}M \cap B_1) \in \mathrm{Syl}_2(\hat{G}M). \tag{4.20}$$

Now the assumption $\tau \notin \hat{G}M > M$ implies that there exist some $1 \leq c_1 \leq c$ and some $j = 0,1$ such that $\hat{G}M \cap B_1 = \langle \delta_0^{2^{c-c_1}}\tau^j\rangle \cong C_{2^{c_1}}$. Set $q_1 := p^{2^{c-c_1}f_0}$, $\delta_1 = \delta_0^{2^{c-c_1}}$, so that $q = q_1^{2^{c_1}}$. Again we can write $\delta_1\tau^j = \delta' x_1^{-i_1}$ for some $\delta' \in \hat{G}$ and $i_1 \in \mathbb{Z}$. Thus $\delta' = \delta_1\tau^j x_1^{i_1} \in \hat{G} \cap B$ and so $\delta' \in \hat{R}$ by (4.16).

Recall that $\alpha \in \mathbb{F}_q^\times$ has order $r = (q-1)_2$. As $c_1 \geq 1$, $q - 1 = q_1^{2^{c_1}} - 1$ is divisible by $q_1 - (-1)^j$. Hence there is a 2-power $1 \leq l < r$ such that α^l has order $(q_1 - (-1)^j)_2 \geq 2$. Our choice of l implies that $\delta_1\tau^j(\alpha^l) = \alpha^l$, and so the element s^l, with s defined in (4.18), is $\delta_1\tau^j$-invariant. It then follows by [NTT2, Corollary 2.5] that the characters χ_{s^l} and θ_l (as defined in (b)) are $\delta_1\tau^j$-invariant. Since θ_l is invariant under $M/Z \leq H$, we see by (4.20) and Proposition 3.11(i), (ii), that θ_l is an R-invariant irreducible character of odd degree in $B_0(S)$, whence θ_l is σ_e-fixed by the hypothesis. The latter implies by (4.19) (with $k = 2^e+1$) and Proposition 3.11(ii) that s^l is L-conjugate to s^{2^e+1}. Using (4.19) again with $k = 2^e+1$, we see that $|\alpha^l| \leq 2^e$, and so

$$(q_1 - (-1)^j)_2 \leq 2^e. \tag{4.21}$$

Now we return to the $\hat{R}$-invariant character λ of $\hat{P}/\hat{P}'$. As $\delta' \in \hat{R}$, $\lambda = \lambda^{\delta'}$ and so $K = \mathrm{Ker}(\lambda)$ contains $\delta'(x)x^{-1}$ for all $x \in \hat{P}$. Note that $\delta'(x_i) = \delta_1\tau^j(x_i) = x_i^{(-1)^j q_1}$, and so $\hat{P}' \leq K \ni (x_i x_j^{-1})^{q_1 - (-1)^j}$ for all $1 \leq i \neq j \leq n$. It follows from (4.21) that $K \ni (x_i x_j^{-1})^{2^e}$ for all such i, j. Furthermore, by [NT1, (3.5)] we have

$$t_k^{2^e} = (x_{2k-1}x_{2k}^{-1})^{r/2 \cdot 2^{e-1}} = ((x_{2k-1}x_{2k}^{-1})^{2^e})^{r/4} \in K.$$

Hence, we conclude as in (d) that $\hat{P}/K$ is 2^e-good, and so λ is σ_e-fixed, as desired. $\Box$

Lemma 4.6. *Let X be a normal abelian subgroup of a finite group Y. Suppose $\bar{j}$ is a central involution in $W = Y/X$ and it acts on X as the inversion $x \mapsto x^{-1}$. Then*

(i) *If $2 \nmid |X|$, then Y splits over X and $Y \cong X \rtimes W$.*
(ii) *If $p > 2$, then $P \in \mathrm{Syl}_p(Y)$ splits over $Q := \mathbf{O}_p(X)$. More precisely, $P = Q \rtimes R$ with R isomorphic to a Sylow p-subgroup of W.*

Proof. (i) We may assume that $\bar{j} = jX$ for some involution $j \in Y$, and $jvj^{-1} = v^{-1}$ for all $v \in X$. Since $\bar{j} \in \mathbf{Z}(W)$, $j^Y \subseteq jX$. Conversely, given any $u \in X$ we can find $v \in X$ such that $v^2 = u$, and so $v^{-1}jv = ju$. Thus $j^Y = jX$, whence

$$|\mathbf{C}_Y(j)| = |Y|/|jX| = |Y/X|.$$

Now $\mathbf{C}_Y(j) \cap X = 1$, and we conclude that $Y = X \rtimes \mathbf{C}_Y(j)$.

(ii) Let $T := \mathbf{O}_{p'}(X)$ so that $X = T \times Q$. Applying (i) to Y/T, we see that Y/T splits over QT/T. Hence $PT/T = QT/T \rtimes (R_1/T)$ for some subgroup R_1 of Y, and R_1/T is isomorphic to a Sylow p-subgroup of W. Now $R_1 = TR$ for $R \in \mathrm{Syl}_p(R_1)$, $Q \lhd Y$, and

$$Q \cap R \leq Q \cap (RT \cap QT) = Q \cap T = 1.$$

As $|P| = |Q| \cdot |R|$, it follows that $P = Q \rtimes R$ (after a suitable conjugation). $\Box$

Proposition 4.7. *Let $S = [\mathcal{G}^F, \mathcal{G}^F]$ be as in (4.1). Suppose in addition that $\mathcal{G}^F$ is an exceptional group of Lie type and $p > 2$. Then Theorem 2.3 holds in this case.*

Proof. (i) First we note that, since $p > 2$, condition (a) of Proposition 4.2 is satisfied, after a suitable conjugation in $\mathrm{Aut}(S)$. Next, $\mathcal{G}$ admits γ-stable Sylow d-tori $\mathcal{S}$, by [MTe, Theorem 25.11], which can be chosen to be a direct product of γ-stable d-tori of rank $\varphi(d)$, see [MTe, Proposition 25.7]. Assume furthermore that $p \nmid |\mathbf{Z}((\mathcal{G}^*)^{F^*})|$ and that a Sylow p-subgroup of $\mathcal{G}^F$ is contained in $\mathcal{S}^F$. Then conditions (b)–(d) of Proposition 4.2 hold, with $R = 1$ and $R^* = 1$, and so we are done.

(ii) Here we assume in addition that $p > 3$; in particular, $p \nmid |\mathbf{Z}((\mathcal{G}^*)^{F^*})|$. Note that in the case $p = 5|(q^2+1)$ and $\mathcal{G}^F = E_8(q)$, we can put $Q \in \mathrm{Syl}_5(S)$ in a subgroup $X \cong \mathrm{SU}_5(q^2)$ by [LSS, Table 5.1], and $\mathrm{SU}_5(q^2+1)$ is 5-good by Lemma 3.14, whence we are done. Aside from this case, one checks that the assumptions in (i) are fulfilled, except when there is $\epsilon = \pm$ such that $p|(q-\epsilon)$ and furthermore $(p, \mathcal{G}^F) = (5, E_6^\epsilon(q))$ or $p \in \{5, 7\}$ and $\mathcal{G}^F = E_7(q)$ or $E_8(q)$. Suppose that we are in these cases, but $p \neq 5$ if $\mathcal{G}^F = E_8(q)$. Then $d = 1$ if $\epsilon = +$ and $d = 2$ if $\epsilon = -$. By the main result of [LSS] we can find a γ-stable connected reductive subgroup $\mathcal{D}$ of $\mathcal{G}$ such that $p \nmid [\mathcal{G}^F : \mathcal{D}^F]$, with $[\mathcal{D}^F, \mathcal{D}^F]$ being a central extension of

- $\mathrm{PSL}_2(q) \times \mathrm{PSL}_6^\epsilon(q)$ when $(p, \mathcal{G}^F) = (5, E_6^\epsilon(q))$,
- $\mathrm{PSL}_8^\epsilon(q)$ if $\mathcal{G}^F = E_7(q)$, and
- $\mathrm{PSL}_9^\epsilon(q)$ if $\mathcal{G}^F = E_8(q)$.

Putting Q in $\mathcal{D}^F$ and working in $\mathcal{D}^F$, we see that condition (b) of Proposition 4.2 holds, with $R \cong C_p$. Taking $\mathcal{H} = \mathcal{G}^*/\mathbf{Z}(\mathcal{G}^*) \cong \mathcal{G}$, we then see that condition (c) of Proposition 4.2 holds, with $R^* \cong C_p$. Hence we are done by using Proposition 4.2.

(iii) From now we will assume that $p|(q-\alpha)$ for a unique $\alpha = \pm$, and either $p = 3$, or $\mathcal{G}^F = E_8(q)$ and $p = 5$. If $\mathcal{G}^F = G_2(q)$, then we can put Q in a subgroup $\mathrm{SL}_3^\alpha(q)$, cf. [LSS, Table 5.1], and see that Q is 3-good by Lemma 3.14.

To handle the next cases, we will view $S = L/\mathbf{Z}(L)$, where $L := (\mathcal{G}^*)^{F^*}$, the corresponding group of Lie type of simply connected type, and aim to show that L is p-good (in most cases). To do this, we will use a certain subgroup $K < L$ of p'-index in L and described in [GL, Table 4-1]. We also use the fact that $3 \nmid |X/[X,X]|$ for $X = \mathrm{SL}_3^\alpha(q)$, and set $M := \mathbf{O}^{\ell'}(K)$.

First we handle the case $(L,p) = (E_8(q),5)$. Then $K \cong M \cdot C_5$ with $M \cong \mathrm{SL}_5^\alpha(q) \circ \mathrm{SL}_5^\alpha(q)$. Now M is 5-good by Lemma 3.14, $5 \nmid |M/M'|$, and $K/M \cong C_5$ is also 5-good. Hence K is 5-good by Corollary 3.6.

From now on we may assume $p = 3$. Let $L = {}^3D_4(q)$. Then $K \cong M \cdot C_3$ and $M \cong \mathrm{SL}_3^\alpha(q)$. Now M is 3-good by Lemma 3.14, $3 \nmid |M/M'|$ as noted above, and $K/M \cong C_3$ is also 3-good. Hence K is 3-good by Corollary 3.6.

Next let $L = F_4(q)$. Then $K \cong M \cdot C_3$ and $M \cong \mathrm{SL}_3^\alpha(q) \circ \mathrm{SL}_3^\alpha(q)$. Now M is 3-good by Lemma 3.14, $3 \nmid |M/M'|$, and we can finish as above.

Suppose $L = E_6^\epsilon(q)_{\mathrm{sc}}$. If $\alpha \neq \epsilon$, then we can choose $K \cong F_4(q)$ which is already shown to be 3-good. Assume next that $\alpha = \epsilon$. Then we can take $K \cong M \cdot (C_3 \times C_3)$ and $M \cong \mathrm{SL}_3^\alpha(q) \circ \mathrm{SL}_3^\alpha(q) \circ \mathrm{SL}_3^\alpha(q)$, and conclude that K and L are 3-good as above.

Suppose $L = E_8(q)$. Then we can take $K \cong M \cdot C_3$ and $M \cong \mathrm{SL}_3^\alpha(q) \circ E_6^\alpha(q)_{\mathrm{sc}}$, which is already shown to be 3-good, and so we are again done.

(iv) Here we consider the case $\mathcal{G}^F = E_7(q)_{\mathrm{ad}}$ and $p = 3$ as in (iii). As $|\mathcal{G}^F/S| = \gcd(2, q-1)$, condition (a) of Proposition 4.2 holds; write $F = \gamma^m$ and $q = q_1^m$ for $m := |G/S|$. Recall that $d = 1$ if $\alpha = +$ and $d = 2$ if $\alpha = -$. Note that there exists a γ-stable maximal torus $\mathcal{T}_1$ of $\mathcal{G}$ such that $\mathcal{T}_1$ is a Sylow 1-torus and $\mathbf{N}_{\mathcal{G}^F}(\mathcal{T}_1) = \mathcal{T}_1^F \cdot W$ where $W = W(E_7)$; furthermore, γ acts on $X(\mathcal{T}_1)$ via $\nu \mapsto q_1\nu$ and on $\mathcal{T}_1$ via $t \mapsto t^{q_1}$. Let $\bar{j}$ denote the central involution of W and let $\mathcal{T}_2 = x\mathcal{T}_1x^{-1}$ with $x \in \mathcal{G}$ and $x^{-1}\gamma(x)\mathcal{T}_1 = \bar{j}$. By [C, Proposition 3.3.4], γ acts on $X(\mathcal{T}_2)$ via $\nu \mapsto -q_1\nu$, whence γ acts on $\mathcal{T}_2$ via $t \mapsto t^{-q_1}$. Thus $\mathcal{T}_2$ is a Sylow 2-torus. We have shown that $\mathcal{T}_d$ is a Sylow d-torus and γ acts on $\mathcal{T}_d$ via $t \mapsto t^{\alpha q_1}$.

By [LSS, Table 5.1], see also [GL, Table 4-1], there is a γ-stable Levi subgroup $\mathcal{D} = \mathbf{C}_{\mathcal{G}}(\mathcal{T})$ such that $\mathcal{D}^F = (C_{q-\alpha} \circ E_6^\alpha(q)_{\mathrm{sc}}) \cdot C_3$. Here, $[\mathcal{D},\mathcal{D}]$ is of type E_6 and $\mathcal{T} = \mathbf{Z}(\mathcal{D})^\circ$ is a one-dimensional γ-stable d-torus. By [MTe, Theorem 25.11], we may assume that $\mathcal{T} < \mathcal{T}_d$, and so γ acts on $\mathcal{T}$ via $t \mapsto t^{\alpha q_1}$. Choosing $s \in \mathcal{T}^F$ of order $(q_1 - \alpha)_3$, we then see that $\gamma(s) = s$. As $s \in \mathbf{Z}(\mathcal{D}^F)$, s is 3-central in $\mathcal{G}^F$. We can also view s as a γ^*-stable 3-element in $(\mathcal{G}^*)^{F^*}$. By Proposition 3.11(i), (iii), $\theta := (\chi_s)_S$ is a P-invariant p'-degree character in $B_0(S)$, whence it is σ_e-fixed and so s and s^{1+p^e} are conjugate in $(\mathcal{G}^*)^{F^*}$. This in turn implies by Lemma 3.12 that $s^{p^e} = 1$, and so

$$(q_1 - \alpha)_3 \leq 3^e. \tag{4.22}$$

Note that $N := [\mathcal{D},\mathcal{D}]^F \cong E_6^\alpha(q)_{\mathrm{sc}}$ is perfect. Next, since $[\mathcal{D},\mathcal{D}]$ is connected, the map

$$g \mapsto g[\mathcal{D},\mathcal{D}]$$

yields a γ-equivariant homomorphism from $\mathcal{D}^F$ onto $(\mathcal{D}/[\mathcal{D},\mathcal{D}])^F$ with kernel N, and so we obtain a γ-equivariant isomorphism

$$\mathcal{D}^F/N \cong (\mathcal{D}/[\mathcal{D},\mathcal{D}])^F.$$

As $\mathcal{D} = [\mathcal{D},\mathcal{D}]\mathcal{T}$, we then see that γ acts on the quotient $\mathcal{D}^F/N$ (of order $q - \alpha$) via the map $y \mapsto y^{\alpha q_1}$.

Now we may assume that $Q \in \mathrm{Syl}_3(S)$ is contained in $\mathbf{O}^{3'}(\mathcal{D}^F) > N$, and consider any P-invariant linear character λ of Q. As shown in (iii), N is 3-good. Moreover, (4.22) and the described action of γ on $\mathcal{D}^F/N$ show that any P-invariant linear character of $\mathbf{O}_3(\mathcal{D}^F/N)$ is σ_e-fixed. It follows by Lemma 3.5 that λ is σ_e-fixed, as desired. □

Proposition 4.8. *Let S be as in (4.1), $p = 2$, and suppose that $\mathcal{G}^F = E_6^\epsilon(q)_{\mathrm{ad}}$ with $4|(q - \epsilon)$ for some $\epsilon = \pm$. Then Theorem 2.3 holds in this case.*

Proof. (a) Recall that $\mathcal{G}^*$ is a simple, simply connected, algebraic group of type E_6 defined over $\mathbb{F}_\ell$, and let $\delta : \mathcal{G}^* \to \mathcal{G}^*$ denote the standard Frobenius endomorphism of $\mathcal{G}^*$ defined by this $\mathbb{F}_\ell$-structure. Let $\mathcal{T}$ be a δ-stable maximal torus of $\mathcal{G}^*$ contained in an δ-stable Borel subgroup $\mathcal{B}$. As shown in part (a) of the proof of [NT1, Proposition 4.3], there is an involutive graph automorphism τ of $\mathcal{G}^*$ that commutes with δ, stabilizes $\mathcal{T}$, and induces the involutive symmetry ρ of the Dynkin diagram of the root system Φ of $\mathcal{G}^*$ with respect to $\mathcal{T} \subset \mathcal{B}$. In particular, if $q = \ell^f$, then

$$(\mathcal{G}^*)^{\delta^f} \cong E_6(q)_{\mathrm{sc}},\ (\mathcal{G}^*)^{\tau\delta^f} \cong {}^2E_6(q)_{\mathrm{sc}}.$$

Furthermore, there is some $g \in \mathbf{N}_{\mathcal{G}^*}(\mathcal{T})$ such that the conjugation j_g by g induces $-\rho$ on $X(\mathcal{T})$ and commutes with δ. Moreover, there is a $\langle j_g\tau, \delta\rangle$-stable one-dimensional subtorus $\mathcal{T}_0$ of $\mathcal{T}$ such that

$$\mathcal{C} := \mathbf{C}_{\mathcal{G}^*}(\mathcal{T}_0) = \mathcal{T}_0\mathcal{L}$$

is a maximal rank subgroup of type D_5T_1 of $\mathcal{G}^*$, with $\mathcal{L} := [\mathcal{C}, \mathcal{C}]$ simple, simply connected, of type D_5.

(b) Suppose we are in the case $\epsilon = -$. We will use the following facts proved in part (b) of the proof of [NT1, Proposition 4.3]. First, one has

$$H := (\mathcal{G}^*)^{\delta'} \cong {}^2E_6(q)_{\mathrm{sc}},$$

where $\delta' := j_g\tau\delta^f$; furthermore, $\mathcal{T}^{\delta'} \cong C_{q+1}^6$. Denoting $L := \mathcal{L}^{\delta'} < H$ and $C := \mathcal{C}^{\delta'}$, by [LSS, Table 5.1] we have that $L \cong \mathrm{Spin}_{10}^-(q)$ and that

$$2 \nmid [H : C]. \tag{4.23}$$

Recall that $4|(q + 1)$ in the case under consideration, whence $|\mathrm{Out}(S)|_2 = 2$. Suppose first that $S < G$. In this case, as shown in part (b2) of the proof of [NT1, Proposition 4.3], $Q/\mathrm{Ker}(\lambda)$ is elementary abelian for any P-invariant linear character of Q, whence λ is σ_1-fixed.

Next suppose that $G = S$. As $\mathcal{T}$ is a Sylow 2-torus for $(\mathcal{G}^*)^{\delta'}$ and $\mathcal{T}_0 < \mathcal{T}$, we see that $\mathcal{T}_0^{\delta'} \cong C_{q+1}$. Now any generator s of $\mathbf{O}_2(\mathcal{T}_0^{\delta'})$ (of order $(q+1)_2$) centralizes C and so it is 2-central in H by (4.23). Viewing $S = [H^*, H^*]$ for H^* dual to H, we see by Proposition 3.11(i), (iii), that $\theta := (\chi_s)_S$ belongs to $B_0(S)$ and has odd degree. By hypothesis, θ is σ_e-fixed, and so s and s^{1+2^e} are conjugate in H by Proposition 3.11(iii). This in turn implies that $s^{2^e} = 1$ by Lemma 3.12. Thus

$$(q + 1)_2 \le 2^e; \tag{4.24}$$

in particular, $e \ge 2$.

Now, using the decomposition $\mathcal{C} = \mathcal{T}_0\mathcal{L}$ and arguing as in part (iv) of the proof of Proposition 4.7, we can put Q inside $L \cdot C_{(q+1)_2}$, where $C_{(q+1)_2} \cong \mathbf{O}_2((\mathcal{C}/\mathcal{L})^{\delta'})$). The inequality (4.24) clearly implies that $C_{(q+1)_2}$ is 2^e-good. On the other hand, $L \cong \mathrm{Spin}_{10}^-(q)$, and so $L/\langle z^2\rangle \cong \Omega_{10}^-(q)$ is 2-good by [NT1, Proposition 3.7], where $\mathbf{Z}(L) = \langle z\rangle \cong C_4$. As $|z^2| = 2 \le e$, it follows that L is 2^2-good, whence it is 2^e-good as well. We conclude that Q is 2^e-good by Corollary 3.6.

(c) From now on we will assume that $\epsilon = +$. Write $q = p^f$ with $f = f_0 2^a$, where $a \ge 0$ and $2 \nmid f_0$, and let $\delta_0 := \delta^{f_0}$.

Note that $j_g\tau$ normalizes $\mathcal{T}$ and acts via $\nu \mapsto -\nu$ on $X(\mathcal{T})$, whence $j_g\tau$ acts as inversion on $\mathcal{T}$ by [NT1, Lemma 4.2(ii)]. Furthermore, $j_g\tau$ commutes with δ, so $j_g\tau$ acts on $H := (\mathcal{G}^*)^{\delta^f} = E_6(q)_{\mathrm{sc}}$. Without loss of generality, we will view $j_g\tau$ as an automorphism of H and replace $j_g\tau$ by its 2-part τ_0.

Observe that the subgroups $\mathcal{T}_0$, $\mathcal{C}$, $\mathcal{L}$, $L := \mathcal{L}^{\delta^f} < H$, and $C := \mathcal{C}^{\delta^f}$ are all stable under τ_0 and δ. Also, $\mathcal{C}$ is a maximal rank subgroup of type D_5T_1 of $\mathcal{G}^*$, so by [LSS, Table 5.1] we have that

$L \cong \mathrm{Spin}_{10}^{+}(q)$ and that

$$2 \nmid [H : C]. \tag{4.25}$$

We can view S as either $H/\mathbf{Z}(H)$ or $[H^*, H^*]$, where $H^* \cong E_6(q)_{\mathrm{ad}}$, and note that

$$\mathrm{Out}(S) = H^*/S \rtimes \langle \tau_0, \delta \rangle.$$

As shown in part (c) of the proof of [NT1, Proposition 4.3], $\tau_0^2 \in R := \mathbf{O}_2(\mathcal{T}^{\delta^f})$, and (4.25) implies that $\langle C, \tau_0, \delta_0 \rangle$ has odd index in $\langle H, \tau_0, \delta \rangle$. Extend $\langle R, \tau_0, \delta_0 \rangle$ to a Sylow 2-subgroup $\tilde{P}$ of $\langle C, \tau_0, \delta_0 \rangle$. Conjugating inside $\mathrm{Aut}(S)$, and using $2 \nmid |\mathbf{Z}(H)|$, we may assume that $P \leq \tilde{P}$ for $P \in \mathrm{Syl}_2(G)$ and that $G \leq \tilde{P}H$. Then

$$Q = P \cap C \in \mathrm{Syl}_2(C),\ \tilde{P} = Q\langle \tau_0, \delta_0 \rangle,$$

and in fact

$$P/Q \leq \tilde{P}/Q \cong C_2 \times C_{2^a}, \tag{4.26}$$

with C_2 generated by τ_0 and C_{2^a} generated by δ_0.

(d) Let λ be any P-invariant linear character of Q and let $K := \mathrm{Ker}(\lambda)$. If $P \ni \tau_0$, then it was shown in part (c) of the proof of [NT1, Proposition 4.3] that Q/K is elementary abelian, and so λ is σ_1-fixed. From now on we may assume that $P \not\ni \tau_0$. Then (4.26) implies that there is some $j \in \{0, 1\}$ and $0 \leq b \leq a$ such that

$$P = \langle Q, \delta_1 \tau_0^j \rangle,$$

where $\delta_1 := \delta_0^{2^{a-b}}$. Setting $q_1 := p^{f_0 \cdot 2^{a-b}}$, we note that

$$\delta_1 \tau_0^j(t) = t^{(-1)^j q_1} \tag{4.27}$$

for all $t \in \mathcal{T}$. Next, as δ^f acts on $\mathcal{T}$ via $t \mapsto t^q$, $\mathcal{T}_0 < \mathcal{T}$ is a 1-torus and so $\mathcal{T}_0^{\delta^f} \cong C_{q-1}$. Furthermore, the condition $P \not\ni \tau$ implies that $(b, j) \neq (0, 1)$ and so $q - 1 = q_1^{2^b} - 1$ is divisible by $q_1 - (-1)^j$. It follows that we can find $s \in \mathcal{T}_0^{\delta^f} < \mathcal{T}^{\delta^f}$ of order $(q_1 - (-1)^j)_2 \geq 2$. By its choice, $s \in \mathbf{Z}(C)$ and so $2 \nmid [H : \mathbf{C}_H(s)]$ by (4.25). Now Proposition 3.11(i), (iii) implies that $\theta := (\chi_s)_S$ is irreducible, of odd degree, and belongs to $B_0(S)$, if we view S as $[H^*, H^*]$. Also, s is $\delta_1 \tau_0^j$-invariant, whence θ is P-invariant. By the main hypothesis, θ is σ_e-fixed, and so s and s^{1+2^e} are conjugate in H by Proposition 3.11(iii). This in turn implies by Lemma 3.12 that $s^{2^e} = 1$, and so

$$(q_1 - (-1)^j)_2 \leq 2^e. \tag{4.28}$$

Now, if $(q_1 - (-1)^j)_2 = 2$, then it was shown in part (d) of the proof of [NT1, Proposition 4.3] that Q/K is elementary abelian, and so λ is again σ_1-fixed. So we may assume that $e \geq 2$. It was also shown in part (d) of the proof of [NT1, Proposition 4.3] that

$$Q = Q_1 \rtimes R_2,$$

with $Q_1 \in \mathrm{Syl}_2(L)$ and $R_2 \cong C_{(q-1)_2}$ contained in $\mathcal{T}$. Recall that λ is $\delta_1 \tau_0^j$-invariant. Hence $K = \mathrm{Ker}(\lambda)$ contains $\delta_1 \tau_0^j(x) x^{-1}$ for all $x \in Q$. If, in addition, $x \in \mathcal{T}$, then (4.27) and (4.28) imply that $\delta_1 \tau_0^j(x) x^{-1} = x^{(-1)^j (q_1 - (-1)^j)}$ generates $\langle x^{2^e} \rangle$. In particular, $y^{2^e} \in K$ for all $y \in R_2$. On the other hand, if $\mathbf{Z}(L) = \langle z \rangle \cong C_4$, then $L/\langle z^2 \rangle \cong \Omega_{10}^{+}(q)$ is 2-good by [NT1, Proposition 3.7], and so Q_1 is 2^e-good as $e \geq 2$. We conclude by Lemma 3.2 that Q/K is 2^e-good, and so λ is σ_e-fixed, as desired. $\square$

Proposition 4.9. *Theorem 2.3 holds in the case where $S = P\Omega_8^{+}(q)$ and $p = 3 \nmid q$.*

Proof. (i) Recall [GLS, Theorem 2.5.12] that $\mathrm{Aut}(S) \cong \mathcal{G}^F \rtimes (C_f \times C_3)$, where $q = r^f$ for a prime $\ell \neq p$, C_f is generated by the standard Frobenius automorphism δ induced by the field automorphism $x \mapsto x^\ell$, and C_3 is generated by a triality graph automorphism. As $\mathcal{G}^F/S$ is a 2-group, by a suitable conjugation in $\mathrm{Aut}(S)$ we may assume that $G \leq S \rtimes (C_f \times C_3)$. If furthermore $G \leq S \rtimes C_f$, then the arguments in the proof of Proposition 4.3 apply and yield the result. In what follows we will therefore assume that $G = S \rtimes A$ with $P = Q \rtimes A$, $A \leq C_f \rtimes C_3$ but $A \not\leq C_f$.

(ii) Fix an orthonormal basis (e_1, e_2, e_3, e_4) of the Euclidean space $\mathbb{R}^4$ and consider

$$\alpha_1 := e_1 + e_2,\ \alpha_2 := -e_2 + e_3,\ \alpha_3 := -e_3 - e_4,\ \alpha_4 := e_4 - e_3$$

as simple roots for a root system Φ of type D_4. Then one can check that the maps

$$\beta : e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1,\ e_4 \mapsto e_4,\ \tau : \alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_2$$

are commuting isometries of order 3 of Φ, and $\beta \in W(D_4)$, the Weyl group of Φ.

Next we consider $X := \langle e_1, e_2, e_3, e_4 \rangle_{\mathbb{Z}}$ and an integer $r \in \mathbb{Z}$. Then

$$Y_1 := (\beta - 1)X = \langle e_1 - e_2, e_2 - e_3 \rangle_{\mathbb{Z}}.$$

Furthermore,

$$Y_2 := \langle Y_1, 2(r\tau - 1)X \rangle_{\mathbb{Z}} \ni 4(r^2 + r + 1)e_1, 4(r^2 + r + 1)e_4.$$

Since $9 \nmid (r^2 + r + 1)$, it follows that, for any $k \geq 2$, we have that $\exp(X/Y) \leq 3$ (in fact it is generated by $e_1 + Y$), if we set

$$Y := \langle Y_2, 3^k X \rangle_{\mathbb{Z}}.$$

(iii) Let $q \equiv \epsilon (\mathrm{mod}\, 3)$ and let $(q - \epsilon)_3 = 3^k$. Working in a subgroup

$$\Omega_2^\epsilon(q)^4 \rtimes \mathsf{A}_4 < \Omega_8^+(q),$$

we see that $Q \cong C_{3^k}^4 \rtimes C_3$, with δ acting on $C_{3^k}^4$ via $t \mapsto t^n$ for some $n \in \mathbb{Z}$. Since $A \not\leq C_f$, we have that $A \ni \delta^j \tau$, with δ^j acting on $C_{3^k}^4$ via $t \mapsto t^r$ with $r := n^j$.

If $k = 1$, then Q is 3-good, and so we are done. Suppose $k \geq 2$. We view $e_1, \ldots, e_4$ as coroots for $\mathcal{G}$ and identify $X/3^k X \rtimes \langle \beta \rangle$ with Q. Also, let $\lambda \in \mathrm{Irr}(Q)$ be linear and $\delta^j\tau$-invariant. The calculations in (ii) then show that $Q/\mathrm{Ker}(\lambda)$ is 3-good, and so λ is σ_1-fixed, as desired. □

Proof of Theorem 2.3. Theorem 2.3 now follows from Corollary 3.10, Propositions 3.15, 4.3, 4.4, 4.5, 4.7, 4.8, and 4.9 (using the classification of finite simple groups). □

5. Theorem C

In this section we prove Theorem C. We start with some general lemmas.

Lemma 5.1. *Let G be a finite group. Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ has order a power of p and fixes p'-roots of unity. Suppose that $N \lhd G$ has p'-index, and let $\theta \in \mathrm{Irr}(N)$ be σ-invariant. Then every $\chi \in \mathrm{Irr}(G|\theta)$ is σ-invariant.*

Proof. By induction on $|G : N|$ and using the Clifford correspondence, we may assume that θ is G-invariant. Now, let $\chi \in \mathrm{Irr}(G|\theta)$, and let $g \in G$. We want to show that $\chi(g)^\sigma = \chi(g)$. Working with the irreducible constituents of $\chi_{N\langle g \rangle}$, we may assume that G/N is cyclic. Then $|\mathrm{Irr}(G|\theta)| = |G/N|$ is not divisible by p. Let $A = \langle \sigma \rangle$, which is a p-group which acts on the set $\Omega = \mathrm{Irr}(G|\theta)$. Then $|\Omega| \equiv |\Omega_0| \bmod p$, where Ω_0 are the A-invariant elements in Ω. We conclude that σ fixes one extension $\chi \in \mathrm{Irr}(G|\theta)$. By Gallagher's theorem (Corollary 6.17 of [Is]), all the other extensions are products of χ with linear characters of G/N, which are σ-fixed. □

Lemma 5.2. *Let G be a finite group. Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ has order a power of p. Suppose that $N \lhd G$ has p-power index, and let $P \in \mathrm{Syl}_p(G)$. Assume that every linear character of P is σ-invariant. Let $\chi \in \mathrm{Irr}(G)$ be of p'-degree, and assume that $\chi_N = \theta$ is σ-invariant. Then χ is σ-invariant.*

Proof. Let $Q = P \cap N$ and $A = P \times \langle \sigma \rangle$. Then θ_Q is A-invariant, and has p'-degree. By Lemma 2.1(ii), we have that θ_Q contains a linear A-invariant constituent ξ with p'-multiplicity. Then $[\chi_Q, \xi]$ is not divisible by p. By Lemma 2.1(i) (with $H = Q$, $A = P$, and $\psi = \chi_P$), we have that χ_P contains an irreducible constituent τ such that $\tau_Q = \xi$, and $[\chi_P, \tau]$ is not divisible by p. By hypothesis, we have that τ is σ-invariant. Now $[(\tau^G)_N, \theta] = [\xi^N, \theta] = [\theta_Q, \xi]$ is not divisible by p. By Lemma 2.1(i) applied to $\langle \sigma \rangle$, τ^G and $H = N$, we have that τ^G contains an irreducible constituent ρ, σ-invariant, such that $[\rho_N, \theta]$ is not divisible by p. Then $\rho_N = \theta$ (by Corollary (11.29) of [Is]) and $\chi = \lambda\rho$, for some linear $\lambda \in \mathrm{Irr}(G/N)$, by Gallagher's theorem. However, λ is σ-invariant by hypothesis and the fact that $G/N \cong P/Q$, so χ is σ-invariant. ☐

Next we use character triples. The notation we follow is that of [N3].

Theorem 5.3. *Suppose that $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Suppose that θ can be afforded by an absolutely irreducible $\mathbb{F}$-representation, where $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{C}$ is a field. Assume that M/N is a non-trivial perfect normal subgroup of G/N. Then there exists a character triple (M^*, N^*, θ^*) isomorphic to (M, N, θ) such that:*

(i) *G acts as automorphisms on M^*, centralizing N^*, and such that*
$$((mN)^*)^g = (m^g N)^*$$
for $m \in M$ and $g \in G$.

(ii) *M^* is perfect, $N^* \leq \mathbf{Z}(M^*)$, N^* is isomorphic to a finite subgroup of $\mathbb{F}^\times$, and θ^* is faithful.*

(iii) *If $N \leq U \leq M$ and $\nu \in \mathrm{Irr}(U|\theta)$, then $(\nu^*)^g = (\nu^g)^*$ for $g \in G$.*

(iv) *If $N \leq U \leq M$ and $\nu \in \mathrm{Irr}(U|\theta)$, then $\mathbb{F}(\nu) = \mathbb{F}(\nu^*)$.*

Proof. By hypothesis, we have that θ can be afforded by an absolutely irreducible $\mathbb{F}N$-representation $\mathcal{Y}$. Arguing as in Theorem (11.2) of [Is], there exists a projective representation $\mathcal{X}$ of G such that $\chi(g) \in \mathrm{GL}_d(\mathbb{F})$ for $g \in G$ and satisfying conditions (a), (b), (c) of Theorem (11.2) of [Is]. (See the remark after the proof of (11.2) in [Is]. See also the proof of Theorem 5.1 in [NTT1].) If α is the factor set associated to $\mathcal{X}$ we have that $\alpha(g, n) = \alpha(n, g) = 1$ for $n \in N$ and $g \in G$ and $\alpha(gn, hm) = \alpha(g, h)$ for $g, h \in G$ and $n, m \in N$.

Now, let $\mathbb{F}^\times$ be the multiplicative group of $\mathbb{F}$. For any subgroup $H \leq G$, we define $\tilde{H} = H \times \mathbb{F}^\times$. It is straightforward to check that $\tilde{G}$ is a group with multiplication
$$(g_1, f_1)(g_2, f_2) = (g_1 g_2, \alpha(g_1, g_2) f_1 f_2)\,.$$
Also, $\tilde{H}$ is a subgroup of $\tilde{G}$ whenever H is a subgroup of G. We have that the map $\pi : \tilde{G} \to G$ given by $(g, f) \mapsto g$ is an onto homomorphism with kernel $\tilde{1} = 1 \times \mathbb{F}^\times$. Hence, $\tilde{M}$ and $\tilde{N}$ are normal subgroups of $\tilde{G}$. Furthermore, we can check that $N \times 1$ (which we again call N) is a normal subgroup of $\tilde{G}$, $\tilde{1}$ is contained in $\mathbf{Z}(\tilde{G})$ and $\tilde{N} = N \times \tilde{1}$.

Now, $\tilde{N}/N \leq \mathbf{Z}(\tilde{G}/N)$. Let $M_1/N = (\tilde{M}/N)'$. Since $\tilde{M}/\tilde{N}$ is perfect, we have that $\tilde{M} = \tilde{N}M_1$. Also, M_1/N is perfect. Also, M_1/N is finite, by Schur's theorem. (See (IV.2.3) of [Hu].) In particular, M_1 is finite. Let $N_1 = M_1 \cap \tilde{N}$, and $F_1 = N_1 \cap \tilde{1}$, so that $N_1 = F_1 \times N$. Notice that F_1 is a finite subgroup of $\mathbb{F}^\times$.

Now, $M_1 \lhd \tilde{G}$ and thus $\tilde{G}$ acts on M_1 by conjugation. Since $\tilde{1}$ is in the kernel of the action, we have that G acts on M_1.

We define

$$\tilde{\mathcal{X}}(g,f) = f\mathcal{X}(g)\,.$$

It is clear that $\tilde{\mathcal{X}}$ is an $\mathbb{F}$-representation of $\tilde{G}$. Also, $\tilde{\mathcal{X}}(n) = \mathcal{Y}(n)$ for $n \in N$. Now, define the linear character $\lambda \in \mathrm{Irr}(N_1)$ by $\lambda(n,f) = \bar{f}$, where $\bar{f}$ is the complex conjugate of f. Let $\theta_1 = 1 \times \theta \in \mathrm{Irr}(N_1)$. Let τ be the character of the representation of $\mathcal{X}_{M_1}$. Note that τ is $\tilde{G}$-invariant and has its values in $\mathbb{F}$. Also, $\tau_{N_1} = \bar{\lambda}\theta_1$, where again $\bar{\lambda}$ is the complex conjugate of λ and $\tau_N = \theta$. Thus $\tau \in \mathrm{Irr}(M_1)$.

Now, the map $x \mapsto \pi(x)$ is an onto group homomorphism from M_1 to M with kernel F_1. Since $F_1 \leq \ker(\theta_1)$, by Lemma (11.26) of [Is], we have that (M_1, N_1, θ_1) and (M, N, θ) are isomorphic character triples. This isomorphism commutes with G-conjugation and preserves field of values of characters. Now, by the remark after Lemma (11.27) of [Is], we have that (M_1, N_1, λ) and (M_1, N_1, θ_1) are isomorphic character triples. Furthermore, if $\nu \in \mathrm{Irr}(U \,|\, \lambda)$, we have that the image of ν under this isomorphism is $\nu\tau_U$. Clearly, we have that $\mathbb{F}(\tau_U\nu) \subseteq \mathbb{F}(\nu)$. On the other hand, if $\sigma \in \mathrm{Gal}(\mathbb{F}(\nu)/\mathbb{F}(\tau_U\nu))$, then

$$\tau_U\nu = (\tau_U\nu)^\sigma = \tau_U^\sigma\nu^\sigma = \tau_U\nu^\sigma\,,$$

using that τ is $\mathbb{F}$-valued. By the uniqueness in Gallagher's theorem, we conclude that $\nu^\sigma = \nu$, and $\mathbb{F}(\tau_U\nu) \subseteq \mathbb{F}(\nu)$. Also, this bijection $\nu \mapsto \tau_U\nu$ commutes with G-action.

Finally, (M_1, N_1, λ) and $(M_1/N, N_1/N, \lambda)$ are isomorphic by Lemma (11.26) of [Is] (with an isomorphism that preserves fields of values and commutes with G-action). Now, write $M^* = M_1/N$, $N^* = N_1/N$ and $\theta^* = \lambda$. $\square$

We prove Theorem C by assuming a slightly weaker version of Conjecture A for almost quasi-simple groups.

Conjecture 5.4. *Let $\sigma = \sigma_e$, as in Conjecture A. Suppose that S is a perfect group such that $S/\mathbf{Z}(S)$ is a simple group, and $|\mathbf{Z}(S)|$ has order not divisible by p. Suppose that a p-group P acts by automorphisms on S centralizing $\mathbf{Z}(S)$. Let Q be a P-invariant Sylow p-subgroup of S. Assume that every linear P-invariant character of Q is σ-invariant. Then every P-invariant $\chi \in \mathrm{Irr}_{p'}(S)$ is σ-invariant.*

This is exactly how we will use Conjecture 5.4.

Corollary 5.5. *Assume Conjecture 5.4, and let $\sigma = \sigma_e$. Suppose that M is a perfect group acted on by a p-group P. Suppose that $|Z|$ is not divisible by p, where $Z = \mathbf{Z}(M)$ is centralized by P. Assume that M/Z is a direct product of non-abelian simple groups transitively permuted by P. Let Q be a P-invariant Sylow p-subgroup of M, and assume that every linear P-invariant character of Q is σ-invariant. Then every P-invariant $\chi \in \mathrm{Irr}_{p'}(M)$ is σ-invariant.*

Proof. Write

$$M/Z = S_1/Z \times \ldots \times S_a/Z\,,$$

where S_i/Z is simple and the S_i's are transitively permuted by P. Hence, for $i \neq j$, we have that $[S_i, S_j] = 1$. Also S_i' is perfect, by elementary group theory, and S_i is the central product of S_i' and $Z_i = Z \cap S_i$. Write $Q_i = Q \cap S_i \in \mathrm{Syl}_p(S_i)$. Since Z_i is a p'-group, we also have that $Q_i \in \mathrm{Syl}_p(S_i')$. Let P_1 be the stabilizer of S_1 in P. Let $\{u_1, \ldots, u_a\}$ be a transversal for the right cosets of P_1 in P.

Let $\theta \in \mathrm{Irr}(Z)$. By Lemma (4.1.ii) of [NTT1], there is a natural bijection

$$(\psi_1, \ldots, \psi_a) \mapsto \psi_1 \cdots \psi_a$$

from $\mathrm{Irr}(S_1|\theta) \times \ldots \times \mathrm{Irr}(S_a|\theta)$ onto $\mathrm{Irr}(M|\theta)$. Furthermore, if $\psi \in \mathrm{Irr}(S_1|\theta)$, then ψ is P_1-invariant if and only if $\psi^{u_1} \cdots \psi^{u_a}$ is P-invariant. Also, the map commutes with σ. In the same way, we have a natural bijection $\mathrm{Irr}(Q_1) \times \cdots \times \mathrm{Irr}(Q_a) \to \mathrm{Irr}(Q)$ that commutes with σ, and such that $\psi_1 \times \ldots \times \psi_a$ is P-invariant if and only if ψ_1 is P_1-invariant. Finally, observe that since S_i is the central product of S_i' and Z, we have that $\mathrm{Irr}(S_i|\theta) = \{\nu \cdot \theta \mid \nu \in \mathrm{Irr}(S_i'|\theta_i)\}$, where $\theta_i = \theta_{Z_i}$. Now the proof easily follows by applying Conjecture 5.4 to each S_i'. □

In our next theorem we will use the following extension result, whose proof uses elementary but non-trivial character theory.

Theorem 5.6. *Suppose that G is a finite group, $K = \mathbf{O}^p(G)$, and let $P \in \mathrm{Syl}_p(G)$. Let $P \leq V \leq G$ and $U = V \cap K$. If $\theta \in \mathrm{Irr}(U)$ has p'-degree and is P-invariant, then θ extends to V.*

Proof. This is Theorem 2.6 of [NT2]. □

The following includes Theorem C.

Theorem 5.7. *Assume Conjecture 5.4. Let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$ such that the exponent of P/P' is less than or equal to p^e. Let $L \triangleleft G$ and suppose that $\theta \in \mathrm{Irr}(L)$ has p'-degree, extends to LP, and is σ-invariant. Then all $\mathrm{Irr}_{p'}(G|\theta)$ are σ-invariant.*

Proof. We argue by double induction, first on $|G|$, and then on $|G : L|$. By the Clifford correspondence, we may assume that θ is G-invariant. Let $\chi \in \mathrm{Irr}_{p'}(G|\theta)$. We want to show that $\chi^\sigma = \chi$. Notice that if $P \leq H < G$, then every p'-degree irreducible character of H is σ-invariant by induction.

Let K/L be a chief factor of G. We claim that we may assume that $G = KP$. Suppose that $KP < G$. Let $\tau \in \mathrm{Irr}_{p'}(PK)$ be under χ. Then τ is σ-invariant by induction, since we are assuming that $KP < G$. Now, $\tau_K \in \mathrm{Irr}(K)$ (because $|KP : K|$ and $\tau(1)$ are coprime) is σ-invariant, has p'-degree, and extends to KP. Since $|G : K| < |G : L|$, by induction we are done. Hence, the claim is proved, and we assume that $G = PK$.

We have now that $\chi_K = \psi \in \mathrm{Irr}(K)$ is σ-invariant of p'-degree. By Lemma 5.2, it suffices to show that ψ is σ-invariant.

Suppose first that K/L is a p-group. Then G/L is a p-group, and Lemma 5.2 applies. Assume next that K/L is a p'-group. By Lemma 5.1, we have that ψ is σ-invariant, so again we are done.

So we are left with the case where K/L is perfect of order divisible by p.

Now, let $M = \mathbf{O}^p(G) = \mathbf{O}^p(K)$, and let $N = M \cap L$. Notice that $ML = K$ and that $M/N \cong K/L$ is a chief factor of G. Let $Q = P \cap M \in \mathrm{Syl}_p(M)$. Let $\theta_1 = \theta_L \in \mathrm{Irr}(N)$. Let $\mathbb{F} = \mathbb{Q}_{|G|_{p'}}(\theta_1)$. Then θ_1 can be afforded by an absolutely irreducible $\mathbb{F}$-representation by Corollary (10.13) of [Is] (in the case $p = 2$ we also use [Is, Corollary (10.2.h)]). By Theorem 5.3, there exists a character triple (M^*, N^*, θ_1^*), where M^* is perfect, $N^* \leq \mathbf{Z}(M^*)$, θ_1^* is faithful, among some further properties. Now, $\chi_M = \psi_M \in \mathrm{Irr}(M) = \tau$ has p'-degree and lies over θ_1. Notice then that τ^* is a p'-degree character over θ_1^* (use Lemma (11.24) of [Is]). Now, $(\tau^*)_{N^*} = e\theta_1^*$. Since M^* is perfect, the determinant of τ^* is trivial. Hence θ_1^* has p'-order, so N^* is a p'-group (using that θ_1^* is faithful).

Recall that by Theorem 5.3, we have that G acts on the group M^* centralizing N^*, and that M/N and M^*/N^* are G-isomorphic. Now, write $(QN)^* = Q^* \times N^*$, where we notice that Q^* is a P-invariant Sylow p-subgroup of M^* (because QN is P-invariant). Let $\lambda^* \in \mathrm{Irr}(Q^*)$ be linear P-invariant. We prove that λ^* is σ-invariant. We have that $\nu^* = \lambda^* \times \theta_1^*$ is linear P-invariant. Let $\nu \in \mathrm{Irr}(QN|\theta_1)$ be the preimage under the character triple isomorphism. Notice that ν has p'-degree and is also P-invariant (by Theorem 5.3(iii)). By Theorem 5.6, we have that ν extends to some $\epsilon \in \mathrm{Irr}(NP)$. Since $NP < G$, we have that ϵ has p'-degree, and by induction, we have that it is σ-invariant. Therefore so is $\epsilon_{NQ} = \nu$. Now $\mathbb{F} = \mathbb{Q}_{|G|_{p'}}(\theta_1)$, where $\theta_1 = \theta_N$. Since θ is σ-fixed and σ fixes p'-roots of unity, we conclude that σ fixes every element of $\mathbb{F}$. Since ν is σ-invariant, then σ also fixes every element of $\mathbb{F}(\nu) = \mathbb{F}(\nu^*)$ (by Theorem 5.3(iv)). We conclude that ν^* is σ-invariant, and therefore so is λ^*. By Corollary 4.5, we have that every p'-degree irreducible character of M^* is σ-invariant. Hence, τ^* is σ-invariant. Again, by Theorem 5.3(iv), we have that $\mathbb{F}(\tau) = \mathbb{F}(\tau^*)$, so we conclude that τ is σ-invariant. Now, $\chi_M = \tau$ and we use Lemma 5.2 to obtain that χ is σ-invariant. □

We conclude this section by providing some evidence in support of Conjecture 5.4.

Theorem 5.8. *If $p > 2$, then Conjecture 5.4 holds if $S/\mathbf{Z}(S)$ is an alternating group, a sporadic group, or a simple group of Lie type in the same characteristic p.*

Proof. First suppose that $S/\mathbf{Z}(S)$ is a simple group of Lie type in the same characteristic p. Since $p \nmid |\mathbf{Z}(S)|$, we can find a simple, simply connected algebraic group $\mathcal{G}$ over a field of characteristic p and a Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$ such that S is a quotient of $G = \mathcal{G}^F$ by a central subgroup. Let $\mathbb{K} := \mathbb{Q}(\exp(2\pi i/|G|_{p'}))$. Since $4||G|$ and $p > 2$, note that $\mathbb{K}$ contains $\sqrt{-1}$. Now, by Theorem 1.3 and Proposition 10.12 of [TZ], $\mathbb{Q}(\chi) \subseteq \mathbb{K}(\exp(2\pi i/p))$ for all $\chi \in \mathrm{Irr}(G)$; in particular, χ is σ_1-fixed. Hence Conjecture 5.4 holds for S in this case.

Next assume that $S/\mathbf{Z}(S) \cong \mathsf{A}_n$ for some $n \geq 5$ and let $\mathbb{K} := \mathbb{Q}(\exp(2\pi i/|S|_{p'}))$. Again, as $p > 2$, we see that $\mathbb{K} \ni \sqrt{-1}$. It is a classical result of Schur (see eg. Theorems 8.6 and 8.7 of [HH] for the case $S = 2\mathsf{A}_n$) that for any $\chi \in \mathrm{Irr}(S)$ and any $g \in S$, $\chi(g) \in \mathbb{K}(\sqrt{m})$ for some positive integer m (which may depend on g), if $|\mathbf{Z}(S)| \leq 2$. If $|\mathbf{Z}(S)| > 2$, then $S \in \{3\mathsf{A}_m, 6\mathsf{A}_m\}$ with $m = 6$ or 7, and $p > 3$. In this case, using eg. [GAP] one can check that $\chi(g) \in \mathbb{K}$ for all $g \in S$ and $\chi \in \mathrm{Irr}(G)$. It follows that $\chi(g) \in \mathbb{K}(\exp(2\pi i/p))$ for all $g \in S$, and so χ is again σ_1-fixed.

Finally, the cases where $S/\mathbf{Z}(S)$ is a sporadic simple group can be verified using [GAP]. □

References

[BFMNST] A. Beltrán, M. Felipe, G. Malle, A. Moreto, G. Navarro, L. Sanus, R. M. Solomon, and Pham Huu Tiep, Nilpotent and abelian Hall subgroups in finite groups, *Trans. Amer. Math. Soc.* **368** (2016), 2497–2513.

[Br] R. Brauer, Representations of finite groups, in: '*Lectures on Modern Mathematics*', vol. I, ed. by T. Saaty, John Wiley & Sons, 1963.

[C] R. Carter, '*Finite Groups of Lie type: Conjugacy Classes and Complex Characters*', Wiley, Chichester, 1985.

[DM] F. Digne and J. Michel, '*Representations of Finite Groups of Lie Type*', London Mathematical Society Student Texts **21**, Cambridge University Press, 1991.

[GAP] The GAP group, 'GAP - *groups, algorithms, and programming*', Version 4.4, 2004, http://www.gap-system.org.

[GL] D. Gorenstein and R. Lyons, The local structure of finite groups of characteristic 2 type, *Mem. Amer. Math. Soc.* **42** (1983).

[GLS] D. Gorenstein, R. Lyons, and R. Solomon, '*The Classification of the Finite Simple Groups*', Number 3. Part I. Chapter A, vol. **40**, *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[HH] P. N. Hoffman and J. F. Humphreys, '*Projective Representations of the Symmetric Groups*', Oxford Science Publications, Clarendon Press, Oxford, 1992.

[Hu] B. Huppert, '*Endliche Gruppen. I*', Springer-Verlag, Berlin et al, 1967.

[Is] I. M. Isaacs, '*Character Theory of Finite Groups*', AMS-Chelsea, Providence, 2006.

[IN] I. M. Isaacs and G. Navarro, Characters of p'-degree of p-solvable groups, *J. Algebra* **246** (2001), 394–413.

[LSS] M. W. Liebeck, J. Saxl and G. M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, *Proc. London Math. Soc.* **65** (1991), 297–325.

[M] G. Malle, The Navarro–Tiep Galois conjecture for $p = 2$, *Archive der Math.* (to appear).

[MTe] G. Malle and D. Testerman, '*Linear Algebraic Groups and Finite Groups of Lie Type*', Cambridge Studies in Advanced Mathematics, **133**, Cambridge University Press, 2011.

[N1] G. Navarro, '*Characters and Blocks of Finite Groups*', Cambridge University Press, 1998.
[N2] G. Navarro, The McKay conjecture and Galois automorphisms, *Annals of Math.* **160** (2004), 1129–1140.
[N3] G. Navarro, '*Character Theory and the McKay conjecture*', Cambridge University Press, 2018.
[NSV] G. Navarro, B. Späth and C. Vallejo, A reduction theorem for the Galois–McKay conjecture, (in preparation).
[NT1] G. Navarro and Pham Huu Tiep, Real groups and Sylow 2-subgroups, *Adv. Math.* **299** (2016), 331–360.
[NT2] G. Navarro and Pham Huu Tiep, Representations of odd degree, *Math. Ann.* **365** (2016), 1155–1185.
[NTT1] G. Navarro, Pham Huu Tiep, and A. Turull, p-rational characters and self-normalizing Sylow p-subgroups, *Represent. Theory* **11** (2007), 84–94.
[NTT2] G. Navarro, Pham Huu Tiep, and A. Turull, Brauer characters with cyclotomic field of values, *J. Pure Appl. Alg.* **212** (2008), 628–635.
[SF] A. A. Schaeffer-Fry, Actions of Galois automorphisms on Harish-Chandra series and Navarro's self-normalizing Sylow 2-subgroup conjecture, https://arxiv.org/pdf/1707.03923.pdf, *Trans. Amer. Math. Soc.* (to appear).
[TZ] Pham Huu Tiep and A. E. Zalesski, Unipotent elements of finite groups of Lie type and realization fields of their complex representations, *J. Algebra* **271** (2004), 327–390.

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