ON REAL AND RATIONAL CHARACTERS IN BLOCKS

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Abstract. The principal p-block of a finite group G contains only one real-valued irreducible ordinary character exactly when $G/\mathbf{O}_{p'}(G)$ has odd order. For $p \neq 3$, the same happens with rational-valued characters. We also prove an analogue for p-Brauer characters with $p \geq 3$.

1. Introduction

Let G be a finite group and let p be a prime. Richard Brauer partitioned the set of the irreducible complex characters of G into p-blocks, which are permuted by complex conjugation. The study of real (self-conjugate) blocks was initiated by Brauer himself in [2]. R. Gow extensively studied the real blocks for $p = 2$ in [17], and he showed that, unless they consist of a single character, they should contain an even number of real-valued irreducible characters. Here we study real-valued irreducible characters in principal blocks (those containing the trivial representation) for every prime, and this is our first main result. Recall that $\mathbf{O}_{p'}(G)$ is the largest normal subgroup of G of order not divisible by p.

Theorem A. *Let G be a finite group, let p be a prime and let B be the principal p-block of G. Then the only irreducible complex character in B which is real-valued is trivial if and only if $G/\mathbf{O}_{p'}(G)$ has odd order. If this is not the case and p is odd, then B has at least three real-valued irreducible characters, no two of which are algebraic conjugates.*

Outside principal blocks, there are various examples of simple and quasi-simple groups having blocks of maximal defect with exactly one real-valued ordinary irreducible character, like $G = \mathrm{PSL}_2(11)$ for $p = 3$, or $G = 2\mathsf{A}_8$ for $p = 5$, so the natural object of our investigation here is principal blocks.

Once we know that the principal block of G has at least two real-valued irreducible characters, we produce a third one by a general argument (Theorem 3.4). Many examples

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show that three is the best that we can obtain in general, for instance $G = \mathsf{A}_5$ for $p = 3$ or $G = M_{11}$ for $p = 5$. For $p = 2$, this number is two, as shown by $G = \mathrm{PSL}_2(11)$.

Theorem A can be significantly improved, especially if $p \neq 3$, in which case we may replace real-valued with rational-valued characters.

Theorem B. *Let G be a finite group, let p be a prime and let B be the principal p-block of G. Then the following conditions are equivalent:*

(a) *If $p \neq 3$, the only irreducible complex character which is rational valued in B is trivial; and if $p = 3$, the only irreducible complex character which is real valued and p-rational in B is trivial.*
(b) *$G/\mathbf{O}_{p'}(G)$ has odd order.*

For $p = 3$, the group $\mathrm{PSL}_2(27)$ has no non-trivial rational character in its principal 3-block. Also, contrary to the case of Theorem A, where we showed that there are no principal blocks with exactly two real-valued characters for p odd, we notice that A_5, for instance, has exactly two rational-valued characters in its principal 5-block.

Another natural hypothesis that looks plausible in Theorem A is the following: can we add the condition of p'-degree on our real characters? (Real characters of p'-degree were first studied in [25].) The answer is no, however. There are solvable groups of even order with only one block such that the trivial character is the only p'-degree real character. For instance, for $p = 7$, the group $G = VH$, where $H = (C_2)^3 \rtimes C_7$, and V an irreducible 7-module of H of dimension 7. For $p = 3$, $\mathrm{PSL}_2(27) \rtimes C_3$ provides an example with no non-trivial real-valued irreducible character of p'-degree.

Our proofs of Theorems A and B rely on the Classification of Finite Simple Groups. But we do have some general arguments for some special cases that may have independent interest, and that avoid the use of the Deligne-Lusztig theory in some cases. (See Section 3.)

Finally, we consider real-valued Brauer characters in the principal block, a much more difficult territory. After a clever argument by T. Breuer that handles the sporadic groups, using Brauer trees for cyclic Sylow subgroups, and relying on the Deligne-Lusztig theory and deep results on modular representations of finite group of Lie type, we have been able to prove the following modular version of Theorem A.

Theorem C. *Let G be a finite group, let p be an odd prime and let B be the principal p-block of G. Then the only irreducible p-Brauer character which is real-valued in B is trivial if and only if $G/\mathbf{O}_{p'}(G)$ has odd order.*

As shown by $\mathrm{SL}_3(2)$, Theorem C does not hold for $p = 2$.

2. Complex Characters in the Principal Block

We use the notation of [24] for characters and [30] for blocks. For instance, $\mathbb{Q}(\theta)$ is the field of values of the character θ, and $B_0(G)$ is the principal p-block of G. Recall that $\chi \in \mathrm{Irr}(G)$ is p-rational if $\mathbb{Q}(\chi)$ is contained in some cyclotomic field $\mathbb{Q}_n$ with $p \nmid n$.

Theorem 2.1. *Let p be any prime, and let S be a non-abelian simple group of order divisible by p.*

(a) *If* $p \neq 3$*, then* S *has an irreducible non-trivial rational-valued character* γ *in the principal block of* S*.*
(b) *If* $p = 3$*, then* S *has an irreducible non-trivial real-valued* p*-rational character* γ *in the principal block of* S*.*

Proof. The cases where S is one of the 26 sporadic simple groups or $S \cong {}^2F_4(2)'$ can be checked directly using [13]. We will now prove the theorem for the remaining simple groups.

First assume that $p > 2$. If $S = \mathsf{A}_n$, then the statements follow from the proof of [36, Lemma 5.3]. If S is a simple group of Lie type and furthermore $S \not\cong \mathrm{PSL}_2(3^{2a+1})$ for any $a \geq 1$ if $p = 3$, then the statements follow from Theorems 5.4 and 5.5 of [36]. If $p = 3$ and $S \cong \mathrm{PSL}_2(3^{2a+1})$, then it is easy to check using the character table of S, see eg. [9, Theorem 38.1], that $B_0(S)$ contains a non-trivial real-valued p-rational irreducible character (say of degree $3^{2a+1} + 1$).

It remains to consider the case $p = 2$. By [33, Theorem 9.5], $\mathrm{Irr}(S)$ contains a non-trivial rational-valued character γ of odd degree. Let γ° denote its restriction to odd-order elements in S. By Fong's Theorem (2.30) of [30], if $1_S \neq \psi \in \mathrm{IBr}_2(S)$ is real-valued, then $2 | \psi(1)$. Since $2 \nmid \gamma(1)$, it follows that γ° contains 1_S as an irreducible constituent. Hence γ belongs to $B_0(S)$, and so we are done. □

We continue with the following lemma.

Lemma 2.2. *Let* G *be a finite group, and suppose that* $N \lhd G$ *has odd index. Let* p *be any prime. Let* θ *be an irreducible ordinary (modular) real-valued character of* N *in the principal* p*-block. Then there exists an irreducible ordinary (modular) real-valued character* η *of* G *over* θ *in the principal* p*-block such that* $\mathbb{Q}(\theta) = \mathbb{Q}(\eta)$*, if* θ *and* η *are ordinary.*

Proof. We know that there exists a unique irreducible real-valued character η of G over θ. (This follows by Corollary 2.2 of [33] if $\theta \in \mathrm{Irr}(N)$, and by Lemma 5 of [32] if $\theta \in \mathrm{IBr}(N)$.) By uniqueness, it follows that $\mathbb{Q}(\theta) = \mathbb{Q}(\eta)$, if θ and η are ordinary. It remains to prove that η belongs to $B_0(G)$. We argue by induction on $|G : N|$, and we may easily assume that G/N is cyclic of prime order q. If $q = p$, then η lies in the principal p-block of G, because there is only one block covering $B_0(N)$. In this case, we are done. So we may assume that $q \neq p$. Let $P \in \mathrm{Syl}_p(G)$, and let $K = N\mathbf{C}_G(P)$. If $K = N$, then, again $B_0(G)$ is the only block covering $B_0(N)$ by Lemma 3.1 of [35], and we are done.

Hence, we may assume that $G = N\mathbf{C}_G(P)$. By [1], we have that θ has a unique extension γ to G in the principal block of G. By uniqueness, γ is real-valued, $\eta = \gamma$, and the proof is finished. □

In the proof of Theorem 2.3 below, we will use the fact that the only finite groups that do not possess non-trivial rational-valued irreducible characters are the groups of odd order. A proof of this result, which uses the Classification of Finite Simple Groups, can be found as Theorem 8.2 in [33].

Theorem 2.3. *Let* G *be a finite group, let* p *be a prime and let* B *be the principal block of* G*. Then the following conditions are equivalent:*

(a) *If $p \neq 3$ or G is p-solvable, the only irreducible complex character which is rational-valued in B is trivial; and if $p = 3$, the only irreducible complex character which is real valued and p-rational in B is trivial.*
(b) *We have that $G/\mathbf{O}_{p'}(G)$ has odd order.*

Proof. Assume (b). Then G is p-solvable, and we know that $\mathrm{Irr}(B) = \mathrm{Irr}(G/\mathbf{O}_{p'}(G))$ by Fong's Theorem 10.20 of [30]. Since $G/\mathbf{O}_{p'}(G)$ is a group of odd order, then it does not have non-trivial real-valued irreducible characters, by a well-known result of Burnside. This shows (a).

To prove the converse, we argue by induction on $|G|$. We may assume that G has no odd index normal subgroups, by Lemma 2.2. If $1 < N$ is a normal subgroup of G, then by using induction in G/N and the previous sentence, we have that G/N has order not divisible by p. Hence, we may also assume that $\mathbf{O}_{p'}(G) = 1$, and we want to show that G has odd order. Notice that we may also assume that G has a unique minimal normal subgroup N.

Suppose that N is a p-group. Then G is p-solvable, has a unique p-block (by Fong's Theorem) with only one rational character (by hypothesis). We conclude that G has odd order by Theorem 8.2 in [33].

Hence, we may assume that G has a unique minimal normal subgroup N, non-abelian, of order divisible by p, and $p \nmid |G/N|$. Let $P \in \mathrm{Syl}_p(G)$. Then all the irreducible characters of $G/N\mathbf{C}_G(P)$ are in the principal p-block of G by Lemma 3.1 of [35]. Thus $G/N\mathbf{C}_G(P)$ is a group with no rational characters. Therefore, this is a group of odd order by Theorem 8.2 in [33]. By the second paragraph, we conclude that $G = N\mathbf{C}_G(P)$. Therefore we can apply the Alperin's theory of isomorphic blocks ([1]). That is, if $\gamma \in \mathrm{Irr}(N)$ is in the principal block, then there exists $\hat{\gamma} \in \mathrm{Irr}(G)$ that extends γ, lies in the principal block of G, and has the same field of values of γ. Using that N is a direct product of simple groups of order divisible by p, we use Theorem 2.1, to finish the proof. □

To finish this section, we show that it suffices to prove Theorem C for simple groups. Since the proof is quite similar to the proof of Theorem 2.3, we simply give a sketch of it.

Theorem 2.4. *Let G be a finite group, let p be an odd prime and let B be the principal block of G. Assume that every non-abelian simple group of order divisible by p has at least two real-valued irreducible p-Brauer characters in its principal p-block. Then B has exactly one real-valued irreducible p-Brauer character if and only if $G/\mathbf{O}_{p'}(G)$ has odd order.*

Proof. First, notice that the only groups with exactly one p-Brauer real-valued irreducible character are the groups of odd order. This follows from Brauer's Lemma on character tables.

If $G/\mathbf{O}_{p'}(G)$ has odd order, then G is a p-solvable group, and we know that $\mathrm{Irr}(B) = \mathrm{Irr}(G/\mathbf{O}_{p'}(G))$ by Fong's Theorem 10.20 of [30]. Since $G/\mathbf{O}_{p'}(G)$ is a group of odd order, then it does not have non-trivial real-valued irreducible Brauer characters.

To prove the converse, we argue by induction on $|G|$. As in the proof of Theorem 2.3, we may also assume that G has a unique minimal normal subgroup N of order divisible by

p, and that G/N has order not divisible by p. If N is a p-group, then G is p-solvable and $\mathrm{IBr}(G)$ has a unique real-valued irreducible Brauer character. Thus $|G|$ has odd order.

Hence, we may assume that N is non-abelian. Let $P \in \mathrm{Syl}_p(G)$. Then all the irreducible characters of $G/N\mathbf{C}_G(P)$ are in the principal p-block of G by Lemma 3.1 of [35]. Thus $G/N\mathbf{C}_G(P)$ is a group with no real-valued Brauer characters. Therefore, it is a group of odd order and we conclude that $G = N\mathbf{C}_G(P)$. Since N is a direct product of non-abelian simple groups, by hypothesis we know that N has a non-trivial irreducible Brauer character in its principal block. We finally apply Alperin's theory of isomorphic blocks ([1]), this time for Brauer characters. ☐

3. Some General Arguments

In the first part of this section, we give a general argument for a block to contain a real-valued character. When χ is an irreducible character, we let $\nu(\chi)$ denote the Frobenius-Schur indicator of χ. Also, if $g \in G$ then we let $\mathrm{Sq}(g)$ be the set of square roots of g, that is the set $\{x \in G \mid x^2 = g\}$. Recall that

$$|\mathrm{Sq}(g)| = \sum_{\chi \in \mathrm{Irr}(G)} \nu(\chi)\chi(g)$$

by Lemma 4.4 of [24].

Theorem 3.1. *Let G be a finite group, and x be a p-element of G of odd order such that $\mathbf{C}_G(x)$ has even order. Let y be a 2-element of $\mathbf{C}_G(x)$ of maximal order. Then xy has no square root in G, and there is a non-trivial real-valued $\chi \in \mathrm{Irr}(G)$ with $\chi(xy) \neq 0$.*

Proof. Suppose that $u \in G$ has $u^2 = xy$. Let w be the p-part of u and z be the p'-part of u. Then $w^2 = x$ and $z^2 = y$. However, we then have $\mathbf{C}_G(w) \leq \mathbf{C}_G(x)$, so that $z \in \mathbf{C}_G(x)$. But then $|\langle z \rangle| = 2|\langle y \rangle|$, contrary to the fact that y is a 2-element of $\mathbf{C}_G(x)$ of maximal order. Therefore

$$0 = |\mathrm{Sq}(xy)| = \sum_{\chi \in \mathrm{Irr}(G)} \nu(\chi)\chi(xy)\,.$$

The trivial character contributes 1 to the latter sum, so there must be a non-trivial irreducible character χ with $\nu(\chi) \neq 0$ and $\chi(xy) \neq 0$. In particular, χ is real-valued by Theorem 4.5(c) of [24]. ☐

Corollary 3.2. *Let G be a finite group, let p be an odd prime, and let x be a p-element of G such that $\mathbf{C}_G(x)$ has even order. Then G has a p-block B whose defect group D contains a conjugate of x, and which contains a non-trivial real-valued irreducible character.*

Proof. Let y be a 2-element of $\mathbf{C}_G(x)$ of maximal order. By Theorem 3.1, we know that there is a non-trivial real-valued irreducible character χ of G with $\chi(xy) \neq 0$. Then the p-block B which contains χ must have a defect group D which contains a conjugate of x (by Corollary 5.9 of [30]). ☐

Corollary 3.3. *Let G be a finite simple group of Lie type in characteristic p. Then the principal p-block of G contains a non-trivial real-valued irreducible character.*

Proof. Recall G has only two blocks, the principal block and the p-block of defect zero consisting of the Steinberg character St, and that this character is real-valued and has degree $|G|_p$. Hence, the result follows from Corollary 3.2 if G contains an element of order $2p$. Suppose then that G contains no element of order $2p$, and that the principal p-block contains no non-trivial real-valued irreducible character. Then every conjugacy class of involutions of G has size divisible by $|U|$, where U is a Sylow p-subgroup of G. Now, we know that

$$\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(1) = 1 + \nu(\mathsf{St})\mathsf{St}(1)$$

is the number of square roots of the identity in G. Since this number is positive, then $\nu(\mathsf{St}) = 1$ and G has exactly $|U|$ involutions. Hence G has a single conjugacy class of involutions of size $|U|$. This is impossible in a finite simple group by a well-known theorem of Burnside (Theorem 3.9 of [24]). □

In the second part of this section, we show the following.

Theorem 3.4. *Suppose that p is odd. Let G be a finite group and let B be the principal p-block of G. Assume that $\mathrm{Irr}(B)$ contains a non-trivial real-valued irreducible character α. Then $\mathrm{Irr}(B)$ contains a non-trivial real-valued irreducible character β different from α, and β may be chosen not algebraically conjugate to α.*

Proof. Let G^0 be the set of p-regular elements of G. We claim that

$$\sum_{g\in G^0} |\mathrm{Sq}(g)| = |G^0|\,.$$

Notice that if $g \in G^0$, then $\mathrm{Sq}(g) \subseteq G^0$, using that p is odd. Also, g^2 is p-regular and $g \in \mathrm{Sq}(g^2)$. This easily implies that

$$G^0 = \bigcup_{g\in G^0} \mathrm{Sq}(g)$$

is a disjoint union, and the claim is proved. Again, recall that

$$|\mathrm{Sq}(g)| = \sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(g)\,.$$

Hence, if $a_\chi := \sum_{g\in G^0} \chi(g)$ for $\chi \in \mathrm{Irr}(G)$, we deduce that

$$|G^0| = \sum_{g\in G^0}\left(\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(g)\right) = \sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)a_\chi\,.$$

By Brauer's Corollary 3.25 of [30], we know that χ is in the principal block of G if and only if $a_\chi \neq 0$. Then

$$0 = \sum_{1_G\neq\chi\in\mathrm{Irr}(B)} \nu(\chi)a_\chi\,,$$

using that $\nu(1_G)a_1 = |G^0|$. Now, since $\nu(\chi) \neq 0$ if and only if χ is real-valued (Theorem 4.5 of [24]), and we know that $\nu(\alpha)a_\alpha \neq 0$, necessarily there exists $\beta \in \mathrm{Irr}(B) - \{\alpha, 1_G\}$ such that $\nu(\beta) \neq 0$. Since $\nu(\alpha)a_\alpha$ is unchanged if we replace α by an algebraic conjugate, we may choose β not to be an algebraic conjugate of α, as claimed. □

Notice that the proof of Theorem 3.4 actually establishes the conclusion that if $G/\mathbf{O}_{p'}(G)$ has even order, then $\mathrm{Aut}(G)$ has at least three orbits on real-valued irreducible characters in the principal p-block of G. This is because the quantity $\nu(\chi)a_\chi$ only depends on the $\mathrm{Aut}(G)$-orbit of $\chi \in \mathrm{Irr}(G)$.

Now, we can combine Theorem 2.3 with Theorem 3.4 to complete the proof of Theorem A.

Corollary 3.5. *Let G be a finite group, let p be an odd prime and assume that $G/\mathbf{O}_{p'}(G)$ has even order. Then the principal p-block of G has at least three real-valued irreducible characters. Furthermore, if there are exactly three such characters, they are all rational-valued.*

We should mention that this result would not work for Brauer characters (as shown by any dihedral group of order $2p$).

4. Real Brauer Characters in the Principal Block

We start with the case where Sylow p-subgroups are cyclic.

Theorem 4.1. *Let G be a finite group such that $P \in \mathrm{Syl}_p(G)$ is cyclic, where p is odd. Assume that $G/\mathbf{O}_{p'}(G)$ even order. Then the principal p-block of G contains at least one non-trivial real-valued irreducible Brauer character.*

Proof. We know by Theorem A that there is $\chi \in \mathrm{Irr}(G)$, non-trivial, real-valued in the principal block B of G. We show first that we may assume that χ is a non-exceptional character. Let $N = \mathbf{N}_G(P)$, $C = \mathbf{C}_G(P)$, and write $e = |N : C|$. (Notice that $e > 1$ by a well-known group theoretical argument and our hypothesis.) By the cyclic defect theory [6], we know that $|\mathrm{IBr}(B)| = e$. If $\phi \in \mathrm{IBr}(B)$, then $\bar{\phi} \in \mathrm{IBr}(B)$ (because B is a real block), and therefore, by counting we may assume that e is odd. We know that if Δ is a complete set of representatives of N-action on $\mathrm{Irr}(P) - \{1_P\}$, then the exceptional characters in B can be written as $\{\chi_\lambda \,|\, \lambda \in \Delta\}$, and that χ_λ is uniquely determined by the N-conjugacy class of λ. By Dade's construction, we have that

$$\overline{\chi_\lambda} = \chi_\mu \,,$$

where μ is N-conjugate to $\bar{\lambda}$. Now, if $\chi = \chi_\lambda$ for some λ, then it follows that λ and $\bar{\lambda}$ are N-conjugate. Therefore $\lambda^x = \bar{\lambda}$ for some $x \in N$. Since x^2 stabilizes λ and N/C has odd order, it follows that $\lambda = \bar{\lambda}$, but this implies that $\lambda = 1_P$, a contradiction. Hence, χ is a non-exceptional character.

By Corollary 3.5, we know that the Brauer tree of the principal p-block has at least three real vertices (including the exceptional vertex). Complex conjugation acts on the principal p-block: there is a unique path between any two real vertices of the Brauer tree, and this unique path must be fixed under the action of complex conjugation (both vertex-wise and edge-wise). A fixed edge (under complex conjugation) of the Brauer tree corresponds to a real Brauer irreducible character, so there are at least two real Brauer irreducible characters. □

In what follows, we use the notation χ° to denote the restriction of a complex character χ of a finite group G to the set of p'-elements of G. For a p-block B and a set X of irreducible complex characters belonging to B, we say that $X^\circ := \{\chi^\circ \mid \chi \in X\}$ is a *spanning set* for B if

$$\langle \chi^\circ \mid \chi \in X \rangle_{\mathbb{C}} = \langle \varphi \mid \varphi \in \mathrm{IBr}_p(G) \cap B \rangle_{\mathbb{C}}.$$

Similarly, X° is called a *generating set* for B if

$$\langle \chi^\circ \mid \chi \in X \rangle_{\mathbb{Z}} = \langle \varphi \mid \varphi \in \mathrm{IBr}_p(G) \cap B \rangle_{\mathbb{Z}}.$$

Finally, if X° is a $\mathbb{Z}$-basis of $\langle \varphi \mid \varphi \in B \cap \mathrm{IBr}_p(G) \rangle_{\mathbb{Z}}$, then X° is called a *basic set* for B.

Lemma 4.2. *Suppose that the principal p-block $B_0(G)$ contains exactly l irreducible Brauer characters. Then $B_0(G)$ contains a nontrivial real-valued irreducible Brauer character, if at least one of the following conditions holds.*

(i) $2|l$.

(ii) (Thomas Breuer) *The restrictions of $\chi + \bar{\chi}$ to p'-classes in G, where χ runs over all complex irreducible characters belonging to $B_0(G)$, span a $\mathbb{C}$-space R of dimension $\geq l/2 + 1$.*

(iii) *$B_0(G)$ has a spanning set X°, where $X \subseteq \mathrm{Irr}(G) \cap B_0(G)$, and the number of non-real-valued characters in X is at most $l/2 - 1$.*

(iv) *$B_0(G)$ has a spanning set X°, where $X \subseteq \mathrm{Irr}(G) \cap B_0(G)$ and X is stable under complex conjugation, and furthermore the number of non-real-valued characters in X is at most $l - 2$.*

Proof. Let C denote the $\mathbb{C}$-span of χ° for all $\chi \in \mathrm{Irr}(G) \cap B_0(G)$. It is well known that $\mathrm{IBr}_p(G) \cap B_0(G)$ is a basis of C, and so $\dim C = l$.

(i) This is obvious, since $B_0(G)$ is stable under complex conjugation.

(ii) Assume the contrary: 1_G is the unique real-valued irreducible Brauer character in $B_0(G)$. Recall again that $B_0(G)$ is stable under complex conjugation. Let $\{\varphi_i, \bar{\varphi}_i \mid 1 \leq i \leq (l-1)/2\}$ denote the set of non-principal irreducible Brauer characters in $B_0(G)$. For any $\chi \in \mathrm{Irr}(G) \cap B_0(G)$ we have

$$\chi^\circ = a \cdot 1_G + \sum_{i=1}^{(l-1)/2} (b_i \varphi_i + c_i \bar{\varphi}_i)$$

for some $a, b_i, c_i \in \mathbb{C}$. It follows that

$$\chi^\circ + \bar{\chi}^\circ = 2a \cdot 1_G + \sum_{i=1}^{(l-1)/2} (b_i + c_i)(\varphi_i + \bar{\varphi}_i),$$

and so $\dim R \leq (l+1)/2$, a contradiction.

(iii) By assumption, $\langle X^\circ \rangle_{\mathbb{C}}$ contains $\mathrm{IBr}_p(G) \cap B_0(G)$, and the latter set is linearly independent over $\mathbb{C}$. Hence X° spans the space C defined above, and we can choose a basis $\{\chi_1^\circ, \ldots, \chi_l^\circ\}$ of C, where $\chi_i \in X$ and $\chi_i = \bar{\chi}_i$ exactly when $1 \leq i \leq m \leq l$. By the assumption $l - m \leq l/2 - 1$, and so $m \geq l/2 + 1$. Now the space R (as defined in (ii)) contains $\chi_1^\circ, \dots, \chi_m^\circ$, which are linearly independent over $\mathbb{C}$. Thus $\dim R \geq m \geq l/2+1$, and we are done by (ii).

(iv) By assumption, we can write

$$X = \{\chi_1, \dots, \chi_m, \chi_{m+1}, \bar{\chi}_{m+1}, \dots, \chi_{m+n}, \bar{\chi}_{m+n}\},$$

where χ_i is real-valued exactly when $1 \leq i \leq m$, $m+2n = l$, and $2n \leq l-2$. As in (iii), the space R (as defined in (ii)) contains

$$\chi_1^\circ, \dots, \chi_m^\circ, \chi_{m+1}^\circ + \bar{\chi}_{m+1}^\circ, \dots, \chi_{m+n}^\circ + \bar{\chi}_{m+n}^\circ;$$

in particular, R has codimension at most n in $\langle X^\circ \rangle_{\mathbb{C}} = C$. It follows that

$$\dim R \geq l - n \geq l - (l-2)/2 = l/2+1,$$

and so we are again done by (ii). □

We will also need the following simple observation:

Lemma 4.3. *Let $\mathcal{E} = B_1 \cup B_2 \cup \ldots \cup B_m$ be a union of p-blocks of a finite group G, where B_1 is stable under complex conjugation. Let $Y \subseteq \mathrm{Irr}(G)$ be stable under complex conjugation and suppose that Y° is a spanning set, respectively, a generating set, or a basic set, for $\mathcal{E}$. Then there is a subset $X \subseteq Y$ stable under complex conjugation such that X° is a spanning set, a generating set, or a basic set for B_1, respectively.*

Proof. Write $Y = \cup_{i=1}^m Y_i$, where Y_i consists of the characters in Y that belong to B_i, and let $X = Y_1$. Since Y and B_1 are stable under complex conjugation, so is X. Now we show that X° is a spanning set for B_1. Let G^0 denote the set of p'-elements in G, and consider the Hermitian product

$$(f,g)' = \sum_{x \in G^0} f(x)\overline{g(x)}$$

on the space of complex-valued functions on G^0. Consider any $\varphi \in \mathrm{IBr}_p(G) \cap B_1$. Then we can write $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1 = \sum_{\chi \in X} a_\chi \chi^\circ, \ \ \varphi_2 = \sum_{\rho \in Y \smallsetminus X} b_\rho \rho^\circ,$$

and $a_\chi, b_\rho \in \mathbb{C}$. By [9, Theorems 60.5 and 48.8], $(\psi_1, \psi_2)' = 0$ whenever $\psi_1 \in \mathrm{IBr}_p(G) \cap B_1$ and $\psi_2 \in \mathrm{IBr}_p(G) \cap \cup_{i=2}^m B_i$. It follows that

$$(\varphi_2, \varphi_2)' = (\varphi - \varphi_1, \varphi_2)' = 0,$$

whence $\varphi_2 = 0$ and $\varphi = \varphi_1$ is a $\mathbb{C}$-combination of X°. In the case Y° is a generating set for $\mathcal{E}$, then we can choose $a_\chi, b_\rho \in \mathbb{Z}$, and the same argument shows that X° is a generating set for B_1. Finally, if Y° is linearly independent then so is X°. □

To handle exceptional groups of Lie type, we will use the following statement:

Lemma 4.4. *Let $\mathcal{G}$ be a simple, simply connected algebraic group with a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$, and let $G := \mathcal{G}^F \cong F_4(q)$, $E_6(q)_{\mathrm{sc}}$, ${}^2E_6(q)_{\mathrm{sc}}$, $E_7(q)_{\mathrm{sc}}$, or $E_8(q)$. Let $p \nmid 2q$ be a prime, and let St denote the Steinberg character of G. Then St° contains a nontrivial real-valued $\varphi \in \mathrm{IBr}_p(G)$ as an irreducible constituent, if at least one of the following conditions holds:*

(i) *$2 \nmid q$ and $p \nmid (q+1)$.*

(ii) $2|q$, $p|(q+1)$, *and* $G \not\cong {}^2E_6(q)_{\rm sc}$.

(iii) $2|q$, *and either* $p \nmid (q+1)$ *or* $G \cong {}^2E_6(q)_{\rm sc}$, *and, moreover, there is a real* p'*-element* $g \in G$ *such that* $\mathsf{St}(g) = \pm 1$.

Proof. In the case of (i), the desired conclusion follows from the fact that $2 \nmid \mathsf{St}(1)$ and St° does not contain 1_G by [20, Theorem B]. Similarly, in the case of (ii), the statement follows from the fact that $2|\mathsf{St}(1)$ and St° contains 1_G with multiplicity 1 by [20, Theorem B].

In the case of (iii), observe by [20, Theorem B] that St° does not contain 1_G. Suppose now that St° does not contain any real-valued irreducible constituent. Then we can write

$$\mathsf{St}^\circ = \sum_{i=1}^{n} a_i(\psi_i + \bar{\psi}_i),\ \ a_i \in \mathbb{Z}_{>0}$$

for some $n \geq 1$ and non-real-valued $\psi_i \in \mathrm{IBr}_p(G)$. As g is real and $\mathsf{St}(g) = \pm 1$, we have

$$\sum_{i=1}^{n} a_i\psi_i(g) = \mathsf{St}(g)/2 = \pm 1/2,$$

a contradiction. □

We will now prove Theorem C for simple groups.

Theorem 4.5. *Let* p *be any odd prime, and let* S *be a finite non-abelian simple group of order divisible by* p. *Then the principal* p*-block of* S *contains a non-trivial, real-valued, irreducible Brauer character* φ.

Proof. (A) First we consider the case $S = \mathsf{A}_n$ with $n \geq \max(p, 5)$. If $p|n$ then one can take φ to be afforded by the heart of the natural permutation module of S (of dimension $n-2$). Suppose $p|(n-1)$; in particular $n \geq 6$. Consider the irreducible p-Brauer character ψ of S_n labeled by the partition $(n-2, 2)$, with p-core (1). Thus ψ belongs to $B_0(\mathsf{S}_n)$. It is not hard to check that ψ is irreducible over A_n (see eg. the proof of [18, Lemma 6.1]), and so we can take $\varphi := \psi_{\mathsf{A}_n}$.

Assume now that $p \nmid n(n-1)$. In this case, the p-core, say (r) with $2 \leq r \leq p-1$, of the partition (n) is not self-associate. It follows that any irreducible Brauer character ψ in $B_0(\mathsf{S}_n)$ restricts irreducibly to A_n. Now choose ψ labeled by the partition $(n-r, r)$, with p-core (r) and so belonging to $B_0(\mathsf{S}_n)$, and take $\varphi := \psi_{\mathsf{A}_n}$.

(B) Assume $S = {}^2F_4(2)'$ or S is one of 26 sporadic simple groups. In many cases, the Brauer character tables of S are all known and one can easily check the statement using [13]. In the cases where the Brauer character table of S is still undetermined, Thomas Breuer has checked the statement using [13] and Lemma 4.2(ii).

(C) Let S be a finite simple group of Lie type in characteristic p. It is well known, see [23], that the only irreducible p-Brauer character of S that does not belong to $B_0(S)$ is St°, where St denotes the Steinberg character. Furthermore, unless S is $\mathrm{PSL}^\epsilon_n(q)$ with $n \geq 3$, $E^\epsilon_6(q)$, or $P\Omega^\epsilon_{4m+2}(q)$ with $m \geq 2$ for some $\epsilon = \pm$, all semisimple elements in S are real, see [38, Proposition 3.1], in which case one can take φ to be any irreducible Brauer character of S different from 1_S and St°.

Suppose S is any of the aforementioned three exceptions. Then we can view S as $\mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$ for a suitable simple, simply connected algebraic group $\mathcal{G}$ in characteristic p

and a (generalized) Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$. Then we can choose ψ to be afforded by the restriction to $\mathcal{G}^F$ of the rational, restricted, irreducible module $L(\varpi)$ of $\mathcal{G}$, with ϖ chosen as follows. If $S = \mathrm{PSL}^\epsilon_n(q)$, then $\mathcal{G}$ is of type SL_n, and $\varpi = \varpi_1 + \varpi_{n-1}$ (see [34, p. 794]). If $S = P\Omega^\epsilon_{4m+2}(q)$, then $\mathcal{G}$ is of type D_{2m+1}, and $\varpi = \varpi_{2m}$ (then φ has degree $(2m+1)(4m+1)$, see [34, p. 795]). If $S = E^\epsilon_6(q)$, then $\mathcal{G}$ is of type E_6, and $\varpi = \varpi_2$ (then φ has degree 78 if $p \neq 3$ and 77 if $p = 3$, see [34, p. 795]). In all these cases, one can check that ψ is indeed trivial at $\mathbf{Z}(\mathcal{G}^F)$ (see [34, pp. 794, 795]) and real-valued, and so we can φ to be ψ viewed as a Brauer character of S.

(D) From now on we may assume that S is a simple group of Lie type in characteristic $r \neq p$. Here we consider the case where S is a classical group. First assume that $S = \mathrm{PSL}^\epsilon_n(q)$ with $\epsilon = \pm$ and $p | \gcd(n, q-\epsilon)$. Then, if $\epsilon = +$, one can take φ to be afforded by the heart of the permutation module of S acting on the set of 1-spaces of $\mathbb{F}_q^n$ (so that $\varphi(1) = (q^n - q)/(q-1) - 1)$. Assume now that $\epsilon = -$. As shown, for instance in p. (ii) of the proof of [10, Theorem 7.2], the unipotent (Weil) character χ of degree $(q^n + q(-1)^n)/(q+1)$ belongs to $B_0(S)$, is rational-valued; furthermore, its reduction modulo p is $\varphi + e \cdot 1_S$ with $e = 0$ or 1 and $\varphi \in \mathrm{IBr}_p(S)$. Certainly, φ has the desired properties.

In all other cases, we can view $S = G/\mathbf{Z}(G)$, where $G = \mathcal{G}^F$ and $\mathcal{G}$ is a simple, simply connected algebraic group in characteristic p and $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism. The assumptions on S, p now ensure that p is a good prime not dividing $|\mathbf{Z}(\mathcal{G})_F| = |\mathbf{Z}(G)|$ (where $\mathbf{Z}(\mathcal{G})_F$ is the largest quotient of $\mathbf{Z}(\mathcal{G})$ on which F acts trivially). Hence, by [14, Theorem A], $B_0(G)$ has a basic set X°, where X consists of (certain) unipotent characters of G. As $p \nmid |\mathbf{Z}(G)|$, the same is true for $B_0(S)$. Furthermore, as mentioned in the proof of [36, Theorem 5.4], all unipotent characters of G are rational-valued. It follows that all irreducible Brauer characters in $B_0(S)$ are rational-valued. It is well known (see also Theorem 2.1) that $B_0(S)$ contains a non-trivial irreducible complex character. Hence some $\varphi \in B_0(S)$ must be non-trivial.

(E) Now we consider the case S is an exceptional group of Lie type in characteristic $r \neq p$. By Theorem 4.1, we may assume that Sylow p-subgroups of S are non-cyclic. In particular, we are done if $S = {}^2B_2(q)$ or ${}^2G_2(q)$. Next suppose that $S = G_2(q)$ and $p|(q^2-1)$. Then, according to [21, Anhang C], we can take φ of degree q^3 if $p = 3|(q-1)$, $q^3 - 1$ if $p = 3|(q+1)$, and $q(q^2-1)^2/3$ if $p > 3$. Similarly, if $S = {}^2F_4(q)$ with $q > 2$, then by [19] we can take $\varphi = \chi_4^\circ - e \cdot 1_G$, with $e \in \{0, 1\}$ and $\chi_4(1) = q(q^2-q+1)(q^4-q^2+1)$. If $S = {}^3D_4(q)$, then all unipotent characters are rational-valued, and so we can argue as in (D), using [14, Theorem A].

(F) From now on we may assume that S is of type F_4, E_6, ${}^2E_6$, E_7, or E_8, and let $l := |\mathrm{IBr}_p(S) \cap B_0(S)|$. Then we view $S = G/\mathbf{Z}(G)$, where $G := \mathcal{G}^F$ for some simple, simply connected algebraic group $\mathcal{G}$ and a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$. According to the main result of [4], the union $\mathcal{E}_p(G, (1))$ of all rational Lusztig series $\mathcal{E}(G, (t))$, where t runs over all p-elements in the dual group G^*, is a union of p-blocks that contains $B_0(G)$. We also let d denote the order of q modulo p.

Here we consider the case p is a *good* prime for $\mathcal{G}$. Then, $p \nmid |\mathbf{Z}(G)|$, and so $B_0(G) = B_0(S)$ and the set $\mathcal{E}(G, (1))$ of unipotent characters of G is a basic set for $\mathcal{E}_p(G, (1))$ by [14, Theorem A]. Note that $\mathcal{E}(G, (1))$ is stable under complex conjugation. Hence, by

Lemma 4.3, $B_0(S)$ has a basic set X°, where X is a certain subset of unipotent characters of G that is stable under complex conjugation.

By Theorem 4.1 we may assume that $d \in \{1, 2, 3, 4, 6\}$, together with additional possibilities $d \in \{5, 8, 10, 12\}$ if $S = E_8(q)$. Then, as listed in [28, Table 2] and [3, Table 3], either $2|l$, or $l \geq 21$ and $G \in \{F_4(q), E_6(q)_{\rm sc}, {}^2E_6(q)_{\rm sc}\}$, or $l \geq 45$ and $G = E_8(q)$. We are done in the first case by Lemma 4.2(i). In the cases of $E_6(q)_{\rm sc}$ and ${}^2E_6(q)_{\rm sc}$, note that the number of non-real-valued unipotent characters of G is at most 2, since there are only one pair of unipotent characters of the same degree [5, §13.9], so we are done by Lemma 4.2(iii). Similarly, if $G = E_8(q)$, then the number of non-real-valued unipotent characters of G is at most 22, since there are 9 pairs and one 4-tuple of unipotent characters of equal degrees [5, §13.9], so we are done by Lemma 4.2(iv). Suppose that $G = F_4(q)$. Then the number of non-real-valued unipotent characters of L is at most 20, since there are 10 pairs of unipotent characters of equal degrees [5, §13.9]. As $d \in \{1, 2, 3, 4, 6\}$, note that at least 4 of these characters have p-defect 0 and so cannot be contained in X. Thus the number of non-real-valued characters in X is at most 16, and we are again done by Lemma 4.2(iv).

(G) Finally, we handle the case where p is a *bad* prime for $\mathcal{G}$.

(i) Let $\Phi_m(q)$ denote the $m^{\rm th}$ cyclotomic polynomial in q. By [29, Lemma 2.3], we can find a regular semisimple p'-element $g \in F_4(q) \leq G$ of order dividing $\Phi_{12}(q)$ if $\mathcal{G}$ is of type F_4 or E_6. Similarly, we can find a regular semisimple p'-element $g \in G$, of order dividing $\Phi_{14}(q)$ if $\mathcal{G}$ is of type E_7 and dividing $\Phi_{24}(q)$ if $\mathcal{G}$ is of type E_8. In all cases, g is real by [38, Proposition 3.1]. Furthermore, the Steinberg character St is rational, and $|\mathsf{St}(g)|$ is the r-part of the order of the maximal torus $\mathbf{C}_G(g)$, whence $\mathsf{St}(g) = \pm 1$.

Since $p = 3$ or 5, we have that $d \in \{1, 2, 4\}$. Then [3, Table 1] implies that d is regular for G (in the sense of Springer), that is, there is an F-stable Sylow d-torus $\mathcal{T}_d$ of $\mathcal{G}$ such that $\mathcal{T} := \mathbf{C}_{\mathcal{G}}(\mathcal{T}_d)$ is a maximal torus of $\mathcal{G}$. In particular, $\zeta := 1_{\mathcal{T}^F}$ has central p-defect, in the sense of [11]. Also, the pair $(\mathcal{T}^F, \zeta)$ is now d-cuspidal. Furthermore, the Lusztig reduction ${}^*R^{\mathcal{G}}_{\mathcal{T}}$ sends 1_G to ζ, see [7, Corollary 12.7], whence 1_G is an irreducible constituent of $R^{\mathcal{G}}_{\mathcal{T}}(\zeta)$ by the adjointness of Lusztig functors. Moreover, if $D_{\mathcal{G}}$ and $D_{\mathcal{T}}$ denote the Alvis-Curtis duality functors for $\mathcal{G}^F$ and $\mathcal{T}^F$, cf. [7, Chapter 8], then

$$ {}^*R^{\mathcal{G}}_{\mathcal{T}}(\mathsf{St}) = {}^*R^{\mathcal{G}}_{\mathcal{T}} \circ D_{\mathcal{G}}(1_G) = \pm D_{\mathcal{T}} \circ {}^*R^{\mathcal{G}}_{\mathcal{T}}(1_G) = \pm D_{\mathcal{T}}(\zeta) = \pm\zeta. $$

Thus both 1_G and St belong to the set X_1 of irreducible constituents of $R^{\mathcal{G}}_{\mathcal{T}}(\zeta)$, which is clearly stable under complex conjugation. It now follows from [11, Theorem A] that all characters in X_1 belong to $B_0(G)$. In particular, applying Lemma 4.4, we are done in the case $2|q$, or if $2 \nmid q$ but $p \nmid (q+1)$.

(ii) We may now assume that $2 \nmid q$ and $d = 2$. Assume in addition that $\mathcal{G} \neq F_4$. Let $\phi = \phi_{6,1}$, respectively $\phi'_{2,4}$, $\phi_{7,1}$, $\phi_{8,1}$ be the (unique unipotent) character of smallest degree > 1 of G, see [5, §13.9]. The uniqueness of ϕ implies that ϕ is rational-valued. Observe that ϕ does not belong to the d-Harish Chandra series labeled by $(\mathcal{L}^F, \zeta)$ where $\mathcal{L}$ is not a torus, see [3, Table 2]. It follows that ϕ belongs to the d-Harish Chandra series labeled by $(\mathcal{L}^F, \zeta)$ with $\mathcal{L}$ being a torus, and so $\zeta = 1_{\mathcal{L}^F}$ has central defect. As in (i), this in turn implies that ϕ belongs to $B_0(G)$, and to $B_0(S)$ when viewed as a character of S. Let φ denote a nontrivial irreducible constituent of ϕ°. By the main result of [22], any nontrivial

$\psi \in \mathrm{IBr}_p(G)$ has degree $\geq \phi(1)-4 \geq 5$; in particular, $\varphi(1) \geq \phi(1)-4$. It follows that $\phi^\circ = \varphi + e \cdot 1_G$ with $0 \leq e \leq 4$, whence φ is rational-valued.

(iii) Finally, we consider the case $S = G = F_4(q)$, $2 \nmid q$, and $p = 3|(q+1)$. In particular, r is a *good* prime for $\mathcal{G}$. Let $N_1 := |\mathcal{E}(G,(1))|$, and let N_2 denote the number of irreducible Brauer characters in $\mathcal{E}_p(G,(1))$. The results of [16, §6], see also [8, §4.1] and the proof of [28, Proposition 6.10], imply that $N_1 = 37$, that the unipotent characters in $\mathcal{E}(G,(1))$ form a spanning set for $\mathcal{E}_p(G,(1))$, and furthermore $N_2 = 35$. By Lemma 4.3, there is a subset $X \subseteq \mathcal{E}(G,(1))$ stable under complex conjugation such that X° is a spanning set for $B_0(G)$. As $X_1 \subseteq \mathcal{E}(G,(1))$, we see that $X_2 := X \cup X_1$ is also a spanning set of $B_0(G)$ that is stable under complex conjugation.

Note that the dimension of the $\mathbb{C}$-space of linear dependence relations between χ°, $\chi \in \mathcal{E}(G,(1))$, is $N_1 - N_2 = 2$. Clearly, the dimension of the $\mathbb{C}$-space of linear dependence relations between χ°, $\chi \in X_2$, is ≤ 2, whence

$$l(B_0(G)) \geq |X_2| - (N_1 - N_2) \geq |X_1| - 2.$$

The size of X_1 (which is the conjugacy class number of the corresponding relative Weyl group), is known to be 25, see eg. [28, Table 2]. Thus $l(B_0(G))) \geq 23$. On the other hand, inspecting [5, §13.9] as in (F), we see that the total number of non-real-valued characters in X_2 is ≤ 18, since at least one pair of unipotent characters of $F_4(q)$ of equal degree has p-defect zero and so does not belong to X_2. So we are done by Lemma 4.2(iv). □

Proof of Theorem C. It follows immediately from Theorem 2.4 and Theorem 4.5. □

Remark 4.6. Meinolf Geck has pointed out to us that the results of [16, §6] can now be proved to hold for all p, q, r, see [15, Proposition 7.12]. Given this, the argument in (G)(iii) of the proof of Theorem 4.5 can also be applied to other exceptional groups of adjoint type.

Finally, we recall that the finite groups with exactly two real-valued irreducible Brauer characters were studied in [31].

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