A PARTIAL LAPLACIAN AS AN INFINITESIMAL GENERATOR ON THE WASSERSTEIN SPACE

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ABSTRACT. In this manuscript, we consider special linear operators which we term partial Laplacians on the Wasserstein space, and which we show to be partial traces of the Wasserstein Hessian. We verify a distinctive smoothing effect of the "heat flows" they generated for a particular class of initial conditions. To this end, we will develop a theory of Fourier analysis and conic surfaces in metric spaces. We then identify a measure which allows for an integration by parts for a class of Sobolev functions. To achieve this goal, we solve a recovery problem on the set of Sobolev functions on the Wasserstein space.

1. INTRODUCTION

A fundamental result in stochastic analysis is that the Laplace operator is the infinitesimal generator of Brownian motion. That is, for any twice continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ with bounded second derivatives,

$$\mathrm{Tr(Hess\ f)(x)} = \lim_{\mathrm{t}\to 0^+} \frac{\mathbb{E}\mathrm{f(x}+\sqrt{2}\mathrm{W_t)} - \mathrm{f(x)}}{\mathrm{t}}$$

for each $x \in \mathbb{R}^d$. Here, of course, $(W_t, t \geq 0)$ is a standard d–dimensional Brownian motion and $\mathrm{Tr(Hess\ f)} \equiv \Delta \mathrm{f}$ is the Laplace operator. This property is also closely related to the fact that

$$v(x,t) = \mathbb{E}f(x+\sqrt{2}W_t)$$

is a solution in $(0,\infty) \times \mathbb{R}^d$, of the heat equation

$$\partial_t v = \Delta v. \tag{1.1}$$

These results can be lifted from $\mathbb{R}^d$ to the space of Borel probability measures on $\mathbb{R}^d$ with finite second moments. We denote this space as $\mathcal{P}_2(\mathbb{R}^d)$ and endow it with the so–called Wasserstein metric and the differential structure amply studied in [1].

In a greater generality, the lift from $\mathbb{R}^d$ to $\mathcal{P}_2(\mathbb{R}^d)$ which appears in Mean Fields Games [3] [4], is to find $\mathcal{V} : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

$$\begin{aligned} &\partial_t \mathcal{V} + \int_{\mathbb{R}^d} \nabla_x \mathcal{V} \cdot \nabla_\omega \mathcal{V} \mu(dy) + \frac{1}{2}|\nabla_q \mathcal{V}|^2 + \mathcal{F}(q,\mu) - \int_{\mathbb{R}^d} \mathrm{Tr}\left[\nabla^2_\omega \mathcal{V}\right] \mu(dy)\mu(dy') \\ &= 2\Delta_q \mathcal{V} + 2\int_{\mathbb{R}^d} \Big(\nabla_y \cdot [\nabla_\omega \mathcal{V}(t,q,\mu)(y)] + \nabla_q \cdot [\nabla_\omega \mathcal{V}(t,q,\mu)](y)\Big)\mu(dy). \end{aligned} \tag{1.2}$$

Here, $\nabla_\omega \mathcal{V}$ denotes the intrinsic Wasserstein gradient (cf. Definition 2.3) introduced by Ambrosio–Gigli–Savaré [1]. This leads to an intrinsic definition of $\nabla^2_\omega \mathcal{V}$, the second order Wasserstein gradient (cf. Definition 3.3). Equation (1.2) poses more challenges than we are currently prepared to address. So for now we address the smoothing effects (cf. Theorem

6.4) of two of the three mechanisms at work in (1.2), leaving out the mechanism induced by the underlying Lagrangian. The latter mechanism was showed in [4] to preserve smoothness property when appropriate monotonicity conditions are imposed on $\mathcal{F}$ and $\mathcal{V}(0,\cdot)$.

Given $U:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$, the intrinsic Wasserstein gradient and Hessian can be understood through the lift of U to the Hilbert space $L^2(B,\mathbb{R}^d)$, where $B\subset\mathbb{R}^d$ is an open ball of unit volume. Indeed, define

$$\hat{U}(X)=U(m)$$

whenever $X\in L^2(B,\mathbb{R}^d)$ is the law of m. When $\hat{U}$ is continuously differentiable in a neighborhood of X, it is shown in [3] that $\nabla_{L^2}\hat{U}(X)$, the Hilbert gradient of $\hat{U}$ at X can be written as the composition of a Borel map $\nabla_\omega U[m]:\mathbb{R}^d\mapsto\mathbb{R}^d$ and X : $\nabla_\omega U[m]\circ X=\nabla_{L^2}\hat{U}(X)$. The map $\nabla_\omega U[m]$ being uniquely determined m–almost everywhere, was proposed in [3] as the Wasserstein gradient of U at m. The result in [3] have been improved in the sense that [12] requires only the sole differentiability property of $\hat{U}$ at X, to obtain the factorization $\nabla_\omega U[m]\circ X=\nabla_{L^2}\hat{U}(X)$. In fact, the study in [12] allows to show a much stronger result: if ξ is the element of minimal norm of the intrinsic Wasserstein sub–differential of U at m and ζ is the element of minimal norm of $\hat{U}$ at X then $\xi\circ X=\zeta$. Similarly, one can relate $\mathrm{Hess}_{L^2}\hat{U}$, the Hessian of $\hat{U}$ at X to $\mathrm{Hess}\,U$, the intrinsic Wasserstein Hessian of U at m without any requirement that $\mathrm{Hess}_{L^2}\hat{U}$ needs to exists in a neighborhood of X. In this work it has often been more advantageous to use $\nabla_\omega U$ as originally defined in [1]. This imposes the use of $\mathrm{Hess}\,U$ rather than that of $\mathrm{Hess}_{L^2}\hat{U}$.

In the sequel, we consider the following stochastic process on the Wasserstein space:

$$t\mapsto\mathbb{B}^m_t:=(\mathbf{id}+\sqrt{2}W_t)_\#m.$$

Here and below, $(W_t)_{t\geq 0}$ is a standard d–dimensional Brownian motion starting at 0. The expression $T_\#\mu$ is the push-forward measure defined for every Borel probability measure μ on $\mathbb{R}^d$ and Borel map $T:\mathbb{R}^d\to\mathbb{R}^d$ via the formula

$$T_\#\mu(A):=\mu(T^{-1}(A)).$$

If $U:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is bounded and twice differentiable at m and $\mathrm{Hess}\,U[m]$ denotes the Wasserstein Hessian of U at m, we define

$$\triangle_w U:=\sum_{i=1}^d \mathrm{Hess}\,U[m](e_i,e_i). \tag{1.3}$$

Here, e_i is the Wasserstein gradient of the i–th moment $m\mapsto\int_{\mathbb{R}^d}x_i m(dx)$. It is a constant vector field, and $\{e_i\}_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$. One readily checks that $\{e_1,\cdots,e_d\}$ is an orthonormal family in $\overline{\nabla C_c^\infty(\mathbb{R}^d)}^{L^2(m)}$ of the tangent space at m.

It is known that (cf. [4] [6]) the function

$$(t,m)\in(0,\infty)\times\mathcal{P}_2(\mathbb{R}^d)\to V(t,m):=\mathbb{E}\Big(U\big(\mathbb{B}^m_t\big)\Big)$$

satisfies

$$\partial_t V=\triangle_w V,\qquad V(0,\cdot)=U. \tag{1.4}$$

When restricted to finitely many symmetric products of $\mathbb{R}^d$, the partial Wasserstein Laplacian operator coincides with classical finite dimensional operators. For instance, suppose U is differentiable in a neighborhood of $\mu := 1/k \sum_{j=1}^k \delta_{x_j}$ for some $x_1, \cdots, x_k \in \mathbb{R}^d$ and U is twice differentiable at μ as stated in Theorem 3.2. If we define v on $(\mathbb{R}^d)^k$ by

$$v(x) \equiv v(x_1, \cdots, x_k) := U\Big(\frac{1}{k}\sum_{j=1}^k \delta_{x_j}\Big)$$

then unless $k = 1$,

$$\triangle_w U\Big[\frac{1}{k}\sum_{j=1}^k \delta_{x_j}\Big] = \sum_{j=1}^k \triangle_{x_j} v(x) + \sum_{j\neq l} \mathrm{div}_{x_j}\big(\nabla_{x_l} v(x)\big) \neq \sum_{j=1}^k \triangle_{x_j} v(x).$$

For $\epsilon > 0$ we also set

$$\sigma_t^{\epsilon,\beta}[m] := (\mathbf{id} + \sqrt{2\beta} W_t)_\#(G_t^\epsilon * m), \tag{1.5}$$

where, G_t^ϵ is the heat kernel for the heat equation given by

$$G_t^\epsilon(z) = \frac{1}{\sqrt{4\pi\epsilon t}^d} \exp\Big(-\frac{|z|^2}{4\epsilon t}\Big). \tag{1.6}$$

The Wasserstein partial Laplacian is the sum of two operators, one being nonpositive with a trivial kernel when restricted to the set of k–polynomial function. These are functions on $\mathcal{P}_2(\mathbb{R}^d)$ of the form

$$m \mapsto F_\Phi[m] := \frac{1}{k}\int_{(\mathbb{R}^d)^k} \Phi(x) m(dx_1)\cdots m(dx_k),$$

$\Phi \in C((\mathbb{R}^d)^k)$ being a symmetric function that grows at most quadratically at infinity. The set of such Φ's is denoted as $\mathrm{Sym}[\mathrm{R}^\mathrm{k}]$ and the set of F_Φ's is denoted as $\mathcal{S}ym[k](\mathbb{R})$. The latter alluded operator, which has a smoothing effect, associates to any smooth function $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the function

$$m \to O[m] := \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_w U[m](x)\big) m(dx).$$

Given $\epsilon > 0$ and a twice continuously differentiable function U, the function

$$(t, m) \in (0, \infty) \times \mathcal{P}_2(\mathbb{R}^d) \to V^\epsilon(t, m) := \mathbb{E}\Big(U\big(\sigma_t^\epsilon[m]\big)\Big) \tag{1.7}$$

solves the initial value differential equation

$$\partial_t V^\epsilon = (\triangle_w + \epsilon O)V^\epsilon, \qquad V^\epsilon(0, \cdot) = U.$$

Our study of the partial Laplacian operator will be mainly restricted to the set of k–polynomials and their graded sums. Although at a first glance the sets of k–polynomials may appear to be too small, by the Stone Weierstrass Theorem, the subalgebra they generate is a dense subset of $C(\mathcal{K})$ for the uniform convergence (cf. Remark 2.7). Here, $\mathcal{K}$ is any locally compact subset of $\mathcal{P}_2(\mathbb{R}^d)$. As a consequence

$$\bigoplus_k \Big[\mathcal{S}ym[k](\mathbb{R}) \cap C_c\big((\mathbb{R}^d)^k\big)\Big], \tag{1.8}$$

the $\mathbb{N}$–graded sums of the set of k–polynomials, generates a subalgebra which is a dense subset of $C(\mathcal{K})$.

The nonnegative real numbers are contained in the spectrum of $-\triangle_w$. For any $\beta \geq 0$, it is shown that the intersection of the kernel of $\triangle_w + \beta\text{Id}$ with $\mathcal{S}ym[k](\mathbb{C})$, is represented by a general conical surface, contained in the symmetric k–product of $\mathbb{R}^d$. The surface in question is the quotient space

$$\Big\{(\xi_1, \cdots, \xi_k) \in (\mathbb{R}^d)^k \;\; 4\pi^2\Big|\sum_{j=1}^k \xi_j\Big|^2 = \beta \;\Big\}/P_k, \tag{1.9}$$

where P_k is the set of permutations of k letters. Note when $k > 1$ and $\beta = 0$, the surface degenerates into a linear space, and so, it has infinitely many elements, which means that the kernel of $\triangle_w$ has infinitely elements. More serious is the fact that the surface is unbounded, which precludes the Wasserstein Laplacian operator to have a smoothing property, unless restricted to an appropriate set of functions. Solving the simplest case of Poisson equation on the Wasserstein space amount to, given

$$a \in \mathcal{S}ym[k](\mathbb{C}) \cap L^2((\mathbb{R}^d)^k) \cap L^1((\mathbb{R}^d)^k),$$

solving

$$-4\pi^2\Big|\sum_{j=1}^k \xi_j\Big|^2 b(\xi_1, \cdots, \xi_k) = a(\xi_1, \cdots, \xi_k).$$

This, obviously is not an elliptic equation as when $k > 1$ and $\beta = 0$, the surface in (1.9) does not reduce to the null vector.

This manuscript starts a Fourier analysis on the set of probability measures of finite second moments, the so–called the Wasserstein space. We later introduce measures on the infinite dimensional metric space, which allow us to integrate by parts products of special functions defined on the Wasserstein space. In this manuscript, we also address the following natural and useful question: suppose we know that a function $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is of the form $F = F_\Phi$ for a symmetric function $\Phi : (\mathbb{R}^d)^k \to \mathbb{R}$. Can we reconstruct Φ? We can convince ourselves that the problem reduces to expressing Φ as a sum, up to a multiplication constant, of the so–called k–th defects of F. When we do not require any differentiability property of F, we reconstruct Φ by providing a polarization isomorphism based on the inclusion–exclusion principle, without any differentiation operations. Our arguments was inspired by works on vector spaces, which can be traced back to [14] in a particular case, followed by generalization in [27].

For each $s \geq 0$, we define $H^s(\mathcal{P}_2(\mathbb{R}^d))$, a space of functions on the Wasserstein space, in the spirit of the Sobolev functions. It has the virtue that whenever $U \in H^s(\mathcal{P}_2(\mathbb{R}^d))$, then $V(t, \cdot)$ given in (1.7) not only solves the previous differential equation but

$$V(t, \cdot) \in H^l\big(\mathcal{P}_2(\mathbb{R}^d)\big) \quad \forall\, l \;\geq 0.$$

We identify a set in $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d)) \subset H^s(\mathcal{P}_2(\mathbb{R}^d))$ such that choosing U_0 in that set, even if U_0 is not three times differentiable, when $s, \epsilon > 0$, then $\triangle_{w,\epsilon} V(t, \cdot)$ becomes twice differentiable (cf. Theorem 6.4). This is an improved smoothing effect in the m variable.

The functions F in $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ are pointwise sums of the infinite series $\sum_{k=1}^\infty 1/k! F_{\Phi_k}$, where $\Phi_k \in L^2((\mathbb{R}^d)^k)$ is a symmetric function which grows at most super linearly at ∞ and is such that its inverse Fourier transform a_k, satisfies (4.2). Under more stringent assumptions on Φ_k, so that $F \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$, we prove that F_{Φ_k} is uniquely determined by a specific projection operator π_k (cf. Remark 5.5) defined on a subset of the graded sum in (1.8) .

A bilinear form

$$\langle \cdot \,;\, \cdot \rangle_{H^0} : \mathcal{H}^0\big(\mathcal{P}_2(\mathbb{R}^d)\big) \times \mathcal{H}^0\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$$

which involving functions, their gradients and Laplacians, is provided in Proposition 4.10. Under appropriate conditions on $F, G \in \mathcal{H}^0(\mathcal{P}_2(\mathbb{R}^d))$, we assert

$$-\langle \triangle_w F; G\rangle_{H^0} = \left\langle \int_{\mathbb{R}^d} \nabla_w F[m](x) m(dx); \int_{\mathbb{R}^d} \nabla_w G[m](x) m(dx) \right\rangle_{H^0}.$$

In some cases, this turns into a integration by parts formula involving signed Radon measures $\mathbb{P}^{k,R}$ on $\mathcal{M}^2 := \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$. When $F = \Phi$, $G = F_\Psi$ where Φ, Ψ are k–symmetric functions of class C^3 and supported by the ball of radius R, Theorem 7.3 shows the above to be equivalent to

$$-\int_{\mathcal{M}^2} \triangle_w F_\Phi[m_1] F_\Psi[m_2] d\mathbb{P}^{k,R} = \int_{\mathcal{M}^2} D_2(\nabla_w F_\Phi, \nabla_w G_\Psi) d\mathbb{P}^{k,R}.$$

Here D_2 is the bilinear function

$$D_2(\nabla_w F, \nabla_w G)(m_1, m_2) := \int_{\mathbb{R}^{2d}} \langle \nabla_w F[m_1](q_1); \nabla_w G[m_2](q_2)\rangle m_1(dq_1) m_2(dq_2).$$

In the recent years, there have been many attempts to construct a "full" Laplacian on the Wasserstein space. In [24], von Renesse and Sturm studied a canonical diffusion process on the Wasserstein space, when the underlying space is the one–dimensional torus. Then in [25], Sturm constructed entropic measures on Wasserstein spaces $\mathcal{P}(M)$, where the underlying set M is a compact manifold of finite dimension. Unlike the case when M is a one–dimensional set, the closability of the Dirichlet form associated to the entropic measures, remains to–date, an outstanding open question. We end this introduction by drawing the attention of the reader to a (far from being exhaustive) literature which studies infinite dimensional Laplacian operators on flat spaces. The first one due to Levy [18], relies on a concept of the mean of a function on a Hilbert space, to propose a Laplacian operator. No meaningful subset of the domain of definition of this operator was known until a later studied by Dorfman [9]. This author proves, when the Hessian has the form $\text{Hess}U(x) = r(x)I + T(x)$, where r is uniformly continuous and T satisfies a so–called N–property, then U belongs to the domain of definition of Levy's Laplacian operator. Other definitions of Laplacian operators on a Hilbert space appeared in the literature. For instance, [26] considers a Hilbert space $\mathbb{D}$ and a nuclear space $\mathbb{L}$ and defined Laplacians on subsets of $L^2(\mathbb{L}^*)$. The previously mentioned studies raised many new challenging questions, which are not resolved in this manuscript but we hope the current study may shed some light on the obstacle to overcome would a full Wasserstein Laplacian be identified. We rather took a different turn by connecting some bilinear forms and measure to a partial trace operator which is nothing but the infinitesimal generators of stochastic paths on the Wasserstein space. The study in the pioneering work by Cardaliaguet et. al. [4], concerned with the so–called *master equation* in mean field game systems, incorporated terms which turned out to be $\triangle_{w,\epsilon} U$ (cf. also [6] [7] [15] [16]). These *master equations* are first or second order non–local Hamilton–Jacobi equations on the Wasserstein space in which the presence of $\triangle_{w,\epsilon} U$ was instrumental for the well–posedness of the *master equation*. Games with both individual noise and common noise in [4] correspond to the case where $\epsilon > 0$ and regularity properties of initial conditions are preserved over time. Identifying conditions under which regularity properties of an initial value can be improved remain a question which we hope will be better understood by studying properties of various Laplacians. This is a main motivation of our work.

2. Notation and Preliminaries

2.1. **Notation.** In this manuscript, if $(\mathcal{S}, \text{dist})$ is a metric space, the domain of $U : \mathcal{S} \to \mathbb{R} \cup \{\pm\infty\}$ is the set $\text{dom}(U)$ of $m \in \mathcal{S}$ such that $U[m] \in \mathbb{R}$.

A function $\rho : [0, +\infty) \to [0, +\infty)$ is called a modulus if ρ continuous, nondecreasing, sub–additive, and $\rho(0) = 0$. It is a modulus of continuity for $U : \mathcal{S} \to \mathbb{R}$ if $|U(s_2) - U(s_1)| \leq \rho(\text{dist}(s_1, s_2))$ for any $s_1, s_2 \in \mathcal{S}$.

We denote $\mathbb{R}^d$ as $\mathbb{M}$ because it is more convenient to write expressions such as $(\mathbb{R}^d)^2$ than $(\mathbb{R}^d)^2$. This notation is also meant to emphasize the fact that most of our results proven in this manuscript are valid on spaces more general than $\mathbb{R}^d$. Throughout this manuscript, $\mathcal{P}_2(\mathbb{R}^d)$ denotes the set Borel probability measures on $(\mathbb{R}^d)$, of finite second moments. This is a length space when endowed with W_2, the Wasserstein distance.

Given $m, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ we denote as $\Gamma(m, \nu)$ the set of Borel measures γ on $(\mathbb{R}^d)^2$, which have m as their first marginal and ν as their second marginal. We denote as $\Gamma_0(m, \nu)$, the set of $\gamma \in \Gamma(m, \nu)$ such that

$$W_2^2(m, \nu) = \int_{(\mathbb{R}^d)^2} |x - y|^2 \gamma(dx, dy).$$

We denote the first (resp. second) projection of $(\mathbb{R}^d)^2$ onto $(\mathbb{R}^d)$ as π^1 (resp. π^2)

$$\pi^1(x, y) = x, \quad (\text{resp. } \pi^2(x, y) = y).$$

Let $L^2(m)$ denote the set of Borel maps $\zeta : (\mathbb{R}^d) \to (\mathbb{R}^d)$ such that $\|\zeta\|_m^2 := \int_{(\mathbb{R}^d)} |\zeta(x)|^2 m(dx) < \infty$. This is a Hilbert space with the inner product $\langle \cdot ; \cdot \rangle_m$ such that

$$\langle \zeta_1 ; \zeta_2 \rangle_m = \int_{(\mathbb{R}^d)} \langle \zeta_1(x), \zeta_2(x) \rangle m(dx).$$

Let $T_m\mathcal{P}_2(\mathbb{R}^d)$ denote the closure of $\nabla C_c^\infty((\mathbb{R}^d))$ in $L^2(m)$, and let us denote the orthogonal projection of $L^2(m)$ onto $T_m\mathcal{P}_2(\mathbb{R}^d)$ as π_m. The union of all the sets $\{m\} \times L^2(m)$ is denoted as $\mathcal{T}\mathcal{P}_2(\mathbb{R}^d)$ and by an abuse of language, is referred to as the tangent bundle of $\mathcal{P}_2(\mathbb{R}^d)$.

Let P_k denote the set of permutations of k letters. If $a \in \mathbb{C}$ we denote its complex conjugate as a^*.

Let $\text{Sym}[(\mathrm{R}^\mathrm{d})^\mathrm{k}]$ be the set of $\Phi \in C((\mathbb{R}^d)^k)$ such that there exists $C > 0$ such that for any $x = (x_1, \cdots, x_k) \in (\mathbb{R}^d)^k$,

$$|\Phi(x)| \leq (1 + |x|^2) \quad \text{and} \quad \Phi(x) = \Phi(x^\sigma) \quad \text{for any} \quad \sigma \in P_k.$$

In other words, Φ is well–defined on the k–symmetric product of $(\mathbb{R}^d)$. In this case, we call Φ is symmetric. Let $\text{Sym}_2[(\mathrm{R}^\mathrm{d})^\mathrm{k}]$ be the set of $\Phi \in \text{Sym}[(\mathrm{R}^\mathrm{d})^\mathrm{k}] \cap C^2((\mathbb{R}^d)^k)$ such that $\nabla^2\Phi$ is bounded and uniformly continuous on $(\mathbb{R}^d)^k$ and $\nabla^2\Phi$ has a modulus of continuity which is a concave function. Similarly, we define $\text{Sym}[\mathbb{C}^\mathrm{k}]$ and $\text{Sym}_2[\mathbb{C}^\mathrm{k}]$ when the functions are taking complex values.

When $\Phi \in \text{Sym}[(\mathrm{R}^\mathrm{d})^\mathrm{k}]$, the function $F_\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ set to be

$$F_\Phi[m] := \frac{1}{k} \int_{(\mathbb{R}^d)^k} \Phi(x_1, \cdots, x_k) m(dx_1) \cdots m(dx_k). \tag{2.1}$$

is well–defined. We denote as $\mathcal{S}ym[k](\mathbb{R})$ the set of F_Φ such that $\Phi \in \mathrm{Sym}[(\mathrm{\mathbb{R}^d})^{\mathrm{k}}]$ and denote as $\mathcal{S}ym_2[k](\mathbb{R})$ the set of F_Φ such that $\Phi \in \mathrm{Sym}_2[(\mathrm{\mathbb{R}^d})^{\mathrm{k}}]$.

The Fourier transform of $\Phi \in \mathrm{Sym}[(\mathrm{\mathbb{R}^d})^{\mathrm{k}}]$ is the function $\widehat{\Phi} \in \mathcal{S}ym[k](\mathbb{R})$ defined by

$$\widehat{\Phi}(\xi) = \int_{(\mathbb{R}^d)^k} \exp\Big(-2\pi \sum_{j=1}^{k} \langle \xi_j, x_j \rangle\Big) \Phi(x) dx.$$

We denote the Fourier inverse of $A \in L^2\big((\mathbb{R}^d)^k\big)$ as $\check{A}$.

2.2. **Preliminaries.** Let $U : \mathcal{P}_2(\mathbb{R}^d) \to [-\infty, \infty]$ denote a function with values in the extended real line. The recent work [12], shows two notions of Wasserstein subgradient which appeared in the literature to be equivalent. For that reason, we recall once more these definitions and state in Remark 2.2 that they are equivalent.

Definition 2.1. *Let $m \in \mathrm{dom}(U)$ and let $\zeta \in L^2(m)$.*

(i) *We call ζ a subgradient of U at m and write $\zeta \in \partial_. U[m]$ if for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$*

$$U[\nu] - U[m] \geq \inf_{\gamma \in \Gamma_0(m,\nu)} \int_{(\mathbb{R}^d)^2} \zeta(x) \cdot (y - x) \gamma(dx, dx) + o\big(W_2(m, \nu)\big), \tag{2.2}$$

(ii) *We call ζ a supergradient of U at m and write $\zeta \in \partial^{\cdot} U[m]$ if $-\zeta \in \partial_.(-U)[m]$.*

Remark 2.2. *Let m and ζ be as in Definition 2.1.*

(i) *It has recently been shown* [12] *that $\zeta \in \partial_. U[m]$ if and only if* (2.2) *holds when we replace the "inf" by "sup".*
(ii) *It is well–known that in case both $\partial_. U[m]$ and $\partial^{\cdot} U[m]$ are not empty then they coincide.*

Definition 2.3. *Let $m \in \mathrm{dom}(U)$.*

(i) *We say that U is differentiable at m if both $\partial_. U[m]$ and $\partial^{\cdot} U[m]$ are nonempty. In this case, we set $\partial U[m] = \partial^{\cdot} U[m]$.*
(ii) *If $\partial_. U[m]$ is nonempty, then it is a closed convex set in the Hilbert space $L^2(m)$ and so, it has a unique element of minimal norm. As customary done in convex analysis, we denote this element as $\nabla_w U[m]$ and refer to it as the Wasserstein gradient of U.*
(iii) *Thanks to Remark 2.2 (ii), if $\partial^{\cdot} U[m]$ is not empty, there is no confusion referring to its unique element of minimal norm as the Wasserstein gradient of U at m.*
(iv) *If $\nabla_w U[m]$ exists, $dU[m] : L^2(m) \to \mathbb{R}$ denotes the linear form $\zeta \mapsto \zeta \cdot (U[m]) := \langle \nabla_w U[m]; \zeta \rangle_m$.*

Remark 2.4. *Note that if $\phi \in C_c^\infty((\mathbb{R}^d))$, then according to Definition 2.3, the Wasserstein gradient of F_ϕ is $\nabla \phi$.*

Remark 2.5. *Assume $\rho : [0, \infty) \to [0, \infty)$ is a concave modulus.*

(i) *Then $\rho(t)/t$ is monotone nonincreasing and so, for any $t \geq 0$ and $\epsilon > 0$, we have $\rho(t) \leq \rho(\epsilon) + t/\epsilon \rho(\epsilon)$.*
(ii) *If $m, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \Gamma_0(m, \nu)$ then*

$$\int_{(\mathbb{R}^d)^2} |x - y| \rho\big(|x - y|\big) \gamma(dx, dy) \leq \rho\big(W_2^{\frac{1}{2}}(m, \nu)\big) W_2(m, \nu) \Big(W_2^{\frac{1}{2}}(m, \nu) + 1\Big)$$

Proof. (i) Mollifying ρ if necessary, it is not a loss of generality to assume that ρ is of class C^1. We have $t^2(\rho(t)/t)' = \rho'(t)t - \rho(t)$. But $R(t) := -\rho(t)$ is convex and so, for $t > 0$ we have $R(0) - R(t) \geq R'(t)(0-t)$. This is equivalent to $t^2(\rho(t)/t)' \leq 0$ which proves the first part of the remark. If $s \in [0, \epsilon]$ and $t \in [\epsilon, \infty)$ then

$$\rho(s) \leq \rho(\epsilon) \leq \rho(\epsilon) + \frac{s}{\epsilon}\rho(\epsilon) \quad \text{and so,} \quad \frac{\rho(t)}{t} \leq \frac{\rho(\epsilon)}{\epsilon} \leq \frac{\rho(\epsilon)}{t} + \frac{\rho(\epsilon)}{\epsilon}.$$

This proves (i).

(ii) Let $\gamma \in \Gamma_0(m, \nu)$. By (i) for any $\epsilon > 0$,

$$\int_{(\mathbb{R}^d)^2} |x-y| \rho\big(|x-y|\big)\gamma(dx, dy) \leq \rho(\epsilon)\Big(\frac{W_2^2(m,\nu)}{\epsilon} + \int_{(\mathbb{R}^d)^2} |x-y|\gamma(dx,dy)\Big).$$

We apply Cauchy–Schwarz inequality to obtain

$$\int_{(\mathbb{R}^d)^2} |x-y| \rho\big(|x-y|\big)\gamma(dx, dy) \leq \rho(\epsilon) W_2(m,\nu)\Big(\frac{W_2(m,\nu)}{\epsilon} + 1\Big).$$

We conclude the proof by setting $\epsilon := W_2^{1/2}(m, \nu)$. □

Remark 2.6. *Let $m \in \mathcal{P}_2(\mathbb{R}^d)$, $\zeta \in \nabla C_c^\infty(\mathbb{M})$. For $t \in [0,1]$ we set*

$$r_t(x,y) = (1-t)x + ty, \quad E(x,y) := \int_0^1 \big|\nabla\zeta(r_t(x,y)) - \nabla\zeta(x)\big|^2 dt \qquad \forall x, y \in (\mathbb{R}^d).$$

Define

$$\epsilon_\zeta(r) := \sup_{W_2(m,\nu) \leq r} \sup_{\gamma \in \Gamma(m,\nu)} \int_{(\mathbb{R}^d)^2} (1+|x|)^2 E(x,y)\gamma(dx,dy), \qquad r > 0.$$

Then we have $\lim_{r\to 0^+} \epsilon_\zeta(r) = 0$.

Since $(\mathbb{R}^d)$ is not a compact set, the space $\mathcal{P}_2(\mathbb{R}^d)$ is not a locally compact space (cf. e.g. [1]). Suppose $\phi : (\mathbb{R}^d) \to [0, \infty]$ is a lower semicontinuous monotone nondecreasing function

$$\phi : (\mathbb{R}^d) \to [0, \infty], \quad \lim_{|x|\to\infty} \frac{\phi(x)}{|x|^2} = \infty.$$

Consider the locally compact set

$$\mathcal{P}_\phi((\mathbb{R}^d)) := \Big\{ m \in \mathcal{P}_2(\mathbb{R}^d) \;\Big|\; \int_{(\mathbb{R}^d)} \phi(x) m(dx) < \infty \Big\}. \tag{2.3}$$

Remark 2.7. *The subalgebra generated by $\big\{F_\Phi \mid \Phi \in C_c((\mathbb{R}^d))\big\}$, separates points in $\mathcal{P}_\phi((\mathbb{R}^d))$ and vanishes nowhere. Hence, by the Stone Weierstrass Theorem, it is a dense subset of $C\big(\mathcal{P}_\phi((\mathbb{R}^d))\big)$ for the uniform convergence.*

3. Traces of second order derivative: the partial Laplacian operators

Let $U : \mathcal{P}_2(\mathbb{R}^d) \to [-\infty, \infty]$ denote a function with values in the extended real line.

3.1. **The Wasserstein Laplacian operator.** Consistent with Levi–Civita connection in [17], we have the following definition.

Definition 3.1. *Suppose U is differentiable in a neighborhood of $m \in \mathrm{dom}(U)$ and for any $\zeta \in C_c^\infty((\mathbb{R}^d), (\mathbb{R}^d))$, $\nu \mapsto \zeta \cdot (U[\nu])$ is differentiable at m.*

(i) *We define $\bar{\mathrm{H}}\mathrm{ess}\, U[m] : \nabla C_c^\infty((\mathbb{R}^d)) \times \nabla C_c^\infty((\mathbb{R}^d)) \to \mathbb{R}$ if the following exists:*

$$\bar{\mathrm{H}}\mathrm{ess}\, U[m](\zeta_1, \zeta_2) = \zeta_1 \cdot \Big(\zeta_2 \cdot \big(U[m]\big)\Big) - \Big(\bar{\nabla}_{\zeta_1}\zeta_2\Big) \cdot (U[m])$$

for any $\zeta_1, \zeta_2 \in \nabla C_c^\infty((\mathbb{R}^d))$. Here, $\bar{\nabla}_{\zeta_1}\zeta_2 = \nabla\zeta_2\zeta_1$.

(ii) *If there is a constant C such that $|\mathrm{Hess}\, U[m](\zeta_1, \zeta_2)| \le C \|\zeta_1\|_m \|\zeta_2\|_m$ for all $\zeta_1, \zeta_2 \in \nabla C_c^\infty((\mathbb{R}^d))$ then $\bar{\mathrm{H}}\mathrm{ess}[m]$ has a unique extension onto $T_m\mathcal{P}_2(\mathbb{R}^d) \times T_m\mathcal{P}_2(\mathbb{R}^d)$ which we denote as $\mathrm{Hess}\, U[m]$. In that case, we say that U has a Hessian at m.*

Given two nonnegative functions $\rho, \epsilon : [0, \infty) \to \mathbb{R}$ (depending on m) such that $\lim_{t\to 0^+} \epsilon(t) = 0$ and ρ is a concave modulus, in what follows, we denote $\Upsilon : [0, \infty) \times [0, \infty) \to \mathbb{R}$ as follows:

$$\Upsilon(s, t) := (t + s)\Big(\rho(t) + \epsilon(s)\Big).$$

Theorem 3.2. *Suppose U is differentiable in a neighborhood of $m \in \mathrm{dom}(U)$, and the map $x \mapsto \nabla_w U[\nu](x)$ admits an extension which is continuous for ν in a neighborhood of m. Assume further that there exists a constant C_m such that*

$$|\nabla_w U[\nu](x)| \le C_m(1 + |x|) \tag{3.1}$$

for any $x \in (\mathbb{R}^d)$ and any ν in the neighborhood of m. Suppose $\rho, \epsilon : [0, \infty) \to \mathbb{R}$ are nonnegative function (depending on m) such that $\lim_{t\to 0^+} \epsilon(t) = 0$ and ρ is a concave modulus. Suppose there are Borel bounded matrix valued functions $\tilde{A}[m] : (\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ and $A_{mm} : (\mathbb{R}^d)^2 \to \mathbb{R}^{d\times d}$ satisfying the following properties: for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ we have

$$\sup_{\gamma\in\Gamma_0(m,\nu)} \Big|\nabla_w U[\nu](y) - \nabla_w U[m](x) - P_\gamma[m](x, y)\Big| \le \Upsilon\Big(W_2(\mu, \nu), |x - y|\Big).$$

Here, for $\gamma \in \mathcal{P}((\mathbb{R}^d)^2)$ and $x, y \in (\mathbb{R}^d)$, we have set

$$P_\gamma[m](x, y) := \tilde{A}[m](x)(y - x) + \int_{(\mathbb{R}^d)^2} A_{mm}(x, a)(b - a)\gamma(da, db). \tag{3.2}$$

Then, U has a Hessian at m and

$$\mathrm{Hess}\, U[m](\zeta_1, \zeta_2) = \int_{(\mathbb{R}^d)} \langle \tilde{A}[m](x)\zeta_1(x), \zeta_2(x)\rangle m(dx) + \int_{(\mathbb{R}^d)^2} \langle A_{mm}(x, a)\zeta_1(a), \zeta_2(x)\rangle m(dx)m(da)$$

for $\zeta_1, \zeta_2 \in T_m\mathcal{P}_2(\mathbb{R}^d)$.

Proof. Fix $\zeta_1, \zeta_2 \in \nabla C_c^\infty((\mathbb{R}^d))$. We are to show that the map $\nu \mapsto \Lambda(\nu) := dU[\nu](\zeta_2)$ is differentiable at m and then show that $\zeta_1 \cdot \big(dU[\nu](\zeta_2)\big)$ exists.

Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\gamma \in \Gamma_0(m, \nu)$ and set $A_\nu := \nabla_w U[\nu]$. Since ζ_2 is of compact support, its first and second derivatives are bounded, and so, we may choose bounded vector fields $v, w \in C((\mathbb{R}^d)^2, (\mathbb{R}^d))$ such that for any $x, y \in (\mathbb{R}^d)$ we have

$$\zeta_2(y) = \zeta_2(x) + \nabla\zeta_2(x)(y - x) + |y - x|^2 v(x, y), \quad \zeta_2(y) - \zeta_2(x) = |y - x| w(x, y). \tag{3.3}$$

Note we in fact have the identity

$$|y-x|^2 v(x,y) := \int_0^1 \Big(\nabla\zeta_2\big((1-t)x+ty\big) - \nabla\zeta_2(x)\Big)(y-x)dt \tag{3.4}$$

By assumption, for each $\gamma \in \mathcal{P}((\mathbb{R}^d)^2)$, there exists $l(\gamma,\nu,y) \in (\mathbb{R}^d)$ such that $|l(\gamma,\nu,y)| \leq 1$ and

$$A_\nu(y) - A_m(x) - P_\gamma[m](x,y) = l(\gamma,\nu,y)\Upsilon\Big(W_2(\mu,\nu), |x-y|\Big). \tag{3.5}$$

We have then by definition of the map Λ that

$$\begin{aligned}\Lambda(\nu) - \Lambda(m) &= \int_{(\mathbb{R}^d)^2} \big(\langle A_\nu(y), \zeta_2(y)\rangle - \langle A_m(x), \zeta_2(x)\rangle\big)\gamma(dx,dy)\\ &= \int_{(\mathbb{R}^d)^2} \big(\langle A_\nu(y) - A_m(x), \zeta_2(y)\rangle + \langle A_m(x), \zeta_2(y)-\zeta_2(x)\rangle\big)\gamma(dx,dy).\end{aligned}$$

This, combined with (3.3) yields

$$\begin{aligned}\Lambda(\nu) - \Lambda(m) &= \int_{(\mathbb{R}^d)^2} \langle A_\nu(y) - A_m(x), \zeta_2(x) + |y-x| w(x,y)\rangle\gamma(dx,dy)\\ &\quad+ \int_{(\mathbb{R}^d)^2} \langle A_m(x), \nabla\zeta_2(x)(y-x) + |y-x|^2 v(x,y)\rangle\big)\gamma(dx,dy)\\ &= I + II. \end{aligned}\tag{3.6}$$

Set

$$R := \Big| II - \int_{(\mathbb{R}^d)^2} \langle \nabla\zeta_2^T(x) A_m(x), (y-x)\rangle\gamma(dx,dy)\Big|.$$

We use first (3.4) and second (3.1) to conclude that

$$\begin{aligned}R &\leq \int_{(\mathbb{R}^d)^2} \big|A_m(x)\big| \int_0^1 \big|\nabla\zeta_2\big((1-t)x+ty\big) - \nabla\zeta_2(x)\big||y-x|dt\gamma(dx,dy)\\ &\leq W_2(m,\nu)\sqrt{\int_{(\mathbb{R}^d)^2} \big|A_m(x)\big|^2 \int_0^1 \big|\nabla\zeta_2\big(r_t(x,y)\big) - \nabla\zeta_2(x)\big|^2 dt\,\gamma(dx,dy)}\\ &\leq C_m W_2(m,\nu)\sqrt{\int_{(\mathbb{R}^d)^2} (1+|x|)^2 \int_0^1 \big|\nabla\zeta_2\big(r_t(x,y)\big) - \nabla\zeta_2(x)\big|^2 dt\gamma(dx,dy)}\end{aligned}\tag{3.7}$$

Using the notation of Remark 2.6, the previous inequality reads off

$$R \leq C_m W_2(m,\nu)\sqrt{\epsilon_{\zeta_2}\big(W_2(m,\nu)\big)}, \qquad \lim_{s\to 0^+} \sqrt{\epsilon_{\zeta_2}(s)} = 0. \tag{3.8}$$

By (3.5)

$$\begin{aligned}I &= \int_{(\mathbb{R}^d)^2} \Big\langle \tilde{A}[m](x)(y-x) + \int_{(\mathbb{R}^d)^2} A_{mm}(x,a)(b-a)\gamma(da,db), \zeta_2(x)\Big\rangle\gamma(dx,dy)\\ &\quad+ \int_{(\mathbb{R}^d)^2} \Big\langle \tilde{A}[m](x)(y-x) + \int_{(\mathbb{R}^d)^2} A_{mm}(x,a)(b-a)\gamma(da,db), |y-x|w(x,y)\Big\rangle\gamma(dx,dy)\\ &\quad+ \int_{(\mathbb{R}^d)^2} \Big\langle l(\gamma,\nu,y)\Upsilon\Big(W_2(m,\nu), |x-y|\Big), \zeta_2(y)\Big\rangle\gamma(dx,dy) = III + IV + V.\end{aligned}\tag{3.9}$$

We have by Cauchy–Schwarz inequality that

$$|V| \le \|\zeta_2\|_{L^\infty} \int_{(\mathbb{R}^d)^2} \Upsilon\big(W_2(m,\nu), |x-y|\big)\gamma(dx,dy). \tag{3.10}$$

By Jensen's inequality

$$\int_{(\mathbb{R}^d)^2} |x-y|\epsilon\big(W_2(m,\nu)\big)\gamma(dx,dy) \le W_2(m,\nu)\epsilon\big(W_2(m,\nu)\big). \tag{3.11}$$

Since ρ is concave, we may apply first Jensen's inequality and second use the fact that it is monotone nondecreasing to obtain

$$\int_{(\mathbb{R}^d)^2} \rho(|x-y|)W_2(m,\nu)\gamma(dx,dy) \le W_2(m,\nu)\rho\Big(\int_{(\mathbb{R}^d)^2} |x-y|\gamma(dx,dy)\Big) \le W_2(m,\nu)\rho\big(W_2(m,\nu)\big). \tag{3.12}$$

We combine (3.10)-(3.12) and use Remark 2.5 to infer

$$|V| \le \|\zeta_2\|_{L^\infty} W_2(m,\nu)\Big(2\epsilon(W_2(m,\nu)) + \rho\Big(W_2(m,\nu)\Big) + \rho\Big(W_2^{\frac{1}{2}}(m,\nu)\Big)\Big(W_2^{\frac{1}{2}}(m,\nu)+1\Big)\Big) \tag{3.13}$$

Checking also that

$$\|\tilde{A}[m](x)(y-x) + \int_{(\mathbb{R}^d)^2} A_{mm}(x,a)(b-a)\gamma(da,db)\|_\gamma \le \big(\|\tilde{A}[m]\|_{L^\infty(m)} + \|A_{mm}\|_{L^2(m\otimes m)}\big)W_2(m,\nu),$$

we conclude

$$|IV| \le \big(\|\tilde{A}[m]\|_{L^\infty(m)} + \|A_{mm}\|_{L^2(m\otimes m)}\big)W_2^2(m,\nu). \tag{3.14}$$

We combine (3.6) (3.8) (3.9) (3.13) and (3.14) and make the substitution $(a,b) \leftrightarrow (x,y)$ to obtain

$$\begin{aligned}\Lambda(\nu) - \Lambda(m) = &\int_{(\mathbb{R}^d)^2} \langle \tilde{A}[m]^T(x)\zeta_2(x) + \nabla\zeta_2^T(x)A_m(x), (y-x)\rangle\gamma(dx,dy) \\ &+ \int_{(\mathbb{R}^d)^2} \Big\langle \int_{(\mathbb{R}^d)^2} (y-x), A_{mm}^T(a,x)\zeta_2(a)\Big\rangle \gamma(da,db)\gamma(dx,dy) + o\big(W_2(m,\nu)\big).\end{aligned}$$

Thus, Λ is differentiable at m and

$$\nabla_w\Lambda[m](x) = \tilde{A}[m]^T(x)\zeta_2(x) + \nabla\zeta_2^T(x)A_m(x) + \int_{(\mathbb{R}^d)} A_{mm}^T(a,x)\zeta_2(a)m(da)$$

Consequently,

$$\begin{aligned}\zeta_1 \cdot (\Lambda[m]) = &\int_{(\mathbb{R}^d)} \langle \tilde{A}[m]^T(x)\zeta_2(x), \zeta_1(x)\rangle m(dx) + \int_{(\mathbb{R}^d)^2} \langle A_{mm}(a,x)\zeta_1(x), \zeta_2(a)\rangle m(da)m(dx) \\ &+ \int_{(\mathbb{R}^d)} \langle A_m(x), \nabla\zeta_2(x)\zeta_1(x)\rangle m(dx).\end{aligned} \tag{3.15}$$

Making the substitution $a \leftrightarrow x$ (3.15) and using definition 3.1, we yield the following:

$$\bar{\mathrm{H}}\mathrm{ess}\, U[m](\zeta_1,\zeta_2) = \int_{(\mathbb{R}^d)} \langle \tilde{A}[m](x)\zeta_1(x), \zeta_2(x)\rangle m(dx) + \int_{(\mathbb{R}^d)^2} \langle A_{mm}(x,a)\zeta_1(a), \zeta_2(x)\rangle m(dx)m(da). \tag{3.16}$$

Obviously, there is a constant C such that $|\bar{\mathrm{H}}\mathrm{ess}\, U[m](\zeta_1,\zeta_2)| \le C\,\|\zeta_1\|_m\,\|\zeta_2\|_m$. Thus $\mathrm{Hess}\, U[m]$ is well–defined and (3.16) remains valid for $\zeta_1, \zeta_2 \in T_m\mathcal{P}_2(\mathbb{R}^d)$. □

Definition 3.3. *Under the assumptions of Theorem 3.2, we say that U is twice differentiable at $m \in \mathrm{dom}(U)$. If A_{mm} satisfies (3.2), so does $\pi_m\big(A_{mm}\big)$ which is the matrix whose rows are the orthogonal projections onto $T_m\mathcal{P}_2(\mathbb{R}^d)$ of the rows of A_{mm}. Note that although A_{mm} may not be unique, $\pi_m\big(A_{mm}\big)$ is uniquely determined. In the sequel, we tacitly assume that $A_{mm} \equiv \pi_m\big(A_{mm}\big)$.*

(i) *We define the Wasserstein Laplacian operator $\triangle_w$ such that $\triangle_w U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, is given by*

$$(\triangle_w U)[m] = \sum_{i=1}^d \mathrm{Hess}\, U[m](e_i, e_i),$$

where e_i is the Wasserstein gradient of the i–th moment $m \mapsto \int_{\mathbb{R}^d} x_i m(dx)$.

(ii) *Given $\epsilon \geq 0$, we call $\triangle_{w,\epsilon}$ such that $\triangle_{w,\epsilon} U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the ϵ–perturbation of $\triangle_w$ such that*

$$(\triangle_{w,\epsilon} U)[m] = \sum_{i=1}^d \Big(\mathrm{Hess}\, U[m](e_i, e_i) + \epsilon \int_{(\mathbb{R}^d)} \langle \tilde{A}[m](x) e_i(x), e_i(x)\rangle m(dx)\Big)$$

(iii) *We define the second order Wasserstein gradient of U at m to be $\pi_m\big(A_{mm}\big)$ and we denote it as $\nabla_w^2 U[m]$.*

(iv) *Suppose further that U is twice differentiable in a neighborhood of m. If $(x, \nu) \mapsto \tilde{A}[\nu](x)$ and $(x, y, \nu) \mapsto \pi_\nu\big(A_{\nu\nu}\big)(x, y)$ are continuous, we say that U is twice continuously differentiable on that neighborhood.*

Proposition 3.4. *Let U be as in Theorem 3.2, which in particular means that we have fixed $m \in \mathcal{P}_2(\mathbb{R}^d)$ such that U is differentiable in a neighborhood of $m \in \mathrm{dom}(U)$ and U is twice differentiable at m. Let $T > 0$ and suppose $\sigma \in AC_2(0, T, \mathcal{P}_2(\mathbb{R}^d))$ is a path which has a velocity of minimal norm $\mathbf{v} \in C^1((0, T) \times (\mathbb{R}^d))$ which is bounded and has bounded first order time and space derivatives. If $s \in (0, T)$ and $m = \sigma_s$ then*

$$\frac{d^2}{dt^2} U(\sigma_t))\Big|_{t=s} = \mathrm{Hess}\, U[\sigma_s](\mathbf{v}_s, \mathbf{v}_s) + \langle \partial_t \mathbf{v}_s + \nabla \mathbf{v}_s \mathbf{v}_s ; \nabla_w U[\sigma_s]\rangle_{\sigma_s}$$

Proof. We skip the proof of this proposition since it is similar to that of Theorem 3.2. The only new ingredient to use here is the following additional remark: if $\gamma_h \in \Gamma_0(\sigma_t, \sigma_{t+h})$ and $\pi^1, \pi^2 : (\mathbb{R}^d)^2 \to (\mathbb{R}^d)$ denote the standard projections then

$$\lim_{h \to 0} \Big(\pi^1, \frac{\pi^2 - \pi^1}{h}\Big)_{\#} \gamma_h = (\mathbf{id}, \mathbf{v}_t)_{\#} \sigma_t \qquad \text{in} \quad \mathcal{P}_2((\mathbb{R}^d)^2).$$

□

Remark 3.5. *Suppose the assumptions in Theorem 3.2 holds, i.e. U is twice differentiable at $m \in \mathrm{dom}(U)$. Then*

(i) *$A_m := \nabla_w U[m]$ is differentiable on $(\mathbb{R}^d)$ and its gradient (w.r.t. the x variable) is $\tilde{A}[m]$, whose rows belong to $T_m\mathcal{P}_2(\mathbb{R}^d)$.*

(ii) *We have*

$$\triangle_w U[m] = \int_{(\mathbb{R}^d)} \mathrm{div}_x\big(\nabla_w U[m](x)\big) m(dx) + \int_{(\mathbb{R}^d)^2} \mathrm{Tr}\Big(\nabla_w^2 U[m](x, a)\Big) m(dx) m(da)$$

and

$$(\triangle_{w,\epsilon}U)[m] = (1+\epsilon)\int_{(\mathbb{R}^d)} \mathrm{div}_x\big(\nabla_w U[m](x)\big)m(dx) + \int_{(\mathbb{R}^d)^2} \mathrm{Tr}\Big(\nabla^2_w U[m](x,a)\Big)m(dx)m(da)$$

(iii) *Note that the expressions in (ii a) continue to make sense if we merely assume that*

$$\mathrm{div}_x\big(A_m(x)\big) \in L^1((\mathbb{R}^d), m) \quad \textit{and} \quad \mathrm{Tr}\Big(\nabla^2_w U[m](\cdot,\cdot)\Big) \in L^1(({\mathbb{R}^d})^2, m\otimes m).$$

3.2. **Particular case: Hessians of functions belonging to $\mathcal{S}ym[k](\mathbb{R})$.** Let $\Phi \in \mathrm{Sym}_2[(\mathbb{R}^d)^{\mathrm{k}}]$. Set $A_m = \nabla_{x_1}\Phi$ when $k=1$ and for $x_1 \in (\mathbb{R}^d)$ set

$$A_m(x_1) = \int_{(\mathbb{R}^d)^{k-1}} \nabla_{x_1}\Phi(x_1,\cdots,x_k)m(dx_2)\cdots m(dx_k) \quad \text{if} \quad k\geq 2.$$

If $x_1, x_2 \in (\mathbb{R}^d)$ we set

$$A_{mm}(x_1,x_2) := \begin{cases} 0 & \text{if} \quad k=1, \\ \nabla^2_{x_2x_1}\Phi(x_1,x_2) & \text{if} \quad k=2, \\ (k-1)\int_{(\mathbb{R}^d)^{k-2}} \nabla^2_{x_2x_1}\Phi(x)m(dx_3)\cdots m(dx_k) & \text{if} \quad k\geq 3 \end{cases}$$

For $\gamma \in \mathcal{P}_2((\mathbb{R}^d)^2)$ we set

$$P_\gamma[m](x_1,y_1) := \nabla_{x_1}A_m(x_1)(y_1-x_1) + \int_{(\mathbb{R}^d)^2} A_{mm}(x_1,x_2)(y_2-x_2)\gamma(dx_2,dy_2),$$

Then we have the following lemma for special functions of the form F_Φ and referee the reader to [8] for related calculations.

Lemma 3.6. *For any $m\in\mathcal{P}_2(\mathbb{R}^d)$, the following hold.*

(i) *The function F_Φ is differentiable in the sense of Wasserstein at any $m\in\mathcal{P}_2(\mathbb{R}^d)$ and $\nabla_w F_\Phi[m] = A_m \in T_m\mathcal{P}_2(\mathbb{R}^d)$.*

(ii) *Further assume that $\nabla^2\Phi$ has a modulus of continuity $\rho : [0,\infty)\to[0,\infty)$ which is concave. If $\nu\in\mathcal{P}_2(\mathbb{R}^d)$, then there exists $\tilde\rho, \epsilon_2 : [0,\infty)\to\mathbb{R}$ are nonnegative function (depending on m and Φ) such that $\lim_{t\to0^+}\tilde\rho(t) = \lim_{t\to0^+}\epsilon_2(t) = 0$ and $\tilde\rho$ is a concave modulus and*

$$\sup_{\gamma\in\Gamma_0(m,\nu)}\Big|A_\nu(y_1) - A_m(x_1) - P_\gamma[m,\nu](x_1,y_1)\Big| \leq |x-y|\tilde\rho\big(|x_1-y_1|\big) + W_2(m,\nu)\epsilon_2\big(W_2(m,\nu)\big)\,.$$

Proof. Note that since $\nabla^2\Phi$ is bounded there exists a constant $C>0$ such that

$$|\Phi(x)|\leq C(1+|x|^2), \quad |\nabla\Phi(x)|\leq C(1+|x|), \quad |\nabla^2\Phi(x)|\leq C \qquad \forall \quad x\in(\mathbb{R}^d). \tag{3.17}$$

The second inequality in (3.17) ensures that for any $m\in\mathcal{P}_2(\mathbb{R}^d)$, $\nabla_{x_1}\Phi\in L^1(m^{\otimes(k-1)})$ and so, A_m is well–defined. Furthermore, Φ is bounded. Similarly, the second and third inequalities in (3.17) ensure that $\nabla_{x_1}A_m$ and A_{mm} are well–defined and bounded.

(i) The proof of (i) is easier when $k=1$. We assume in the sequel that $k\geq 2$. Using the fact that Φ is symmetric, for any $i\in\{2,\cdots,k\}$ if σ is the permutation such that $\sigma(1)=i$, $\sigma(i)=1$ and $\sigma(j)=j$ for any $j\notin\{1,i\}$ we have

$$\nabla_{x_1}\Phi(x) = \nabla_{x_i}\Phi(x^\sigma). \tag{3.18}$$

Applying Taylor expansion, thanks to the third inequality in (3.17) there exists a uniformly continuous function $f : (\mathbb{R}^d)^k \times (\mathbb{R}^d)^k \to \mathbb{R}$ bounded by C such that

$$\Phi(y) = \Phi(x) + \sum_{i=1}^k \nabla_{x_i}\Phi(x)\cdot(y_i - x_i) + \frac{1}{2} f(x,y)|x-y|^2. \tag{3.19}$$

Let $m, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma \in \Gamma_0(m,\nu)$. By changing variables

$$\int_{(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2} \nabla_{x_i}\Phi(x)\cdot(y_i - x_i)\gamma(dx_1,dy_1)\gamma(dx_i,dy_i) = \int_{(\mathbb{R}^d)^2} \nabla_{x_i}\Phi(x^\sigma)\cdot(y_1-x_1)\gamma(dx_i,dy_i)\gamma(dx_1,dy_1).$$

Using (3.18) we conclude that

$$\int_{(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2} \Big(\nabla_{x_i}\Phi(x)\cdot(y_i-x_i) - \nabla_{x_1}\Phi(x)\cdot(y_1-x_1)\Big)\gamma(dx_1,dy_1)\gamma(dx_i,dy_i) = 0. \tag{3.20}$$

We have

$$F_\Phi[\nu] - F_\Phi[m] = \frac{1}{k}\int_{(\mathbb{R}^d)^k\times(\mathbb{R}^d)^k} (\Phi(y)-\Phi(x))\gamma^{\otimes k}(dx,dy).$$

This, together with (3.19) and (3.20) implies

$$F_\Phi[\nu] - F_\Phi[m] = \int_{(\mathbb{R}^d)^2} A_m(x_1)\cdot(y_1-x_1)\gamma(dx_1,dy_1) + \frac{1}{2}\int_{(\mathbb{R}^d)^k\times(\mathbb{R}^d)^k} f(x,y)|x-y|^2\gamma^{\otimes k}(dx,dy) \tag{3.21}$$

and so,

$$\Big|F_\Phi[\nu] - F_\Phi[m] - \int_{(\mathbb{R}^d)^2} A_m(x_1)\cdot(y_1-x_1)\gamma(dx_1,dx_2)\Big| \le \frac{Ck}{2}W_2^2(m,\nu).$$

This proves that $A_m \in \partial F_\Phi[m]$. Note that A_m is the gradient of

$$x_1 \mapsto \Phi_1(x_1) := \int_{(\mathbb{R}^d)^{k-1}} \Phi(x_1,\cdots,x_k)m(dx_2)\cdots m(dx_k)$$

which is a bounded function with bounded first derivatives. Thus, $A_m \in T_m\mathcal{P}_2(\mathbb{R}^d)$. For any $\zeta \in T_m\mathcal{P}_2(\mathbb{R}^d)$ we have $\pi_m(\zeta - A_m) = 0$, which means that $\pi_m(\zeta) = \pi_m(A_m) = A_m$. In particular $\|\zeta\|_m \ge \|\pi_m(\zeta)\|_m = \|A_m\|_m$, which proves that A_m is the element of minimal norm in $\partial F_\Phi[m]$.

(ii) Since the proof of (ii) is easier in the case $k=2$ compared to the case when $k \ge 3$, we assume in the sequel that $k \ge 3$.

Let $i \in \{3,\cdots,k\}$ and let σ be the permutation such that $\sigma(2)=i$, $\sigma(i)=2$ and $\sigma(j)=j$ for any $j \notin \{2,i\}$. Given $x = (x_1,\cdots,x_k) \in (\mathbb{R}^d)^k$, using the fact that Φ is symmetric, we have

$$(\nabla^2_{x_2x_1}\Phi)(x) = (\nabla^2_{x_1x_2}\Phi)(x) = (\nabla^2_{x_1x_i}\Phi)(x^\sigma) = (\nabla^2_{x_ix_1}\Phi)(x^\sigma). \tag{3.22}$$

For any $y = (y_1,\cdots,y_k) \in (\mathbb{R}^d)^k$,

$$(\nabla_{x_1}\Phi)(y) - (\nabla_{x_1}\Phi)(x) = \int_0^1 \sum_{i=1}^k (\nabla^2_{x_ix_1}\Phi)(x+t(y-x))(y_i-x_i)dt. \tag{3.23}$$

Let $m, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma \in \Gamma_0(m,\nu)$. The change of variables which exchanges x_2 with x_i is used to obtain

$$\int_{(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2} \Big((\nabla^2_{x_i x_1}\Phi)(x)(y_i - x_i) - (\nabla^2_{x_i x_1}\Phi)(x^\sigma)(y_2 - x_2) \Big) \gamma(dx_2, dy_2)\gamma(dx_i, dy_i) = 0.$$

Combining the latter with (3.22) we infer

$$\int_{(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2} \Big((\nabla^2_{x_i x_1}\Phi)(x)(y_i - x_i) - (\nabla^2_{x_2 x_1}\Phi)(x)(y_2 - x_2) \Big) \gamma(dx_2, dy_2)\gamma(dx_i, dy_i) = 0. \tag{3.24}$$

Set

$$e_t(x,y) := \int_0^1 \sum_{i=1}^k \Big((\nabla^2_{x_i x_1}\Phi)(x + t(y-x)) - (\nabla^2_{x_i x_1}\Phi)(x) \Big)(y_i - x_i) dt$$

By (3.23)

$$A_\nu(y_1) - A_m(x_1) = \int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^{k-1}} \int_0^1 \sum_{i=1}^k (\nabla^2_{x_i x_1}\Phi)(x + t(y-x))(y_i - x_i) dt \gamma(dx_2, dy_2) \cdots \gamma(dx_k, dy_k)$$

This, together with (3.24) yields

$$\begin{aligned} A_\nu(y_1) - A_m(x_1) &= \nabla_{x_1} A_m(x_1)(y_1 - x_1) + \int_{(\mathbb{R}^d)^2} A_{mm}(x_1, x_2)(y_2 - x_2)\gamma(dx_2, dy_2) \\ &\quad + \int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^{k-1}} \int_0^1 e_t(x,y) dt \gamma(dx_2, dy_2) \cdots \gamma(dx_k, dy_k). \end{aligned} \tag{3.25}$$

Let $\rho : [0,\infty) \to [0,\infty)$ be a concave function, modulus of continuity of $\nabla^2\Phi$. Then there exists a constant C_k depending only on k and d such that

$$|e_t(x,y)| \leq C_k \sum_{i=1}^k \rho(|x_i - y_i|)|x_i - y_i|.$$

Thus, if we use the notation $\hat{x}^1 = (x_2, \cdots, x_k)$, we have

$$\begin{aligned} \int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^{k-1}} \int_0^1 |e_t(x,y)| dt \gamma^{\otimes(k-1)}(d\hat{x}^1, d\hat{y}^1) &\leq C_k \sum_{i=1}^k \int_{(\mathbb{R}^d)^2} \rho(|x_i - y_i|)|x_i - y_i|\gamma(dx_i, dy_i) \\ &= C_k \rho(|x_1 - y_1|)|x_1 - y_1| \\ &\quad + C_k \sum_{i=2}^k \int_{(\mathbb{R}^d)^2} \rho(|a-b|)|a-b|\gamma(da, db). \end{aligned} \tag{3.26}$$

Thanks to Remark 2.5 we conclude

$$\begin{aligned} \int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^{k-1}} \int_0^1 |e_t(x,y)| dt \gamma^{\otimes(k-1)}(d\hat{x}^1, d\hat{y}^1) &\leq C_k \rho(|x_1 - y_1|)|x_1 - y_1| \\ &\quad + (k-1) C_k \rho\big(W_2^{\frac{1}{2}}(m,\nu)\big) W_2(m,\nu) \Big(W_2^{\frac{1}{2}}(m,\nu) + 1 \Big). \end{aligned} \tag{3.27}$$

This, together with (3.25) and (3.27) proves (ii), after setting setting $\tilde{\rho}(t) = C_k \rho(t)$ and $\epsilon_2(t) = (k-1) C_k \rho(t) \Big(t^{\frac{1}{2}} + 1 \Big)$. □

Remark 3.7. *Using the notation in Lemma 3.6, we obtain the following.*

(i) *First*

$$\nabla_{x_1}\big(A_m(x_1)\big) = \int_{(\mathbb{R}^d)^{k-1}} \nabla^2_{x_1x_1}\Phi(x_1,\cdots,x_k)m(dx_2)\cdots m(dx_k)$$

is a symmetric matrix such that each row is an element of $T_m\mathcal{P}_2(\mathbb{R}^d)$.

(ii) *Second, the rows of* A_{mm} *are in* $T_m\mathcal{P}_2(\mathbb{R}^d)$.

With the above, we obtain the following. Although the expression of $\triangle_{w,\epsilon}F_\Phi$ coincide with that in Section 5.3 in [4], we are putting here for the sake of completeness.

Corollary 3.8. *Further assume that* $\nabla^2\Phi$ *has a modulus of continuity* $\rho : [0,\infty) \to [0,\infty)$ *which is concave. Then,*

(i) F_Φ *is twice differentiable at any* $m \in \mathcal{P}_2(\mathbb{R}^d)$, $\nabla_w U[m] = A_m$ *and* $\nabla^2_w F_\Phi[m] = A_{mm}$.

(ii) *We have* $\triangle_w F_\Phi = F_{\Theta_0}$.

(iii) *Similarly, if* $\epsilon > 0$ *then* $\triangle_{w,\epsilon}F_\Phi = F_{\Theta_\epsilon}$. *Here*

$$\Theta_\epsilon := \Big((1+\epsilon)\sum_{j=1}^k \triangle_{x_j}\Phi + \sum_{j\neq n}\sum_{l=1}^d \frac{\partial^2\Phi}{\partial(x_n)_l\partial(x_j)_l}\Big).$$

Proof. Thanks to Lemma 3.6, we apply Theorem 3.2 to obtain that U is twice differentiable. Remark 3.5 gives an explicit expression of $\triangle_w F_\Phi[m]$ and that of $\triangle_{w,\epsilon}F_\Phi[m]$. We use the symmetric properties of the second derivatives of Φ to obtain

$$\int_{(\mathbb{R}^d)^k} \triangle_{x_1}\Phi m(dx_1)\cdots m(dx_k) = \frac{1}{k}\int_{(\mathbb{R}^d)^k}\sum_{j=1}^k \triangle_{x_j}\Phi m(dx_1)\cdots m(dx_k)$$

and

$$\int_{(\mathbb{R}^d)^k}\sum_{n=1}^d \frac{\partial^2\Phi}{\partial(x_2)_n\partial(x_1)_n}m(dx_1)\cdots m(dx_k) = \frac{1}{k(k-1)}\int_{(\mathbb{R}^d)^k}\sum_{n=1}^d\sum_{j\neq l}\frac{\partial^2\Phi}{\partial(x_j)_n\partial(x_l)_n}m(dx_1)\cdots m(dx_k)$$

to conclude the proof of the Corollary. □

3.3. **Convergence theorem for the Wasserstein Hessian.** In this subsection $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $C_\mu > 0$ and $\mathcal{O}$, an open ball centered at $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Suppose $G_N : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a sequence of continuous functions converging uniformly to $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and G_N is twice differentiable (cf. Definition 3.3) on $\mathcal{O}$. Suppose

$$\nabla_w G_N[m] \in C((\mathbb{R}^d);\mathbb{R}^d), \quad \nabla\big(\nabla_w G_N\big)[m]) \in C((\mathbb{R}^d);\mathbb{R}^{d\times d}), \quad \nabla^2_w G_N[m]) \in C\big((\mathbb{R}^d)^2;\mathbb{R}^{d\times d}\big)$$

for any $m \in \mathcal{O}$ and $|\nabla_w G_N[m](x)| \le C_\mu(1+|x|)$ for any $(N,x,m) \in \mathbb{N}\times(\mathbb{R}^d)\times\mathcal{O}$. Suppose $(x,m) \mapsto \nabla\big(\nabla_w G_N\big)[m](x)$ is continuous bounded on $(\mathbb{R}^d)\times\mathcal{O}$ and $(x,y,m) \mapsto \nabla^2_w G_N[m](x,y)$ is continuous and bounded on $(\mathbb{R}^d)^2\times\mathcal{O}$ independently on N. Suppose

$$\sup_{\gamma\in\Gamma_0(m,\nu)}\Big|G_N[\nu] - G_N[m] - \int_{(\mathbb{R}^d)^2}\nabla_w G_N[m](x)\cdot(y-x)\gamma(dx,dx)\Big| \le C_m W_2^2(m,\nu), \tag{3.28}$$

for any $m,\nu \in \mathcal{O}$, and

$$\sup_{\gamma\in\Gamma_0(m,\nu)}\Big|\nabla_w G_N[\nu](y) - \nabla_w G_N[m](x) - P^N_\gamma[m](x,y)\Big| \le C_m\big(|x-y|^2 + W_2^2(m,\nu)\big) \tag{3.29}$$

Here, for $\gamma \in \mathcal{P}((\mathbb{R}^d)^2)$ and $x, y \in (\mathbb{R}^d)$, we have set

$$P^N_\gamma[m](x,y) := \nabla\big(\nabla_w G_N\big)[m](x)(y-x) + \int_{(\mathbb{R}^d)^2} \nabla^2_w G_N[m](x,a)(b-a)\gamma(da,db),$$

Theorem 3.9. *Suppose* (3.28) *and* (3.29) *hold and*

(a) $\big(\nabla_w G_N\big)_N$ *converges uniformly on* $(\mathbb{R}^d) \times \mathcal{O}$ *to* $(x,m) \mapsto A_m(x)$,
(b) $\big(\nabla\big(\nabla_w G_N\big)\big)_N$ *converges uniformly on* $(\mathbb{R}^d) \times \mathcal{O}$ *to* $(x,m) \mapsto \tilde{A}[m](x)$
(c) $\big(\nabla^2_w G_N\big)_N$ *converges uniformly on* $(\mathbb{R}^d)^2 \times \mathcal{O}$ *to* $(x,y,m) \mapsto A_{mm}(x,y)$.

Then

(i) *G is differentiable on $\mathcal{O}$ and $A = \nabla_w G$.*
(ii) *G is twice continuously differentiable on $\mathcal{O}$, $\tilde{A} = \nabla\big(\nabla_w G\big)$ and $A_{mm} \equiv \nabla^2_w G[m]$.*

Proof. Note that $(x,m) \mapsto A_m(x)$, $\tilde{A}[m](x)$ and $(x,y,m) \mapsto A_{mm}(x,y)$ are continuous and the latter two functions are bounded as limits of bounded functions. Let $m, \nu \in \mathcal{O}$.

(i) For any $\gamma \in \Gamma_0(m,\nu)$ we have

$$\Big|\int_{(\mathbb{R}^d)^2} \Big(\nabla_w G_N[m](x) - A_m(x)\Big)\cdot(y-x)\gamma(dx,dx)\Big| \le \Big\|\nabla_w G_N[m] - A_m\Big\|_{L^\infty} W_2(m,\nu)$$

and so, letting N tend to ∞ in (3.28), we obtain

$$\Big|G[\nu] - G[m] - \int_{(\mathbb{R}^d)^2} A_m(x)\cdot(y-x)\gamma(dx,dx)\Big| \le C_m W_2^2(m,\nu).$$

Since $\gamma \in \Gamma_0(m,\nu)$ is arbitrary, we conclude $A_m \in \partial G[m]$. Observe that since $|\nabla_w G[m](x)| \le C_m(1+|x|)$ for any $(x,m) \in (\mathbb{R}^d) \times \mathcal{O}$, and as $\nabla_w G_N[m] \in T_m\mathcal{P}_2(\mathbb{R}^d)$, we conclude that $A_m \in T_m\mathcal{P}_2(\mathbb{R}^d)$ and so, $A_m = \nabla_w G[m]$.

(ii) Since $\big(\nabla\big(\nabla_w G_N\big)\big)_N$ converges uniformly to $\tilde{A}$, we have $\tilde{A} = \nabla\big(\nabla_w G\big)$. Observe if $\gamma \in \Gamma_0(m,\nu)$ and $x_1, y_1 \in (\mathbb{R}^d)$ then

$$\Big|\int_{(\mathbb{R}^d)^2} \Big(\big(\nabla^2_w G_N[m] - A_{mm}\big)(x_1,x_2)(y_2-x_2)\Big)\gamma(dx_2,dy_2)\Big| \;\le\; \Big\|\nabla_w G_N[m] - A_{mm}\Big\|_{L^\infty} W_2(m,\nu).$$

As above, we conclude that

$$\Big|\nabla_w G[\nu](y) - \nabla_w G[m](x) - P_\gamma[m](x,y)\Big| \le C_m\big(|x-y|^2 + W_2^2(m,\nu)\big)$$

where

$$P_\gamma[m](x,y) := \tilde{A}[m](x)(y-x) + \int_{(\mathbb{R}^d)^2} A_{mm}(x,a)(b-a)\gamma(da,db).$$

Since $\gamma \in \Gamma_0(m,\nu)$ is arbitrary, we apply Theorem 3.2 to obtain that G is twice differentiable at m, $\tilde{A}[m] = \nabla\big(\nabla_w G\big)[m]$ and $A_{mm} = \nabla^2_w G[m]$. The identity $\pi_m\big(\nabla^2_w G_N[m]\big) = \nabla^2_w G_N[m]$ implies $A_{mm} = \pi_m\big(A_{mm}\big)$. We conclude the proof of (ii) by setting $\rho(t) = C_m t$ and $\epsilon_2(t) = C_m t$. □

4. Fourier transform and expansions

4.1. **Polynomial eigenfunctions of the Laplacian operator; Harmonic functions.** Let $\Phi^k_\xi \in \mathrm{Sym}_2[\mathbb{C}^k]$ be defined by

$$\Phi^k_\xi(x) := \frac{1}{k!}\sum_{\sigma\in P_k} \exp\Big(-2\pi i \sum_{j=1}^k \langle \xi_{\sigma(j)}, x_j\rangle\Big), \quad \forall\, \xi = (\xi_1, \cdots, \xi_k) \in (\mathbb{R}^d)^k.$$

The function Φ^k_ξ is obtain as the symmetrization of $x \mapsto \exp\Big(-2\pi i \sum_{j=1}^k \langle \xi_j, x_j\rangle\Big)$.

Set

$$\lambda_k^2(\xi) := 4\pi^2 \Big|\sum_{j=1}^k \xi_j\Big|^2 \quad \text{and} \quad \widehat{m}(\xi_j) := \int_{(\mathbb{R}^d)} \exp\big(-2\pi \langle \xi_j, z\rangle\big) m(dz). \tag{4.1}$$

Note $\widehat{m}$ is the Fourier transform of m.

Lemma 4.1. *The following hold for any* $m \in \mathcal{P}_2(\mathbb{R}^d)$:

- (i) *We have* $F^k_\xi[m] = k^{-1} \prod_{j=1}^k \widehat{m}(\xi_j)$.
- (ii) *We have* $\triangle_w F^k_\xi[m] = -\lambda_k^2(\xi) F^k_\xi[m]$.
- (iii) *We have* $\triangle_{w,\epsilon} F^k_\xi[m] = -\Big(\lambda_k^2(\xi) + 4\pi^2 \epsilon \sum_{j=1}^n |\xi_j|^2\Big) F^k_\xi[m]$.

Proof. We skip the proof of this lemma as it can be readily obtained.

Remark 4.2. *The followings hold.*

- (i) *If* $\lambda_k(\xi) = 0$ *then* F^k_ξ *belongs to the kernel of* $\triangle_w$. *We say that* F^k_ξ *is a harmonic function.*
- (ii) *(***non smooth harmonic functions***) Let* $f \in \mathrm{Sym}_2[(\mathbb{R}^d)]$ *be an even function and set* $\Phi(x, y) = f(x-y)$. *Note that* $\mathrm{div}_x(\nabla_x \Phi) = \mathrm{Tr}(\nabla_{yx}\Phi)$. *Using Corollary 3.8, we conclude that* $\triangle_w F_\Phi[m] \equiv 0$ *and so,* U *is a harmonic function. Starting with* $f \notin C^3((\mathbb{R}^d))$, *we obtain that* $\Phi \notin C^3((\mathbb{R}^d))$ *and so,* F_Φ *is a harmonic function which is not regular up to the third order. However, if* $\epsilon > 0$, $\triangle_{w,\epsilon} F_\Phi \not\equiv 0$ *and* $\triangle_{w,\epsilon}$ *has a smoothing effect.*
- (iii) *A direct consequence of (ii) is that* $\triangle_w$ *is not a smoothing operator (except on* $H^s\big(\mathcal{P}_2(\mathbb{R}^d)\big)$*: cf. Theorem 6.3).*

4.2. H^s**–spaces and spaces of Fourier transforms.** Throughout this subsection, $s \geq 0$ is a real number.

Definition 4.3. *Let* λ_k *be the function defined in* (4.1).

- (i) *We call* $\mathcal{A}$ *the set of sequences of functions* $(a_k)_{k=1}^\infty$ *such that* $a_k : (\mathbb{R}^d)^k \to \mathbb{C}$ *is a Borel function that is symmetric in the sense that* $a_k(\xi) = a_k(\xi^\sigma)$ *for any* $\xi \in (\mathbb{R}^d)^k$ *and any* $\sigma \in P_k$. *In other words,* a_k *is defined on* $(\mathbb{R}^d)^k/P_k$, *the* k*–symmetric product of* $(\mathbb{R}^d)$.
- (ii) *We call* $(a_k)_{k=1}^\infty \in \mathcal{A}$ *the Fourier transform of* $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, *if there exist* $\Phi_k \in \mathcal{S}ym[k](\mathbb{R}) \cap L^2((\mathbb{R}^d)^k)$ *is such that the series*

$$\sum_{k=1}^\infty \frac{1}{k!\; k} \int_{(\mathbb{R}^d)^k} \Phi_k(x) m(dx_1) \cdots m(dx_k)$$

converges to F *and* $\widehat{a}_k = \Phi_k$.

Definition 4.4. *We have the following definition.*

(i) *We call* $\mathcal{A}^s$ *the set of sequences* $A := \big(a_k\big)_{k=1}^{\infty} \subset \mathcal{A}$ *such that*

$$||A||^2_{H^s} := \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} |a_k(\xi)|^2 \big(1 + \lambda_k^2(\xi)\big)^s d\xi < \infty. \tag{4.2}$$

(ii) *If* $B = \big(b_k\big)_{k=1}^{\infty} \in \mathcal{A}^s$*, then the following is a well–defined sesquilinear form (cf. Lemma 4.5):*

$$\langle A; B\rangle_{H^s} := \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) b_k^*(\xi) \big(1 + \lambda_k^2(\xi)\big)^s d\xi$$

Lemma 4.5. *The sesquilinear form* $\langle \cdot; \cdot\rangle_{H^s} : \mathcal{A}^s \times \mathcal{A}^s \to \mathbb{C}$ *is well defined.*

Proof. Let $A, B \in \mathcal{A}^s$ be as in Definition 4.4. Then for any $\lambda > 0$ we have

$$\frac{1}{k!}\Big|\int_{(\mathbb{R}^d)^k} a_k(\xi) b_k^*(\xi) \big(1 + \lambda_k^2(\xi)\big)^s d\xi\Big| \leq \frac{1}{2k!} \int_{(\mathbb{R}^d)^k} \bigg(\frac{|a_k(\xi)|^2}{\lambda^2} + \lambda^2 |b_k(\xi)|^2\bigg) \big(1 + \lambda_k^2(\xi)\big)^s d\xi. \tag{4.3}$$

Therefore, the series produced by the left hand side of (4.3) converges absolutely, which concludes the proof. □

Lemma 4.6. *Suppose* $A := \big(a_k\big)_{k=1}^{\infty}$ *and* $B = \big(b_k\big)_{k=1}^{\infty}$ *belong to* $\mathcal{A}^s$*. Then*

$$|\langle A; B\rangle_{H^s}| \leq ||A||_{H^s} \cdot ||B||_{H^s}, \quad ||A + B||_{H^s} \leq ||A||_{H^s} + ||B||_{H^s}.$$

Proof. Assume without loss of generality that $||B||_{H^0} \neq 0$. By (4.3)

$$2|\langle A; B\rangle_{H^s}| \leq \frac{||A||^2_{H^s}}{\lambda^2} + \lambda^2 ||B||^2_{H^s}.$$

We use $\lambda := ||A||^{\frac{1}{2}}_{H^s} ||B||^{-\frac{1}{2}}_{H^s}$ to obtain the first desired identity. The second desired inequality is a consequence of the identity

$$||A + B||^2_{H^s} = ||A||^2_{H^s} + ||B||^2_{H^s} + \langle A; B\rangle_{H^s} + \langle G; F\rangle_{H^s}.$$

□

Lemma 4.7. *Let* $\Phi \in \mathrm{Sym}[(\mathrm{R}^\mathrm{d})^\mathrm{k}] \cap L^2((\mathbb{R}^d)^k)$ *and let* $\big(a_k\big)_{k=1}^{\infty}, \big(b_k\big)_{k=1}^{\infty} \in \mathcal{A}$ *be such that*

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \Big(|a_k(\xi)|^2 + |b_k(\xi)|^2\Big) d\xi < \infty.$$

(i) *Since the Fourier transform is an isometry of* $L^2((\mathbb{R}^d)^k; \mathbb{C})$ *onto* $L^2((\mathbb{R}^d)^k; \mathbb{C})$*, we obtain* $a := \check{\Phi} \in L^2((\mathbb{R}^d)^k; \mathbb{C})$*. One checks that* a *is symmetric and so, if we further assume that* $\Phi \in L^1((\mathbb{R}^d)^k)$ *then* $a \in \mathcal{S}ym[k](\mathbb{C}) \cap L^\infty((\mathbb{R}^d)^k; \mathbb{C})$*.*

(ii) *Observe for any* $k \geq 1$*,* $a_k, b_k \in L^2((\mathbb{R}^d)^k; \mathbb{C})$*. Let* $\Phi_k, \Psi_k \in \mathcal{S}ym[k](\mathbb{C}) \cap L^2((\mathbb{R}^d)^k; \mathbb{C})$ *be such that* $a_k = \check{\Phi}_k$ *and* $b_k = \check{\Psi}_k$*. For any* $N \geq 1$*,*

$$\sum_{k=1}^{N} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) b_k^*(\xi) d\xi = \sum_{k=1}^{N} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \Phi_k(x) \Psi_k^*(x) dx.$$

(iv) *Further assume for any integer $k \geq 1$, $a_k \in L^1(({\mathbb{R}^d})^k;\mathbb{C})$. Then for any $N \geq 1$*

$$\sum_{k=1}^N \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi)F^k_\xi[m]d\xi = \sum_{k=1}^N \frac{1}{k!\ \ k}\int_{(\mathbb{R}^d)^k} \Phi_k(x)m(dx_1)\cdots m(dx_k)$$

The series converge uniformly on $\mathcal{P}_2(\mathbb{R}^d)$ if there exist constant $C, \delta >$ independent of m and k such that

$$\int_{(\mathbb{R}^d)^k} |a_k(\xi)|d\xi \leq \frac{Ck!}{k^\delta} \tag{4.4}$$

Proof. (i) is straightforward to check. (ii) is a consequence of Plancherel's theorem and the fact that the Fourier transform is an isometry of $L^2(({\mathbb{R}^d})^k;\mathbb{C})$.

(iii) Since $\Phi_k = \widehat{a}_k$, when $a_k \in L^1(({\mathbb{R}^d})^k;\mathbb{C})$, we may use Fubini's theorem to obtain

$$\begin{aligned}\int_{(\mathbb{R}^d)^k} \Phi_k(x)m(dx_1)\cdots m(dx_k) &= \int_{(\mathbb{R}^d)^k} m(dx_1)\cdots m(dx_k)\int_{(\mathbb{R}^d)^k} a_k(\xi_1,\cdots,\xi_k)\exp\Big(-2\pi i\sum_{j=1}^k\langle \xi_j, x_j\rangle\Big)d\xi\\ &= \int_{(\mathbb{R}^d)^k} a_k(\xi_1,\cdots,\xi_k)d\xi\int_{(\mathbb{R}^d)^k}\exp\Big(-2\pi i\sum_{j=1}^k\langle \xi_j, x_j\rangle\Big)m(dx_1)\cdots m(dx_k)\\ &= k\int_{(\mathbb{R}^d)^k} a_k(\xi_1,\cdots,\xi_k)F_\xi[m]d\xi. \end{aligned} \tag{4.5}$$

By the fact that $|F^k_\xi[m]| \leq k^{-1}$, we have

$$\Big|\sum_{k=1}^N \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi)F^k_\xi[m]d\xi\Big| \leq \sum_{k=1}^N \frac{1}{k!\ k}\int_{(\mathbb{R}^d)^k}|a_k(\xi)|d\xi \leq \sum_{k=1}^N \frac{C}{k^{1+\delta}}.$$

This concludes the proof. □

Definition 4.8. *We have the following definition.*

(i) *We call $H^s(\mathcal{P}_2(\mathbb{R}^d))$ the set of $F : \mathcal{P}_2(\mathbb{R}^d) \to [-\infty,\infty]$ for which there exist $(a_k)_{k=1}^\infty \subset \mathcal{A}^s$ such that*

$$\sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi)F^k_\xi[m]d\xi$$

converges to $F[m]$ for any $m \in \mathcal{P}_2(\mathbb{R}^d)$.

(ii) *We define $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ the set of $F : \mathcal{P}_2(\mathbb{R}^d) \to [-\infty,\infty]$ for which there exist $(a_k)_{k=1}^\infty \subset \mathcal{A}^s$, $\delta, C > 0$ such that (4.4) holds for all k natural number and*

$$\sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi)F^k_\xi[m]d\xi \tag{4.6}$$

converges to $F[m]$ for any $m \in \mathcal{P}_2(\mathbb{R}^d)$. Thanks to Remark 5.2, the following definition is meaningful.

From definition, we have

$$\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d)) \subset H^s(\mathcal{P}_2(\mathbb{R}^d)) \cap C(\mathcal{P}_2(\mathbb{R}^d))$$

and the second inclusion results from the fact that the convergence of the series converges uniformly on $m \in \mathcal{P}_2(\mathbb{R}^d)$ (cf. Lemma 4.7 (iv)).

4.3. **Integrations by parts; Hessians in terms of Fourier transforms.** Throughout this subsection $s, \delta > 0$, $\epsilon \geq 0$ and $\big(a_k\big)_{k=1}^\infty \in \mathcal{A}^s$. When needed we shall make various assumptions such as

$$\int_{(\mathbb{R}^d)^k} |a_k(\xi)| \cdot |\xi_1| d\xi, \int_{(\mathbb{R}^d)^k} |a_k(\xi)| \cdot |\xi_1|^2 d\xi \;\leq \frac{Ck!}{k^{1+\delta}}, \tag{4.7}$$

$$\int_{(\mathbb{R}^d)^k} |a_k(\xi)| \cdot |\xi_1| \cdot |\xi_2| d\xi \quad \leq \frac{Ck!}{k^{2+\delta}}. \tag{4.8}$$

or

$$\int_{(\mathbb{R}^d)^k} |a_k(\xi)| \cdot |\xi_1|^3 d\xi \leq \frac{Ck!}{k^{1+\delta}}, \quad \int_{(\mathbb{R}^d)^k} |a_k(\xi)| \cdot |\xi_1|^2 \cdot |\xi_2| d\xi \quad \leq \frac{Ck!}{k^{3+\delta}}. \tag{4.9}$$

When (4.4) is in force then the series

$$U_0[m] := \sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi, \qquad m \in \mathcal{P}_2(\mathbb{R}^d) \tag{4.10}$$

converges uniformly (cf. Lemma 4.7 (iv)).

Corollary 4.9. *Assume* (4.4), (4.7) *and* (4.8) *hold.*

(i) *Then U_0 is continuously differentiable on $\mathcal{P}_2(\mathbb{R}^d)$, and using the notation $\langle \xi, x\rangle$ in place of $\sum_{j=1}^k \langle \xi_j, x_j\rangle$, we have*

$$\nabla_w U_0[m](x_1) \equiv \sum_{k=1}^\infty \frac{-2\pi i}{k!} \int_{(\mathbb{R}^d)^{k-1} \times (\mathbb{R}^d)^k} a_k(\xi) \xi_1 e^{-2\pi i \langle \xi, x\rangle} m(dx_2) \cdots m(dx_k) d\xi \tag{4.11}$$

(ii) *If we further assume* (4.9) *holds, then U_0 is twice continuously differentiable on $\mathcal{P}_2(\mathbb{R}^d)$. We have*

$$\nabla\big(\nabla_w U_0[m](x_1)\big) \equiv \sum_{k=1}^\infty \frac{-4\pi^2}{k!} \int_{(\mathbb{R}^d)^{k-1} \times (\mathbb{R}^d)^k} a_k(\xi) \xi_1 \otimes \xi_1 e^{-2\pi i \langle \xi, x\rangle} m(dx_2) \cdots m(dx_k) d\xi \tag{4.12}$$

and

$$\nabla_w^2 U_0[m](x_1, x_2)\big) \equiv \sum_{k=1}^\infty \frac{-4\pi^2 (k-1)}{k!} \int_{(\mathbb{R}^d)^{k-2} \times (\mathbb{R}^d)^k} a_k(\xi)\, \xi_1 \otimes \xi_2 e^{-2\pi i \langle \xi, x\rangle} m(dx_3) \cdots m(dx_k) d\xi. \tag{4.13}$$

(iii) *Under the same assumptions as in (ii),*

$$\triangle_{w,\epsilon} U_0[m] = \sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) \triangle_{w,\epsilon}(F_\xi^k[m]) d\xi.$$

Proof. (i) Let $m \in \mathcal{P}_2(\mathbb{R}^d)$ and set

$$G_N[m] = \sum_{k=1}^N \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi = \sum_{k=1}^N \frac{1}{k!} \frac{1}{k} \int_{(\mathbb{R}^d)^k} \Phi_k(x) m(dx_1) \cdots m(dx_k),$$

where $\Phi_k = \widehat{a}_k$. Observe

$$\nabla_{x_1}\Phi_k(x) = -2\pi i\int_{(\mathbb{R}^d)^k} a_k(\xi)\xi_1 \exp\Big(-2\pi i\sum_{j=1}^k\langle \xi_j, x_j\rangle\Big)d\xi,$$

Thus, by Lemma (3.6) and the linearity of the Wasserstein gradient,

$$\nabla_w G_N[m](x_1) = -2\pi i\sum_{k=1}^N g_k^0[m](x_1)$$

where

$$g_k^0[m](x_1) := \frac{1}{k!}\int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^k} a_k(\xi)\xi_1\exp\big(-2\pi i\sum_{j=1}^k\langle\xi_j, x_j\rangle\big)m(dx_2)\cdots m(dx_k)d\xi.$$

If $\nu\in\mathcal{P}_2(\mathbb{R}^d)$ and $\gamma\in\Gamma_0(m,\nu)$, the first order expansion of $t\to e^{-2\pi t}$ yields the first order Taylor expansion of F_{Φ_k} around m, given by

$$\Big|F_{\Phi_k}[\nu] - F_{\Phi_k}[m] - \int_{(\mathbb{R}^d)^2}\langle\nabla_w F_{\Phi_k}[m](x_1); y_1 - x_1\rangle\gamma(dx_1,dy_1)\Big| \le C[k]W_2^2(m,\nu).$$

Here

$$C[k] := \frac{2\pi^2}{k}\sum_{j,l=1}^k\int_{(\mathbb{R}^d)^k}|\xi_j||\xi_l||a_k(\xi)|d\xi$$

Since a_k is symmetric,

$$\sum_{j,l=1}^k\int_{(\mathbb{R}^d)^k}|\xi_j||\xi_l||a_k(\xi)|d\xi = k\int_{(\mathbb{R}^d)^k}|\xi_1|^2|a_k(\xi)|d\xi + k(k-1)\int_{(\mathbb{R}^d)^k}|\xi_1||\xi_2||a_k(\xi)|d\xi. \tag{4.14}$$

We combine (4.4), (4.7) and (4.8) and use that $k-1\le k$ to obtain $k^{1+\delta}C[k]\le 4\pi^2 k!\, C$. Thus, by the above first order Taylor expansion of F_{Φ_k} around m we obtain

$$\Big|G_N[\nu] - G_N[m] - \int_{(\mathbb{R}^d)^2}\langle\nabla_w G_N[m](x_1); y_1 - x_1\rangle\gamma(dx_1,dy_1)\Big| \le C_m W_2^2(m,\nu) \tag{4.15}$$

where

$$C_m := \sum_{k=1}^N\frac{1}{k!}C[k] < \infty.$$

But $(x_1, m)\mapsto g_k^0[m](x_1)$ are continuous functions such that

$$\Big|g_k^0[m](x_1)\Big| \le \frac{1}{k!}\int_{(\mathbb{R}^d)^k}|a_k(\xi)\xi_1|d\xi.$$

Thanks to (4.4) and (4.7) we obtain that the series $\big(-2\pi i\sum_{k=1}^N g_k^0[m](x_1)\big)_N$ is a Cauchy sequence for the uniform convergence and so, it converges uniformly to a continuous function given by the function at the right hand side of (4.11), which we denote as A. We let N tend to ∞ in (4.15) to conclude $A\equiv\nabla_w U_0$ and conclude the proof of (i).

For any $n\in\{1,\cdots,k\}$ we have

$$\nabla_{x_n x_1}\Phi_k(x) = -4\pi^2\int_{(\mathbb{R}^d)^k} a_k(\xi)\xi_1\otimes\xi_n\exp\Big(-2\pi i\sum_{j=1}^k\langle\xi_j, x_j\rangle\Big)d\xi, \tag{4.16}$$

Hence by Remark 3.7

$$\nabla\big(\nabla_w G_N[m](x_1)\big) = -4\pi^2 \sum_{k=1}^{N} g_k^1[m](x_1) \tag{4.17}$$

where

$$g_k^1[m](x_1) := \frac{1}{k!} \int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^k} a_k(\xi)\xi_1 \otimes \xi_1 \exp\Big(-2\pi i \sum_{j=1}^{k} \langle \xi_j, x_j\rangle\Big) m(dx_2)\cdots m(dx_k) d\xi$$

Note,

$$\Big| g_k^1[m](x_1)\Big| \le \frac{1}{k!} \int_{(\mathbb{R}^d)^k} |a_k(\xi)|\, |\xi_1|^2 d\xi.$$

Thanks to (4.4) and (4.7) again, we obtain that the series $\big(-4\pi^2 \sum_{k=1}^{N} g_k^1[m](x_1)\big)_N$ is a Cauchy sequence for the uniform convergence and so, it converges uniformly to the continuous function $\tilde{A}$ at the right hand side of (4.12). We will soon see that it is legitimate to denote this limit as $\big(\nabla_w U_0[m](x_1)\big)$.

By Lemma 3.6 and the linearity of ∇_w^2,

$$\nabla_w^2 G_N[m](x_1, x_2)\big) = -4\pi^2 \sum_{k=2}^{N} g_k^2[m](x_1, x_2) \tag{4.18}$$

where

$$g_k^2[m](x_1, x_2) := \frac{k-1}{k!} \int_{(\mathbb{R}^d)^{k-2}\times(\mathbb{R}^d)^k} a_k(\xi)\, \xi_1 \otimes \xi_2 \exp\Big(-2\pi i \sum_{j=1}^{k} \langle \xi_j, x_j\rangle\Big) m(dx_3)\cdots m(dx_k) d\xi.$$

The first order expansion of $t \to e^{-2\pi t}$ yields the first order Taylor expansion of $A_k := \nabla_w F_{\Phi_k}$ around (x_1, m) given by

$$A_k[\nu](y_1) = A_k[m](x_1) - 4\pi^2 k! g_k^1[m](x_1)(y_1 - x_1) - 4\pi^2 k! \int_{(\mathbb{R}^d)^2} g_k^2[m]^1(x_1, x_2)(y_2 - x_2)\gamma(dx_2, dy_2) + B_k.$$

Here, the remainder B_k is such that

$$|B_k| \le 4\pi^2 \sum_{j,l=1}^{k} \int_{(\mathbb{R}^d)^{2(k-1)}\times(\mathbb{R}^d)^k} |a_k(\xi)||\xi_l||\xi_j||y_l - x_l||y_j - x_j|\gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) d\xi.$$

Using the fact that a_k is symmetric, we argue as in (4.14) to express the upper bound on B_k in terms of integrals involving just the variables (ξ_1, ξ_2, ξ_3). We obtain

$$\begin{aligned}
|B_k| \le\ & 4\pi^2 |x_1 - y_1|^2 \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^3 d\xi \\
& + 4\pi^2 (k-1) W_2^2(m, \nu)\Big(\int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1||\xi_2|^2 d\xi + (k-2)\int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1||\xi_2||\xi_3| d\xi\Big) \\
& + 8\pi^2 (k-1)|x_1 - y_1| W_2(m, \nu) \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^2|\xi_2| d\xi
\end{aligned}$$

We use that $2|\xi_2||\xi_3| \leq |\xi_2|^2 + |\xi_3|^2$ and use again the fact that a_k is symmetric and argue as in (4.14) to eliminate the variables ξ_3 from the previous integral. We obtain

$$|B_k| \leq 4\pi^3 |x_1 - y_1|^2 \Big(\int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^3 d\xi + (k-1) \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^2 |\xi_2| d\xi \Big)$$
$$+ 4\pi^2 (k-1) k W_2^2(m,\nu) \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^2 |\xi_2| d\xi$$

We exploit the first order Taylor expansion of $\nabla_w F_{\Phi_k}$ around (x_1, m), use (4.17) and (4.18) to conclude that by linearity that

$$\nabla_w G_N[\nu](y_1) - \nabla_w G_N[m](x_1) = \nabla \big(\nabla_w G_N[m](x_1)\big)(y_1 - x_1)$$
$$+ \int_{(\mathbb{R}^d)^2} \nabla_w^2 G_N[m](x_1, x_2)\big)(y_2 - x_2)\gamma(dx_2, dy_2) + R_N \tag{4.19}$$

where the remainder R_N satisfies

$$|R_N| \leq 4\pi^2 |x_1 - y_1|^2 \sum_{k=1}^N \Big(\frac{1}{k!} \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^3 d\xi + \frac{k-1}{k!} \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^2 |\xi_2| d\xi \Big)$$
$$+ 4\pi^2 \sum_{k=1}^N \frac{(k-1)k}{k!} W_2^2(m,\nu) \int_{(\mathbb{R}^d)^k} |a_k(\xi)||\xi_1|^2 |\xi_2| d\xi$$

We combine (4.4), (4.7), (4.8) and (4.9) to obtain that a universal constant $\bar{C}$ such that

$$|R_N| \leq \bar{C}_m := \sum_{k=1}^\infty \frac{\bar{C}}{k^{1+\delta}}.$$

If necessary, we replace C_m by $\max\{C_m, \bar{C}_m\}$. We use (4.19) to obtain

$$\Big| \nabla_w G_N[\nu](y_1) - \nabla_w G_N[m](x_1) - P_\gamma^N[m](x_1, y_1) \Big| \leq C_m \Big(|x_1 - y_1|^2 + W_2^2(m,\nu) \Big) \tag{4.20}$$

Here,

$$P_\gamma^N[m](x_1, y_1) := \nabla_{x_1} \big(\nabla_w G_N[m](x_1)\big)(y_1 - x_1) + \int_{(\mathbb{R}^d)^2} \nabla_w^2 G_N[m](x_1, x_2)(y_2 - x_2)\gamma(dx_2, dy_2),$$

We have

$$\Big| g_k^2[m](x_1, x_2) \Big| \leq \frac{k-1}{k!} \int_{(\mathbb{R}^d)^k} |a_k(\xi)| \; |\xi_1|^2 d\xi.$$

Once again, thanks to (4.4) and (4.7) we obtain that the series $\big(-4\pi^2 \sum_{k=1}^N g_k^2[m](x_1, x_2)\big)_N$ is a Cauchy sequence for the uniform convergence and so, it converges uniformly to the continuous function $\bar{A}$ at the right hand side of (4.13). We let N tend to ∞ in (4.20) to conclude that

$$\Big| \nabla_w U_0[\nu](y_1) - \nabla_w U_0[m](x_1) - P_\gamma[m](x_1, y_1) \Big| \leq C_m \Big(|x_1 - y_1|^2 + W_2^2(m,\nu) \Big) \tag{4.21}$$

Here,

$$P_\gamma[m](x_1, y_1) := \tilde{A}[m](x_1)(y_1 - x_1) + \int_{(\mathbb{R}^d)^2} \bar{A}[m](x_1, x_2)(y_2 - x_2)\gamma(dx_2, dy_2),$$

We use Theorem 3.9 to obtain that

$$\tilde{A}[m](x_1) = \nabla \big(\nabla_w U_0[m](x_1)\big) \quad \bar{A}[m](x_1, x_2) = \nabla_w^2 U_0[m](x_1, x_2)\big)$$

and conclude the proof of (ii).

(iii) By Corollary 3.8 and the above uniform convergences, we have

$$\triangle_{w,\epsilon} U_0(m) \equiv \sum_{k=1}^{\infty} \triangle_{w,\epsilon}\Big(\frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi \Big) = \sum_{k=1}^{\infty} \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi) \triangle_{w,\epsilon}\big(F_\xi^k[m]\big) d\xi.$$

This concludes the proof. □

In the next proposition, we assume that we are given $(b_k)_{k=1}^\infty \subset \mathcal{A}^0$ be such that (4.4), (4.7), (4.8) , (4.9). We assume that

$$\int_{(\mathbb{R}^d)^k} \big(|a_k(\xi)| + |b_k(\xi)|\big)\lambda_k^2(\xi) d\xi \leq \frac{Ck!}{k^\delta} \tag{4.22}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k!}\int_{(\mathbb{R}^d)^k} |a_k(\xi)|^2 \lambda_k^4(\xi) d\xi < \infty. \tag{4.23}$$

Assume $(b_k)_{k=1}^\infty \subset \mathcal{A}^0$ is such that the analogous of (4.4), (4.7), (4.8) , (4.9). Define

$$F := \sum_{k=1}^{\infty} \frac{1}{k!}\int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi, \qquad G := \sum_{k=1}^{\infty} \frac{1}{k!}\int_{(\mathbb{R}^d)^k} b_k(\xi) F_\xi^k[m] d\xi.$$

Proposition 4.10. *Assume $s = 0$, and $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ (4.4), (4.7), (4.8) , (4.9), and $(a_k)_{k=1}^\infty$ further satisfies (4.13) hold. If (4.22) and (4.23) hold then*

(i) *The following functions belong to the d–cartesian product of $\mathcal{H}^0(\mathcal{P}_2(\mathbb{R}^d))$ by itself:*

$$m \to \int_{(\mathbb{R}^d)} \nabla_w F[m](x_1) m(dx_1), \quad \int_{(\mathbb{R}^d)} \nabla_w G[m](x_1) m(dx_1)$$

(ii) *We have $\triangle_w F \in \mathcal{H}^0(\mathcal{P}_2(\mathbb{R}^d))$.*

(iii) *We have the integration by parts formula:*

$$-\langle \triangle_w F; G\rangle_{H^0} = \Big\langle \int_{(\mathbb{R}^d)} \nabla_w F[m](x) m(dx); \int_{(\mathbb{R}^d)} \nabla_w G[m](x) m(dx) \Big\rangle_{H^0}$$

(iv) *In particular,*

$$-\langle \triangle_w F; F\rangle_{H^0} = \Big\| \int_{(\mathbb{R}^d)} \nabla_w F[m](x) m(dx) \Big\|_{H^0}^2$$

Proof. Corollary 4.9 ensures that F is twice continuously differentiable, G is continuously differentiable. Setting

$$f_k(\xi) := \sum_{j=1}^{k} \xi_j a_k, \quad g_k(\xi) := \sum_{j=1}^{k} \xi_j b_k,$$

we use Lemma 4.1 and Corollary 4.9 to obtain the explicit expressions

$$\int_{(\mathbb{R}^d)} \nabla_w F[m](x_1) m(dx_1) = \sum_{k=1}^{\infty} \frac{-2\pi i}{k!}\int_{(\mathbb{R}^d)^k} f_k(\xi) F_\xi^k[m] d\xi, \tag{4.24}$$

$$\int_{(\mathbb{R}^d)} \nabla_w G[m](x_1) m(dx_1) = \sum_{k=1}^{\infty} \frac{-2\pi i}{k!} \int_{(\mathbb{R}^d)^k} g_k(\xi) F_\xi^k[m] d\xi \tag{4.25}$$

and

$$\triangle_w F[m](x_1) = -\sum_{k=1}^{\infty} \int_{(\mathbb{R}^d)^k} a_k(\xi) \lambda_k^2(\xi) F_\xi^k[m] d\xi \tag{4.26}$$

Combining (4.4) and (4.22) we have

$$\int_{(\mathbb{R}^d)^k} |f_k(\xi)| d\xi \le \int_{(\mathbb{R}^d)^k} |a_k(\xi)| d\xi + \int_{(\mathbb{R}^d)^k} |a_k(\xi)| \lambda_k^2(\xi) d\xi \le \frac{2Ck!}{k^\delta} \tag{4.27}$$

Since $(a_k)_{k=1}^\infty \subset \mathcal{A}^0$

$$\sum_{k=1}^{\infty} \int_{(\mathbb{R}^d)^k} \frac{|f_k(\xi)|^2}{k!} d\xi = \int_{(\mathbb{R}^d)^k} \sum_{k=1}^{\infty} \frac{|a_k(\xi)|^2}{k!} \lambda_k^2(\xi) d\xi < \infty. \tag{4.28}$$

We combine (4.27) and (4.28) to conclude the the proof of (i) for F. Similarly, we conclude the proof of (i) for G.

(ii) is obtained as a consequence of (4.22) and (4.23).

(iii) Since $\triangle_w F \in \mathcal{H}^0(\mathcal{P}_2(\mathbb{R}^d)), G$ we use their expressions to obtain that have

$$\langle \triangle_w F; G \rangle_{H^0} = -\sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) (b_k(\xi))^* \lambda_k^2(\xi) d\xi \tag{4.29}$$

We use the expressions in (4.24) and (4.25) to conclude thet

$$\langle \nabla_w F; \nabla_w G \rangle_{H^0} = \sum_{k=1}^{\infty} \frac{4\pi^2}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) (b_k(\xi))^* \left| \sum_{j=1}^{k} \xi_j \right|^2 d\xi$$

This, together with (4.29) concludes the proof of (iii). □

5. Recovery of k–polynomial of $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ functions

The studies in this section are preliminary useful to express the scalar product on $\mathcal{H}^0(\mathcal{P}_2(\mathbb{M}))$ in terms of a measures on $\mathcal{P}_2(\mathbb{M})$, which we later define in in Section 7 . In this section, we study two types of problems. The first one consists to know if we can write any symmetric function $\Phi_k \in C((\mathbb{R}^d)^k)$ in terms of F_{Φ_k}. The second question is related to the following property about Fourier transforms. Given $(a_k)_{k=1}^\infty \in \mathcal{A}^s$ such that $a_k \in L^1((\mathbb{R}^d)^k)$, define

$$F_N : m \mapsto \sum_{k=1}^{N} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi.$$

for any natural number N. For $k \in \{1, \cdots, N\}$ we are able to recover $\int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi$ from F_N. For instance, the recovering allows to conclude that if $F_N \equiv 0$ then $a_k \equiv 0$ for any $k \in \{1, \cdots, N\}$. In this section, we endeavour to prove a more general statement by allowing

$N = \infty$, at the expense of imposing additional growth conditions on the $\|a_k\|_{L^1}$. Further assume there exist $C, \delta >$ such that (4.4) holds and

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi$$

converges uniformly on $\mathcal{P}_2(\mathbb{R}^d)$ on $\mathcal{P}_2(\mathbb{R}^d)$ (cf. Lemma 4.7) to a function we denote as F . Set

$$\Phi_k := \widehat{a}_k \in \mathrm{Sym}[(\mathrm{R^d})^{\mathrm{k}}] \cap L^2((\mathbb{R}^d)^k) \tag{5.1}$$

Note, Φ_k is continuous and by Riemann–Lebesgue lemma (cf. e.g. [23] Exercise 22 pp. 94)

$$\lim_{|x| \to \infty} \Phi_k(x) = 0. \tag{5.2}$$

5.1. **The inverse of the restriction $\Phi \to F_\Phi$ to polynomial.** A natural question we address in the subsection is the reconstruction of Φ from $F = F_\Phi$. For example, assume $k = 2$ and $F = F_{2\Phi}$. We have

$$F(\delta_{a_1}) = \int_{(\mathbb{R}^d)^2} \Phi(x_1, x_2) \delta_{a_1}(dx_1) \delta_{a_1}(dx_2) = \Phi(a_1, a_1). \tag{5.3}$$

Similarly,

$$F\Big(\frac{\delta_{a_1} + \delta_{a_2}}{2}\Big) = \frac{1}{4}\Big(\Phi(a_1, a_1) + \Phi(a_2, a_2) + 2\Phi(a_1, a_2)\Big) \tag{5.4}$$

We combine (5.3) and (5.4) to obtain the polarization identity

$$\Phi(a_1, a_2) = 2F\Big(\frac{\delta_{a_1} + \delta_{a_2}}{2}\Big) - \frac{1}{2}F\big(\delta_{a_1}\big) - \frac{1}{2}F\big(\delta_{a_2}\big). \tag{5.5}$$

Observe that if $F_{2\Phi} \equiv 0$, (5.5) implies $\Phi \equiv 0$ and so, $\Phi \to F_\Phi$ is an injective map of $C((\mathbb{R}^d)^2/P_2)$. In general, we could determine Φ applying the idea of coming from the construction of polar forms, either by a construction using differentiation or the inclusion–exclusion principle. To avoid differentiating, we chose here to use the inclusion–exclusion principle.

Given two positive integers $1 \le r \le k$ we defined the index set of multi–indexes

$$C_r^k := \{(i_1, \cdots, i_r) \mid i_1, \cdots, i_r \in \{1, \cdots, k\}, i_1 < i_2 < \cdots < i_r\}$$

Now, given $x = (x_1, x_2, \cdots, x_k) \in (\mathbb{R}^d)^k$, and for a given multi–index $(i_1, \cdots .i_r) = I \in C_r^k$, we define m_{x_I} as follows:

$$m_{x_I} := \frac{1}{r} \sum_{j=1}^{r} \delta_{x_{i_j}}.$$

Given a continuous function $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we define

$$O_k(F)(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{r=1}^{k} \bigg((-1)^{k-r} r^k \sum_{I \in C_r^k} F(m_{x_I}) \bigg).$$

Theorem 5.1. *The map kO_k is the inverse map of $\Phi \to F_\Phi$. In other words, we have*

$$\Phi(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{r=1}^{k} \bigg((-1)^{k-r} r^k \sum_{I \in C_r^k} F_{k\Phi}(m_{x_I}) \bigg) = O_k(F_{k\Phi})(x_1, \cdots, x_k)$$

for any $x_1, \cdots, x_k \in (\mathbb{R}^d)$. In other words, $O_k(F_\Phi) = \Phi/k$.

Proof. Note first that for any $F = F_{k\Phi}$ such that $\Phi \in C((\mathbb{R}^d)^k/P_k)$, $O_k(F)$ is continuous and symmetric in the sense that it is defined on the quotient space $(\mathbb{R}^d)^k/P_k$. Let $\mathcal{M}_c((\mathbb{R}^d))$ denote the set of signed Radon measures of compact support on $(\mathbb{R}^d)$. This is a vector space which contains the set of Radon probability measures on $(\mathbb{R}^d)$. We define $\alpha : \mathcal{M}_c^k((\mathbb{R}^d)) \to \mathbb{R}$ by

$$\alpha(m_1, ..., m_k) := \int_{(\mathbb{R}^d)^k} \Phi(x_1, x_2, ..., x_k) m_1(dx_1) m_2(dx_2) ... m_k(dx_k),$$

for $m_1, \cdots, m_k \in \mathcal{M}_c((\mathbb{R}^d))$. This is a k–multilinear form and so

$$m \to \tilde{\alpha}(m) := \alpha(m, ..., m)\,.$$

is a k–homogeneous functional on $\mathcal{M}_c((\mathbb{R}^d))$. We apply the polarization identity coming from the inclusion–exclusion principle, which goes back to [19] [20] [21] (cf. [22] for a recent and simple proof, and [14] [27] for a formulation in terms of n-th defects of F). We obtain

$$\alpha(m_1, ..., m_k) = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \sum_{I \in C_r^k} \tilde{\alpha}\left(\sum_{i \in I} m_i \right). \tag{5.6}$$

Setting $m_i = \delta_{x_i}$, using the definition of α and the fact that α is k–multilinear, we have

$$\Phi(x_1, x_2, ..., x_k) = \alpha(\delta_{x_1}, \cdots, \delta_{x_k}) = r^k \alpha\Big(\frac{\delta_{x_1}}{r}, \cdots, \frac{\delta_{x_k}}{r}\Big).$$

This, together with (5.6) yields

$$\Phi(x_1, x_2, ..., x_k) = \frac{r^k}{k!} \sum_{r=1}^{k} (-1)^{k-r} \sum_{I \in C_r^k} \tilde{\alpha}\left(m_{x_I}\right).$$

Since $F_{k\Phi}$ and $\tilde{\alpha}$ coincide on the set of Radon probability measure, we conclude the proof of the theorem. □

5.2. **One dimensional analytical extension.** For any $\lambda \in (0,1)$, $m \in \mathcal{P}_2(\mathbb{R}^d)$ and $y \in (\mathbb{R}^d)^k$, we have

$$\begin{aligned} & F_{\Phi_k}[\lambda m + (1-\lambda)\delta_y] \\ = \;& \frac{1}{k} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \lambda^l (1-\lambda)^{k-l} \int_{(\mathbb{R}^d)^l} \Phi_k(x_1, .., x_l, \underbrace{y, ..., y}_{k-l \text{ times}}) m(dx_1) \cdots m(dx_l) \end{aligned} \tag{5.7}$$

Now since

$$\|\Phi_k\|_{L^\infty} \le \|\widehat{a}_k\|_{L^1} < \infty \tag{5.8}$$

we may apply the dominated convergence theorem and use (5.2) to obtain that all the terms in (5.7), except corresponding to $k = l$, tend to 0 as $|y| \to \infty$. Thus,

$$\lim_{|y| \to \infty} F_{\Phi_k}\left(\lambda m + (1-\lambda)\delta_y\right) = \lambda^k F_{\Phi_k}[m]. \tag{5.9}$$

Remark 5.2. *Suppose* $(a_k)_{k=1}^\infty \in \mathcal{A}^s$ *satisfies* (4.4) *so that* (5.1) *holds and* $\Phi_k \in C_0((\mathbb{R}^d)^k)$.

(i) *For any* $(\lambda, m) \in (0,1) \times \mathcal{P}_2(\mathbb{R}^d)$

$$\lim_{|y| \to \infty} F\left(\lambda m + (1-\lambda)\delta_y\right) = \sum_{k=1}^{\infty} \frac{1}{k!} \lambda^k F_{\Phi_k}[m] =: \mathcal{F}[\lambda, m]. \tag{5.10}$$

(ii) *Hence,* $\lambda \mapsto \mathcal{F}[\lambda, m]$ *admits a extension denoted the same way, which is continuously differentiable at* 0*. For any* $l \geq 1$*,*

$$\frac{\partial^l \mathcal{F}}{\partial \lambda^l}[\lambda, m]\bigg|_{\lambda=0} = F_{\Phi_l}[m]\,. \tag{5.11}$$

Proof. (i) By (5.7) and (5.8) we obtain for any integers $1 \leq M < N$,

$$\begin{aligned}\sum_{k=M}^{N} \frac{1}{k!}\Big|F_{\Phi_k}[\lambda m + (1-\lambda)\delta_y] - \lambda^k F_{\Phi_k}[m]\Big| \quad &\leq \quad \sum_{k=M}^{N} \frac{1}{k!\; k} \sum_{l=0}^{k-1} \frac{k!}{l!(k-l)!} \lambda^l (1-\lambda)^{k-l} \big\|\widehat{a}_k\big\|_{L^1} \\ &\leq \quad \sum_{k=M}^{N} \frac{\big\|\widehat{a}_k\big\|_{L^1}}{k!\; k}.\end{aligned}$$

This, together with (4.4) implies

$$\sum_{k=M}^{\infty} \frac{1}{k!}\Big|F_{\Phi_k}[\lambda m + (1-\lambda)\delta_y] - \lambda^k F_{\Phi_k}[m]\Big| \leq \sum_{k=M}^{\infty} \frac{C}{k^{\delta+1}}$$

Thus by (5.9),

$$\limsup_{|y|\to\infty}\Big|F\left(\lambda m + (1-\lambda)\delta_y\right) - \mathcal{F}[\lambda, m]\Big| \leq \sum_{k=M}^{\infty} \frac{C}{k^{\delta+1}}.$$

We let M tend to ∞ to obtain (i).

(ii) Observe the domain of convergence of the analytic function $(\sum_{k=1}^{\infty} z/k^{1+\delta})$ in the complex plane $\mathbb{C}$ is the unit disk. Since by (4.4)

$$\frac{1}{k!} \sup_{\mathcal{P}_2(\mathbb{R}^d)} \big|F_{\Phi_k}\big| \leq \frac{C}{k^{1+\delta}}$$

we conclude $\mathcal{F}[\cdot, m]$ extends to an analytic function on the unit disk. Therefore, it is differentiable at 0 and one checks that (ii) holds. □

5.3. **Projections of a subset of $\bigoplus_k \mathcal{S}ym[k](\mathbb{R})$ onto $\mathcal{S}ym[k](\mathbb{C})$.**

Definition 5.3. *For any natural number* k*, thanks to Remark 5.2, we may define the following operator* $\pi_k : \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d)) \to \mathcal{S}ym[k](\mathbb{R})$ *as follow:*

$$\pi_k(F)[m] := \frac{\partial^l \mathcal{F}}{\partial \lambda^l}[\lambda, m]\bigg|_{\lambda=0} \quad \forall\, m \in \mathcal{P}_2(\mathbb{R}^d).$$

Then the following Corollary is a direct consequence of Remark 5.2.

Corollary 5.4. *Suppose* $(a_k)_{k=1}^{\infty} \in \mathcal{A}^s$ *satisfies* (4.4) *so that* (5.1) *holds and* $\Phi_k := \widehat{a}_k \in C_0((\mathbb{R}^d)^k)$*. Let* $F \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ *be the continuous function obtained as the uniform limit of the series*

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi = \sum_{k=1}^{\infty} \frac{1}{k!} F_{\Phi_k}[m] \quad \forall m \in \mathcal{P}_2(\mathbb{R}^d).$$

(i) *We have*

$$\pi_k(F) = \frac{1}{k!} F_{\Phi_k} \quad \text{and} \quad F = \sum_{k=1}^{\infty} \pi_k(F).$$

(ii) *In particular* $\pi_k \circ \pi_k(F) = \pi_k(F)$.

(iii) *By (i) and Theorem 5.1 we obtain*

$$\Phi_k = k\; k! \;\; O_k(\pi_k(F)).$$

(iv) *Using (iii) and the Fourier transform inverse formula we have*

$$a_k(\xi) = k\; k! \;\; \widehat{O_k(\pi_k(F))}(-\xi).$$

Because of (i) and (ii) in Corollary 5.4 we refer π_k in Definition 5.3 as projection operator.

Remark 5.5. *(Sufficient conditions for uniqueness of Fourier coefficients) Suppose* $(a_k)_{k=1}^{\infty} \in \mathcal{A}^s$ *satisfies* (4.4) *and let* $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{C}$ *be defined by*

$$F[m] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi \quad \forall m \in \mathcal{P}_2(\mathbb{R}^d).$$

If $F \equiv 0$ *then for any natural number* k, *we have* $a_k \equiv 0$

Proof. The remark is obtained as a direct consequence of Corollary 5.4. □

Definition 5.6. *Let* $s \geq 0$ *and let* λ_k *be the function defined in* (4.1).

(i) *Let* $(a_k)_{k=1}^{\infty}, (b_k)_{k=1}^{\infty} \in \mathcal{A}^s$ *be such that* (4.4). *Let* $F, G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{C}$ *be defined by*

$$F[m] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F_\xi^k[m] d\xi \quad \text{and} \quad G[m] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} b_k(\xi) F_\xi^k[m] d\xi \quad \forall m \in \mathcal{P}_2(\mathbb{R}^d).$$

By Remark 5.5, $(a_k)_{k=1}^{\infty}, (b_k)_{k=1}^{\infty}$ *are uniquely determined. We define*

$$\langle F; G \rangle_{H^s} = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) b_k^*(\xi) \Big(1 + \lambda_k^2(\xi)\Big)^s d\xi$$

and

$$\|F\|_{H^s}^2 = \langle F; F \rangle_{H^s}.$$

(ii) *Let* $(V_k^n)_{k=1}^{\infty}, (W_k^n)_{k=1}^{\infty} \in \mathcal{A}^s$ *for* $n = 1, \cdots, d$ *and let* $V_k \in L^2((\mathbb{R}^d)^k; \mathbb{C}^d)$ *(resp.* $W_k \in L^2((\mathbb{R}^d)^k; \mathbb{C}^d)$*) be the vector field whose components* $(V_k^n)_{n=1}^d$, *(resp.* $(W_k^n)_{n=1}^d$ *satisfy* (4.4). *Let*

$$V[m] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} V_k(\xi) F_\xi^k[m] d\xi, \quad W[m] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} W_k(\xi) F_\xi^k[m] d\xi.$$

We define

$$\langle V; W \rangle_{H^s} = \sum_{n=1}^{d} \langle V^n; W^n \rangle_{H^s}, \quad \|V\|_{H^s}^2 = \langle V; V \rangle_{H^s}.$$

Definition 5.7. *(Conditional continuity of $O_k \circ \pi_k$) Let $s \geq 0$ and let $N \geq 1$ be an integer. Suppose there exists $(a_k)_k \in \mathcal{A}^s$ such that $a_k \equiv 0$ for any $k > N$,*

$$a_k \in L^1((\mathbb{R}^d)^k/P_k;\mathbb{C}) \quad \text{and} \quad \Phi_k := \widehat{a}_k \in \mathrm{Sym}[(\mathbb{R}^d)^k] \quad \forall k \in \{1, \cdots, N\}. \tag{5.12}$$

We say that $F \in \mathcal{H}^s_N(\mathcal{P}_2(\mathbb{R}^d))$ provided that

$$F = \sum_{k=1}^N \frac{1}{k!} F_{\Phi_k}. \tag{5.13}$$

Remark 5.8. *Let $N \geq 1$ be an integer. Suppose $(a_k)_k \in \mathcal{A}^s$ is such that 5.12 holds. If there exists an integer N such that $a_k \equiv 0$ for any $k > N$ then there are constants $C, \delta > 0$ such that (4.4). In other words, $\mathcal{H}^s_N(\mathcal{P}_2(\mathbb{R}^d)) \subset \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$.*

Lemma 5.9. *Let $s \geq 0$ be a real number and N be a natural number. There exists a constant C_N such that for any natural number $k \leq N$ and any $F \in \mathcal{H}^s_N(\mathcal{P}_2(\mathbb{R}^d))$ we have*

$$\|O_k \circ \pi_k(F)\|_{C((\mathbb{R}^d)^k)} \leq C_N \sup_m |F[m]|.$$

In other words, if we endow $\mathcal{H}^s_N(\mathcal{P}_2(\mathbb{R}^d))$ with the supremum norm then $O_k \circ \pi_k : \mathcal{H}^s_N(\mathcal{P}_2(\mathbb{R}^d)) \to C((\mathbb{R}^d)^k/P_k)$ is continuous.

Proof. Let $(a_k)_k \in \mathcal{A}^s$ be such that (5.12) holds and $a_k \equiv 0$ for any $k > N$. Suppose F satisfies (5.13) where $\Phi_k := \widehat{a}_k$. Note

$$kO_k \circ \pi_k(F) = \Phi_k$$

and so, we need to estimate the norms of the Φ_k in terms of the norm of F. Set

$$\Psi_N(x_1, \cdots, x_N) := \frac{N}{k\ k!} \frac{k!(N-k)!}{N!} \sum_{I \in C^N_k} \Phi_k(x_I).$$

Observe that $\Psi_N \in C((\mathbb{R}^d)^N/P_N)$ and $F = F_{\Psi_N}$. By Theorem 5.1 $\Psi_N = NO_N(F)$ and so,

$$|\Psi_N| \leq \frac{N^{N+2}}{N!} \sup_m |F[m]|. \tag{5.14}$$

Recall that as $a_k \in L^1$, (5.2) holds:

$$\lim_{|(x_1,\cdots,x_k)| \to \infty} \Phi(x_1, \cdots, x_k) = 0$$

and so, $\lim_{|x_I| \to \infty} \Psi_N(x_1, \cdots, x_N)$ exists. We conclude that

$$\lim_{x_1 \to \infty} \Psi_N(x_1, \cdots, x_N) = \frac{\Phi_1(x_1)}{N}.$$

This, together with (5.14) implies

$$\frac{1}{N!} \|\Phi_1\|_{C((\mathbb{R}^d))} \leq \frac{N^{N+2}}{N!} \sup_m |F[m]|. \tag{5.15}$$

Observe

$$\lim_{x_1, x_2 \to \infty} \Psi_N(x_1, \cdots, x_N) - \frac{\sum_{j=1}^2 \Phi_1(x_j)}{N!} = \frac{N}{22!} \frac{2!(N-2)!}{N!} \Phi_2(x_1, x_2).$$

Hence,

$$\frac{N}{2\ 2!} \frac{2!(N-2)!}{N!} \|\Phi_2\|_{C((\mathbb{R}^d)^2)} \leq \|\Psi_N\|_{C((\mathbb{R}^d)^N)} + \frac{2}{N!} \|\Phi_1\|_{C((\mathbb{R}^d))}$$

We combine (5.14) and (5.15) to conclude that

$$\frac{N}{2\ 2!}\frac{2!(N-2)!}{N!}\|\Phi_2\|_{C((\mathbb{R}^d)^2)} \leq \frac{N^{N+2}+2N^{N+2}}{N!}\sup_m |F[m]|.$$

We repeat the same procedure $(N-2)$ times to conclude the proof of the Lemma. □

6. A stochastic process on the Wasserstein space and its smoothing effects.

One of the main focus of our work is the smoothing effects of $\triangle_{w,\frac{\epsilon}{\beta}}$: under mild additional assumptions on $U_0 \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ we will witness a smoothing effect on $U(t,\cdot)$.

6.1. **Heat equation.** Assume $U_0 : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is twice continuously differentiable (cf. Definition 3.3). We assume for any $m \in \mathcal{P}_2(\mathbb{R}^d)$, $\nabla\big(\nabla_w U_0[m]\big) : (\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ and $\nabla^2_w U_0[m] : (\mathbb{R}^d)\times(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ are uniformly bounded and have a concave modulus of continuity independent of m. We fix $\epsilon, \beta > 0$ and define

$$U(t,m) := \mathbb{E}\Big(U_0\big(\sigma_t^{\epsilon,\beta}[m]\big)\Big) \qquad \forall (t,m) \in (0,\infty)\times\mathcal{P}_2(\mathbb{R}^d).$$

where $\sigma_t^{\epsilon,\beta}[m]$ is defined as in (1.5). The arguments starting three lines after (81) in [4] lead to the following statements:

- (i) U is continuously differentiable on $(0,\infty)\times\mathcal{P}_2(\mathbb{R}^d)$ and for any $t>0$, $U(t,\cdot)$ is twice continuously differentiable on $\mathcal{P}_2(\mathbb{R}^d)$.
- (ii) U satisfies the heat equation

$$\partial_t U = \beta\triangle_{w,\frac{\epsilon}{\beta}} U \quad \text{on} \quad (0,\infty)\times\mathcal{P}_2(\mathbb{R}^d), \quad U(0,\cdot) = U_0.$$

The reader should not be misled by the fact that in the statement of Theorem 4.3 [4], the authors refer to a smooth Hamiltonian $H : (\mathbb{R}^d)^2 \to \mathbb{R}$ such that $\nabla_{pp}H(x,p) > 0$, whereas $H \equiv 0$ in the current manuscript (see statements starting three lines after (81) [4]). Statements similar to the above ones have also been made in [15] [16].

However when the initial condition U_0 is less regular that $U_0 \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ (see Definition 4.8), in Theorems 6.3 and 6.4, we reach conclusions much stronger than those in [4] [15] [16].

Recall that G_t^ϵ denote the heat kernel for the heat equation, given in (1.6). If $\beta > 0$ and $\epsilon \geq 0$ recall that $\sigma_t^{\epsilon,\beta}$ is defined in (1.5). In Lemma 4.1 we obtained a family $\{F_\xi^k, \mid \xi \in (\mathbb{R}^d)^k\}_{k=1}^\infty$ of eigenfunctions of $\triangle_{w,\epsilon}$ with eigenvalues

$$-\lambda^2_{k,\epsilon}(\xi) := -4\pi^2\Big(\Big|\sum_{j=1}^k \xi_j\Big|^2 + \epsilon\sum_{j=1}^n |\xi_j|^2\Big). \tag{6.1}$$

What is obvious is that

$$V_\xi^k(t,m) := \exp\Big(-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)\ F_\xi^k \tag{6.2}$$

satisfies the heat equation

$$\partial_t V_\xi^k = \beta\triangle_{w,\frac{\epsilon}{\beta}} V_\xi^k, \qquad V_\xi^k(0,\cdot) = F_\xi^k.$$

In fact, more is true in the space $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ for a given $s \geq 0$.

Remark 6.1. *For $s \geq 0$, the set $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ is dense in a Hilbert space $\mathbb{H}^s$ which we identify with a closed subspace of $\mathcal{A}^s$. For $\epsilon \geq 0$ and $\beta > 0$, $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is closable in $\mathbb{H}^s$ and the closure of $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ generates a strongly continuous (in fact, contractive) semigroup on $\mathbb{H}^s$, i.e. there exists a unique mild solution to the Cauchy problem in $\mathbb{H}$,*

$$\partial_t V = \beta\triangle_{w,\frac{\epsilon}{\beta}} V, \qquad V(0,\cdot) \in \mathbb{H}^s.$$

We next list facts which allow to support the above statement:

(i) First, the domain of $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is dense in the separable Hilbert space $\mathbb{H}^s$. Then the identity $\langle \beta\triangle_{w,\frac{\epsilon}{\beta}} F; G\rangle_{H^s} = \langle F; \beta\triangle_{w,\frac{\epsilon}{\beta}} G\rangle_{H^s}$ for F, G in the domain shows that $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is a symmetric operator in $\mathbb{H}^s$. Therefore the operator admits a closed maximal symmetric extension, which we still denote as $\beta\triangle_{w,\frac{\epsilon}{\beta}}$.

(ii) Next, we check $-\langle \beta\triangle_{w,\frac{\epsilon}{\beta}} F; F\rangle_{H^s} \geq 0$ for all F in the domain of the extended operator. Now one may check, by definition and closeness, that the range of $I - \beta\triangle_{w,\frac{\epsilon}{\beta}}$ is $\mathbb{H}^s$. By Minty's Theorem, $-\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is maximal monotone. Therefore, together with (i), $-\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is self-adjoint.

(iii) Therefore we have, for all $\lambda > 0$, $\lambda I - \beta\triangle_{w,\frac{\epsilon}{\beta}}$ is a bijection from the domain to $\mathbb{H}^s$, and that its inverse is bounded and satisfies $\|(\lambda I - \beta\triangle_{w,\frac{\epsilon}{\beta}})^{-1}\|_{\mathcal{L}(\mathbb{H}^s)} \leq \lambda^{-1}$. We may now apply Theorems II.3.5 and II.3.8 in [10] *(see also* [2]*) on $\mathbb{H}^s$ to conclude that the closure of $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ is indeed an infinitesimal generator of a strongly continuous semigroup in $\mathbb{H}^s$.*

From now on, we focus on the space $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ instead of $\mathbb{H}^s$. In fact, under appropriate convergence conditions, the superposition

$$V(t,m) := \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) \exp\bigl(-\beta \lambda^2_{k,\frac{\epsilon}{\beta}}(\xi) t\bigr) F^k_\xi[m] d\xi \tag{6.3}$$

will be shown to converges in $C(\mathcal{P}_2(\mathbb{R}^d))$ norm, and hence the solution to the heat equation exists in $C(\mathcal{P}_2(\mathbb{R}^d))$. It must satisfy $V(t,m) = \mathbb{E}\Bigl(U_0\bigl[\sigma^{\epsilon,\beta}_t[m]\bigr]\Bigr)$. Hence,

$$\partial_t V = \beta\triangle_{w,\frac{\epsilon}{\beta}} V, \qquad V(0,\cdot) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} a_k(\xi) F^k_\xi[m] d\xi := U_0. \tag{6.4}$$

In order to study properties of $\triangle_{w,\frac{\epsilon}{\beta}} V$, we later write it as the superposition of functions. We leave it as an exercise to the reader to check that

$$\mathbb{E}\Bigl(F^k_\xi[\sigma^\epsilon_t]\Bigr) = \exp\Bigl(-\beta \lambda^2_{k,\frac{\epsilon}{\beta}}(\xi) t\Bigr) F^k_\xi[m].$$

As a consequence, one obtains the following lemma.

Lemma 6.2. *We have $V^k_\xi = U^k_\xi$ where V^k_ξ is given in* (6.2) *and $U^k_\xi(t,m) := \mathbb{E}\Bigl(F^k_\xi\bigl[\sigma^{\epsilon,\beta}_t[m]\bigr]\Bigr)$.*

In the remainder of this section, $s, \delta > 0$, $\epsilon \geq 0$ and $\bigl(a_k\bigr)_{k=1}^\infty \in \mathcal{A}^s$. Assume (4.4) holds and let U_0 be as in (4.10). For $t > 0$, set

$$V(t,m) := \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} b_k(t,\xi) F^k_\xi[m] d\xi$$

where for $t > 0$ and $\epsilon \geq 0$,

$$b_k(t,\xi) := a_k(\xi) \exp\Big(-\beta \lambda^2_{k,\frac{\epsilon}{\beta}}(\xi) t\Big). \tag{6.5}$$

and $\lambda^2_{k,\epsilon}(\xi)$ is given by (6.1). We will at some point need a stronger assumption than (4.4):

$$\int_{(\mathbb{R}^d)^k} |a_k(\xi)| d\xi \leq \frac{Ck!}{k^{3+\delta}}, \tag{6.6}$$

We now hope to study the smoothing effect of the heat equation. To illustrate this effect, we provide the following two theorems which give, under mild assumptions, twice continuous differentiability when $\varepsilon = 0$ (c.f. Theorem 6.3); and that the smoothness is further improved (i.e. $\beta \Delta^r_{w,\frac{\varepsilon}{\beta}} V(t,\cdot)$ is twice continuously differentiable) when $\varepsilon > 0$ (c.f. Theorems 6.3) and 6.4), when the initial condition U_0 only sits in a space of lower regularity $\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$.

Theorem 6.3 (superposition)**.** *Assume $U_0 \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$ for some $s > 0$ and (4.4) holds. Then the followings hold:*

(i) *For any $l, t > 0$, the series $V(t,\cdot) \in H^l(\mathcal{P}_2(\mathbb{R}^d))$ and converges uniformly.*
(ii) *We have $V(t,m) = \mathbb{E}\Big(U_0(\sigma^\epsilon_t[m])\Big)$*
(iii) *If $\epsilon = 0$, (4.4), (4.7), (4.8) and (4.9) hold, then for $t > 0$, $V(t,\cdot)$ is twice continuously differentiable and*

$$\beta \triangle_{w,\frac{\epsilon}{\beta}} V(t,m) = \sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} b_k(t,\xi) \beta \triangle_{w,\frac{\epsilon}{\beta}} F^k_\xi[m] d\xi. \tag{6.7}$$

Furthermore, V satisfies the heat equation (6.4).
(iv) *If $\epsilon > 0$ and (6.6) holds then $V(t,\cdot)$ is twice continuously differentiable, (6.7) holds, and V satisfies the heat equation (6.4)*

Proof. (i) We need to prove (i) only for $l > s$. Fix $l > s$ and let $n \geq l$ be integer and set

$$\bar{C} := \inf_{a \geq 0} \frac{(n)! + (at)^n}{(n)!(1+a)^n} > 0.$$

We have

$$|b_k(t,\xi)|^2 \Big(1 + \lambda^2_{k,0}(\xi)\Big)^l \leq |a_k(\xi)|^2 \frac{\Big(1 + \lambda^2_{k,0}(\xi)\Big)^l}{\left(1 + \frac{\Big(\beta \lambda^2_{k,\frac{\epsilon}{\beta}}(\xi) t\Big)^n}{(n)!}\right)} \leq |a_k(\xi)|^2 \frac{\Big(1 + \lambda^2_{k,0}(\xi)\Big)^l}{\Big(1 + \frac{(\lambda^2_{k,0}(\xi)t)^n}{(n)!}\Big)}$$

and so,

$$|b_k(t,\xi)|^2 \Big(1 + \lambda^2_{k,0}(\xi)\Big)^l \leq \frac{|a_k(\xi)|^2}{\bar{C}}. \tag{6.8}$$

Thus,

$$\sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)|^2 \Big(1 + \lambda^2_{k,0}(\xi)\Big)^l d\xi \leq \sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \frac{|a_k(\xi)|^2}{\bar{C}} d\xi < \infty.$$

Similarly,

$$\sum_{k=N+1}^{\infty}\frac{1}{k!}\Big|\int_{(\mathbb{R}^d)^k} a_k(\xi)\exp\big(-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\big)F^k_\xi[m]d\xi\Big| \leq \sum_{k=N+1}^{\infty}\frac{1}{k!}\int_{(\mathbb{R}^d)^k}|a_k(\xi)|d\xi \leq \sum_{k=N+1}^{\infty}\frac{C}{k^{1+\delta}}.$$

These prove (i).

(ii) By Lemma 6.2, given an integer $N>1$ we have

$$\mathbb{E}\Big(\sum_{k=1}^{N}\frac{1}{k!}a_k(\xi)F^k_\xi\big[\sigma^\epsilon_t[m]\big]\Big) = \sum_{k=1}^{N}\frac{1}{k!}b_k(t,\xi)F^k_\xi[m].$$

and so,

$$\mathbb{E}\Big(\int_{(\mathbb{R}^d)^k}\sum_{k=1}^{N}\frac{1}{k!}a_k(\xi)F^k_\xi\big[\sigma^\epsilon_t[m]\big]d\xi\Big) = \sum_{k=1}^{N}\frac{1}{k!}\int_{(\mathbb{R}^d)^k}b_k(t,\xi)F^k_\xi[m]d\xi. \tag{6.9}$$

By (i), the expression on the right hand side of (6.9) converges uniformy to $V(t,m)$. Since $|F^k_\xi|\leq 1$

$$\sum_{k=1}^{N}\frac{1}{k!}\Big|\int_{(\mathbb{R}^d)^k}a_k(\xi)F^k_\xi\big[\sigma^\epsilon_t[m]\big]d\xi\Big| \leq \sum_{k=1}^{N}\frac{1}{k!}\int_{(\mathbb{R}^d)^k}|a_k(\xi)|d\xi.$$

We apply the Lebesgue dominated convergence theorem to obtain

$$\mathbb{E}\Big(\int_{(\mathbb{R}^d)^k}\sum_{k=1}^{\infty}\frac{1}{k!}a_k(\xi)F^k_\xi\big[\sigma^\epsilon_t[m]\big]d\xi\Big) = \sum_{k=1}^{\infty}\mathbb{E}\Big(\int_{(\mathbb{R}^d)^k}\frac{1}{k!}a_k(\xi)F^k_\xi\big[\sigma^\epsilon_t[m]\big]d\xi\Big).$$

In conclusion, we have proven that letting N tend to ∞ in (6.9) yields

$$\mathbb{E}\Big(U_0(\sigma^\epsilon_t[m])\Big) = V(t,m).$$

Under the sole assumptions in (4.4), (4.7),

$$\Big|\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)|a_k(\xi)|\exp\Big(-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)F^k_\xi[m]\Big| \leq \frac{\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)}{\exp\Big(\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)}|a_k(\xi)| \leq \frac{|a_k(\xi)|}{kt}.$$

Thus,

$$\sum_{k=1}^{\infty}\frac{1}{k!}\Big|\int_{(\mathbb{R}^d)^k}\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)a_k(\xi)\exp\Big(-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)F^k_\xi[m]d\xi\Big| \leq \sum_{k=1}^{\infty}\frac{C}{t\,k^{1+\delta}}.$$

This proves the uniform convergence of the following series:

$$\partial_t V(t,m) = -\sum_{k=1}^{\infty}\frac{1}{k!}\int_{(\mathbb{R}^d)^k}\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)a_k(\xi)\exp\Big(-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)F^k_\xi[m]d\xi. \tag{6.10}$$

Since $F^k_\xi[m]$ is an eigenfunction of $\beta\triangle_{w,\frac{\epsilon}{\beta}}$ and $-\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)$ is the associate eigenvalue, (6.10) reads off

$$\partial_t V(t,m) = \sum_{k=1}^{\infty}\frac{1}{k!}\int_{(\mathbb{R}^d)^k}b_k(t,\xi)\beta\triangle_{w,\frac{\epsilon}{\beta}}F^k_\xi[m]d\xi. \tag{6.11}$$

(iii) Suppose $\epsilon = 0$, (4.4), (4.7), (4.8) and (4.9) hold. The identity $|b_k| \leq |a_k|$ yields

$$\int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| \leq \frac{Ck!}{k^\delta}, \quad \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)|\big(|\xi_1| + |\xi_1|^2\big) d\xi \leq \frac{2Ck!}{k^{1+\delta}}, \tag{6.12}$$

$$\int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| \cdot |\xi_1| \, |\xi_2| d\xi \quad \leq \frac{Ck!}{k^{2+\delta}} \tag{6.13}$$

and

$$\int_{(\mathbb{R}^d)^k} |b_k(\xi)| \cdot |\xi_1|^3 d\xi \leq \frac{Ck!}{k^{1+\delta}}, \quad \int_{(\mathbb{R}^d)^k} |b_k(\xi)| \cdot |\xi_1|^2 \cdot |\xi_2| d\xi \quad \leq \frac{Ck!}{k^{3+\delta}}. \tag{6.14}$$

Thanks to Corollary 4.9, (6.12) , (6.13) and (6.14) yield

$$\sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} b_k(t,\xi) \beta \triangle_{w,\frac{\epsilon}{\beta}} F_\xi^k[m] d\xi = \beta \triangle_{w,\frac{\epsilon}{\beta}} \Big(\sum_{k=1}^\infty \frac{1}{k!} \int_{(\mathbb{R}^d)^k} b_k(t,\xi) F_\xi^k[m] d\xi \Big).$$

This, together with (6.11) completes the proof of (iii).

(iv) Suppose $\epsilon > 0$ and (6.6) holds. We have

$$\left(1 + \Big(4\pi^2 \epsilon t \sum_{j=1}^k |\xi_j|^2\Big) + \frac{1}{2}\Big(4\pi^2 \epsilon t \sum_{j=1}^k |\xi_j|^2\Big)^2\right) |b_k(t,\xi)| \leq |a_k(\xi)|. \tag{6.15}$$

This, together with (6.6), yields

$$\min\{1, 4\pi^2 \epsilon t\} \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)|\big(1 + |\xi_1| + |\xi_1|^2\big) d\xi \leq 3 \int_{(\mathbb{R}^d)^k} |a_k(\xi)| d\xi \leq \frac{3Ck!}{k^{3+\delta}} \tag{6.16}$$

Similarly, using the fact that $2|\xi_1||\xi_2| \leq |\xi_1|^2 + |\xi_2|^2$ we obtain

$$8\pi^2 \epsilon t \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| \cdot |\xi_1| \cdot |\xi_2| d\xi \leq 4\pi^2 \epsilon t \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| (|\xi_1|^2 + |\xi_2|^2) d\xi$$

and so,

$$8\pi^2 \epsilon t \int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| \cdot |\xi_1| \cdot |\xi_2| d\xi \leq \int_{(\mathbb{R}^d)^k} |a_k(\xi)| d\xi \leq \frac{Ck!}{k^{3+\delta}}. \tag{6.17}$$

We exploit the crude estimate

$$|\xi_1|^2 |\xi_2| + |\xi_1|^3 \leq \frac{3}{2}\Big(1 + |\xi_1|^4 + |\xi_2|^2\Big) \tag{6.18}$$

But

$$\min\{1, 4\pi^2 \epsilon t, 8\pi^4 \epsilon^2 t^2\}\Big(1+|\xi_1|^4+|\xi_2|^2\Big) \leq \left(1+\Big(4\pi^2 \epsilon t \sum_{j=1}^k |\xi_j|^2\Big)+\frac{1}{2}\Big(4\pi^2 \epsilon t \sum_{j=1}^k |\xi_j|^2\Big)^2\right) \tag{6.19}$$

Thanks to (6.6) and (6.15), we combine (6.18) and (6.19) to obtain a constant $\bar{\epsilon}(t) \in (0,\infty)$ depending only on ϵ and $t > 0$ such that

$$\int_{(\mathbb{R}^d)^k} |b_k(t,\xi)| \cdot |\xi_1|^3 \cdot |\xi_1|^2 |\xi_2| d\xi \leq \frac{\bar{\epsilon}(t)\ k!}{k^{3+\delta}}. \tag{6.20}$$

Thanks to (6.16), (6.17) and (6.20), we may apply Corollary 4.9 to obtain that $V(t,\cdot)$ is twice continuously differentiable and the identity in (6.7) holds. This, together with (6.11), proves (iv). $\square$

The next theorem considers initial conditions belonging to $\mathcal{H}^s\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ with their Fourier coefficients satisfying (6.6). For some parameters, we show the heat equation drastically improved the regularity properties of the value function at time $t > 0$.

Theorem 6.4 (smoothing effects)**.** *Suppose $\epsilon, s > 0$, and let $\big(a_k\big)_{k=1}^{\infty} \in \mathcal{A}^s$ be such that for any $k \geq 1$, $a_k \in L^2((\mathbb{R}^d)^k; \mathbb{C})$ and (6.6) holds. Let U_0 and $V(t,\cdot)$ be the uniformly convergent series in Theorem 6.3. Then for any $t > 0$, $\Big(\beta\triangle_{w,\frac{\epsilon}{\beta}}\Big)^r V(t,\cdot)$ is twice continuously differentiable for all $r \in \mathbb{N}$.*

Proof. The statement is clear for $r = 0$ from Theorem 6.3. We start with $r = 1$. Recall that b_k is given by (6.5). Set

$$c_k(t,\xi) := -\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi) b_k(t,\xi) = -\frac{a_k(\xi)\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)}{\exp\big(\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\big)}$$

By Theorem 6.3

$$\beta\triangle_{w,\frac{\epsilon}{\beta}} V(t,m) := \sum_{k=1}^{\infty} \frac{1}{k!}\int_{(\mathbb{R}^d)^k} c_k(t,\xi) F^k_\xi[m] d\xi.$$

We have

$$|c_k(t,\xi)| \leq \frac{|a_k(\xi)|}{t}, \tag{6.21}$$

and setting $|\xi|^2 = \sum_{j=1}^k |\xi_j|^2$ we obtain

$$|c_k(t,\xi)|\cdot|\xi|^4 = \frac{|a_k(\xi)|\cdot\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)\ |\xi|^4}{\exp\Big(\frac{\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t}{2}\Big)\cdot\exp\Big(\frac{\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t}{2}\Big)} \leq \frac{|a_k(\xi)|\cdot\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)|\xi|^4}{\exp\Big(\frac{\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t}{2}\Big)\cdot\exp\Big(2\pi^2\epsilon t|\xi|^2\Big)}$$

Thus,

$$|c_k(t,\xi)|\cdot|\xi|^4 \leq \frac{|a_k(\xi)|}{\pi^4 t^3\epsilon^2} \tag{6.22}$$

We use the crude estimate

$$1 + |\xi_1| + |\xi_1|^2 + |\xi_1||\xi_2| + |\xi_1|^2|\xi_2| + |\xi_1|^3 \leq 6(1+|\xi|^4).$$

to conclude that

$$\int_{(\mathbb{R}^d)^k} |c_k(t,\xi)|\Big(1 + |\xi_1| + |\xi_1|^2 + |\xi_1||\xi_2| + |\xi_1|^2|\xi_2| + |\xi_1|^3\Big)d\xi \leq 6\int_{(\mathbb{R}^d)^k} |c_k(t,\xi)|(1+|\xi|^4)d\xi$$

This, together with (6.22) and (6.21) implies there is a constant $\bar{\epsilon}(t) \in (0,\infty)$ depending only on ϵ and $t > 0$ such that

$$\int_{(\mathbb{R}^d)^k} |c_k(t,\xi)|\Big(1 + |\xi_1| + |\xi_1|^2 + |\xi_1||\xi_2| + |\xi_1|^2|\xi_2| + |\xi_1|^3\Big)d\xi \leq \frac{\epsilon(t)k!}{k^{3+\delta}}. \tag{6.23}$$

Thanks to Corollary 4.9, (6.23) implies $\beta\triangle_{w,\frac{\epsilon}{\beta}} V(t,\cdot)$ is twice differentiable.

To arrive at the statement with $r>1$, one only need to inductively follow the same argument as above: apply Corollary 4.9 together with noticing $2|\xi_1||\xi_2| \le |\xi_1|^2+|\xi_2|^2$ and

$$\frac{|a_k(\xi)|\cdot\Big(\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)\Big)^r\ |\xi|^4}{\exp\Big(\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t\Big)} \le \frac{|a_k(\xi)|\cdot\Big(\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)\Big)^r\ |\xi|^4}{\left(1+\frac{1}{r!}\left(\frac{\beta\lambda^2_{k,\frac{\epsilon}{\beta}}(\xi)t}{2}\right)^r\right)\cdot\exp\Big(2\pi^2\epsilon t|\xi|^2\Big)}$$

□

7. Measures concentrated on locally compact subset of $\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$.

As amply explained in the last paragraph of the introduction, there has been some efforts to construct Laplacians on either Hilbert spaces [18] [26] or other metric spaces such as the Wasserstein space [13] [24] [25]. These Laplacians are expected to have associated measures consistent with Dirichlet forms on the space of functions, and for instance allow for integrations by parts. The goal of the section is to initiate a study which we hope will shade some light on the search of the appropriate measures corresponding to the partial Laplacians.

The set $\mathcal{M}^2 := \mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$ is a metric space when endowed with the metric $\mathbb{W}$ defined by

$$\mathbb{W}^2(m,\tilde m) := W_2^2(m_1,\tilde m_1)+W_2^2(m_2,\tilde m_2)$$

for $m=(m_1,m_2)$, $\tilde m=(\tilde m_1,\tilde m_2)\in\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$. If $R>0$ we denote as B_R the ball in $(\mathbb{R}^d)$ centered at the origin and of radius R. Note the infinite dimensional ball in $\mathcal{M}^2$, of radius $R>0$, centered at $\delta_{(0,0)}$ is

$$\big\{(m_1,m_2)\in\mathcal{M}^2 \mid \int_{(\mathbb{R}^d)}|x|^2(m_1+m_2)(dx)\le R\big\}.$$

Let $\mathbb{S}_R$ denote the set of $m\in\mathcal{P}_2(\mathbb{R}^d)$ such that the support of m is contained in B_R and let $\chi_R:\mathcal{P}_2(\mathbb{R}^d)\to\{0,1\}$ be the function which assumes the value 1 on $\mathbb{S}_R$ and the value 0 on the complement of $\mathbb{S}_R$.

Fix a natural number k, If $r,p\le k$ are natural numbers and let $I\in C^k_r$ and $J\in C^k_p$. The map $x\to(m_{x_I},m_{x_J})$ is a $(r^{-1}+p^{-1})$–Lipschitz map of $(\mathbb{R}^d)$ onto $\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$. Thus, it maps compact sets into compact sets and bounded sets into bounded set.

Let $\mathbb{P}^{I,J,R}$ denote the push forward of the Lebesgue measure on B^k_R by the map $x\to(m_{x_I},m_{x_j})$. If $H:\mathcal{M}^2\to[0,\infty)$ is continuous then

$$\int_{\mathcal{M}^2}H(m_1,m_2)\mathbb{P}^{I,J,R}(dm_1,dm_2)=\int_{B^k_R}H(m_{x_I},m_{x_J})dx_1\cdots dx_k.$$

We define the signed Borel regular measure

$$\mathbb{P}^{k,R} := \frac{k^2}{(k!)^3}\sum_{r=1}^k\sum_{p=1}^k(-1)^{r+p}r^kp^k\sum_{I\in C^k_r}\sum_{J\in C^k_p}\mathbb{P}^{I,J,R}.$$

Proposition 7.1. *Let $r,p\le k$ be integers and let $I\in C^k_r$, $J\in C^k_p$ be multi–indexes.*

(i) *The measure $\mathbb{P}^{I,J,R}$ are of finite total mass and are supported by $\mathbb{S}_R$.*
(ii) *The signed measure is $\mathbb{P}^{k,R}$ of finite total variations and is supported by $\mathbb{S}_R$.*

Proof. Since (i) implies (ii), it suffices to show (i). Setting $H \equiv 1$ on $\mathcal{M}^2$ we have

$$\mathbb{P}^{I,J,R}\big(\mathcal{M}^2\big) = \mathcal{L}^{kd}(B_R^k) < \infty. \tag{7.1}$$

Observe that if $x \in B_R^k$ then $m_{x_I}, m_{x_J} \in \mathbb{S}_R$ and so, $\chi_R(m_{x_I})\chi_R(m_{x_J}) = 1$. Thus,

$$\int_{\mathcal{M}^2} \chi_R(m_1)\chi_R(m_2)\mathbb{P}^{I,J,R}(dm_1, dm_2) = \int_{B_R^k} \chi_R(m_{x_I})\chi_R(m_{x_J})dx_1 \cdots dx_k = \mathcal{L}^{kd}(B_R^k).$$

This, together with (7.1) proves (i). □

Proposition 7.2. *If $\Phi, \Psi \in L^2((\mathbb{R}^d)^k/P_k)$ are supported by B_R then*

$$\Big\langle F_\Phi; F_\Psi \Big\rangle_{H^0} = \int_{\mathcal{M}^2} F_\Phi[m_1]F_\Psi[m_2]\mathbb{P}^{k,R}(dm_1, dm_2)$$

Proof. By Lemma 4.7, since Φ and Ψ are supported by B_R we have

$$\langle F_\Phi; F_\Psi \rangle_{H^0} = \frac{1}{k!}\int_{B_R^k} \Phi(x)\Psi(x)dx.$$

By Theorem 5.1 we obtain

$$\begin{aligned}\langle F_\Phi; F_\Psi \rangle_{H^0} &= \frac{k^2}{(k!)^3}\int_{B_R^k} \sum_{r=1}^k \Big((-1)^{k-r} r^k \sum_{I \in C_r^k} F_\Phi(m_{x_I})\Big) \sum_{s=1}^k \Big((-1)^{k-s} s^k \sum_{J \in C_s^k} F_\Psi(m_{x_J})\Big)dx \\ &= \sum_{r,s=1}^k \frac{k^2}{(k!)^3}(-1)^{r+s} r^k s^k \sum_{I \in C_r^k}\sum_{J \in C_s^k} \int_{\mathcal{M}^2} F_\Phi(m_1)F_\Psi(m_2)\mathbb{P}^{I,J}(dm_1, dm_2).\end{aligned}$$

This concludes the proof. □

Theorem 7.3. *If $R > 0$ and Let $\Phi, \Psi \in C_c^3((\mathbb{R}^d)^k/P_k)$ then*

$$-\int_{\mathcal{M}^2} \triangle_w F_\Phi[m_1]F_\Psi[m_2]d\mathbb{P}^{k,R} = \int_{\mathcal{M}^2} D_2(\nabla_w F_\Phi, \nabla_w G_\Psi)d\mathbb{P}^{k,R}$$

where

$$D_2(\nabla_w F, \nabla_w G)(m_1, m_2) := \int_{(\mathbb{R}^d)^2} \langle \nabla_w F[m_1](q_1); \nabla_w G[m_2](q_2)\rangle m_1(dq_1)m_2(dq_2).$$

Proof. Let $m \in \mathbb{S}_R$. Set

$$f := \sum_{j=1}^k \nabla_{x_j}\Phi = \big[f^1, \cdots, f^d\big]^T, \qquad g := \sum_{j=1}^k \nabla_{x_j}\Psi = \big[g^1, \cdots, g^d\big]^T$$

The material presented in Subsection 3.2 allows to obtain

$$\int_{(\mathbb{R}^d)} \nabla_w F_\Phi[m](x_1)m(dx_1) = \frac{1}{k}\int_{(\mathbb{R}^d)^k} f(x_1, \cdots, x_k)m(dx_1)\cdots m(dx_k) = \begin{bmatrix} F_{f^1}[m] \\ \vdots \\ F_{f^d}[m] \end{bmatrix} \tag{7.2}$$

Similarly,

$$\int_{(\mathbb{R}^d)} \nabla_w F_\Psi[m](x_1) m(dx_1) = \frac{1}{k}\int_{(\mathbb{R}^d)^k} g(x_1,\cdots,x_k) m(dx_1)\cdots m(dx_k) = \begin{bmatrix} F_{g^1}[m] \\ \vdots \\ F_{g^d}[m] \end{bmatrix} \tag{7.3}$$

By Proposition 7.2

$$\begin{aligned} \left\langle \int_{(\mathbb{R}^d)} \nabla_w F_\Phi(m,x) m(dx); \int_{(\mathbb{R}^d)} \nabla_w F_\Psi(m,x) m(dx) \right\rangle_{H^0} &= \sum_{n=1}^d \langle F_{f^n}; F_{g^n}\rangle_{H^0} \\ &= \sum_{n=1}^d \int_{\mathcal{M}^2} F_{f^n}[m_1] F_{g^n}[m_2] \mathbb{P}^{k,R}(dm_1, dm_2). \end{aligned}$$

In other words,

$$\left\langle \int_{(\mathbb{R}^d)} \nabla_w F_\Phi(m,x) m(dx); \int_{(\mathbb{R}^d)} \nabla_w F_\Psi(m,x) m(dx) \right\rangle_{H^0} = \int_{\mathcal{M}^2} D_2(\nabla_w F, \nabla_w G)(m_1,m_2) \mathbb{P}^{k,R}(dm_1, dm_2) \tag{7.4}$$

By Corollary 3.8 (ii), $\triangle_w F_\Phi = F_\Theta$ where

$$\Theta(x_1,\cdots,x_k) = \sum_{j,l=1}^k \sum_{n=1}^d \frac{\partial^2 \Phi}{\partial (x_j)_n \partial (x_l)_n}.$$

This, together with Proposition 7.2 implies

$$\langle \triangle_w F_\Phi; F_\Psi\rangle_{H^0} = \langle F_\Theta; F_\Psi\rangle_{H^0} = \int_{\mathcal{M}^2} F_\Theta[m_1] F_\Psi[m_2] \mathbb{P}^{k,R}(dm_1, dm_2).$$

Hence,

$$\langle \triangle_w F_\Phi; F_\Psi\rangle_{H^0} = \int_{\mathcal{M}^2} \triangle_w F_\Phi[m_1] F_\Psi[m_2] \mathbb{P}^{k,R}(dm_1, dm_2) \tag{7.5}$$

Thanks to Proposition 4.10, (7.4) and (7.5) yield the desired result. □

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