Controlled Singular Volterra Integral Equations and Pontryagin Maximum Principle *

Ping Lin† and Jiongmin Yong ‡

Abstract. This paper is concerned with a class of (controlled) singular Volterra integral equations, which could be used to describe problems involving memories. The well-known fractional order ordinary differential equations of the Riemann–Liouville or Caputo types are strictly special cases of the equations studied in this paper. Well-posedness and some regularity results in proper spaces are established for such kind of equations. For an associated optimal control problem, by using an extended Liapounoff's type theorem and the spike variation technique, we establish a Pontryagin's type maximum principle for optimal controls. Different from the existing literature of optimal controls for fractional differential equations, our method enables us to deal with the problem without assuming regularity conditions on the controls, the convexity condition on the control domain, and some additional unnecessary conditions on the nonlinear terms of the integral equation and the cost functional.

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1 Introduction

Consider the following controlled Volterra integral equation:

$$y(t)=\eta(t)+\int_0^t\frac{f\big(t,s,y(s),u(s)\big)}{(t-s)^{1-\beta}}ds,\qquad \text{a.e. } t\in[0,T]. \tag{1.1}$$

In the above, $\eta(\cdot)$ and $f(\cdot\,,\cdot\,,\cdot\,,\cdot)$ are given maps, called the *free term* and the *generator* of the *state equation*, respectively, $y(\cdot)$ is called the *state trajectory* taking values in the Euclidean space $\mathbb{R}^n$, and $u(\cdot)$ is called the *control* taking values in some separable metric space U. To measure the performance of the control, we introduce the cost functional

$$J(u(\cdot))=\int_0^T g\big(t,y(t),u(t)\big)dt+\sum_{j=1}^m h^j\big(y(t_j)\big), \tag{1.2}$$

with the two terms on the right hand representing the *running cost* and the *prespecified instant costs* (at $0\leqslant t_1<t_2<\cdots<t_m\leqslant T$), respectively.

Equations like (1.1) can be used to describe some dynamics involving memories. In the classical situations of optimal control for Volterra integral equations, people usually assume that $\beta=1$, namely, the term $\frac{1}{(t-s)^{1-\beta}}$ is absent. Relevant works can be traced back to those by Vinokurov in the later 1960s [46], followed by the works of Angell [4], Kamien–Muller [28], Medhin [34], Carlson [15], Burnap–Kazemi [12], and some recent works by de la Vega [19], Belbas [6, 7], and Bonnans–de la Vega–Dupuis [9]. On the other hand, in the past several decades, fractional (order) differential equations have attracted quite a few researchers' attention due to some very interesting applications in physics, chemistry, engineering, population dynamics, finance and

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†School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China. E-mail address: linp258 @nenu.edu.cn.

‡Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA. E-mail address: jiongmin.yong @ucf.edu.

other sciences; See Oldham–Spanier [38] for some early examples of diffusion processes, Torvik–Bagley [45], Caputo [13], and Caputo–Mainardi [14] for modeling of the mechanical properties of materials, Benson [8] for the advection and the dispersion of solutes in natural porous or fractured media, Chern [16], Diethelm–Freed [23] for the modeling behavior of viscoelastic and viscoplastic materials under external influences, Scalas–Gorenflo–Mainardi [42] for the mathematical models in finance, Das–Gupta [18], Demirci–Unal–Özalp [20], Arafa–Rida–Khalil [5], Diethelm [22] for some population and epidemic models, Metzler et al. [35] for the relaxation in filled polymer networks, and Okyere et al. [37] for an SIR model of population. An extensive survey on fractional differential equations can be found in the book by Kilbas–Srivastava–Trujillo [32]. We also refer the readers to [17, 21, 39, 40, 43] for fractional (order) differential equations and their applications. In the recent years, optimal control problems have been studied for fractional differential equations by a number of authors. We mention the works of Agrawal [1, 2], Agrawal–Defterli–Baleanu [3], Bourdin [11], Frederico–Torres [24], Hasan–Tangpong–Agrawal [26] and Kamocki [29, 30].

The most popular fractional differential equations are those in the sense of Riemann–Liouville or in the sense of Caputo (See Section 3 for some details). It turns out that these equations (of the order no more than 1, for scalar functions) are equivalent to Volterra integral equations with the integrand being singular along $s = t$, and the free term $\eta(\cdot)$ being possibly discontinuous (blowing up) at $t = 0$. More precisely, the corresponding controlled state equation of form (1.1), with $\beta \in (0, 1)$, could have the feature that

$$\eta(t) = \frac{c}{t^{1-\beta}} \text{ (or } c\text{)}, \qquad f(t, s, y, u) = \widetilde{f}(s, y, u), \qquad 0 \leqslant s < t \leqslant T, \quad \forall\ (y, u), \tag{1.3}$$

for some map $\widetilde{f}(\cdot\,, \cdot\,, \cdot)$ and constant $c \in \mathbb{R}$. The singularity of the integrand makes the optimal control problems for fractional differential equations different from the classical problems for Volterra integral equations as in the above-mentioned literature.

The purpose of this paper is to study an optimal control problem for a singular Volterra integral equation of form (1.1). We see that the integrand $(t, s) \mapsto \frac{f(t,s,y,u)}{(t-s)^{1-\beta}}$ has singularity along $t = s$ and we also only assume certain integrability of the map $s \mapsto f(t, s, y, u)$. Moreover, the free term $\eta(\cdot)$ can be (unboundedly) discontinuous. Since the free term $\eta(\cdot)$ will be an L^p function in general, we expect, under suitable conditions, that the state trajectory $y(\cdot)$ will also be in L^p space. On the other hand, in the cost functional, we need $y(t_j)$ to be defined.[1] Therefore, we need to have the continuity of the state trajectory at the points t_j. To this end, we impose suitable conditions on the generator $f(t, s, y, u)$, and extend some classical tools to be applicable to our situation, including the extended Gronwall's inequality. Concerning the optimal control problem, we will establish a Pontryagin's maximum principle. Among others, a major difficulty that we encounter is the control set U which is merely a separable metric space. Therefore, we could only use the spike variation. Due to the singularity in the integral kernel, we need to extend a crucial lemma found in [33] applicable to the current singular integral equation case.

The rest of the paper is organized as follows. In Section 2, necessary preliminaries will be presented. Some results are interesting by themselves. Well-posedness of the state equation, together with the continuity of the solutions, will be established in Section 3. Section 4 is devoted to a proof of Pontryagin's type maximum principle for our optimal control problem of singular integral equations. As a special case, the maximum principles for fractional differential equations in the sense of Riemann–Liouville, and Caputo, will be briefly described. Some concluding remarks will be collected in Section 5.

2 Preliminary

In this section, we will present some preliminary results which will be useful later. First of all, let $T > 0$ be a fixed time horizon. We introduce the following spaces:

$$L^p(0, T; \mathbb{R}^n) = \Big\{\varphi : [0, T] \to \mathbb{R}^n \ \Big|\ \varphi(\cdot) \text{ is measurable, } \|\varphi(\cdot)\|_p \equiv \Big(\int_0^T |\varphi(t)|^p dt\Big)^{\frac{1}{p}} < \infty\Big\}, \quad 1 \leqslant p < \infty,$$
$$L^\infty(0, T; \mathbb{R}^n) = \Big\{\varphi : [0, T] \to \mathbb{R}^n \ \Big|\ \varphi(\cdot) \text{ is measurable, } \|\varphi(\cdot)\|_\infty \equiv \operatorname*{esssup}_{t\in[0,T]} |\varphi(t)| < \infty\Big\}.$$

[1]Practically, such kind of cost may often appear; e.g., they could be monthly, quarterly, semi-annually, or annually performance checks.

Also, we define

$$L^{p+}(0,T;\mathbb{R}^n)=\bigcup_{r>p}L^r(0,T;\mathbb{R}^n),\quad 1\leqslant p<\infty,\quad L^{p-}(0,T;\mathbb{R}^n)=\bigcap_{r<p}L^r(0,T;\mathbb{R}^n),\quad 1<p<\infty,$$

and

$$C([0,T];\mathbb{R}^n)=\big\{\varphi:[0,T]\to\mathbb{R}^n\bigm|\varphi(\cdot)\text{ is continuous}\big\}.$$

When $\mathbb{R}^n=\mathbb{R}$, we simply write $L^p(0,T)$ and so on, omitting $\mathbb{R}$.

Next, we denote $\Delta=\big\{(t,s)\in[0,T]^2\bigm|0\leqslant s<t\leqslant T\big\}$. Note that the "diagonal line" $\{(t,t)\,|\,t\in[0,T]\}$ is not contained in Δ. Thus if $\varphi:\Delta\to\mathbb{R}^n$ with $(t,s)\mapsto\varphi(t,s)$ being continuous, then $\varphi(\cdot\,,\cdot)$ could be unbounded as $|t-s|\to0$. Throughout this paper, we denote $t_1\vee t_2=\max\{t_1,t_2\}$ and $t_1\wedge t_2=\min\{t_1,t_2\}$, for any $t_1,t_2\in\mathbb{R}$. In particular, $t^+=t\vee0$. The characteristic function of any set E is denoted by $\mathbf{1}_E(\cdot)$. For two sets E_1 and E_2 in $\mathbb{R}$ and a function $\varphi(\cdot):E_1\to\mathbb{R}$, we denote

$$\varphi(t)\mathbf{1}_{E_2}(t)\equiv\begin{cases}\varphi(t), & t\in E_1\cap E_2,\\ 0, & t\in\mathbb{R}\setminus(E_1\cap E_2).\end{cases}$$

We call a continuous and strictly increasing function $\omega(\cdot):[0,\infty)\to[0,\infty)$ a *modulus of continuity* if $\omega(0)=0$. Before going further, let us recall the Young's inequality for convolution (Theorem 3.9.4 in [10]).

Lemma 2.1. *Let $p,q,r\geqslant1$ satisfy $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}$. Then for any $f(\cdot)\in L^q(\mathbb{R}^n)$, $g(\cdot)\in L^r(\mathbb{R}^n)$,*

$$\|f(\cdot)*g(\cdot)\|_{L^p(\mathbb{R}^n)}\leqslant\|f(\cdot)\|_{L^q(\mathbb{R}^n)}\|g(\cdot)\|_{L^r(\mathbb{R}^n)}.\tag{2.1}$$

Corollary 2.2. *Let $\beta\in(0,1)$, $1\leqslant r<\frac{1}{1-\beta}$, and $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}$, $p,q\geqslant1$. Then for any $a<b$, $0<\delta\leqslant b-a$, and $\varphi(\cdot)\in L^q(a,b)$,*

$$\Big(\int_a^{a+\delta}\Big|\int_a^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds\Big|^pdt\Big)^{\frac{1}{p}}\leqslant\Big(\frac{\delta^{1-r(1-\beta)}}{1-r(1-\beta)}\Big)^{\frac{1}{r}}\|\varphi(\cdot)\|_{L^q(a,b)}.\tag{2.2}$$

Proof. Hereafter, we denote $\theta_\delta(t)=t^{\beta-1}\mathbf{1}_{(0,\delta]}(t)$, for all $t\in\mathbb{R}$. Let $\varphi(\cdot)\in L^q(a,b)$. Then

$$\{[\varphi(\cdot)\mathbf{1}_{[a,b]}(\cdot)]*\theta_\delta(\cdot)\}(t)=\int_a^b\frac{\varphi(s)}{(t-s)^{1-\beta}}\mathbf{1}_{(0,\delta]}(t-s)ds=\int_{a\vee(t-\delta)}^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds,\qquad\text{a.e. }t\in[a,b+\delta].$$

For $t\notin[a,b+\delta]$, the value of the above convolution is 0, and

$$\{[\varphi(\cdot)\mathbf{1}_{[a,b]}(\cdot)]*\theta_\delta(\cdot)\}(t)=\int_a^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds,\qquad\text{a.e. }t\in[a,a+\delta].$$

Hence, by Lemma 2.1, for $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}$ with $1\leqslant r<\frac{1}{1-\beta}$, we have

$$\begin{aligned}&\Big(\int_a^{a+\delta}\Big|\int_a^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds\Big|^pdt\Big)^{\frac{1}{p}}=\|\{[\varphi(\cdot)\mathbf{1}_{[a,b]}(\cdot)]*\theta_\delta(\cdot)\}\|_{L^p(a,a+\delta)}\\&\leqslant\|\{[\varphi(\cdot)\mathbf{1}_{[a,b]}(\cdot)]*\theta_\delta(\cdot)\}\|_{L^p(\mathbb{R})}\leqslant\|\varphi(\cdot)\|_{L^q(a,b)}\|\theta_\delta(\cdot)\|_{L^r(0,\delta)}=\Big(\frac{\delta^{1-r(1-\beta)}}{1-r(1-\beta)}\Big)^{\frac{1}{r}}\|\varphi(\cdot)\|_{L^q(a,b)}.\end{aligned}$$

This proves (2.2). □

We now present the following result.

Lemma 2.3. *Let $\beta\in(0,1)$ and $p\geqslant1$. Suppose $\varphi:\Delta\to\mathbb{R}^n$ is measurable with $\varphi(0,\cdot)\in L^p(0,T;\mathbb{R}^n)$, and for some modulus of continuity $\omega(\cdot)$, together with some $\bar\varphi(\cdot)\in L^p(0,T)$, it holds that*

$$|\varphi(t,s)-\varphi(t',s)|\leqslant\omega(|t-t'|)\bar\varphi(s),\qquad t,t'\in[0,T],\ s\in[0,T]\backslash\{t,t'\}.\tag{2.3}$$

Define

$$\psi(t)=\int_0^t\frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds,\qquad \text{a.e. } t\in[0,T]. \tag{2.4}$$

Then $\psi(\cdot)\in L^p(0,T;\mathbb{R}^n)$ *and*

$$\|\psi(\cdot)\|_p\leqslant\frac{T^\beta}{\beta}\Big(\|\varphi(0,\cdot)\|_p+\omega(T)\|\bar\varphi(\cdot)\|_p\Big). \tag{2.5}$$

In addition, if $p>\frac{1}{\beta}$*, then* $t\mapsto\psi(t)$ *is continuous on* $[0,T]$ *and*

$$|\psi(t)|\leqslant K\Big(\|\varphi(0,\cdot)\|_p+\|\bar\varphi(\cdot)\|_p\Big),\quad t\in[0,T]. \tag{2.6}$$

Hereafter, $K>0$ *stands for an absolute constant which could be different from line to line.*

Proof. By our condition,

$$|\varphi(t,s)|\leqslant|\varphi(0,s)|+\omega(T)\bar\varphi(s)\equiv\widehat\varphi(s),\qquad (t,s)\in\Delta,\ s\neq0,$$

with $\widehat\varphi(\cdot)\in L^p(0,T)$. Thus, by Corollary 2.2,

$$\|\psi(\cdot)\|_p=\Big(\int_0^T\Big|\int_0^t\frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds\Big|^pdt\Big)^{\frac1p}\leqslant\Big[\int_0^T\Big(\int_0^t\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}}ds\Big)^pdt\Big]^{\frac1p}\leqslant\frac{T^\beta}{\beta}\|\widehat\varphi(\cdot)\|_p,$$

proving (2.5). Next, let $p>\frac1\beta$ which is equivalent to $\kappa\equiv(1-\beta)\frac{p}{p-1}<1$. Pick any $t_0\in(0,T)$ and let $\delta>0$ small so that $[t_0-2\delta,t_0+2\delta]\subseteq[0,T]$. For any $t_0-\delta<t<t'<t_0+\delta$, we look at the following:

$$\begin{aligned}
&|\psi(t)-\psi(t')|=\Big|\int_0^t\frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds-\int_0^{t'}\frac{\varphi(t',s)}{(t'-s)^{1-\beta}}ds\Big|\\
&\leqslant\int_0^{t-\delta}|\varphi(t,s)|\Big(\frac{1}{(t-s)^{1-\beta}}-\frac{1}{(t'-s)^{1-\beta}}\Big)ds+\int_0^{t-\delta}\frac{|\varphi(t,s)-\varphi(t',s)|}{(t'-s)^{1-\beta}}ds\\
&\quad+\int_{t-\delta}^t\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}}ds+\int_{t-\delta}^{t'}\frac{\widehat\varphi(s)}{(t'-s)^{1-\beta}}ds\\
&\leqslant(t'-t)^{1-\beta}\int_0^{t-\delta}\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}(t'-s)^{1-\beta}}ds+\omega(|t-t'|)\int_0^{t-\delta}\frac{\bar\varphi(s)}{(t'-s)^{1-\beta}}ds\\
&\quad+\|\widehat\varphi(\cdot)\|_p\Big(\int_{t-\delta}^t\frac{ds}{(t-s)^\kappa}\Big)^{\frac{p-1}{p}}+\|\widehat\varphi(\cdot)\|_p\Big(\int_{t-\delta}^{t'}\frac{ds}{(t'-s)^\kappa}\Big)^{\frac{p-1}{p}}\\
&\leqslant\frac{(t'-t)^{1-\beta}}{\delta^{2(1-\beta)}}\|\widehat\varphi(\cdot)\|_1+\omega(|t-t'|)\Big(\frac{T^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}\|\bar\varphi(\cdot)\|_p+\|\widehat\varphi(\cdot)\|_p\Big[\Big(\frac{\delta^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}+\Big(\frac{(t'-t+\delta)^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}\Big].
\end{aligned}$$

Hence, for any $\varepsilon>0$, we first take $\delta>0$ sufficiently small so that

$$\|\widehat\varphi(\cdot)\|_p\Big[\Big(\frac{\delta^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}+\Big(\frac{(3\delta)^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}\Big]<\frac{\varepsilon}{2}.$$

Since the modulus of continuity $\omega(\cdot)$ is continuous and $\omega(0)=0$, we can take $\bar\delta\in(0,\delta)$ even smaller so that

$$\frac{\bar\delta^{1-\beta}}{\delta^{2(1-\beta)}}\|\widehat\varphi(\cdot)\|_1+\omega(\bar\delta)\Big(\frac{T^{1-\kappa}}{1-\kappa}\Big)^{\frac{p-1}{p}}\|\bar\varphi(\cdot)\|_p<\frac{\varepsilon}{2}.$$

Combining the above, we see that $\psi(\cdot)$ is continuous at t_0. The points 0 and T can be treated similarly. Finally,

$$\begin{aligned}
|\psi(t)|&\leqslant\int_0^t\frac{|\varphi(0,s)|+\omega(T)\bar\varphi(s)}{(t-s)^{1-\beta}}ds\leqslant\Big(\|\varphi(0,\cdot)\|_p+\omega(T)\|\bar\varphi(\cdot)\|_p\Big)\Big(\int_0^t\frac{ds}{(t-s)^{(1-\beta)\frac{p}{p-1}}}\Big)^{\frac{p-1}{p}}\\
&=\Big(\|\varphi(0,\cdot)\|_p+\omega(T)\|\bar\varphi(\cdot)\|_p\Big)\Big(\frac{p-1}{\beta p-1}t^{\frac{\beta p-1}{p-1}}\Big)^{\frac{p-1}{p}}\leqslant K\Big(\|\varphi(0,\cdot)\|_p+\|\bar\varphi(\cdot)\|_p\Big),\ t\in[0,T].
\end{aligned}$$

This proves (2.6). $\square$

The above lemma show that for any $p \in [1,\infty)$, under condition (2.3), one has $\psi(\cdot) \in L^p(0,T;\mathbb{R}^n)$. To guarantee the continuity of $\psi(\cdot)$ on $[0,T]$, we need $\bar\varphi(\cdot) \in L^p(0,T)$ for $p > \frac{1}{\beta}$.

The following result is an extended Gronwall's inequality with a singular kernel.

Lemma 2.4. *Let $\beta \in (0,1)$ and $q > \frac{1}{\beta}$. Let $L(\cdot), a(\cdot), y(\cdot)$ be nonnegative functions with $L(\cdot) \in L^q(0,T)$ and $a(\cdot), y(\cdot) \in L^{\frac{q}{q-1}}(0,T)$. Suppose*

$$y(t) \leqslant a(t) + \int_0^t \frac{L(s)y(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T]. \tag{2.7}$$

Then there exists a constant $K > 0$ such that

$$y(t) \leqslant a(t) + K\int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T]. \tag{2.8}$$

Proof. Since $L(\cdot) \in L^q(0,T)$ and $y(\cdot) \in L^{\frac{q}{q-1}}(0,T)$, we have that $L(\cdot)y(\cdot) \in L^1(0,T)$. Hence, the integral on the right-hand side of (2.7) is well-defined, as a function in $L^1(0,T)$. Now, we observe the following:

$$\begin{aligned} y(t) &\leqslant a(t) + \int_0^t \frac{L(s)y(s)}{(t-s)^{1-\beta}}ds \leqslant a(t) + \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds + \int_0^t \frac{L(s)}{(t-s)^{1-\beta}}\int_0^s \frac{L(\tau)y(\tau)}{(s-\tau)^{1-\beta}}d\tau ds \\ &\leqslant a(t) + \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds + \int_0^t L(\tau)\Big[\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds\Big]y(\tau)d\tau, \qquad \text{a.e. } t \in [0,T]. \end{aligned}$$

Let $r = \dfrac{s-\tau}{t-\tau}$. Then $s = \tau + (t-\tau)r$ and $ds = (t-\tau)dr$. Thus,

$$\begin{aligned} &\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds \leqslant \Big(\int_\tau^t L(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_\tau^t \frac{ds}{(t-s)^{(1-\beta)\frac{q}{q-1}}(s-\tau)^{(1-\beta)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}} \\ &\leqslant \|L(\cdot)\|_q \frac{1}{(t-\tau)^{2(1-\beta)-\frac{q-1}{q}}}\Big(\int_0^1 \frac{dr}{(1-r)^{(1-\beta)\frac{q}{q-1}}r^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}. \end{aligned}$$

Since $q > \frac{1}{\beta}$ which is equivalent to

$$0 < 1 - \frac{(1-\beta)q}{q-1} = \frac{q-1-q+\beta q}{q-1} = \frac{\beta q - 1}{q-1}, \tag{2.9}$$

we obtain

$$\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds \leqslant \frac{\|L(\cdot)\|_q}{(t-\tau)^{2(1-\beta)-\frac{q-1}{q}}}B\Big(\frac{\beta q-1}{q-1}, \frac{\beta q-1}{q-1}\Big)^{\frac{q-1}{q}} \equiv \frac{c_1}{(t-\tau)^{1-\beta_1}},$$

with $B(\cdot\,,\cdot)$ being the Beta function and

$$c_1 = \|L(\cdot)\|_q B\Big(\frac{\beta q-1}{q-1}, \frac{\beta q-1}{q-1}\Big)^{\frac{q-1}{q}}, \qquad \beta_1 = 1 - \Big(2(1-\beta) - \frac{q-1}{q}\Big) = \beta + \Big(\beta - \frac{1}{q}\Big) > \beta.$$

Consequently,

$$\begin{aligned} y(t) &\leqslant a(t) + \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds + c_1\int_0^t \frac{L(s)y(s)}{(t-s)^{1-\beta_1}}ds \\ &\leqslant a(t) + \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds + c_1\int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_1}}ds + c_1\int_0^t \frac{L(s)}{(t-s)^{1-\beta_1}}\int_0^s \frac{L(\tau)y(\tau)}{(s-\tau)^{1-\beta}}d\tau ds \\ &= a(t) + \sum_{i=0}^1 c_i\int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_1\int_0^t L(\tau)\Big[\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds\Big]y(\tau)d\tau, \qquad \text{a.e. } t \in [0,T] \end{aligned}$$

with $c_0 = 1$ and $\beta_0 = \beta$. Let $r = \frac{s-\tau}{t-\tau}$. Then $s = \tau + (t-\tau)r$ and $ds = (t-\tau)dr$. Thus,

$$\begin{aligned}&\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds \leqslant \Big(\int_\tau^t L(s)^q ds\Big)^{\frac{1}{q}} \Big(\int_\tau^t \frac{ds}{(t-s)^{(1-\beta_1)\frac{q}{q-1}}(s-\tau)^{(1-\beta)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}} \\ &= \|L(\cdot)\|_q \frac{1}{(t-\tau)^{2-\beta-\beta_1-\frac{q-1}{q}}} \Big(\int_0^1 \frac{dr}{(1-r)^{(1-\beta_1)\frac{q}{q-1}} r^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.\end{aligned}$$

Since $q > \frac{1}{\beta} > \frac{1}{\beta_1}$, we have

$$0 < 1 - \frac{(1-\beta_1)q}{q-1} = \frac{q-1-q+\beta_1 q}{q-1} = \frac{\beta_1 q - 1}{q-1}. \tag{2.10}$$

Hence, we obtain

$$c_1 \int_\tau^t \frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds \leqslant \frac{c_1\|L(\cdot)\|_q}{(t-\tau)^{2-\beta-\beta_1-\frac{q-1}{q}}} B\Big(\frac{\beta q-1}{q-1}, \frac{\beta_1 q-1}{q-1}\Big)^{\frac{q-1}{q}} \equiv \frac{c_2}{(t-\tau)^{1-\beta_2}},$$

with

$$c_2 = c_1\|L(\cdot)\|_q B\Big(\frac{\beta q-1}{q-1}, \frac{\beta_1 q-1}{q-1}\Big)^{\frac{q-1}{q}}, \qquad \beta_2 = 1 - \Big(2-\beta-\beta_1-\frac{q-1}{q}\Big) = \beta + 2\Big(\beta - \frac{1}{q}\Big) > \beta.$$

Consequently,

$$y(t) \leqslant a(t) + \sum_{i=0}^{1} c_i \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_2 \int_0^t \frac{L(s)y(s)}{(t-s)^{1-\beta_2}}ds, \qquad \text{a.e. } t \in [0,T].$$

By induction, we are able to show that

$$y(t) \leqslant a(t) + \sum_{i=0}^{k-1} c_i \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_k \int_0^t \frac{L(s)y(s)}{(t-s)^{1-\beta_k}}ds, \qquad \text{a.e. } t \in [0,T].$$

with $\beta_i = \beta + i\Big(\beta - \frac{1}{q}\Big)$, $0 \leqslant i \leqslant k$, and recursively defined $c_i \geq 0$:

$$c_i = c_{i-1}\|L(\cdot)\|_q B\Big(\frac{\beta q-1}{q-1}, \frac{\beta_{i-1} q-1}{q-1}\Big)^{\frac{q-1}{q}}, \qquad 1 \leqslant i \leqslant k.$$

We let $k \geqslant 1$ be the smallest integer that $\beta_k \geqslant 1$. The above implies

$$y(t) \leqslant a(t) + \sum_{i=0}^{k-1} c_i \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_k T^{\beta_k - 1} \int_0^t L(s)y(s)ds, \qquad \text{a.e. } t \in [0,T].$$

Then by a standard argument in proving a usual Gronwall's inequality, and using

$$\frac{1}{(t-s)^{1-\beta_i}} \leqslant \frac{K}{(t-s)^{1-\beta}}, \qquad 0 \leqslant i \leqslant k,\ 0 \leqslant s < t \leqslant T,$$

we obtain (2.8). ◻

3 State Equation

In this section, we discuss our state equation (1.1), together with the cost functional (1.2). Let U be a separable metric space with the metric ρ, which could be a non-empty bounded or unbounded set in $\mathbb{R}^m$ with the metric induced by the usual Euclidean norm. With the Borel σ-field, U is regarded as a measurable space. Let $u_0 \in U$ be fixed. For any $p \geqslant 1$, we define

$$\mathscr{U}^p[0,T] = \big\{u : [0,T] \to U \bigm| u(\cdot) \text{ is measurable},\ \rho(u(\cdot), u_0) \in L^p(0,T;\mathbb{R})\big\}.$$

3.1 Well-posedness in L^p space and continuity of the state trajectory

We introduce the following assumptions for the generator $f(\cdot,\cdot,\cdot,\cdot)$ of our state equation.

(H1) Let $f:\Delta\times\mathbb{R}^n\times U\to\mathbb{R}^n$ be a map with $(t,s)\mapsto f(t,s,y,u)$ being measurable, $y\mapsto f(t,s,y,u)$ being continuously differentiable and $(y,u)\mapsto f(t,s,y,u)$ being continuous. There are nonnegative functions $L_0(\cdot),L(\cdot)$ with

$$L_0(\cdot)\in L^{(\frac{p}{1+\beta p}\vee 1)+}(0,T),\qquad L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T), \tag{3.1}$$

for some $p\geqslant 1$ (with the convention that $\frac{1}{0}=\infty$) and for some $\beta\in(0,1)$, $u_0\in U$.

$$|f(t,s,0,u_0)|\leqslant L_0(s),\qquad (t,s)\in\Delta, \tag{3.2}$$

$$|f(t,s,y,u)-f(t,s,y',u')|\leqslant L(s)\big[|y-y'|+\rho(u,u')\big],\qquad (t,s)\in\Delta,\ \ y,y'\in\mathbb{R}^n,\ \ u,u'\in U. \tag{3.3}$$

We note that (3.2)–(3.3) imply

$$|f(t,s,y,u)|\leqslant L_0(s)+L(s)\big[|y|+\rho(u,u_0)\big],\qquad (t,s,y,u)\in\Delta\times\mathbb{R}^n\times U. \tag{3.4}$$

Further, one sees that $L(\cdot)$ belongs to a smaller space than that to which $L_0(\cdot)$ belongs.

We now present the well-posedness of the state equation (1.1) in L^p spaces.

Theorem 3.1. *Let* (H1) *hold with some $p\geqslant 1$ and $\beta\in(0,1)$. Then for any $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$ and $u(\cdot)\in\mathscr{U}^p[0,T]$,* (1.1) *admits a unique solution $y(\cdot)\equiv y(\cdot\,;\eta(\cdot),u(\cdot))\in L^p(0,T;\mathbb{R}^n)$, and the following estimate holds*

$$\|y(\cdot)\|_p\leqslant\|\eta(\cdot)\|_p+K\Big(1+\|\rho(u(\cdot),u_0)\|_{L^p(0,T)}\Big). \tag{3.5}$$

If $(\eta_1(\cdot),u_1(\cdot))$, $(\eta_2(\cdot),u_2(\cdot))\in L^p(0,T;\mathbb{R}^n)\times\mathscr{U}^p[0,T]$ and $y_1(\cdot)$, $y_2(\cdot)$ are the solutions of (1.1) *corresponding to $(\eta_1(\cdot),u_1(\cdot))$ and $(\eta_2(\cdot),u_2(\cdot))$, respectively, then*

$$\|y_1(\cdot)-y_2(\cdot)\|_p\leqslant K\Big\{\|\eta_1(\cdot)-\eta_2(\cdot)\|_p+\Big[\int_0^T\Big(\int_0^t\frac{|f(t,s,y_1(s),u_1(s))-f(t,s,y_1(s),u_2(s))|}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac{1}{p}}\Big\}. \tag{3.6}$$

Proof. Fix any $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$ and $u(\cdot)\in\mathscr{U}^p[0,T]$. For any $z(\cdot)\in L^p(0,S;\mathbb{R}^n)$ with $0<S\leqslant T$, define

$$\mathscr{T}[z(\cdot)](t)=\eta(t)+\int_0^t\frac{f(t,s,z(s),u(s))}{(t-s)^{1-\beta}}ds,\qquad \text{a.e. } t\in[0,S].$$

Then by Corollary 2.2, for any $q\geqslant 1$, $0\leqslant\varepsilon<\frac{\beta}{1-\beta}$, with $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{1+\varepsilon}$,

$$\begin{aligned}\|\mathscr{T}[z(\cdot)]\|_{L^p(0,S;\mathbb{R}^n)}&\leqslant\|\eta(\cdot)\|_{L^p(0,S;\mathbb{R}^n)}+\Big(\int_0^S\Big|\int_0^t\frac{f(t,s,z(s),u(s))}{(t-s)^{1-\beta}}ds\Big|^p dt\Big)^{\frac{1}{p}}\\&\leqslant\|\eta(\cdot)\|_p+\Big[\int_0^S\Big(\int_0^t\frac{L_0(s)+L(s)\big[|z(s)|+\rho(u(s),u_0)\big]}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac{1}{p}}\\&\leqslant\|\eta(\cdot)\|_p+\Big(\frac{S^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}}\|L_0(\cdot)+L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\|_{L^q(0,S)}.\end{aligned} \tag{3.7}$$

We consider three cases.

Case 1. $p>\frac{1}{1-\beta}$. In this case, $\frac{1}{\beta}>\frac{p}{p-1}$ and $\frac{p}{1+\beta p}>1$. For any $\varepsilon\in(0,\frac{\beta}{1-\beta})$, which is equivalent to $(1-\beta)(1+\varepsilon)<1$, we have (noting $p>\frac{1}{1-\beta}$ for the current case)

$$\frac{1}{q}=\frac{1}{p}+1-\frac{1}{1+\varepsilon}<\frac{1}{p}+1-\frac{1}{1+\frac{\beta}{1-\beta}}=\frac{1}{p}+\beta<1,\qquad \frac{1}{q}-\frac{1}{p}=1-\frac{1}{1+\varepsilon}<\beta.$$

Thus,

$$q \searrow \frac{p}{1+\beta p} > 1, \qquad \frac{pq}{p-q} \searrow \frac{1}{\beta}, \qquad \text{as } \varepsilon \nearrow \frac{\beta}{1-\beta}.$$

Since $L_0(\cdot) \in L^{\frac{p}{1+\beta p}+}(0,T)$ and $L(\cdot) \in L^{\frac{1}{\beta}+}(0,T)$, we could find ε close enough to $\frac{\beta}{1-\beta}$ so that $L_0(\cdot) \in L^q(0,T)$ and $L(\cdot) \in L^{\frac{pq}{p-q}}(0,T)$. Then

$$\|L_0(\cdot)+L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\|_{L^q(0,S)} \leqslant \|L_0(\cdot)\|_{L^q(0,T)} + \|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,T)} \|\,|z(\cdot)|+\rho(u(\cdot),u_0)\|_{L^p(0,S)}, \tag{3.8}$$

which leads to

$$\begin{aligned}
&\|\mathscr{T}[z(\cdot)]\|_{L^p(0,S;\mathbb{R}^n)} \\
&\leqslant \|\eta(\cdot)\|_p + \Big(\tfrac{S^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}} \Big\{\|L_0(\cdot)\|_{L^q(0,T)} + \|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,T)} \|\,|z(\cdot)|+\rho(u(\cdot),u_0)\|_{L^p(0,S)}\Big\}.
\end{aligned} \tag{3.9}$$

Consequently, $\mathscr{T} : L^p(0,S;\mathbb{R}^n) \to L^p(0,S;\mathbb{R}^n)$, for any $S \in (0,T]$. Next, let $\delta \in (0,T]$ be undetermined, and let $z_1(\cdot),\ z_2(\cdot) \in L^p(0,\delta;\mathbb{R}^n)$. We look at the following: (again making use of Corollary 2.2)

$$\begin{aligned}
&\|\mathscr{T}[z_1(\cdot)] - \mathscr{T}[z_2(\cdot)]\|_{L^p(0,\delta;\mathbb{R}^n)} \equiv \Big(\int_0^\delta |\mathscr{T}[z_1(\cdot)](t) - \mathscr{T}[z_2(\cdot)](t)|^p dt\Big)^{\frac{1}{p}} \\
&\leqslant \Big[\int_0^\delta \Big|\int_0^t \frac{f(t,s,z_1(s),u(s)) - f(t,s,z_2(s),u(s))}{(t-s)^{1-\beta}} ds\Big|^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \Big[\int_0^\delta \Big(\int_0^t \frac{L(s)|z_1(s)-z_2(s)|}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \leqslant \Big(\frac{\delta^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}} \|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,T)} \|z_1(\cdot)-z_2(\cdot)\|_{L^p(0,\delta;\mathbb{R}^n)}.
\end{aligned}$$

Taking $\delta \in (0,T]$ such that $\Big(\frac{\delta^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}} \|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,T)} < 1$, one sees that $\mathscr{T} : L^p(0,\delta;\mathbb{R}^n) \to L^p(0,\delta;\mathbb{R}^n)$ is a contraction. Hence, it admits a unique fixed point $y(\cdot)$ on $L^p(0,\delta;\mathbb{R}^n)$, which is the unique solution of the state equation (1.1) on $[0,\delta]$. Next, we look (1.1) on $[\delta,2\delta]$. For any $z(\cdot) \in L^p(\delta,2\delta;\mathbb{R}^n)$, define

$$\mathscr{T}[z(\cdot)](t) = \eta(t) + \int_0^\delta \frac{f(t,s,y(s),u(s))}{(t-s)^{1-\beta}} ds + \int_\delta^t \frac{f(t,s,z(s),u(s))}{(t-s)^{1-\beta}} ds, \qquad \text{a.e. } t \in [\delta,2\delta].$$

Denote $\widetilde{z}(\cdot) = y(\cdot)\mathbf{1}_{[0,\delta)}(\cdot) + z(\cdot)\mathbf{1}_{[\delta,2\delta]}(\cdot) \in L^p(0,2\delta;\mathbb{R}^n)$. Then similar to (3.7)–(3.9) (with $S = 2\delta$),

$$\begin{aligned}
&\Big\|\eta(\cdot) + \int_0^\delta \frac{f(\cdot\,,s,y(s),u(s))}{(\cdot-s)^{1-\beta}} ds\Big\|_{L^p(\delta,2\delta;\mathbb{R}^n)} = \Big[\int_\delta^{2\delta} \Big|\eta(t) + \int_0^\delta \frac{f(t,s,y(s),u(s))}{(t-s)^{1-\beta}} ds\Big|^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \Big(\int_\delta^{2\delta} |\eta(t)|^p dt\Big)^{\frac{1}{p}} + \Big[\int_\delta^{2\delta} \Big(\int_0^\delta \frac{|f(t,s,y(s),u(s))|}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_p + \Big[\int_\delta^{2\delta} \Big(\int_0^\delta \frac{L_0(s) + L(s)\big[|y(s)| + \rho(u(s),u_0)\big]}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_p + \Big[\int_0^{2\delta} \Big(\int_0^t \frac{L_0(s)}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} + \Big[\int_0^{2\delta} \Big(\int_0^t \frac{L(s)\big[|\widetilde{z}(s)| + \rho(u(s),u_0)\big]}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_p + K\Big(1 + \|y(\cdot)\|_{L^p(0,\delta;\mathbb{R}^n)} + \|z(\cdot)\|_{L^p(\delta,2\delta;\mathbb{R}^n)} + \|\rho(u(\cdot),u_0)\|_{L^p(0,T)}\Big).
\end{aligned}$$

Hence, similarly to the proof of (3.9), we can get $\mathscr{T} : L^p(\delta,2\delta;\mathbb{R}^n) \to L^p(\delta,2\delta;\mathbb{R}^n)$. Let $z_1(\cdot),\ z_2(\cdot) \in L^p(\delta,2\delta;\mathbb{R}^n)$. By Corollary 2.2, we have

$$\begin{aligned}
&\|\mathscr{T}[z_1(\cdot)] - \mathscr{T}[z_2(\cdot)]\|_{L^p(\delta,2\delta;\mathbb{R}^n)} \equiv \Big(\int_\delta^{2\delta} |\mathscr{T}[z_1(\cdot)](t) - \mathscr{T}[z_2(\cdot)](t)|^p dt\Big)^{\frac{1}{p}} \\
&= \Big[\int_\delta^{2\delta} \Big|\int_\delta^t \frac{f(t,s,z_1(s),u(s)) - f(t,s,z_2(s),u(s))}{(t-s)^{1-\beta}} ds\Big|^p dt\Big]^{\frac{1}{p}} \leqslant \Big[\int_\delta^{2\delta} \Big(\int_\delta^t \frac{L(s)|z_1(s)-z_2(s)|}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \Big(\tfrac{\delta^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}} \|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,T)} \|z_1(\cdot)-z_2(\cdot)\|_{L^p(\delta,2\delta;\mathbb{R}^n)}.
\end{aligned}$$

Then we obtain the existence and uniqueness of the solution to the state equation on $[\delta, 2\delta]$. By induction, we can get the solution $y(\cdot)$ in $[0,\delta]$, $[\delta, 2\delta]$, $\cdots$, $\big[([\frac{T}{\delta}]-1)\delta, [\frac{T}{\delta}]\delta\big]$, $\big[[\frac{T}{\delta}]\delta, T\big]$.

Now, let $(\eta_1(\cdot), u_1(\cdot))$, $(\eta_2(\cdot), u_2(\cdot)) \in L^p(0,T;\mathbb{R}^n) \times \mathscr{U}^p[0,T]$ and $y_1(\cdot)$, $y_2(\cdot)$ be the corresponding solutions. Then

$$\begin{aligned}|y_1(t)-y_2(t)| &\leqslant |\eta_1(t)-\eta_2(t)| + \int_0^t \frac{|f(t,s,y_1(s),u_1(s)) - f(t,s,y_1(s),u_2(s))|}{(t-s)^{1-\beta}}ds \\ &\quad + \int_0^t \frac{|f(t,s,y_1(s),u_2(s)) - f(t,s,y_2(s),u_2(s))|}{(t-s)^{1-\beta}}ds \\ &\leqslant |\eta_1(t)-\eta_2(t)| + \int_0^t \frac{|f(t,s,y_1(s),u_1(s)) - f(t,s,y_1(s),u_2(s))|}{(t-s)^{1-\beta}}ds + \int_0^t \frac{L(s)|y_1(s)-y_2(s)|}{(t-s)^{1-\beta}}ds \\ &\equiv a(t) + \int_0^t \frac{L(s)|y_1(s)-y_2(s)|}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t\in[0,T].\end{aligned}$$

Hence, by Lemma 2.4,

$$|y_1(t)-y_2(t)| \leqslant a(t) + K\int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t\in[0,T], \tag{3.10}$$

for some constant $K>0$. Consequently, similar to (3.7),

$$\begin{aligned}&\|y_1(\cdot)-y_2(\cdot)\|_p \leqslant \|a(\cdot)\|_p + K\Big[\int_0^T\Big(\int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac1p} \\ &\leqslant K\Big\{\|\eta_1(\cdot)-\eta_2(\cdot)\|_p + \Big[\int_0^T\Big(\int_0^t \frac{|f(t,s,y_1(s),u_1(s)) - f(t,s,y_1(s),u_2(s))|}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac1p}\Big\},\end{aligned}$$

proving the stability estimate. We can use the similar argument to prove (3.5).

Case 2. $1<p\leqslant \frac{1}{1-\beta}$. In this case,

$$\frac1\beta \leqslant \frac{p}{p-1}, \qquad \frac{p}{1+\beta p} \leqslant 1.$$

Also, since $1-\beta \leqslant \frac1p < 1$, for any $\varepsilon\in(0,p-1)$, the following holds:

$$1-\beta \leqslant \frac1p < \frac{1}{1+\varepsilon}.$$

This implies $(1-\beta)(1+\varepsilon)<1$. Then

$$\frac1p < \frac1q = \frac1p + 1 - \frac{1}{1+\varepsilon} \nearrow 1, \qquad \frac{p-q}{pq} = \frac1q - \frac1p = 1 - \frac{1}{1+\varepsilon} \nearrow \frac{p-1}{p} \qquad \text{as } \varepsilon \nearrow p-1.$$

Consequently, by $L_0(\cdot)\in L^{1+}(0,T)$ and $L(\cdot)\in L^{\frac{p}{p-1}+}(0,T)$, we could find ε close enough to $p-1$ so that $L_0(\cdot)\in L^q(0,T)$ and $L(\cdot)\in L^{\frac{pq}{p-q}}(0,T)$. Then the rest of the proof will be the same as Case 1.

Case 3. $p=1$. In this case, the condition reads $L_0(\cdot)\in L^{1+}(0,T)$ and $L(\cdot)\in L^\infty(0,T)$. Then we take $\varepsilon=0$, and (3.9) reads

$$\|\mathscr{T}[z(\cdot)]\|_{L^1(0,S;\mathbb{R}^n)} \leqslant \|\eta(\cdot)\|_1 + \frac{S^\beta}{\beta}\Big\{\|L_0(\cdot)\|_{L^1(0,T)} + \|L(\cdot)\|_{L^\infty(0,T)}\|\,|z(\cdot)| + \rho(u(\cdot),u_0)\|_{L^1(0,S)}\Big\}. \tag{3.11}$$

The rest of the proof is similar to that of Case 1. □

Let us make some comments and observations on the above theorem. First of all, the above theorem gives some sufficient conditions under which for $(\eta(\cdot),u(\cdot))\in L^p(0,T;\mathbb{R}^n)\times\mathscr{U}^p[0,T]$, equation (1.1) admits a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. The conditions we imposed in (H1) are compatibility conditions of the integrability for the free term $\eta(\cdot)$, the control $u(\cdot)$, and the coefficients $L_0(\cdot)$ and $L(\cdot)$. From the above, we see that if $(\eta(\cdot), u(\cdot)) \in L^p(0,T)\times\mathcal{U}^p[0,T]$ with $p>\frac{1}{1-\beta}$, then by assuming $L(\cdot)\in L^{\frac{1}{\beta}+}(0,T)$ (in the current case, $\frac{p}{p-1}<\frac{1}{\beta}$), the equation has a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. This is the case, in particular, if $\eta(\cdot)\in L^\infty(0,T;\mathbb{R}^n)$ and U is bounded (under the metric ρ). On the other hand, if $1\leqslant p\leqslant\frac{1}{1-\beta}$, that is, say, the free term and/or the control have weaker integrability, then we need to strengthen the integrability of $L(\cdot)$ from $L^{\frac{1}{\beta}+}$ to $L^{\frac{p}{p-1}+}$ (in the current case, $\frac{p}{p-1}\geqslant\frac{1}{\beta}$) to get L^p solution $y(\cdot)$. But, the integrability of $L_0(\cdot)$ is only required to be $L^{1+}(0,T)$. Finally, since we have used the contraction mapping theorem to establish the well-posedness of the state equation, one can see that the solution to the state equation can be obtained by a Picard iteration.

Let us present an example from which we could get some feeling about the above result.

Example 3.2. Let $f(t,s,y,u)=\dfrac{\sqrt{|s-1|^{2\delta-2}+y^2}}{|s-1|^{1-\alpha}}$, $\forall(t,s,y,u)\in\Delta\times\mathbb{R}\times U$, $s\neq1$. Consider the following Volterra integral equation

$$y(t)=\frac{1}{|t-1|^{1-\gamma}}+\int_0^t\frac{\sqrt{(s-1)^{2\delta-2}+y(s)^2}}{|s-1|^{1-\alpha}(t-s)^{1-\beta}}ds,\qquad\text{a.e. } t\in[0,T],\tag{3.12}$$

for some $\alpha,\beta\in(0,1)$, $\gamma,\delta\in(0,1]$, and with $T>1$. In this case, we can take

$$\eta(t)=\frac{1}{|t-1|^{1-\gamma}},\qquad L_0(s)=\frac{1}{|s-1|^{2-\alpha-\delta}},\qquad L(s)=\frac{1}{|s-1|^{1-\alpha}},\quad t\neq1,\ s\neq1.$$

We see that

$$\eta(\cdot)\in L^p(0,T)\qquad\Longleftarrow\qquad p(1-\gamma)<1\qquad\Longleftrightarrow\qquad p<\frac{1}{1-\gamma}\qquad(\frac{1}{0}\triangleq\infty);$$

$$L_0(\cdot)\in L^{(\frac{p}{1+\beta p}\vee1)+}(0,T)\quad\Longleftarrow\quad\begin{cases}(2-\alpha-\delta)\frac{p}{1+\beta p}<1\quad\Longleftrightarrow\quad 2-\alpha-\beta-\delta<\frac{1}{p}\\ \qquad\qquad\qquad\qquad\quad\Longleftrightarrow\quad p<\dfrac{1}{(2-\alpha-\beta-\delta)^+},\\ 2-\alpha-\delta<1\quad\Longleftrightarrow\quad\alpha+\delta>1;\end{cases}$$

$$L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T)\quad\Longleftarrow\quad\begin{cases}\dfrac{1-\alpha}{\beta}<1\quad\Longleftrightarrow\quad\alpha+\beta>1,\\ (1-\alpha)\dfrac{p}{p-1}<1\quad\Longleftrightarrow\quad1-\alpha<1-\dfrac{1}{p}\quad\Longleftrightarrow\quad p>\dfrac{1}{\alpha}.\end{cases}$$

Hence, equation (3.12) has a unique solution $y(\cdot)\in L^p(0,T)$ for any $p\in(\frac{1}{\alpha},\frac{1}{(1-\gamma)\vee(2-\alpha-\beta-\delta)^+})$, provided

$$\alpha+\beta>1,\qquad\alpha+\delta>1.\tag{3.13}$$

We point out that in general, the solution $y(\cdot)$ of the equation (3.12) is not necessarily continuous, even if the free term $\eta(\cdot)$ is continuous. In fact, let $\gamma=1$. Then $\eta(t)\equiv1$ which is continuous. It is seen that the solution $y(\cdot)$ is positive (which can be seen from a Picard iteration). Consequently,

$$\lim_{t\to1}y(t)\geqslant1+\lim_{t\to1}\int_0^t\frac{ds}{|s-1|^{2-\alpha-\delta}(t-s)^{1-\beta}}=\int_0^1\frac{ds}{(1-s)^{3-\alpha-\beta-\delta}}=\infty,$$

provided

$$3-\alpha-\beta-\delta>1\qquad\Longleftrightarrow\qquad\alpha+\beta+\delta<2.\tag{3.14}$$

This will be the case if we take

$$\alpha=\frac{2}{3},\quad\beta=\delta=\frac{1}{2}.$$

In this case, the solution $y(\cdot)\in L^p(0,T)$ exists with $p\in(\frac{3}{2},3)$ and it is discontinuous at $t=1$.

Note that in the above example, the solution $y(\cdot)$ is discontinuous at $t = 1$ only, which is the singularity of $L_0(\cdot)$ and $L(\cdot)$. Since in our optimal control problem, the values $y(t_i)$ of $y(\cdot)$ are needed in the cost functional (1.2), we need to locate the discontinuity points of the solution $y(\cdot)$ *a priori* based on the information of $\eta(\cdot)$, $L_0(\cdot)$, and $L(\cdot)$. To explore the continuity of the state trajectory, let us assume (H1) and let $y(\cdot) \in L^p(0,T;\mathbb{R}^n)$ be the unique solution to the state equation (1.1) which is rewritten here:

$$y(t) = \eta(t) + \int_0^t \frac{f(t,s,y(s),u(s))}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T]. \tag{3.15}$$

Define

$$\psi(t) = \int_0^t \frac{f(t,s,y(s),u(s))}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T].$$

We introduce the following assumption.

(H1)$'$ Let (H1) holds with $p > \frac{1}{\beta}$,

$$L_0(\cdot) \in L^{\frac{1}{\beta}+}(0,T), \qquad L(\cdot) \in L^{\frac{p}{p\beta-1}+}(0,T), \tag{3.16}$$

and

$$|f(t,s,y,u) - f(t',s,y,u)| \leqslant K\omega(|t-t'|)(1+|y|), \qquad (t,s),(t',s) \in \Delta,\ y \in \mathbb{R}^n,\ u \in U, \tag{3.17}$$

for some modulus of continuity $\omega(\cdot)$.

Note that in the case $p > \frac{1}{\beta}$,

$$L^{\frac{1}{\beta}+}(0,T) \subset L^{(\frac{p}{1+\beta p}\vee 1)+}(0,T), \qquad L^{\frac{p}{p\beta-1}+}(0,T) \subset L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T).$$

Hence, (H1)$'$ is stronger than (H1). We have the following result.

Proposition 3.3. *Let* (H1)$'$ *hold. Then* $y(\cdot) - \eta(\cdot) \in C([0,T];\mathbb{R}^n)$.

Proof. By Theorem 3.1, for any $u(\cdot) \in \mathscr{U}^p[0,T]$, (1.1) admits a unique solution $y(\cdot) \in L^p(0,T;\mathbb{R}^n)$. By (3.4),

$$|f(t,s,y(s),u(s))| \leqslant L_0(s) + L(s)\big[|y(s)| + \rho(u(s),u_0)\big] \equiv \widehat{\varphi}(s), \qquad (t,s) \in \Delta. \tag{3.18}$$

Now, by (3.16), we have $L(\cdot) \in L^r(0,T)$ for some $r > \frac{p}{p\beta-1}$. Let $q = \frac{rp}{r+p}$. Then

$$\frac{1}{\beta} < q < p, \quad \frac{pq}{p-q} = r.$$

Consequently,

$$\int_0^T |L(s)|^q\big(|y(s)| + \rho(u(s),u_0)\big)^q ds \leqslant \Big(\int_0^T \big[|y(s)| + \rho(u(s),u_0)\big]^p ds\Big)^{\frac{q}{p}}\Big(\int_0^T L(s)^{\frac{pq}{p-q}}ds\Big)^{\frac{p-q}{p}} < \infty.$$

Hence, $\widehat{\varphi}(\cdot)$ defined by (3.18) is in $L^q(0,T)$ for some $q > \frac{1}{\beta}$. Then our conclusion follows from Lemma 2.3. $\square$

3.2 Special cases

In this subsection, we look at some special cases.

1. Linear Volterra integral equations. Let $p \geqslant 1$. Consider the following linear integral equation:

$$y(t) = \eta(t) + \int_0^t \frac{A(t,s)y(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T], \tag{3.19}$$

where $\beta \in (0,1)$, $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$ and $A : \Delta \to \mathbb{R}^{n\times n}$ is measurable and satisfies

$$|A(t,s)| \leqslant L(s), \qquad (t,s) \in \Delta, \tag{3.20}$$

for some measurable function $L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T)$. Then, by Theorem 3.1, for any $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$, equation (3.19) admits a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. For any $s\in[0,T)$, consider

$$\Phi(t,s)=\frac{A(t,s)}{(t-s)^{1-\beta}}+\int_s^t\frac{A(t,\tau)\Phi(\tau,s)}{(t-\tau)^{1-\beta}}d\tau,\qquad \text{a.e. } t\in(s,T],\tag{3.21}$$

where

$$|A(t,s)|\leqslant L(s),\quad (t,s)\in\Delta,\qquad L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T)\text{ with }1\leqslant p<\frac{1}{1-\beta}.$$

We have the following result.

Proposition 3.4. *For almost all $s\in[0,T)$, equation (3.21) admits a unique solution $\Phi(\cdot,s)$ on $[s,T]$ and the following holds:*

$$\Big(\int_s^T|\Phi(t,s)|^pdt\Big)^{\frac{1}{p}}\leqslant K[1+L(s)],\qquad \text{a.e. } s\in[0,T].\tag{3.22}$$

Consequently,

$$\int_0^T\Big(\int_s^T|\Phi(t,s)|^pdt\Big)^{\frac{1}{p-1}}ds<\infty.\tag{3.23}$$

Moreover, the following holds:

$$\Big(\int_0^T\Big|\int_0^t\Phi(t,s)\eta(s)ds\Big|^pdt\Big)^{\frac{1}{p}}\leqslant\Big[\int_0^T\Big(\int_s^T|\Phi(t,s)|^pdt\Big)^{\frac{1}{p-1}}ds\Big]^{\frac{p-1}{p}}\|\eta(\cdot)\|_p.\tag{3.24}$$

Proof. Fix an $s\in[0,T)$ at which both $A(t,s)$ and $L(s)$ are well-defined. Following the proof of Theorem 3.1 on $[\delta,2\delta]$, we see that equation (3.21) admits a unique solution $\Phi(\cdot\,,s)$ on $[s,T]$. Moreover, the corresponding estimate (3.5) reads

$$\Big(\int_s^T|\Phi(t,s)|^pdt\Big)^{\frac{1}{p}}\leqslant\Big(\int_s^T\frac{|A(t,s)|^p}{(t-s)^{(1-\beta)p}}dt\Big)^{\frac{1}{p}}+K\leqslant\Big(\int_s^T\frac{L(s)^p}{(t-s)^{(1-\beta)p}}dt\Big)^{\frac{1}{p}}+K\leqslant K[1+L(s)],\ s\in(0,T].$$

This proves (3.22). Since $L(\cdot)\in L^{\frac{1}{\beta}\vee\frac{p}{p-1}}(0,T)$, we obtain (3.23).

Now, for any $\eta(\cdot)\in L^p(0,T)$, and any $R>0$, we truncate both $|\Phi(\cdot\,,\cdot)|$ and $|\eta(\cdot)|$ as follows:

$$\big[|\Phi(t,s)|\big]_R=|\Phi(t,s)|\wedge R,\qquad \big[|\eta(s)|\big]_R=|\eta(s)|\wedge R.$$

Then, both $\big[|\Phi(\cdot\,,\cdot)|\big]_R$ and $[|\eta(\cdot)|]_R$ are bounded. Now, thanks to the above estimate, we have

$$\begin{aligned}
&\int_0^T\Big(\int_0^t\big[|\Phi(t,s)|\big]_R\big[|\eta(s)|\big]_Rds\Big)^pdt=\int_0^T\int_0^t\Big(\int_0^t\big[|\Phi(t,\tau)|\big]_R\big[|\eta(\tau)|\big]_Rd\tau\Big)^{p-1}\big[|\Phi(t,s)|\big]_R\big[|\eta(s)|\big]_Rdsdt\\
&=\int_0^T\int_s^T\Big(\int_0^t\big[|\Phi(t,\tau)|\big]_R\big[|\eta(\tau)|\big]_Rd\tau\Big)^{p-1}\big[|\Phi(t,s)|\big]_R\big[|\eta(s)|\big]_Rdtds\\
&\leqslant\int_0^T\Big[\int_s^T\Big(\int_0^t\big[|\Phi(t,\tau)|\big]_R\big[|\eta(\tau)|\big]_Rd\tau\Big)^pdt\Big]^{\frac{p-1}{p}}\Big(\int_s^T\big[|\Phi(t,s)|\big]_R^p\big[|\eta(s)|\big]_R^pdt\Big)^{\frac{1}{p}}ds\\
&\leqslant\Big[\int_0^T\Big(\int_0^t\big[|\Phi(t,\tau)|\big]_R\big[|\eta(\tau)|\big]_Rd\tau\Big)^pdt\Big]^{\frac{p-1}{p}}\int_0^T\big[|\eta(s)|\big]_R\Big(\int_s^T\big[|\Phi(t,s)|\big]_R^pdt\Big)^{\frac{1}{p}}ds\\
&\leqslant\Big[\int_0^T\Big(\int_0^t\big[|\Phi(t,\tau)|\big]_R\big[|\eta(\tau)|\big]_Rd\tau\Big)^pdt\Big]^{\frac{p-1}{p}}\Big(\int_0^T\big[|\eta(s)|\big]_R^pds\Big)^{\frac{1}{p}}\Big[\int_0^T\Big(\int_s^T\big[|\Phi(t,s)|\big]_R^pdt\Big)^{\frac{1}{p-1}}ds\Big]^{\frac{p-1}{p}}.
\end{aligned}$$

Hence,

$$\Big[\int_0^T\Big(\int_0^t\big[|\Phi(t,s)|\big]_R\big[|\eta(s)|\big]_Rds\Big)^pdt\Big]^{\frac{1}{p}}\leqslant\Big[\int_0^T\Big(\int_s^T\big[|\Phi(t,s)|\big]_R^pdt\Big)^{\frac{1}{p-1}}ds\Big]^{\frac{p-1}{p}}\Big(\int_0^T\big[|\eta(s)|\big]_R^pds\Big)^{\frac{1}{p}}.$$

By Fatou's Lemma, sending $R\to\infty$, we obtain (3.24). $\square$

Define

$$y(t)=\eta(t)+\int_0^t\Phi(t,s)\eta(s)ds,\qquad \text{a.e. } t\in[0,T]. \tag{3.25}$$

We calculate the following:

$$\begin{aligned}&\int_0^t\frac{A(t,s)y(s)}{(t-s)^{1-\beta}}ds=\int_0^t\frac{A(t,s)}{(t-s)^{1-\beta}}\big[\eta(s)+\int_0^s\Phi(s,\tau)\eta(\tau)d\tau\big]ds\\ &=\int_0^t\frac{A(t,s)\eta(s)}{(t-s)^{1-\beta}}ds+\int_0^t\int_\tau^t\frac{A(t,s)\Phi(s,\tau)\eta(\tau)}{(t-s)^{1-\beta}}dsd\tau\\ &=\int_0^t\Big(\frac{A(t,s)}{(t-s)^{1-\beta}}+\int_s^t\frac{A(t,\tau)\Phi(\tau,s)}{(t-\tau)^{1-\beta}}d\tau\Big)\eta(s)ds=\int_0^t\Phi(t,s)\eta(s)ds=y(t)-\eta(t),\qquad \text{a.e. } t\in[0,T].\end{aligned}$$

Hence, (3.25) gives a representation for the solution to the linear equation (3.19). We call (3.25) the *variation of constant formula.*

2. Fractional differential equations. Let us first recall some basic notions of fractional integrals and derivatives. For $\alpha\in(0,1)$, let

$$[I^\alpha f(\cdot)](t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\qquad t\geqslant 0, \tag{3.26}$$

and as long as the right-hand side is well-defined, where $\Gamma(\cdot)$ is the Gamma function. We call I^α the α-th order integral operator. Let

$$[D^\alpha y(\cdot)](t)=\frac{d}{dt}[I^{1-\alpha}y(\cdot)](t)\equiv\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{y(s)}{(t-s)^\alpha}ds, \tag{3.27}$$

and

$$[D^\alpha_* y(\cdot)](t)=[D^\alpha(y(\cdot)-y(0))](t)=[D^\alpha(y(\cdot)](t)-\frac{y(0)}{\Gamma(1-\alpha)}t^{-\alpha}. \tag{3.28}$$

In particular, when $y(\cdot)\in AC([0,T];\mathbb{R})$, the set of all absolutely continuous functions defined on $[0,T]$, one has

$$[D^\alpha_* y(\cdot)](t)=[I^{1-\alpha}y'(\cdot)](t)\equiv\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y'(s)}{(t-s)^\alpha}ds, \tag{3.29}$$

We call D^α and D^α_* the α-th order *Riemann-Liouville* and *Caputo differential operators*, respectively. We have the following standard result (see [32], Lemmas 2.5 and 2.22).

Proposition 3.5. *Let $\alpha\in(0,1)$. Then for any $y(\cdot)\in L^1(0,T;\mathbb{R})$ with $[I^{1-\alpha}y(\cdot)](\cdot)\in AC([0,T];\mathbb{R})$.*

$$I^\alpha\{D^\alpha[y(\cdot)]\}(t)=y(t)-\frac{I^{1-\alpha}[y(\cdot)](0)}{\Gamma(\alpha)t^{1-\alpha}},\qquad \text{a.e. } t\in(0,T]; \tag{3.30}$$

and for $y(\cdot)\in AC([0,T];\mathbb{R})$, the set of all absolutely continuous functions,

$$I^\alpha\{D^\alpha_*[y(\cdot)]\}(t)=y(t)-y(0). \tag{3.31}$$

Now, let us consider the following fractional differential equation of Riemann-Liouville type:

$$D^\alpha[y(\cdot)](t)=f(t,y(t),u(t)),\qquad t\in[0,T]. \tag{3.32}$$

Applying the operator I^α to the above, we obtain

$$y(t)=\frac{I^{1-\alpha}[y(\cdot)](0)}{\Gamma(\alpha)t^{1-\alpha}}+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s,y(s),u(s))}{(t-s)^{1-\alpha}}ds,\qquad \text{a.e. } t\in[0,T]. \tag{3.33}$$

We refer the readers to Theorem 3.1 in [32] for the equivalence of (3.32) and (3.33).

Likewise, if we consider the following fractional differential equation of Caputo type:

$$D_*^\alpha[y(\cdot)](t)=f(t,y(t),u(t)),\qquad t\in[0,T],\tag{3.34}$$

applying the operator I^α to the above, we obtain

$$y(t)=y(0)+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s,y(s),u(s))}{(t-s)^{1-\alpha}}ds,\qquad \text{a.e. } t\in[0,T].\tag{3.35}$$

We refer the readers to Theorem 3.24 in [32] for the equivalence of (3.34) and (3.35).

From the above, we see that fractional differential equations of Riemann-Liouville and Caputo types are special cases of (1.1).

3.3 A backward linear Volterra integral equation

In this subsection, we consider the following linear backward Volterra integral equation:

$$\psi(t)=\xi(t)+\int_t^T\frac{A(s,t)^\top\psi(s)}{(s-t)^{1-\beta}}ds,\qquad \text{a.e. } t\in[0,T],\tag{3.36}$$

where $A:\Delta\to\mathbb{R}^{n\times n}$ is measurable and satisfies (3.20) with $L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T)$. Such an equation will play an important role in the next section. Let $p>1$. We claim that for any $\xi(\cdot)\in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$, the above equation admits a unique solution $\psi(\cdot)\in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$. In fact, by our condition, we can find an $r>\frac{1}{\beta}\vee\frac{p}{p-1}$ such that $L(\cdot)\in L^r(0,T)$. By $r>\frac{1}{\beta}$, we can find an $\varepsilon>0$ such that

$$\frac{1}{1+\varepsilon}=1-\frac{1}{r}>1-\beta\quad\Rightarrow\quad(1+\varepsilon)(1-\beta)<1.$$

Then, for any $\psi(\cdot)\in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$, we have (denote $p'=\frac{p}{p-1}$)

$$\begin{aligned}&\Big\|\int_\cdot^T\frac{A(s,\cdot)^\top\psi(s)}{(s-\cdot)^{1-\beta}}ds\Big\|_{p'}\leqslant\Big\{\int_0^T\Big(L(t)\int_t^T\frac{|\psi(s)|}{(s-t)^{1-\beta}}ds\Big)^{p'}dt\Big\}^{\frac{1}{p'}}\\&\leqslant\Big[\int_0^T L(t)^r dt\Big]^{\frac{1}{r}}\Big[\int_0^T\Big(\int_t^T\frac{|\psi(s)|}{(s-t)^{1-\beta}}ds\Big)^{\frac{p'r}{r-p'}}dt\Big]^{\frac{r-p'}{p'r}}\leqslant\Big(\frac{T^{1-(1+\varepsilon)(1-\beta)}}{1-(1+\varepsilon)(1-\beta)}\Big)^{\frac{1}{1+\varepsilon}}\|L(\cdot)\|_r\|\psi(\cdot)\|_{p'}.\end{aligned}$$

Here, the Young's inequality for convolution is used with

$$\frac{1}{1+\varepsilon}+\frac{1}{p'}=\frac{r-p'}{p'r}+1=\frac{1}{p'}-\frac{1}{r}+1\qquad\Longleftrightarrow\qquad\frac{1}{1+\varepsilon}=1-\frac{1}{r}.$$

By a similar argument used in the proof of Theorem 3.1, we get the well-posedness of equation (3.36).

4 Pontryagin's Maximum Principle

In this section, we discuss the optimal control problem for equation (1.1) with cost functional (1.2). To begin with, let us introduce the following assumptions. The conditions assumed are more than sufficient. But for the simplicity of presentation, we prefer to use these stronger conditions.

(H1)$''$ Let (H1)$'$ hold with $(y,u)\mapsto f_y(t,s,y,u)$ being continuous.

(H2) Let $\eta(\cdot)$ be continuous at t_j, $j=1,2,\cdots,m$. Let $h^j:\mathbb{R}^n\to\mathbb{R}$, $j=1,2,\cdots,m$ be continuously differentiable, and $g:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}$ be a map with $t\mapsto g(t,y,u)$ being measurable, $y\mapsto g(t,y,u)$ being continuously differentiable and $(y,u)\mapsto(g(t,y,u),g_y(t,y,u))$ being continuous. There exists a constant $L>0$ such that

$$|g(t,0,u)|+|g_y(t,y,u)|\leqslant L,\qquad(t,y,u)\in[0,T]\times\mathbb{R}^n\times U.$$

Clearly, under (H1)$''$ and (H2), our cost functional (1.2) is well-defined. Hence, we can formulate the following optimal control problem.

Problem (P) Find a $u^*(\cdot) \in \mathscr{U}^p[0,T]$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}^p[0,T]} J(u(\cdot)). \tag{4.1}$$

Any $u^*(\cdot)$ satisfying (4.1) is called an *optimal control* of Problem (P), the corresponding state $y^*(\cdot)$ is called an *optimal state* and $(y^*(\cdot), u^*(\cdot))$ is called an *optimal pair*.

In this section, we shall first give a set of necessary conditions for optimal pairs of Problem (P). Usually, such a result is referred to as a *Pontryagin's maximum principle*. Then, we shall show some examples.

4.1 Pontryagin's maximum principle for Problem (P)

In establishing the Pontryagin's maximum principle for the case that U is not assumed to be convex, we need the following Liapunoff's type theorem (see, Corollary 3.9 of Chapter 4 in [33]).

Lemma 4.1. *Let X be a Banach space. For any $\delta > 0$, let*

$$\mathscr{E}_\delta = \big\{E \subseteq [0,T] \bigm| |E| = \delta T\big\},$$

where $|E|$ stands for the Lebesgue measure of E. Then for any $\mathbf{h}(\cdot) \in C([0,T]; L^1(0,T;X))$,

$$\inf_{E\in\mathscr{E}_\delta} \Big\| \int_0^t \Big(\frac{1}{\delta}\mathbf{1}_E(s) - 1\Big)\mathbf{h}(\cdot,s)ds\Big\|_{C([0,T];X)} = 0. \tag{4.2}$$

The following is an extension of the above result to the case having singularities.

Lemma 4.2. *Let $\varphi : \Delta \to \mathbb{R}^n$ be measurable such that*

$$\begin{cases} |\varphi(0,s)| \leqslant \bar\varphi(s), & 0 < s \leqslant T, \\ |\varphi(t,s) - \varphi(t',s)| \leqslant \omega(|t-t'|)\bar\varphi(s), & (t,s),(t',s) \in \Delta, \end{cases} \tag{4.3}$$

for some $\bar\varphi(\cdot) \in L^q(0,T)$ with $q > \frac{1}{\beta}$, $\beta \in (0,1)$ and some modulus of continuity $\omega : [0,+\infty) \to [0,+\infty)$. Then

$$\inf_{E\in\mathscr{E}_\delta} \sup_{t\in[0,T]} \Big| \int_0^t \Big(\frac{1}{\delta}\mathbf{1}_E(s) - 1\Big)\frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds\Big| = 0. \tag{4.4}$$

Proof. First of all, we extend $\varphi(t,s)$ to be 0 on $[0,T]^2 \setminus \Delta$. By our condition,

$$|\varphi(t,s)| \leqslant |\varphi(0,s)| + \omega(t)\bar\varphi(s) \leqslant [1+\omega(T)]\bar\varphi(s), \qquad (t,s) \in \Delta,\ s \neq 0.$$

Let $\Pi : 0 = t_0 < t_1 < t_2 < \cdots < t_\ell = T$ be a partition of $[0,T]$ and we denote $\|\Pi\| = \max_{1\leqslant i\leqslant \ell}(t_i - t_{i-1})$ to be its mesh size. For each t_i, since $\varphi(t_i,\cdot) \in L^q(0,T)$ (by extending to be 0 for $s=0$ and $s=t_i$), there exist a sequence of continuous functions $\{\varphi_k^i(\cdot)\}_{k=1}^\infty$ such that for any $k = 0,\ 1,\cdots$,

$$\Big(\int_0^T |\varphi(t_i,s) - \varphi_k^i(s)|^q ds\Big)^{\frac{1}{q}} \leqslant \frac{1}{k}, \qquad |\varphi_k^i(s)| \leqslant |\varphi(t_i,s)| \leqslant \big[1+\omega(T)\big]\bar\varphi(s) \equiv \widehat\varphi(s), \quad s \in [0,T]. \tag{4.5}$$

Now, let

$$\varphi_k(t,s) = \sum_{i=1}^{\ell}\Big[\frac{t_i - t}{t_i - t_{i-1}}\varphi_k^{i-1}(s) + \frac{t - t_{i-1}}{t_i - t_{i-1}}\varphi_k^i(s)\Big]\mathbf{1}_{(t_{i-1},t_i]}(t), \quad t,s \in [0,T].$$

Then $(t,s) \mapsto \varphi_k(t,s)$ is continuous and

$$|\varphi_k(t,s)| \leqslant \sum_{i=1}^{\ell}\Big[\frac{t_i - t}{t_i - t_{i-1}}|\varphi_k^{i-1}(s)| + \frac{t - t_{i-1}}{t_i - t_{i-1}}|\varphi_k^i(s)|\Big]\mathbf{1}_{(t_{i-1},t_i]}(t) \leqslant \widehat\varphi(s), \qquad t,s \in [0,T].$$

Further, for any $k=0,\ 1,\cdots,$

$$
\begin{aligned}
|\varphi(t,s)-\varphi_k(t,s)|&\leqslant\sum_{i=1}^{\ell}\Big[\frac{t_i-t}{t_i-t_{i-1}}\Big(|\varphi(t,s)-\varphi(t_{i-1},s)|+|\varphi(t_{i-1},s)-\varphi_k^{i-1}(s)|\Big)\\
&\qquad+\frac{t-t_{i-1}}{t_i-t_{i-1}}\Big(|\varphi(t,s)-\varphi(t_i,s)|+|\varphi(t_i,s)-\varphi_k^{i}(s)|\Big)\Big]\mathbf{1}_{(t_{i-1},t_i]}(t)\\
&\leqslant\omega(\|\Pi\|)\bar\varphi(s)+\sum_{i=1}^{\ell}\Big(\frac{t_i-t}{t_i-t_{i-1}}|\varphi(t_{i-1},s)-\varphi_k^{i-1}(s)|+\frac{t-t_{i-1}}{t_i-t_{i-1}}|\varphi(t_i,s)-\varphi_k^{i}(s)|\Big)\mathbf{1}_{(t_{i-1},t_i]}(t)\\
&\leqslant\omega(\|\Pi\|)\bar\varphi(s)+\sum_{i=1}^{\ell}\Big(|\varphi(t_{i-1},s)-\varphi_k^{i-1}(s)|+|\varphi(t_i,s)-\varphi_k^{i}(s)|\Big)\mathbf{1}_{(t_{i-1},t_i]}(t),\qquad (t,s)\in\Delta,\ t\neq 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s\neq t_i,\ i=1,\ 2,\ \cdots,l.
\end{aligned}
$$

Hence, noting $q>\frac{1}{\beta}$, for any $t\in(0,T]$ and $k=0,\ 1,\cdots,$

$$
\begin{aligned}
&\int_0^t\frac{|\varphi(t,s)-\varphi_k(t,s)|}{(t-s)^{1-\beta}}ds\leqslant\Big(\int_0^t|\varphi(t,s)-\varphi_k(t,s)|^qds\Big)^{\frac{1}{q}}\Big(\int_0^t\frac{1}{(t-s)^{(1-\beta)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}}\\
&\leqslant K\Big(\int_0^t|\varphi(t,s)-\varphi_k(t,s)|^qds\Big)^{\frac{1}{q}}\leqslant K\omega(\|\Pi\|)\|\bar\varphi(\cdot)\|_q+K\sum_{i=1}^{\ell}\Big[\Big(\int_0^T|\varphi(t_{i-1},s)-\varphi_k^{i-1}(s)|^qds\Big)^{\frac{1}{q}}\\
&\qquad+\Big(\int_0^T|\varphi(t_i,s)-\varphi_k^{i}(s)|^qds\Big)^{\frac{1}{q}}\Big]\mathbf{1}_{(t_{i-1},t_i]}(t)\leqslant K\omega(\|\Pi\|)\|\bar\varphi(\cdot)\|_q+\frac{K}{k}.
\end{aligned}
$$

Next, for any $\varepsilon>0$ and k, we define the following

$$
\psi_k^\varepsilon(t)=\int_0^t\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}+\varepsilon}ds,\quad \widehat\psi^\varepsilon(t)=\int_0^t\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}+\varepsilon}ds,\quad \widehat\psi(t)=\int_0^t\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}}ds,\qquad t\in[0,T].
$$

We claim that

$$
\lim_{\varepsilon\to0}\psi_k^\varepsilon(t)=\psi_k(t)\equiv\int_0^t\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}}ds,\qquad\text{uniformly in } t\in[0,T]\text{ and } k=0,\ 1,\cdots.
$$

In fact, for any ε and k,

$$
|\psi_k^\varepsilon(t)-\psi_k(t)|\leqslant\int_0^t\Big(\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}}-\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}+\varepsilon}\Big)ds=\widehat\psi(t)-\widehat\psi^\varepsilon(t),\ t\in[0,T].
$$

Since $\frac{1}{(t-s)^{1-\beta}+\varepsilon}$ is monotone increasing, approaching to $\frac{1}{(t-s)^{1-\beta}}$ for all $(t,s)\in\Delta$, by monotone convergence theorem, we have $\widehat\psi^\varepsilon(t)\nearrow\widehat\psi(t)$ as $\varepsilon\searrow0$, for any $t\in[0,T]$. Further, by Lemma 2.3, $\widehat\psi(\cdot)$ is continuous, by Dini Theorem, the above convergence is uniform in $t\in[0,T]$ and $k=0,\ 1,\cdots$. Since $\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}+\varepsilon}$ is continuous on $\bar\Delta$, by Lemma 4.1, for any $\delta>0$, we may find $E_\delta\in\mathscr{E}_\delta$ such that

$$
\max_{t\in[0,T]}\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}+\varepsilon}ds\Big|\leqslant\delta.
$$

Then

$$
\begin{aligned}
&\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds\Big|\leqslant\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}+\varepsilon}ds\Big|\\
&\quad+\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\Big(\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}+\varepsilon}-\frac{\varphi_k(t,s)}{(t-s)^{1-\beta}}\Big)ds\Big|+\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\varphi(t,s)-\varphi_k(t,s)}{(t-s)^{1-\beta}}ds\Big|\\
&\leqslant\delta+\Big(\frac{1}{\delta}+1\Big)\int_0^t\widehat\varphi(s)\Big(\frac{1}{(t-s)^{1-\beta}}-\frac{1}{(t-s)^{1-\beta}+\varepsilon}\Big)ds+K\Big(\frac{1}{\delta}+1\Big)\Big(\omega(\|\Pi\|)+\frac{1}{k}\Big)\\
&=\delta+\Big(\frac{1}{\delta}+1\Big)\big[\widehat\psi(t)-\widehat\psi^\varepsilon(t)\big]+K\Big(\frac{1}{\delta}+1\Big)\Big(\omega(\|\Pi\|)+\frac{1}{k}\Big),\ t\in[0,T].
\end{aligned}
$$

Consequently, for any small enough $\delta>0$, we let $\varepsilon\to 0$, $k\to\infty$, $\|\Pi\|\to 0$. Then the right-hand side of the above will approach to $\delta>0$ which can be arbitrarily small. This proves our conclusion. $\square$

We now state Pontryagin's maximum principle for Problem (P).

Theorem 4.3. *Let* (H1)$''$ *and* (H2) *hold. Let* $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$ *and* $\eta(\cdot)$ *be continuous at* t_j, $j=1,2,\cdots,m$. *Suppose* $(y^*(\cdot),u^*(\cdot))$ *is an optimal pair of Problem* (P). *Then there exists a solution* $\psi(\cdot)\in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$ *of the following adjoint equation*

$$\begin{aligned}\psi(t)=&-g_y(t,y^*(t),u^*(t))^\top-\sum_{j=1}^m\mathbf{1}_{[0,t_j)}(t)\frac{f_y(t_j,t,y^*(t),u^*(t))^\top}{(t_j-t)^{1-\beta}}h_y^j\big((y^*(t_j)\big)^\top\\&+\int_t^T\frac{f_y(s,t,y^*(t),u^*(t))^\top}{(s-t)^{1-\beta}}\psi(s)ds,\qquad\text{a.e. } t\in[0,T],\end{aligned}\tag{4.6}$$

such that the following maximum condition holds:

$$\begin{aligned}&\int_s^T\psi(t)^\top\frac{f(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}dt-g(s,y^*(s),u^*(s))-\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\\&=\max_{u\in U}\Big[\int_s^T\psi(t)^\top\frac{f(t,s,y^*(s),u)}{(t-s)^{1-\beta}}dt-g(s,y^*(s),u)-\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f(t_j,s,y^*(s),u)}{(t_j-s)^{1-\beta}}\Big],\\&\hspace{8cm}\forall u\in U,\quad\text{a.e. } s\in[0,T].\end{aligned}\tag{4.7}$$

Proof. We split the proof into several steps.

Step 1. A variational inequality. Let $(y^*(\cdot),u^*(\cdot))$ be an optimal pair of Problem (P). Fix any $u(\cdot)\in\mathscr{U}^p[0,T]$. Let

$$\begin{cases}\widehat f(t,s)=f(t,s,y^*(s),u(s))-f(t,s,y^*(s),u^*(s)), & (t,s)\in\Delta,\\ \widehat g(s)=g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s)), & s\in[0,T].\end{cases}\tag{4.8}$$

Taking

$$\varphi(t,s)=\widehat f(t,s),\quad h(t,s)=\widehat g(s),\qquad (t,s)\in\Delta,$$

and making use of Lemmas 4.2 and 4.1, respectively, for any $\delta>0$, we can find an $E_\delta\in\mathscr{E}_\delta$ such that

$$\begin{cases}\Big|\displaystyle\int_0^t\Big(\frac1\delta\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}ds\Big|\leqslant\delta, & t\in[0,T],\\ \Big|\displaystyle\int_0^T\Big(\frac1\delta\mathbf{1}_{E_\delta}(s)-1\Big)\widehat g(s)ds\Big|\leqslant\delta.\end{cases}\tag{4.9}$$

Denote

$$u^\delta(t)=\begin{cases}u^*(t), & t\in[0,T]\setminus E_\delta,\\ u(t), & t\in E_\delta.\end{cases}\tag{4.10}$$

Clearly, $u^\delta(\cdot)\in\mathscr{U}^p[0,T]$. Let $y^\delta(\cdot)=y(\cdot\,;\eta(\cdot),u^\delta(\cdot))$ be the corresponding state. Then

$$y^\delta(t)-y^*(t)=\int_0^t\Big(\frac{f_y^\delta(t,s)}{(t-s)^{1-\beta}}\big[y^\delta(s)-y^*(s)\big]+\mathbf{1}_{E_\delta}(s)\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}\Big)ds,\qquad t\in[0,T],$$

where $\widehat f(\cdot\,,\cdot)$ is given in (4.8), and

$$f_y^\delta(t,s)=\int_0^1 f_y\big(t,s,y^*(s)+\tau[y^\delta(s)-y^*(s)],u^\delta(s)\big)d\tau,\qquad (t,s)\in\Delta.$$

By (H1)$''$, we have

$$
\begin{aligned}
&|\widehat f(t,s)|=|f(t,s,y^*(s),u(s))-f(t,s,y^*(s),u^*(s))|\\
&\qquad\leqslant 2L_0(s)+L(s)\big[2|y^*(s)|+\rho(u(s),u_0)+\rho(u^*(s),u_0)\big]\equiv\varphi(s),\qquad (t,s)\in\Delta,\\
&|f_y^\delta(t,s)|\leqslant L(s),\qquad (t,s)\in\Delta.
\end{aligned}
$$

Clearly, $\varphi(\cdot)\in L^q0,T)$ for some $q\in(\frac{1}{\beta},p)$. Now, we have

$$
|y^\delta(t)-y^*(t)|\leqslant\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}|y^\delta(s)-y^*(s)|ds+\int_0^t\mathbf{1}_{E_\delta}(s)\frac{\varphi(s)}{(t-s)^{1-\beta}}ds,\qquad t\in[0,T].
$$

By the extended Gronwall's inequality (Lemma 2.4), choosing $q'\in(\frac{1}{\beta},q)$,

$$
\begin{aligned}
|y^\delta(t)-y^*(t)|&\leqslant\int_0^t\mathbf{1}_{E_\delta}(s)\frac{\varphi(s)}{(t-s)^{1-\beta}}ds+K\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}\int_0^s\mathbf{1}_{E_\delta}(\tau)\frac{\varphi(\tau)}{(s-\tau)^{1-\beta}}d\tau ds\\
&\leqslant K\Big(\int_0^t\mathbf{1}_{E_\delta}(s)^{q'}\varphi(s)^{q'}ds\Big)^{\frac{1}{q'}}+K\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}\Big(\int_0^s\mathbf{1}_{E_\delta}(\tau)^{q'}\varphi(\tau)^{q'}d\tau\Big)^{\frac{1}{q'}}ds\leqslant K|E_\delta|^{\frac{q-q'}{q'q}}\leqslant K\delta^{\frac{q-q'}{q'q}}\to0,
\end{aligned}
$$

as $\delta\to0$, uniformly in $t\in[0,T]$. Define

$$
Y^\delta(t)=\frac{y^\delta(t)-y^*(t)}{\delta},\qquad t\in[0,T].
$$

Then $Y^\delta(\cdot)$ is the solution of the following:

$$
Y^\delta(t)=\int_0^t\Big(\frac{f_y^\delta(t,s)}{(t-s)^{1-\beta}}Y^\delta(s)+\frac{\mathbf{1}_{E_\delta}(s)}{\delta}\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}\Big)ds,\qquad t\in[0,T].
$$

Now, let $Y(\cdot)$ be the solution to the following *variational equation*:

$$
Y(t)=\int_0^t\Big(\frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}Y(s)+\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}\Big)ds,\qquad t\in[0,T]. \tag{4.11}
$$

We have

$$
|Y(t)|\leqslant\int_0^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds+\int_0^t\frac{L(s)|Y(s)|}{(t-s)^{1-\beta}}ds,\qquad t\in[0,T].
$$

By the extended Gronwall's inequality, noting $q>\frac{1}{\beta}$ and $L(\cdot)\in L^{\frac{1}{\beta}+}(0,T)$,

$$
|Y(t)|\leqslant\int_0^t\frac{\varphi(s)}{(t-s)^{1-\beta}}ds+K\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}\int_0^s\frac{\varphi(\tau)}{(s-\tau)^{1-\beta}}d\tau ds\leqslant K\|\varphi(\cdot)\|_q,\qquad t\in[0,T].
$$

Hence,

$$
\begin{aligned}
Y^\delta(t)-Y(t)&=\int_0^t\frac{f_y^\delta(t,s)}{(t-s)^{1-\beta}}\big[Y^\delta(s)-Y(s)\big]ds+\int_0^t\frac{f_y^\delta(t,s)-f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}Y(s)ds\\
&\qquad+\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}ds\\
&\equiv\int_0^t\frac{f_y^\delta(t,s)}{(t-s)^{1-\beta}}\big[Y^\delta(s)-Y(s)\big]ds+a_1^\delta(t)+a_2^\delta(t),\qquad t\in[0,T].
\end{aligned}
$$

Since

$$
\Big|\frac{f_y^\delta(t,s)-f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}Y(s)\Big|\leqslant\frac{2L(s)}{(t-s)^{1-\beta}}|Y(s)|,\qquad \text{a.e. } s\in[0,t),
$$

with

$$\int_0^t \frac{2L(s)|Y(s)|}{(t-s)^{1-\beta}}ds<\infty,$$

by dominated convergence theorem, we obtain

$$\lim_{\delta\to0}a_1^\delta(t)=0,\qquad t\in[0,T].$$

On the other hand, by (4.9),

$$a_2^\delta(t)\equiv\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\widehat f(t,s)}{(t-s)^{1-\beta}}ds\Big|\leqslant\delta,\qquad t\in[0,T].$$

Thus,

$$|Y^\delta(t)-Y(t)|\leqslant\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}|Y^\delta(s)-Y(s)|ds+|a_1^\delta(t)|+|a_2^\delta(t)|,\qquad t\in[0,T].$$

By the extended Gronwall's inequality again,

$$|Y^\delta(t)-Y(t)|\leqslant|a_1^\delta(t)|+|a_2^\delta(t)|+K\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}\Big(|a_1^\delta(s)|+|a_2^\delta(s)|\Big)ds,\qquad t\in[0,T].$$

Then by the dominated convergence theorem, one obtains

$$\lim_{\delta\to0}|Y^\delta(t)-Y(t)|=0,\qquad t\in[0,T].$$

Also, by the optimality of $(y^*(\cdot),u^*(\cdot))$, one has

$$0\leqslant\frac{J(u^\delta(\cdot))-J(u^*(\cdot))}{\delta}=\int_0^T\Big[g_y^\delta(t)Y^\delta(t)+\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)\widehat g(s)\Big]dt+\sum_{j=1}^m h_y^{j,\delta}Y^\delta(t_j),$$

where $\widehat g(\cdot)$ is given in (4.8), and

$$g_y^\delta(t)=\int_0^1 g_y(t,y^*(t)+\tau[y^\delta(t)-y^*(t)],u^\delta(t))d\tau,\quad t\in[0,T],\;\; h_y^{j,\delta}=\int_0^1 h_y^j\big(y^*(t_j)+\tau[y^\delta(t_j)-y^*(t_j)]\big)d\tau.$$

Thus, using the second inequality in (4.9), together with the convergence $y^\delta(t)\to y^*(t)$, $t\in[0,T]$, we have

$$\begin{aligned}
0&\leqslant\int_0^T\Big(g_y(t,y^*(t),u^*(t))Y(t)+g(t,y^*(t),u(t))-g(t,y^*(t),u^*(t))\Big)dt+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)Y(t_j)\\
&=\int_0^T\Big(g_y(s,y^*(s),u^*(s))Y(s)+g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\Big)ds\\
&\quad+\int_0^T\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f_y(t_j,s,y^*(s),u^*(s))Y(s)}{(t_j-s)^{1-\beta}}+\frac{f(t_j,s,y^*(s),u(s))-f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)ds\\
&=\int_0^T\Big(g_y(s,y^*(s),u^*(s))+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f_y(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)Y(s)ds\\
&\quad+\int_0^T\Big[g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\\
&\qquad+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f(t_j,s,y^*(s),u(s))-f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big]ds,
\end{aligned}$$

with $Y(\cdot)$ being the solution to the variational equation (4.11).

Step 2. Duality. Let $\psi(\cdot)$ be the solution to the adjoint equation (4.6). Then we have

$$
\begin{aligned}
0 \leqslant & \int_0^T \Big(g_y(s,y^*(s),u^*(s)) + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f_y(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)Y(s)ds \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \quad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)\Big]ds \\
= & \int_0^T \Big(-\psi(s) + \int_s^\top \frac{f_y^T(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}\psi(t)dt\Big)^\top Y(s)ds \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \quad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)\Big]ds \\
= & \int_0^T \psi(t)^\top\Big(-Y(t) + \int_0^t \frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}Y(s)ds\Big)dt \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \quad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)\Big]ds \\
= & \int_0^T \Big[-\psi(t)^\top \int_0^t \Big(\frac{f(t,s,y^*(s),u(s)) - f(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}\Big)ds\Big]dt \\
& \quad + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \quad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)\Big]ds \\
= & \int_0^T \Big[\int_s^T -\psi(t)^\top\Big(\frac{f(t,s,y^*(s),u(s)) - f(t,s,y^*(s),u^*(s))}{(t-s)^{1-\beta}}\Big)dt \\
& \quad + g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \quad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\Big(\frac{f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))}{(t_j-s)^{1-\beta}}\Big)\Big]ds \\
\equiv & \int_0^T \big[\Psi(s,u^*(s)) - \Psi(s,u(s))\big]ds,
\end{aligned}
$$

with

$$
\Psi(s,u) = \int_s^T \psi(t)^\top \frac{f(t,s,y^*(s),u)}{(t-s)^{1-\beta}}dt - g(s,y^*(s),u) - \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j)}(s)\frac{f(t_j,s,y^*(s),u)}{(t_j-s)^{1-\beta}}.
$$

Let $u \in U$ be fixed and let $s_0 \in (0,T)$ be a Lebesgue point of $\Psi(s,u^*(s)) - \Psi(s,u)$. Then for any $\delta > 0$, let

$$
u(s) = \begin{cases} u^*(s), & s \in [0,T] \setminus (s_0-\delta, s_0+\delta), \\ u, & s \in (s_0-\delta, s_0+\delta). \end{cases}
$$

Hence, from the above, we obtain

$$
0 \leqslant \frac{1}{2\delta}\int_{s_0-\delta}^{s_0+\delta} [\Psi(s,u^*(s)) - \Psi(s,u)]ds \to \Psi(s_0,u^*(s_0)) - \Psi(s_0,u), \quad \delta \to 0.
$$

Consequently, by the Lebesgue point theorem (see Step 4, Section 4 in Chapter 4 of [33]), we obtain the maximum condition (4.7). □

4.2 Special cases in the sense of Riemann-Liouville and Caputo

In recent years, optimal control problems for fractional differential equations have attracted the attention of some researchers. However, most of the works on maximum principles for fractional differential equations were established by convex perturbation technique. See, for instance, Agrawal [1], Agrawal–Defterli–Baleanu [3], Frederico–Torres [24] and Kamocki [30] in the sense of Riemann-Liouville case, and Agrawal [2], Bourdin [11] and Hasan–Tangpong–Agrawal [26] in the sense of Caputo case. Therefore, the case that U is not assumed to be convex cannot be treated.

Let us take a look at a recent work [30] of Kamocki. The fractional differential equation of Riemann-Liouville type (3.32) with $\alpha \in (0,1)$ was considered. When $p > 1$ and $I^{1-\alpha}[y(\cdot)](0) = 0$, a Pontryagin's maximum principle for Problem (P) was proved. For the case $I^{1-\alpha}[y(\cdot)](0) \neq 0$, maximum principle was obtained only for $1 < p < \frac{1}{1-\alpha}$. Further, among other standard conditions, it was assumed that the set

$$\Big\{(g(t,y,u), f(t,s,y,u)) \;\Big|\; u \in U\Big\}$$

is convex, although U was only assumed to be compact. Of course, spike variation was not used.

It is easy to check that all the above-mentioned results for fractional differential equations are the special cases of what we presented in the previous subsection.

5 Concluding Remarks

In this paper, we have presented some analysis on the singular Volterra integral equations, and established a Pontryagin type maximum principle for an optimal control of such kind of equations. Here are some remarks in order.

• As we have indicated, the fractional differential equations of Riemann-Liouville or Caputo types of order no more than one are fully covered by our results. For fractional differential equations of higher order, similar results can be obtained by properly modifying our approach.

• It is easy to see that all the results that we presented will remain true for non-singular Volterra integral equations.

• We have allowed to have very general singularity in the free term and the generator. Therefore, our results can apply to a much wider class of problems than those covered by fractional differential equations and non-singular Volterra integral equations.

• Further extension is possible. For example, if the terminal state is constrained, or even more generally, for given $0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_k \leqslant T$, one requires that

$$(\tau_0, y(\tau_0), \tau_1, y(t_1), \cdots, \tau_k, y(\tau_k)) \in \Gamma,$$

for some given set Γ. For this, one might need to use Ekeland's variational principle to obtain the maximum principle. Some kind of transversality conditions will be obtained.

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