Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 129 (2019) 381–418

stochastic processes and their applications

www.elsevier.com/locate/spa

Linear–quadratic stochastic two-person nonzero-sum differential games: Open-loop and closed-loop Nash equilibria☆

Jingrui Sun*, Jiongmin Yong

Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Received 16 July 2016; received in revised form 29 November 2017; accepted 1 March 2018
Available online 13 March 2018

Abstract

In this paper, we consider a linear–quadratic stochastic two-person nonzero-sum differential game. Open-loop and closed-loop Nash equilibria are introduced. The existence of the former is characterized by the solvability of a system of forward–backward stochastic differential equations, and that of the latter is characterized by the solvability of a system of coupled symmetric Riccati differential equations. Sometimes, open-loop Nash equilibria admit a closed-loop representation, via the solution to a system of non-symmetric Riccati equations, which could be different from the outcome of the closed-loop Nash equilibria in general. However, it is found that for the case of zero-sum differential games, the Riccati equation system for the closed-loop representation of an open-loop saddle point coincides with that for the closed-loop saddle point, which leads to the conclusion that the closed-loop representation of an open-loop saddle point is the outcome of the corresponding closed-loop saddle point as long as both exist. In particular, for linear–quadratic optimal control problem, the closed-loop representation of an open-loop optimal control coincides with the outcome of the corresponding closed-loop optimal strategy, provided both exist.

MSC: 93E20; 91A23; 49N70; 49N10

Keywords: Stochastic differential equation; Linear–quadratic differential game; Two-person; Nonzero-sum; Nash equilibrium; Riccati differential equation; Closed-loop; Open-loop

☆ This work is supported in part by NSF Grant DMS-1406776.
* Corresponding author.
E-mail addresses: sjr@mail.ustc.edu.cn (J. Sun), jiongmin.yong@ucf.edu (J. Yong).

https://doi.org/10.1016/j.spa.2018.03.002
0304-4149/

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $\{W(t), t \geqslant 0\}$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of $W(\cdot)$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following controlled linear (forward) stochastic differential equation (FSDE, for short) on $[t, T]$:

$$\begin{cases} dX(s) = \big[A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s) + \sigma(s)\big]dW(s), \\ X(t) = x. \end{cases} \tag{1.1}$$

In the above, $X(\cdot)$ is called the *state process* taking values in the n-dimensional Euclidean space $\mathbb{R}^n$ with the *initial pair* $(t, x) \in [0, T) \times \mathbb{R}^n$; for $i = 1, 2$, $u_i(\cdot)$ is called the *control process* of Player i taking values in $\mathbb{R}^{m_i}$. We assume that the *coefficients* $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C(\cdot)$, $D_1(\cdot)$, and $D_2(\cdot)$ are deterministic matrix-valued functions of proper dimensions, and that $b(\cdot)$ and $\sigma(\cdot)$ are $\mathbb{F}$-progressively measurable processes taking values in $\mathbb{R}^n$. For $i = 1, 2$ and $t \in [0, T)$, we define

$$\mathcal{U}_i[t, T] = \Big\{u_i : [t, T] \times \Omega \to \mathbb{R}^{m_i} \,\Big|\, u_i(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \\ \mathbb{E}\int_t^T |u_i(s)|^2 ds < \infty\Big\}.$$

Any element $u_i(\cdot) \in \mathcal{U}_i[t, T]$ is called an *admissible control* of Player i on $[t, T]$. Under some mild conditions on the coefficients, for any initial pair $(t, x) \in [0, T) \times \mathbb{R}^n$ and controls $u_i(\cdot) \in \mathcal{U}_i[t, T]$, $i = 1, 2$, the state equation (1.1) admits a unique solution $X(\cdot) \equiv X(\cdot\,; t, x, u_1(\cdot), u_2(\cdot))$. The cost functional for Player i is defined by the following:

$$\begin{aligned} J^i(t, x; u_1(\cdot), u_2(\cdot)) \triangleq \mathbb{E}\Big\{&\langle G^i X(T), X(T)\rangle + 2\langle g^i, X(T)\rangle \\ &+ \int_t^T \Big[\Big\langle \begin{pmatrix} Q^i(s) & S_1^i(s)^\top & S_2^i(s)^\top \\ S_1^i(s) & R_{11}^i(s) & R_{12}^i(s) \\ S_2^i(s) & R_{21}^i(s) & R_{22}^i(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix}\Big\rangle \\ &+ 2\Big\langle \begin{pmatrix} q^i(s) \\ \rho_1^i(s) \\ \rho_2^i(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix}\Big\rangle\Big]ds\Big\}, \end{aligned} \tag{1.2}$$

where $Q^i(\cdot)$, $S_1^i(\cdot)$, $S_2^i(\cdot)$, $R_{11}^i(\cdot)$, $R_{12}^i(\cdot)$, $R_{21}^i(\cdot)$, and $R_{22}^i(\cdot)$ are deterministic matrix-valued functions of proper dimensions with

$$Q^i(\cdot)^\top = Q^i(\cdot), \quad R_{jj}^i(\cdot)^\top = R_{jj}^i(\cdot), \quad R_{12}^i(\cdot)^\top = R_{21}^i(\cdot), \qquad i, j = 1, 2,$$

where the superscript $^\top$ denotes the transpose of matrices, and G^i is a deterministic symmetric matrix; $q^i(\cdot)$, $\rho_1^i(\cdot)$, and $\rho_2^i(\cdot)$ are allowed to be vector-valued $\mathbb{F}$-progressively measurable processes, and g^i is allowed to be an $\mathcal{F}_T$-measurable random vector. Then we can formally pose the following problem.

Problem (SDG). For any initial pair $(t, x) \in [0, T) \times \mathbb{R}^n$ and $i = 1, 2$, Player i wants to find a control $u_i^*(\cdot) \in \mathcal{U}_i[t, T]$ such that the cost functional $J^i(t, x; u_1(\cdot), u_2(\cdot))$ is minimized.

The above-posed problem is referred to as a linear–quadratic (LQ, for short) stochastic *two-person differential game*. In the case

$$J^1(t, x; u_1(\cdot), u_2(\cdot)) + J^2(t, x; u_1(\cdot), u_2(\cdot)) = 0, \\ \forall (t, x) \in [0, T] \times \mathbb{R}^n, \ \forall u_i(\cdot) \in \mathcal{U}_i[t, T], \quad i = 1, 2, \tag{1.3}$$

the corresponding Problem (SDG) is called an LQ stochastic two-person *zero-sum* differential game. To guarantee (1.3), one usually assumes that

$$\begin{array}{llll} G^1 + G^2 = 0, & g^1 + g^2 = 0, & Q^1(\cdot) + Q^2(\cdot) = 0, & q^1(\cdot) + q^2(\cdot) = 0, \\ S_j^1(\cdot) + S_j^2(\cdot) = 0, & R_{jk}^1(\cdot) + R_{jk}^2(\cdot) = 0, & \rho_j^1(\cdot) + \rho_j^2(\cdot) = 0, & j, k = 1, 2. \end{array} \tag{1.4}$$

We refer the readers to [22,23] (and the references cited therein) for the case of LQ stochastic two-person zero-sum differential games. Recall that in [22], open-loop and closed-loop saddle points were introduced and it was established that the existence of an open-loop saddle point for the problem is equivalent to the solvability of a forward–backward stochastic differential equation (FBSDE, for short), and the existence of a closed-loop saddle point for the problem is equivalent to the solvability of a (differential) Riccati equation. In this paper, we will not assume (1.4) so that (1.3) is not necessarily true. Such a Problem (SDG) is usually referred to as an LQ stochastic two-person *nonzero-sum* differential game, emphasizing that (1.3) is not assumed. We have two main goals in this paper: Establish a theory for Problem (SDG) parallel to that of [22] (for zero-sum case); and study the difference between the closed-loop representation of open-loop Nash equilibria and the outcome of closed-loop Nash equilibria. It turns out that the above-mentioned difference for the non-zero sum case is indicated through the symmetry of the corresponding Riccati equations: One is symmetric and the other is not. On the other hand, we found that the situation in the zero-sum case, which was not discussed in [22], is totally different: The closed-loop representation of an open-loop saddle point coincides with the outcome of the corresponding closed-loop saddle point, when both exist. In particular, for stochastic linear–quadratic optimal control problem, the closed-loop representation of an open-loop optimal control is the outcome of the corresponding closed-loop optimal strategy [21].

Mathematically, posing condition (1.4) makes the structure of the problem much simpler, since with such a condition, only one performance index is needed, for which one player is the minimizer and the other player is the maximizer. However, as we know that in the real life, each player should have his/her own cost functional, and even for the totally hostile situation, the objectives of the opponents might not necessarily be exactly the opposite (zero-sum). Therefore, realistically, it is more meaningful to investigate Problem (SDG) without assuming (1.4). By the way, although we will not discuss such a situation in the current paper, we still would like to point out that sometimes, certain cooperations between the players might result in both players rewarded more.

Static version of nonzero-sum differential games could be regarded as a kind of non-cooperative games for which one can trace back to the work of Nash [16]. For some early works on nonzero-sum differential games, we would like to mention Lukes–Russell [12], Friedman [5], and Bensoussan [2]. In the past two decays, due to the appearance of backward stochastic differential equations (BSDEs, for short), some new and interesting works were published. Among them, we would like to mention [6,7,4,3,20,10,11,8].

The rest of the paper is organized as follows. Section 2 will collect some preliminaries. Among other things, we will recall some known results on LQ optimal control problems. In Section 3, we will introduce open-loop and closed-loop Nash equilibria. A characterization of the existence of open-loop Nash equilibria in terms of the solvability of two coupled FBSDEs will be presented in Section 4. Section 5 is devoted to the discussion of the closed-loop Nash equilibria whose existence is characterized by the solvability of two coupled symmetric Riccati equations. In Section 6, we will present two examples showing the difference between open-loop and closed-loop Nash equilibria. In Section 7, closed-loop representation of open-loop Nash equilibria will be studied, and comparison between the closed-loop representation of open-loop Nash equilibria and the outcome of closed-loop Nash equilibria will be carried out. We will take a deeper look at the situation for LQ zero-sum games in Section 8. Finally, some concluding remarks will be put in order in Section 9.

2. Preliminaries

Let $\mathbb{R}^{n\times m}$ be the space of all $(n \times m)$ matrices and $\mathbb{S}^n \subseteq \mathbb{R}^{n\times n}$ be the set of all $(n \times n)$ symmetric matrices. The inner product $\langle \cdot\,, \cdot\rangle$ on $\mathbb{R}^{n\times m}$ is given by $\langle M, N\rangle \mapsto \mathrm{tr}\,(M^\top N)$, and the induced norm is given by $|M| = \sqrt{\mathrm{tr}\,(M^\top M)}$. We denote by $\mathscr{R}(M)$ the range of a matrix M, and for $M, N \in \mathbb{S}^n$ we use the notation $M \geqslant N$ (respectively, $M > N$) to indicate that $M - N$ is positive semi-definite (respectively, positive definite). Recall that any $M \in \mathbb{R}^{n\times m}$ admits a unique (Moore–Penrose) *pseudo-inverse* $M^\dagger \in \mathbb{R}^{m\times n}$ having the following properties [19]:

$$MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \quad (MM^\dagger)^\top = MM^\dagger, \quad (M^\dagger M)^\top = M^\dagger M.$$

Further, if $M \in \mathbb{R}^{n\times m}$ and $\Psi \in \mathbb{R}^{n\times \ell}$ such that

$$\mathscr{R}(\Psi) \subseteq \mathscr{R}(M),$$

then all the solutions Θ to the linear equation

$$M\Theta = \Psi$$

are given by the following:

$$\Theta = M^\dagger\Psi + (I - M^\dagger M)\Gamma, \qquad \Gamma \in \mathbb{R}^{m\times \ell}.$$

In addition, if $M^\top = M \in \mathbb{S}^n$, then

$$M^\dagger = (M^\dagger)^\top, \quad MM^\dagger = M^\dagger M; \quad \text{and} \quad M \geqslant 0 \iff M^\dagger \geqslant 0.$$

Next, let $T > 0$ be a fixed time horizon. For any $t \in [0, T]$ and Euclidean space $\mathbb{H}$, we introduce the following spaces of deterministic functions:

$$\begin{aligned}
L^p(t, T; \mathbb{H}) &= \Big\{\varphi : [t, T] \to \mathbb{H}\Big| \int_t^T |\varphi(s)|^p ds < \infty\Big\}, \quad 1 \leqslant p < \infty,\\
L^\infty(t, T; \mathbb{H}) &= \Big\{\varphi : [t, T] \to \mathbb{H}\Big|\operatorname*{esssup}_{s\in[t,T]}|\varphi(s)| < \infty\Big\},\\
C([t, T]; \mathbb{H}) &= \Big\{\varphi : [t, T] \to \mathbb{H}\Big|\varphi(\cdot) \text{ is continuous}\Big\}.
\end{aligned}$$

Further, we introduce the following spaces of random variables and stochastic processes: For any $t \in [0, T]$,

$$
\begin{aligned}
&L^2_{\mathcal{F}_t}(\Omega;\mathbb{H})=\Big\{\xi:\Omega\to\mathbb{H}\,\big|\,\xi\text{ is }\mathcal{F}_t\text{-measurable, }\mathbb{E}|\xi|^2<\infty\Big\},\\
&L^2_{\mathbb{F}}(t,T;\mathbb{H})=\Big\{\varphi:[t,T]\times\Omega\to\mathbb{H}\,\big|\,\varphi(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable,}\\
&\qquad \mathbb{E}\int_t^T|\varphi(s)|^2ds<\infty\Big\},\\
&L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H}))=\Big\{\varphi:[t,T]\times\Omega\to\mathbb{H}\,\big|\,\varphi(\cdot)\text{ is }\mathbb{F}\text{-adapted, continuous,}\\
&\qquad \mathbb{E}\Big(\sup_{t\leqslant s\leqslant T}|\varphi(s)|^2\Big)<\infty\Big\},\\
&L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H}))=\Big\{\varphi:[t,T]\times\Omega\to\mathbb{H}\,\big|\,\varphi(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable,}\\
&\qquad \mathbb{E}\Big(\int_t^T|\varphi(s)|ds\Big)^2<\infty\Big\}.
\end{aligned}
$$

We now recall some results on stochastic LQ optimal control problems. Consider the state equation

$$
\begin{cases}
dX(s)=\big[A(s)X(s)+B(s)u(s)+b(s)\big]ds\\
\qquad\qquad +\big[C(s)X(s)+D(s)u(s)+\sigma(s)\big]dW(s), \qquad s\in[t,T],\\
X(t)=x.
\end{cases}
\tag{2.1}
$$

The cost functional takes the following form:

$$
\begin{aligned}
J(t,x;u(\cdot))\triangleq\mathbb{E}\Big\{&\langle GX(T),X(T)\rangle+2\langle g,X(T)\rangle\\
&+\int_t^T\Big[\Big\langle\begin{pmatrix}Q(s)&S(s)^\top\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}X(s)\\u(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\Big\rangle\\
&+2\Big\langle\begin{pmatrix}q(s)\\\rho(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\Big\rangle\Big]ds\Big\}.
\end{aligned}
\tag{2.2}
$$

We adopt the following assumptions.

(S1) The coefficients of the state equation (2.1) satisfy the following:

$$
\begin{cases}
A(\cdot)\in L^1(0,T;\mathbb{R}^{n\times n}), & B(\cdot)\in L^2(0,T;\mathbb{R}^{n\times m}), & b(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\\
C(\cdot)\in L^2(0,T;\mathbb{R}^{n\times n}), & D(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}), & \sigma(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).
\end{cases}
$$

(S2) The weighting coefficients in the cost functional (2.2) satisfy the following:

$$
\begin{cases}
Q(\cdot)\in L^1(0,T;\mathbb{S}^n), & S(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n}), & R(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\
q(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), & \rho(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), & g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n),\quad G\in\mathbb{S}^n.
\end{cases}
$$

Note that under (S1), for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]\equiv L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$, the state equation (2.1) admits a unique strong solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$. Further, if (S2) is also assumed, then the cost functional (2.2) is well-defined for every $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$. Therefore, the following problem is meaningful.

Problem (SLQ). For any given initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, find a $\bar u(\cdot)\in\mathcal{U}[t,T]$ such that

$$
J(t,x;\bar u(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot))\equiv V(t,x).
\tag{2.3}
$$

Any $\bar u(\cdot)\in\mathcal{U}[t,T]$ satisfying (2.3) is called an *open-loop optimal control* of Problem (SLQ) for (t,x); the corresponding $\bar X(\cdot)\equiv X(\cdot\,;t,x,\bar u(\cdot))$ is called an *open-loop optimal state process* and $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an *open-loop optimal pair*. The map $V(\cdot\,,\cdot)$ is called the *value function* of Problem (LQ).

Definition 2.1. Let $(t, x) \in [0, T) \times \mathbb{R}^n$. If there exists a (unique) $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that (2.3) holds, then we say that Problem (SLQ) is (*uniquely*) *open-loop solvable at* (t, x). If Problem (SLQ) is (uniquely) open-loop solvable for every $(t, x) \in [0, T) \times \mathbb{R}^n$, then we say that Problem (SLQ) is (*uniquely*) *open-loop solvable on* $[0, T) \times \mathbb{R}^n$.

The following result is concerned with open-loop optimal controls of Problem (SLQ) for a given initial pair, whose proof can be found in [22] (see also [21]).

Theorem 2.2. *Let* (S1)–(S2) *hold. For a given initial pair* $(t, x) \in [0, T) \times \mathbb{R}^n$*, a state-control pair* $(\bar{X}(\cdot), \bar{u}(\cdot))$ *is an open-loop optimal pair of Problem* (SLQ) *if and only if the following hold:*

(i) *The stationarity condition holds:*

$$B(s)^\top \bar{Y}(s) + D(s)^\top \bar{Z}(s) + S(s)\bar{X}(s) + R(s)\bar{u}(s) + \rho(s) = 0, \quad a.e.\ s \in [t, T],\ a.s.$$

where $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ *is the adapted solution to the following BSDE:*

$$\begin{cases} d\bar{Y}(s) = -\big[A(s)^\top \bar{Y}(s) + C(s)^\top \bar{Z}(s) + Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) + q(s)\big]ds \\ \qquad\qquad + \bar{Z}(s)dW(s), \quad s \in [t, T], \\ \bar{Y}(T) = G\bar{X}(T) + g. \end{cases}$$

(ii) *The map* $u(\cdot) \mapsto J(t, 0; u(\cdot))$ *is convex.*

Next, for any given $t \in [0, T)$, take $\Theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n}) \equiv \mathcal{Q}[t, T]$ and $v(\cdot) \in \mathcal{U}[t, T]$. For any $x \in \mathbb{R}^n$, let us consider the following equation on $[t, T]$:

$$\begin{cases} dX(s) = \big\{[A(s) + B(s)\Theta(s)]X(s) + B(s)v(s) + b(s)\big\}ds \\ \qquad\qquad + \big\{[C(s) + D(s)\Theta(s)]X(s) + D(s)v(s) + \sigma(s)\big\}dW(s), \\ X(t) = x, \end{cases} \tag{2.4}$$

which admits a unique solution $X(\cdot) \equiv X(\cdot\,; t, x, \Theta(\cdot), v(\cdot))$, depending on $\Theta(\cdot)$ and $v(\cdot)$. The above is called a *closed-loop system* of the original state equation (2.1) under *closed-loop strategy* $(\Theta(\cdot), v(\cdot))$. We point out that $(\Theta(\cdot), v(\cdot))$ is independent of the initial state x. With the above corresponding solution $X(\cdot)$, we define

$$\begin{aligned} J(t, x;\ \Theta(\cdot)X(\cdot) + v(\cdot)) = \mathbb{E}\Big\{ &\langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\ &+ \int_t^T \Big[\Big\langle \begin{pmatrix} Q(s) & S(s)^\top \\ S(s) & R(s) \end{pmatrix} \begin{pmatrix} X(s) \\ \Theta(s)X(s) + v(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ \Theta(s)X(s) + v(s) \end{pmatrix}\Big\rangle \\ &+ 2\Big\langle \begin{pmatrix} q(s) \\ \rho(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ \Theta(s)X(s) + v(s) \end{pmatrix}\Big\rangle\Big]ds\Big\}. \end{aligned}$$

Let us recall the following definition.

Definition 2.3. A pair $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \mathcal{Q}[t, T] \times \mathcal{U}[t, T]$ is called a *closed-loop optimal strategy* of Problem (SLQ) on $[t, T]$ if

$$\begin{aligned} &J(t, x;\ \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t, x;\ \Theta(\cdot)X(\cdot) + v(\cdot)), \\ &\qquad \forall\, x \in \mathbb{R}^n,\ \ \forall\, (\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t, T] \times \mathcal{U}[t, T], \end{aligned} \tag{2.5}$$

where $\bar{X}(\cdot) = X(\cdot\,; t, x, \bar{\Theta}(\cdot), \bar{v}(\cdot))$, and $X(\cdot) = X(\cdot\,; t, x, \Theta(\cdot), v(\cdot))$.

We emphasize that the pair $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is required to be independent of the initial state $x \in \mathbb{R}^n$. It is interesting that the following equivalence theorem holds.

Proposition 2.4. *Let* (S1)–(S2) *hold and let* $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$. *Then the following statements are equivalent:*

(i) $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ *is a closed-loop optimal strategy of Problem* (SLQ) *on* $[t,T]$.

(ii) *For any* $x \in \mathbb{R}^n$ *and* $v(\cdot) \in \mathcal{U}[t,T]$,

$$J(t,x;\, \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;\, \bar{\Theta}(\cdot)X(\cdot) + v(\cdot)),$$

where $\bar{X}(\cdot) = X(\cdot\,; t,x,\bar{\Theta}(\cdot),\bar{v}(\cdot))$ *and* $X(\cdot) = X(\cdot\,; t,x,\bar{\Theta}(\cdot),v(\cdot))$.

(iii) *For any* $x \in \mathbb{R}^n$ *and* $u(\cdot) \in \mathcal{U}[t,T]$,

$$J(t,x;\, \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;u(\cdot)), \tag{2.6}$$

where $\bar{X}(\cdot) = X(\cdot\,; t,x,\bar{\Theta}(\cdot),\bar{v}(\cdot))$.

Proof. The implication (i) ⇒ (ii) follows by taking $\Theta(\cdot) = \bar{\Theta}(\cdot)$ in (2.5).

For the implication (ii) ⇒ (iii), take any $u(\cdot) \in \mathcal{U}[t,T]$ and let $X(\cdot) = X(\cdot\,; t,x,u(\cdot))$. Then

$$\begin{aligned} dX(s) = &\Big\{[A(s) + B(s)\bar{\Theta}(s)]X(s) + B(s)[u(s) - \bar{\Theta}(s)X(s)] + b(s)\Big\}ds \\ &+ \Big\{[C(s) + D(s)\bar{\Theta}(s)]X(s) + D(s)[u(s) - \bar{\Theta}(s)X(s)] + \sigma(s)\Big\}dW(s), \end{aligned}$$

with $X(t) = x$. Thus, if let

$$v(\cdot) = u(\cdot) - \bar{\Theta}(\cdot)X(\cdot),$$

we have

$$J(t,x;\, \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;\, \bar{\Theta}(\cdot)X(\cdot) + v(\cdot)) = J(t,x;u(\cdot)),$$

which proves (iii).

For the implication (iii) ⇒ (i), take any $(\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ and let $X(\cdot)$ be the solution to (2.4). Let $u(\cdot) = \Theta(\cdot)X(\cdot) + v(\cdot)$, Then by (iii), we have

$$J(t,x;\, \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;u(\cdot)) = J(t,x;\, \Theta(\cdot)X(\cdot) + v(\cdot)).$$

This completes the proof. □

From the above result, we see that if $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is a closed-loop optimal strategy of Problem (SLQ) on $[t,T]$, then for any fixed initial state $x \in \mathbb{R}^n$, with $\bar{X}(\cdot)$ denoting the state process corresponding to (t,x) and $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$, (2.6) implies that the outcome

$$\bar{u}(\cdot) \equiv \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot) \in \mathcal{U}[t,T]$$

is an open-loop optimal control of Problem (SLQ) for (t,x). Therefore, for Problem (SLQ), the existence of closed-loop strategies on $[t,T]$ implies the existence of open-loop optimal controls for initial pair (t,x) for any $x \in \mathbb{R}^n$. We point out that the situation will be different for two-person differential games. Details will be carried out later.

For closed-loop optimal strategies, we have the following characterization [22,21].

Theorem 2.5. *Let* (S1)–(S2) *hold. Then Problem* (SLQ) *admits a closed-loop optimal strategy on* $[t, T]$ *if and only if the following Riccati equation admits a solution* $P(\cdot) \in C([t, T]; \mathbb{S}^n)$*:*

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = 0, \quad \text{a.e. on } [t, T], \\ \mathscr{R}(B^\top P + D^\top PC + S) \subseteq \mathscr{R}(R + D^\top PD), \qquad \text{a.e. on } [t, T], \\ R + D^\top PD \geqslant 0, \qquad \text{a.e. on } [t, T], \\ P(T) = G, \end{cases}$$

such that

$$(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \in L^2(t, T; \mathbb{R}^{m\times n}),$$

and the adapted solution $(\eta(\cdot), \zeta(\cdot))$ *to the BSDE on* $[t, T]$

$$\begin{cases} d\eta = -\Big\{\big[A - B(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)\big]^\top \eta \\ \qquad\quad + \big[C - D(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)\big]^\top \zeta \\ \qquad\quad + \big[C - D(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)\big]^\top P\sigma \\ \qquad\quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \rho + Pb + q\Big\} ds + \zeta dW, \\ \eta(T) = g, \end{cases}$$

satisfies

$$\begin{cases} B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho \in \mathscr{R}(R + D^\top PD), \quad \text{a.e. } s \in [t, T], \ \text{a.s.} \\ (R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m). \end{cases}$$

In this case, any closed-loop optimal strategy $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ *of Problem* (SLQ) *admits the following representation:*

$$\begin{cases} \bar{\Theta} = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \\ \qquad + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\theta, \\ \bar{v} = -(R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \\ \qquad + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\nu, \end{cases}$$

for some $\theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$*. Further, the value function* $V(\cdot\,, \cdot)$ *is given by*

$$\begin{aligned} V(t, x) = \mathbb{E}\Big\{ &\langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle + \int_t^T \Big[\langle P\sigma, \sigma\rangle + 2\langle \eta, b\rangle + 2\langle \zeta, \sigma\rangle \\ &- \big\langle (R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), \\ &B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho\big\rangle\Big] ds\Big\}. \end{aligned}$$

3. Stochastic differential games

We now return to our Problem (SDG). Recall the sets $\mathcal{U}_i[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m_i})$ of all open-loop controls of Player i ($i = 1, 2$). For notational simplicity, we let $m = m_1 + m_2$ and denote

$$
\begin{aligned}
&B(\cdot) = (B_1(\cdot), B_2(\cdot)), \quad D(\cdot) = (D_1(\cdot), D_2(\cdot)),\\
&S^i(\cdot) = \begin{pmatrix} S^i_1(\cdot) \\ S^i_2(\cdot) \end{pmatrix}, \quad R^i(\cdot) = \begin{pmatrix} R^i_{11}(\cdot) & R^i_{12}(\cdot) \\ R^i_{21}(\cdot) & R^i_{22}(\cdot) \end{pmatrix} \equiv \begin{pmatrix} R^i_1(\cdot) \\ R^i_2(\cdot) \end{pmatrix},\\
&\rho^i(\cdot) = \begin{pmatrix} \rho^i_1(\cdot) \\ \rho^i_2(\cdot) \end{pmatrix}, \quad u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}.
\end{aligned}
$$

Naturally, we identify $\mathcal{U}[t,T] = \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$. With such notations, the state equation becomes

$$
\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{3.1}
$$

and the cost functionals become ($i = 1, 2$)

$$
\begin{aligned}
J^i(t,x;u(\cdot)) = \mathbb{E}\Big\{ &\langle G^i X(T), X(T)\rangle + 2\langle g^i, X(T)\rangle \\
&+ \int_t^T \Big[\Big\langle \begin{pmatrix} Q^i(s) & S^i(s)^\top \\ S^i(s) & R^i(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \Big\rangle \\
&+ 2\Big\langle \begin{pmatrix} q^i(s) \\ \rho^i(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \Big\rangle \Big] ds \Big\}.
\end{aligned} \tag{3.2}
$$

Let us introduce the following standard assumptions:

(G1) The coefficients of the state equation (3.1) satisfy the following:

$$
\begin{cases} A(\cdot) \in L^1(0,T;\mathbb{R}^{n\times n}), \quad B(\cdot) \in L^2(0,T;\mathbb{R}^{n\times m}), \quad b(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n)), \\ C(\cdot) \in L^2(0,T;\mathbb{R}^{n\times n}), \quad D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \quad \sigma(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n). \end{cases}
$$

(G2) The weighting coefficients in the cost functionals (3.2) satisfy the following: For $i = 1, 2$,

$$
\begin{cases} Q^i(\cdot) \in L^1(0,T;\mathbb{S}^n), \quad S^i(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n}), \quad R^i(\cdot) \in L^\infty(0,T;\mathbb{S}^m), \\ q^i(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n)), \quad \rho^i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \quad g^i \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n), \quad G^i \in \mathbb{S}^n. \end{cases}
$$

Under (G1), for any $(t,x) \in [0,T) \times \mathbb{R}^n$ and $u(\cdot) = (u_1(\cdot)^\top, u_2(\cdot)^\top)^\top \in \mathcal{U}[t,T]$, Eq. (3.1) admits a unique solution [22]

$$
X(\cdot) \triangleq X(\cdot\,; t, x, u_1(\cdot), u_2(\cdot)) \equiv X(\cdot\,; t, x, u(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)).
$$

Moreover, the following estimate holds:

$$
\mathbb{E}\Big(\sup_{t\leqslant s\leqslant T} |X(s)|^2\Big) \leqslant K\mathbb{E}\Big\{|x|^2 + \Big(\int_t^T |b(s)|ds\Big)^2 + \int_t^T |\sigma(s)|^2 ds + \int_t^T |u(s)|^2 ds\Big\},
$$

hereafter $K > 0$ represents an absolute constant. Therefore, under (G1)–(G2), the cost functionals $J^i(t,x;u(\cdot)) \equiv J^i(t,x;u_1(\cdot),u_2(\cdot))$ are well-defined for all $(t,x) \in [0,T) \times \mathbb{R}^n$ and all $(u_1(\cdot),u_2(\cdot)) \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$. Having the above, we now introduce the following definition.

Definition 3.1. A pair $(u_1^*(\cdot),u_2^*(\cdot)) \in \mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ is called an *open-loop Nash equilibrium* of Problem (SDG) for the initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$ if

$$
\begin{aligned}
&J^1(t,x;u_1^*(\cdot),u_2^*(\cdot)) \leqslant J^1(t,x;u_1(\cdot),u_2^*(\cdot)), \qquad \forall\, u_1(\cdot)\in\mathcal{U}_1[t,T],\\
&J^2(t,x;u_1^*(\cdot),u_2^*(\cdot)) \leqslant J^2(t,x;u_1^*(\cdot),u_2(\cdot)), \qquad \forall\, u_2(\cdot)\in\mathcal{U}_2[t,T].
\end{aligned} \tag{3.3}
$$

Next, we denote

$$
\mathcal{Q}_i[t,T]=L^2(t,T;\mathbb{R}^{m_i\times n}), \qquad i=1,2.
$$

For any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, $\Theta(\cdot)\equiv(\Theta_1(\cdot)^\top,\Theta_2(\cdot)^\top)^\top\in\mathcal{Q}_1[t,T]\times\mathcal{Q}_2[t,T]$ and any $v(\cdot)\equiv(v_1(\cdot)^\top,v_2(\cdot)^\top)^\top\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, consider the following system:

$$
\begin{cases}
dX(s)=\big\{[A(s)+B(s)\Theta(s)]X(s)+B(s)v(s)+b(s)\big\}ds\\
\qquad\qquad +\ \big\{[C(s)+D(s)\Theta(s)]X(s)+D(s)v(s)+\sigma(s)\big\}dW(s),\\
X(t)=x.
\end{cases} \tag{3.4}
$$

Under (G1), the above admits a unique solution $X(\cdot)\equiv X(\cdot\,;t,x,\Theta_1(\cdot),v_1(\cdot),\Theta_2(\cdot),v_2(\cdot))$. If we denote

$$
u_i(\cdot)=\Theta_i(\cdot)X(\cdot)+v_i(\cdot), \qquad i=1,2, \tag{3.5}
$$

then (3.4) coincides with the original state equation (1.1). We call $(\Theta_i(\cdot),v_i(\cdot))$ a *closed-loop strategy* of Player i, and call (3.4) the *closed-loop system* of the original system under closed-loop strategies $(\Theta_1(\cdot),v_1(\cdot))$ and $(\Theta_2(\cdot),v_2(\cdot))$ of Players 1 and 2. Also, we call $u(\cdot)\equiv(u_1(\cdot)^\top,u_2(\cdot)^\top)^\top$ with $u_i(\cdot)$ defined by (3.5) the outcome of the closed-loop strategy $(\Theta(\cdot),v(\cdot))$. With the solution $X(\cdot)$ to (3.4), we denote

$$
\begin{aligned}
&J^i(t,x;\,\Theta(\cdot)X(\cdot)+v(\cdot))\equiv J^i(t,x;\,\Theta_1(\cdot)X(\cdot)+v_1(\cdot),\,\Theta_2(\cdot)X(\cdot)+v_2(\cdot))\\
&=\mathbb{E}\Big\{\langle G^iX(T),X(T)\rangle+2\langle g^i,X(T)\rangle\\
&\quad+\int_t^T\Big[\Big\langle\begin{pmatrix}Q^i & (S^i)^\top\\ S^i & R^i\end{pmatrix}\begin{pmatrix}X\\ \Theta X+v\end{pmatrix},\begin{pmatrix}X\\ \Theta X+v\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}q^i\\ \rho^i\end{pmatrix},\begin{pmatrix}X\\ \Theta X+v\end{pmatrix}\Big\rangle\Big]ds\Big\}\\
&=\mathbb{E}\Big\{\langle G^iX(T),X(T)\rangle+2\langle g^i,X(T)\rangle\\
&\quad+\int_t^T\Big[\Big\langle\begin{pmatrix}Q^i+\Theta^\top S^i+(S^i)^\top\Theta+\Theta^\top R^i\Theta & (S^i)^\top+\Theta^\top R^i\\ S^i+R^i\Theta & R^i\end{pmatrix}\begin{pmatrix}X\\ v\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\Big\rangle\\
&\quad+2\Big\langle\begin{pmatrix}q^i+\Theta^\top\rho^i\\ \rho^i\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\Big\rangle\Big]ds\Big\}.
\end{aligned} \tag{3.6}
$$

Similarly, one can define $J^i(t,x;\,\Theta_1(\cdot)X(\cdot)+v_1(\cdot),u_2(\cdot))$ and $J^i(t,x;\,u_1(\cdot),\,\Theta_2(\cdot)X(\cdot)+v_2(\cdot))$. We now introduce the following definition.

Definition 3.2. A 4-tuple $(\Theta_1^*(\cdot),v_1^*(\cdot);\,\Theta_2^*(\cdot),v_2^*(\cdot))\in\mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$ is called a *closed-loop Nash equilibrium* of Problem (SDG) on $[t,T]$ if for any $x\in\mathbb{R}^n$ and any 4-tuple $(\Theta_1(\cdot),v_1(\cdot);\,\Theta_2(\cdot),v_2(\cdot))\in\mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$, the following hold:

$$
\begin{aligned}
&J^1(t,x;\,\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\,\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^1(t,x;\,\Theta_1(\cdot)X(\cdot)+v_1(\cdot),\,\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)),
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&J^2(t,x;\,\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\,\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^2(t,x;\,\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\,\Theta_2(\cdot)X(\cdot)+v_2(\cdot)).
\end{aligned} \tag{3.8}
$$

Note that in both (3.7) and (3.8),

$$X^*(\cdot) = X(\cdot\,; t, x, \Theta_1^*(\cdot), v_1^*(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot)),$$

whereas, in (3.7),

$$X(\cdot) = X(\cdot\,; t, x, \Theta_1(\cdot), v_1(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot)),$$

and in (3.8),

$$X(\cdot) = X(\cdot\,; t, x, \Theta_1^*(\cdot), v_1^*(\cdot), \Theta_2(\cdot), v_2(\cdot)).$$

Thus, $X(\cdot)$ appeared in (3.7) and (3.8) are different in general. We emphasize that the closed-loop Nash equilibrium $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is independent of the initial state x. The following result provides some equivalent definitions of closed-loop Nash equilibrium.

Proposition 3.3. *Let* (G1)–(G2) *hold and let* $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$. *Then the following are equivalent:*

(i) $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ *is a closed-loop Nash equilibrium of Problem* (SDG) *on* $[t,T]$.

(ii) *For any* $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$,

$$\begin{aligned}
&J^1(t,x;\, \Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\, \Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^1(t,x;\, \Theta_1^*(\cdot)X(\cdot)+\, v_1(\cdot),\, \Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)),\\
&J^2(t,x;\, \Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\, \Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^2(t,x;\, \Theta_1^*(\cdot)X(\cdot)+\, v_1^*(\cdot),\, \Theta_2^*(\cdot)X(\cdot)+v_2(\cdot)).
\end{aligned}$$

(iii) *For any* $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$,

$$\begin{aligned}
&J^1(t,x;\, \Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\, \Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^1(t,x; u_1(\cdot),\, \Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)),
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&J^2(t,x;\, \Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\, \Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J^2(t,x;\, \Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot), u_2(\cdot)).
\end{aligned}\tag{3.10}$$

Proof. The proof is similar to that of Proposition 2.4. □

If we denote

$$\bar u_i(\cdot) = \Theta_i^*(\cdot)X^*(\cdot)+v_i^*(\cdot), \qquad i=1,2, \tag{3.11}$$

then (3.9)–(3.10) become

$$J^1(t,x; \bar u_1(\cdot), \bar u_2(\cdot)) \leqslant J^1(t,x; u_1(\cdot),\, \Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)), \tag{3.12}$$

$$J^2(t,x; \bar u_1(\cdot), \bar u_2(\cdot)) \leqslant J^2(t,x;\, \Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot), u_2(\cdot)). \tag{3.13}$$

Since in (3.12), $X(\cdot)$ corresponds to $u_1(\cdot)$ and $(\Theta_2^*(\cdot), v_2^*(\cdot))$, one might not have

$$\bar u_2(\cdot) = \Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot).$$

Likewise, one might not have the following either:

$$\bar u_1(\cdot) = \Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot).$$

Hence, comparing this with (3.3), we see that the outcome $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ of the closed-loop Nash equilibrium $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ defined by (3.11) is not an open-loop Nash equilibrium of Problem (SDG) for $(t, X^*(t))$ in general.

On the other hand, if $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (SDG) on $[t,T]$, we may consider the following state equation on $[t,T]$ (denoting $\Theta^*(\cdot) = (\Theta_1^*(\cdot)^\top, \Theta_2^*(\cdot)^\top)^\top$):

$$\begin{cases} dX(s) = \big[(A + B\Theta^*)X + B_1v_1 + B_2v_2 + b\big]ds \\ \qquad\qquad + \big[(C + D\Theta^*)X + D_1v_1 + D_2v_2 + \sigma\big]dW(s), \\ X(t) = x, \end{cases} \tag{3.14}$$

with cost functionals

$$\widetilde{J}^i(t,x; v_1(\cdot), v_2(\cdot)) = J^i(t,x; \Theta_1^*(\cdot)X(\cdot) + v_1(\cdot), \Theta_2^*(\cdot)X(\cdot) + v_2(\cdot)), \qquad i = 1,2. \tag{3.15}$$

Then by (ii) of Proposition 3.3, $(v_1^*(\cdot), v_2^*(\cdot))$ is an open-loop Nash equilibrium of the corresponding (nonzero-sum differential) problem. Such an observation will be very useful below.

4. Open-loop Nash equilibria and FBSDEs

In this section, we discuss the open-loop Nash equilibria for Problem (SDG) in terms of FBSDEs. The main result of this section can be stated as follows.

Theorem 4.1. *Let* (G1)–(G2) *hold and let* $(t,x) \in [0,T) \times \mathbb{R}^n$ *be given. Then* $u^*(\cdot) \equiv (u_1^*(\cdot)^\top, u_2^*(\cdot)^\top)^\top \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$ *is an open-loop Nash equilibrium of Problem* (SDG) *for* (t,x) *if and only if the following two conditions hold:*

(i) *The adapted solution* $(X^*(\cdot), Y_i^*(\cdot), Z_i^*(\cdot))$ *to the FBSDE on* $[t,T]$

$$\begin{cases} dX^*(s) = \big[A(s)X^*(s) + B(s)u^*(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X^*(s) + D(s)u^*(s) + \sigma(s)\big]dW(s), \\ dY_i^*(s) = -\big[A(s)^\top Y_i^*(s) + C(s)^\top Z_i^*(s) + Q^i(s)X^*(s) \qquad i = 1,2, \\ \qquad\qquad + S^i(s)^\top u^*(s) + q^i(s)\big]ds + Z_i^*(s)dW(s), \\ X^*(t) = x, \qquad Y_i^*(T) = G^iX^*(T) + g^i, \end{cases} \tag{4.1}$$

satisfies the following stationarity condition:

$$B_i(s)^\top Y_i^*(s) + D_i(s)^\top Z_i^*(s) + S_i^i(s)X^*(s) + R_i^i(s)u^*(s) + \rho_i^i(s) = 0, \\ \text{a.e. } s \in [t,T], \ \text{a.s.} \tag{4.2}$$

(ii) *For* $i = 1,2$*, the following convexity condition holds:*

$$\mathbb{E}\Big\{\int_t^T \Big[\big\langle Q^i(s)X_i(s), X_i(s)\big\rangle + 2\big\langle S_i^i(s)X_i(s), u_i(s)\big\rangle + \big\langle R_{ii}^i(s)u_i(s), u_i(s)\big\rangle\Big]ds \\ + \big\langle G^iX_i(T), X_i(T)\big\rangle\Big\} \geqslant 0, \qquad \forall\, u_i(\cdot) \in \mathcal{U}_i[t,T], \tag{4.3}$$

where $X_i(\cdot)$ *is the solution to the following FSDE on* $[t,T]$:

$$\begin{cases} dX_i(s) = \big[A(s)X_i(s) + B_i(s)u_i(s)\big]ds + \big[C(s)X_i(s) + D_i(s)u_i(s)\big]dW(s), \\ X_i(t) = 0. \end{cases} \tag{4.4}$$

Or, equivalently, the map $u_i(\cdot) \mapsto J^i(t,x;u(\cdot))$ *is convex (for* $i = 1,2$*).*

Proof. For a given $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u^*(\cdot)\in\mathcal{U}[t,T]$, let $(X^*(\cdot),Y_1^*(\cdot),Z_1^*(\cdot))$ be the adapted solution to FBSDE (4.1) with $i=1$. For any $u_1(\cdot)\in\mathcal{U}_1[t,T]$ and $\varepsilon\in\mathbb{R}$, let $X^\varepsilon(\cdot)$ be the solution to the following perturbed state equation on $[t,T]$:

$$\begin{cases} dX^\varepsilon(s)=\big\{A(s)X^\varepsilon(s)+B_1(s)[u_1^*(s)+\varepsilon u_1(s)]+B_2(s)u_2^*(s)+b(s)\big\}ds\\ \qquad\qquad +\ \big\{C(s)X^\varepsilon(s)+D_1(s)[u_1^*(s)+\varepsilon u_1(s)]+D_2(s)u_2^*(s)+\sigma(s)\big\}dW(s),\\ X^\varepsilon(t)=x.\end{cases}$$

Then denoting $X_1(\cdot)$ the solution of (4.4) with $i=1$, we have $X^\varepsilon(\cdot)=X^*(\cdot)+\varepsilon X_1(\cdot)$ and

$$\begin{aligned}
&J^1(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot),u_2^*(\cdot))-J^1(t,x;u_1^*(\cdot),u_2^*(\cdot))\\
&=\varepsilon\mathbb{E}\Big\{\big\langle G^1[2X^*(T)+\varepsilon X_1(T)],X_1(T)\big\rangle+2\big\langle g^1,X_1(T)\big\rangle\\
&\quad+\int_t^T\Big[\Big\langle\begin{pmatrix}Q^1&(S_1^1)^\top&(S_2^1)^\top\\ S_1^1&R_{11}^1&R_{12}^1\\ S_2^1&R_{21}^1&R_{22}^1\end{pmatrix}\begin{pmatrix}2X^*+\varepsilon X_1\\ 2u_1^*+\varepsilon u_1\\ 2u_2^*\end{pmatrix},\begin{pmatrix}X_1\\ u_1\\ 0\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}q^1\\ \rho_1^1\end{pmatrix},\begin{pmatrix}X_1\\ u_1\end{pmatrix}\Big\rangle\Big]ds\Big\}\\
&=2\varepsilon\mathbb{E}\Big\{\big\langle G^1X^*(T)+g^1,X_1(T)\big\rangle+\int_t^T\Big[\big\langle Q^1X^*+(S^1)^\top u^*+q^1,X_1\big\rangle\\
&\quad+\big\langle S_1^1X^*+R_1^1u^*+\rho_1^1,u_1\big\rangle\Big]ds\Big\}\\
&\quad+\varepsilon^2\mathbb{E}\Big\{\big\langle G^1X_1(T),X_1(T)\big\rangle+\int_t^T\Big[\big\langle Q^1X_1,X_1\big\rangle+2\big\langle S_1^1X_1,u_1\big\rangle+\big\langle R_{11}^1u_1,u_1\big\rangle\Big]ds\Big\}.
\end{aligned}$$

On the other hand, applying Itô's formula to $s\mapsto\langle Y_1^*(s),X_1(s)\rangle$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big\{\big\langle G^1X^*(T)+g^1,X_1(T)\big\rangle+\int_t^T\Big[\big\langle Q^1X^*+(S^1)^\top u^*+q^1,X_1\big\rangle\\
&\quad+\big\langle S_1^1X^*+R_1^1u^*+\rho_1^1,u_1\big\rangle\Big]ds\Big\}\\
&=\mathbb{E}\int_t^T\Big\{\big\langle-\big[A^\top Y_1^*+C^\top Z_1^*+Q^1X^*+(S^1)^\top u^*+q^1\big],X_1\big\rangle+\big\langle Y_1^*,AX_1+B_1u_1\big\rangle\\
&\quad+\big\langle Z_1^*,CX_1+D_1u_1\big\rangle+\big\langle Q^1X^*+(S^1)^\top u^*+q^1,X_1\big\rangle+\big\langle S_1^1X^*+R_1^1u^*+\rho_1^1,u_1\big\rangle\Big\}ds\\
&=\mathbb{E}\int_t^T\langle B_1^\top Y_1^*+D_1^\top Z_1^*+S_1^1X^*+R_1^1u^*+\rho_1^1,u_1\rangle ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&J^1(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot),u_2^*(\cdot))-J^1(t,x;u_1^*(\cdot),u_2^*(\cdot))\\
&=2\varepsilon\mathbb{E}\int_t^T\langle B_1^\top Y_1^*+D_1^\top Z_1^*+S_1^1X^*+R_1^1u^*+\rho_1^1,u_1\rangle ds\\
&\quad+\varepsilon^2\mathbb{E}\Big\{\big\langle G^1X_1(T),X_1(T)\big\rangle+\int_t^T\Big[\big\langle Q^1X_1,X_1\big\rangle+2\big\langle S_1^1X_1,u_1\big\rangle+\big\langle R_{11}^1u_1,u_1\big\rangle\Big]ds\Big\}.
\end{aligned}$$

It follows that

$$J^1(t,x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J^1(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot),u_2^*(\cdot)),\qquad\forall\,u_1(\cdot)\in\mathcal{U}_1[t,T],\ \forall\,\varepsilon\in\mathbb{R},$$

if and only if (4.3) holds for $i=1$, and

$$B_1^\top Y_1^*+D_1^\top Z_1^*+S_1^1X^*+R_1^1u^*+\rho_1^1=0,\qquad\text{a.e. } s\in[t,T],\ \text{a.s.}\tag{4.5}$$

Similarly,

$$J^2(t,x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J^2(t,x;u_1^*(\cdot),u_2^*(\cdot)+\varepsilon u_2(\cdot)),\qquad \forall\, u_2(\cdot)\in\mathcal{U}_2[t,T],\ \forall\,\varepsilon\in\mathbb{R},$$

if and only if (4.3) holds for $i=2$, and

$$B_2^\top Y_2^*+D_2^\top Z_2^*+S_2^2X^*+R_2^2u^*+\rho_2^2=0,\qquad \text{a.e. } s\in[t,T],\ \text{a.s.} \tag{4.6}$$

Combining (4.5)–(4.6), we obtain (4.2). □

Remark 4.2. (i) Note that (4.1); $i=1,2$, are two coupled FBSDEs, and these two FBSDEs are coupled through the relation (4.2). In fact, from (4.2), we see that

$$\begin{pmatrix}R_{11}^1 & R_{12}^1\\ R_{21}^2 & R_{22}^2\end{pmatrix}\begin{pmatrix}u_1^*\\ u_2^*\end{pmatrix}=-\begin{pmatrix}B_1^\top Y_1^*+D_1^\top Z_1^*+S_1^1X^*+\rho_1^1\\ B_2^\top Y_2^*+D_2^\top Z_2^*+S_2^2X^*+\rho_2^2\end{pmatrix}.$$

Thus, say, in the case that the coefficient matrix of u^* is invertible, one has

$$\begin{pmatrix}u_1^*\\ u_2^*\end{pmatrix}=-\begin{pmatrix}R_{11}^1 & R_{12}^1\\ R_{21}^2 & R_{22}^2\end{pmatrix}^{-1}\begin{pmatrix}B_1^\top Y_1^*+D_1^\top Z_1^*+S_1^1X^*+\rho_1^1\\ B_2^\top Y_2^*+D_2^\top Z_2^*+S_2^2X^*+\rho_2^2\end{pmatrix}.$$

Plugging the above into (4.1), we see the coupling between the two coupled FBSDEs (with $i=1,2$).

(ii) An easily verifiable condition for the convexity of the map $u_i(\cdot)\mapsto J^i(t,x;u(\cdot))$ is

$$G^i\geqslant 0,\quad R_{ii}^i(s)>0,\quad Q^i(s)-S_i^i(s)^\top R_{ii}^i(s)^{-1}S_i^i(s)\geqslant 0;\quad \text{a.e. } s\in[t,T]. \tag{4.7}$$

This can be seen by completing the square: For any $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^{m_i}$,

$$\begin{aligned}&\langle Q^i(s)x,x\rangle+2\langle S_i^i(s)x,u\rangle+\langle R_{ii}^i(s)u,u\rangle\\ &\quad=\langle[Q^i(s)-S_i^i(s)^\top R_{ii}^i(s)^{-1}S_i^i(s)]x,x\rangle\\ &\qquad+\langle R_{ii}^i(s)[u+R_{ii}^i(s)^{-1}S_i^i(s)x],u+R_{ii}^i(s)^{-1}S_i^i(s)x\rangle\\ &\quad\geqslant 0,\quad \text{a.e. } s\in[t,T].\end{aligned}$$

Note that the conditions (4.3); $i=1,2$, are independent, and both can be regarded as the convexity condition for certain Problem (SLQ) with appropriate state equation and cost functional. For more discussion on the convexity, we refer the interested readers to [21].

To conclude this section, let us write FBSDE (4.1) and stationarity condition (4.2) more compactly. For this, we introduce the following:

$$\begin{aligned}&\mathbf{A}(\cdot)=\begin{pmatrix}A(\cdot) & 0\\ 0 & A(\cdot)\end{pmatrix},\quad \mathbf{B}(\cdot)=\begin{pmatrix}B(\cdot) & 0\\ 0 & B(\cdot)\end{pmatrix}\equiv\begin{pmatrix}B_1(\cdot) & B_2(\cdot) & 0 & 0\\ 0 & 0 & B_1(\cdot) & B_2(\cdot)\end{pmatrix},\\ &\mathbf{C}(\cdot)=\begin{pmatrix}C(\cdot) & 0\\ 0 & C(\cdot)\end{pmatrix},\quad \mathbf{D}(\cdot)=\begin{pmatrix}D(\cdot) & 0\\ 0 & D(\cdot)\end{pmatrix}\equiv\begin{pmatrix}D_1(\cdot) & D_2(\cdot) & 0 & 0\\ 0 & 0 & D_1(\cdot) & D_2(\cdot)\end{pmatrix},\\ &\mathbf{Q}(\cdot)=\begin{pmatrix}Q^1(\cdot) & 0\\ 0 & Q^2(\cdot)\end{pmatrix},\quad \mathbf{S}(\cdot)=\begin{pmatrix}S^1(\cdot) & 0\\ 0 & S^2(\cdot)\end{pmatrix},\quad \mathbf{R}(\cdot)=\begin{pmatrix}R^1(\cdot) & 0\\ 0 & R^2(\cdot)\end{pmatrix},\\ &\mathbf{q}(\cdot)=\begin{pmatrix}q^1(\cdot)\\ q^2(\cdot)\end{pmatrix},\quad \boldsymbol{\rho}(\cdot)=\begin{pmatrix}\rho^1(\cdot)\\ \rho^2(\cdot)\end{pmatrix},\quad \mathbf{G}=\begin{pmatrix}G^1 & 0\\ 0 & G^2\end{pmatrix},\quad \mathbf{g}=\begin{pmatrix}g^1\\ g^2\end{pmatrix}.\end{aligned}$$

Then

$$\begin{cases}\mathbf{A}(\cdot)\in L^1(0,T;\mathbb{R}^{2n\times 2n}), \quad \mathbf{B}(\cdot)\in L^2(0,T;\mathbb{R}^{2n\times 2m}),\\ \mathbf{C}(\cdot)\in L^2(0,T;\mathbb{R}^{2n\times 2n}), \quad \mathbf{D}(\cdot)\in L^\infty(0,T;\mathbb{R}^{2n\times 2m}),\\ \mathbf{Q}(\cdot)\in L^1(0,T;\mathbb{S}^{2n}), \quad \mathbf{S}(\cdot)\in L^2(0,T;\mathbb{R}^{2m\times 2n}), \quad \mathbf{R}(\cdot)\in L^\infty(0,T;\mathbb{S}^{2m}),\\ \mathbf{q}(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^{2n})), \quad \boldsymbol{\rho}(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{2m}), \quad \mathbf{G}\in\mathbb{S}^{2n}, \quad \mathbf{g}\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^{2n}).\end{cases}$$

Further, let

$$\mathbf{J}=\begin{pmatrix}I_{m_1} & 0\\ 0 & 0\\ 0 & 0\\ 0 & I_{m_2}\end{pmatrix}\equiv\begin{pmatrix}I_{m_1} & 0_{m_1\times m_2}\\ 0_{m_2\times m_1} & 0_{m_2\times m_2}\\ 0_{m_1\times m_1} & 0_{m_1\times m_2}\\ 0_{m_2\times m_1} & I_{m_2}\end{pmatrix}\in\mathbb{R}^{2m\times m}, \qquad \mathbf{I}_k=\begin{pmatrix}I_k\\ I_k\end{pmatrix}\in\mathbb{R}^{2k\times k}.$$

Clearly, one has

$$\mathbf{B}(\cdot)\mathbf{J}\equiv\begin{pmatrix}B_1(\cdot) & B_2(\cdot) & 0 & 0\\ 0 & 0 & B_1(\cdot) & B_2(\cdot)\end{pmatrix}\begin{pmatrix}I_{m_1} & 0\\ 0 & 0\\ 0 & 0\\ 0 & I_{m_2}\end{pmatrix}=\begin{pmatrix}B_1(\cdot) & 0\\ 0 & B_2(\cdot)\end{pmatrix},$$

$$\mathbf{D}(\cdot)\mathbf{J}\equiv\begin{pmatrix}D_1(\cdot) & D_2(\cdot) & 0 & 0\\ 0 & 0 & D_1(\cdot) & D_2(\cdot)\end{pmatrix}\begin{pmatrix}I_{m_1} & 0\\ 0 & 0\\ 0 & 0\\ 0 & I_{m_2}\end{pmatrix}=\begin{pmatrix}D_1(\cdot) & 0\\ 0 & D_2(\cdot)\end{pmatrix},$$

$$\mathbf{J}^\top\mathbf{S}(\cdot)\equiv\begin{pmatrix}I_{m_1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_2}\end{pmatrix}\begin{pmatrix}S_1^1(\cdot) & 0\\ S_2^1(\cdot) & 0\\ 0 & S_1^2(\cdot)\\ 0 & S_2^2(\cdot)\end{pmatrix}=\begin{pmatrix}S_1^1(\cdot) & 0\\ 0 & S_2^2(\cdot)\end{pmatrix},$$

$$\mathbf{J}^\top\mathbf{R}(\cdot)\equiv\begin{pmatrix}I_{m_1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_2}\end{pmatrix}\begin{pmatrix}R_1^1(\cdot) & 0\\ R_2^1(\cdot) & 0\\ 0 & R_1^2(\cdot)\\ 0 & R_2^2(\cdot)\end{pmatrix}=\begin{pmatrix}R_1^1(\cdot) & 0\\ 0 & R_2^2(\cdot)\end{pmatrix},$$

$$\mathbf{J}^\top\boldsymbol{\rho}(\cdot)\equiv\begin{pmatrix}I_{m_1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_2}\end{pmatrix}\begin{pmatrix}\rho_1^1(\cdot)\\ \rho_2^1(\cdot)\\ \rho_1^2(\cdot)\\ \rho_2^2(\cdot)\end{pmatrix}=\begin{pmatrix}\rho_1^1(\cdot)\\ \rho_2^2(\cdot)\end{pmatrix}.$$

With the above notation, FBSDE (4.1) can be written as (suppressing s and dropping $*$)

$$\begin{cases}dX=\big(AX+Bu+b\big)ds+\big(CX+Du+\sigma\big)dW,\\ d\mathbf{Y}=-\big(\mathbf{A}^\top\mathbf{Y}+\mathbf{C}^\top\mathbf{Z}+\mathbf{Q}\mathbf{I}_nX+\mathbf{S}^\top\mathbf{I}_mu+\mathbf{q}\big)ds+\mathbf{Z}dW,\\ X(t)=x, \qquad \mathbf{Y}(T)=\mathbf{G}\mathbf{I}_nX(T)+\mathbf{g},\end{cases}\tag{4.8}$$

where

$$\mathbf{Y}(\cdot)=\begin{pmatrix}Y_1(\cdot)\\ Y_2(\cdot)\end{pmatrix}, \qquad \mathbf{Z}(\cdot)=\begin{pmatrix}Z_1(\cdot)\\ Z_2(\cdot)\end{pmatrix}$$

and the stationarity condition (4.2) can be written as

$$\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{Y}+\mathbf{D}^\top\mathbf{Z}+\mathbf{S}\mathbf{I}_nX+\mathbf{R}\mathbf{I}_mu+\boldsymbol{\rho}\big)=0,\qquad \text{a.e. } s\in[t,T],\ \text{a.s.} \tag{4.9}$$

Keep in mind that (4.8) is a coupled FBSDE with the coupling given through (4.9).

5. Closed-loop Nash equilibria and Riccati equations

We now look at closed-loop Nash equilibria for Problem (SDG). Again, for simplicity of notation, we will suppress the time variable s as long as no confusion arises. First, we present the following result which is a consequence of Theorem 4.1.

Proposition 5.1. *Let* (G1)–(G2) *hold. Suppose that* $(\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot))\in\mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$ *is a closed-loop Nash equilibrium of Problem* (SDG) *on* $[t,T]$. *Denote* $\Theta^*(\cdot)\equiv(\Theta_1^*(\cdot)^\top,\Theta_2^*(\cdot)^\top)^\top$ *and let* $\mathbb{X}(\cdot)$ *be the solution to the* $\mathbb{R}^{n\times n}$*-valued SDE*

$$\begin{cases}d\mathbb{X}=(A+B\Theta^*)\mathbb{X}ds+(C+D\Theta^*)\mathbb{X}dW, & s\in[t,T],\\ \mathbb{X}(t)=I.\end{cases} \tag{5.1}$$

Then for $i=1,2$, *the adapted solution* $(\mathbb{Y}_i(\cdot),\mathbb{Z}_i(\cdot))$ *to the* $\mathbb{R}^{n\times n}$*-valued BSDE on* $[t,T]$

$$\begin{cases}d\mathbb{Y}_i=-\Big\{(A+B\Theta^*)^\top\mathbb{Y}_i+(C+D\Theta^*)^\top\mathbb{Z}_i\\ \qquad\qquad+\big[Q^i+(\Theta^*)^\top S^i+(S^i)^\top\Theta^*+(\Theta^*)^\top R^i\Theta^*\big]\mathbb{X}\Big\}ds+\mathbb{Z}_idW,\\ \mathbb{Y}_i(T)=G^i\mathbb{X}(T),\end{cases} \tag{5.2}$$

satisfies

$$B_i^\top\mathbb{Y}_i+D_i^\top\mathbb{Z}_i+(S_i^i+R_i^i\Theta^*)\mathbb{X}=0,\qquad \textit{a.e. } s\in[t,T],\ \textit{a.s.} \tag{5.3}$$

Proof. Let us consider state equation (3.14) with the cost functionals defined by (3.15). Denoting $v(\cdot)=(v_1(\cdot)^\top,v_2(\cdot)^\top)^\top$, by an argument similar to (3.6), we have:

$$\begin{aligned}&\widetilde{J}^i(t,x;v(\cdot))\equiv J^i(t,x;\Theta^*(\cdot)X(\cdot)+v(\cdot))\\ &=\mathbb{E}\Big\{\langle G^iX(T),X(T)\rangle+2\langle g^i,X(T)\rangle\\ &\quad+\int_t^T\Big[\Big\langle\begin{pmatrix}Q^i+(\Theta^*)^\top S^i+(S^i)^\top\Theta^*+(\Theta^*)^\top R^i\Theta^* & (S^i)^\top+(\Theta^*)^\top R^i\\ S^i+R^i\Theta^* & R^i\end{pmatrix}\begin{pmatrix}X\\ v\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\Big\rangle\\ &\quad+2\Big\langle\begin{pmatrix}q^i+(\Theta^*)^\top\rho^i\\ \rho^i\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\Big\rangle\Big]ds\Big\}.\end{aligned}$$

We know by (ii) of Proposition 3.3 that $v^*(\cdot)\equiv(v_1^*(\cdot)^\top,v_2^*(\cdot)^\top)^\top$ is an open-loop Nash equilibrium for the problem with the state equation (3.14) and with the cost functionals $\widetilde{J}^i(t,x;v(\cdot))$ for any initial pair (t,x). Thus, according to Theorem 4.1, we have for $i=1,2$,

$$B_i^\top Y_i^*+D_i^\top Z_i^*+(S_i^i+R_i^i\Theta^*)X^*+R_i^iv^*+\rho_i^i=0,\qquad \text{a.e. } s\in[t,T],\ \text{a.s.} \tag{5.4}$$

with $X^*(\cdot)$ being the solution to the closed-loop system:

$$\begin{cases}dX^*=\big[(A+B\Theta^*)X^*+Bv^*+b\big]ds\\ \qquad\quad+\big[(C+D\Theta^*)X^*+Dv^*+\sigma\big]dW, & s\in[t,T],\\ X^*(t)=x,\end{cases} \tag{5.5}$$

and $(Y_i^*(\cdot), Z_i^*(\cdot))$ being the adapted solution to the following BSDE:

$$\begin{cases} dY_i^* = -\Big\{(A+B\Theta^*)^\top Y_i^* + (C+D\Theta^*)^\top Z_i^* \\ \qquad\qquad + \big[Q^i + (\Theta^*)^\top S^i + (S^i)^\top\Theta^* + (\Theta^*)^\top R^i\Theta^*\big]X^* \\ \qquad\qquad + (S^i + R^i\Theta^*)^\top v^* + q^i + (\Theta^*)^\top\rho^i\Big\}ds + Z_i^* dW, \qquad s\in[t,T], \\ Y_i^*(T) = G^i X^*(T) + g^i. \end{cases} \tag{5.6}$$

Since $(\Theta^*(\cdot), v^*(\cdot))$ is independent of x and (5.4)–(5.6) hold for all $x\in\mathbb{R}^n$, by subtracting solutions corresponding to x and 0, the latter from the former, we see that for any $x\in\mathbb{R}^n$, the adapted solution $(X(\cdot), Y_i(\cdot), Z_i(\cdot))$ $(i=1,2)$ to the following FBSDE on $[t,T]$:

$$\begin{cases} dX = (A+B\Theta^*)Xds + (C+D\Theta^*)XdW, \qquad s\in[t,T], \\ dY_i = -\Big\{(A+B\Theta^*)^\top Y_i + (C+D\Theta^*)^\top Z_i \\ \qquad\qquad + \big[Q^i + (\Theta^*)^\top S^i + (S^i)^\top\Theta^* + (\Theta^*)^\top R^i\Theta^*\big]X\Big\}ds + Z_i dW, \\ X(t) = x, \qquad Y_i(T) = G^i X(T), \end{cases}$$

satisfies

$$B_i^\top Y_i + D_i^\top Z_i + (S_i^i + R_i^i\Theta^*)X = 0, \qquad \text{a.e. } s\in[t,T], \text{ a.s.}$$

The desired result then follows easily. □

Now we are ready to present the main result of this section, which characterizes the closed-loop Nash equilibrium of Problem (SDG).

Theorem 5.2. *Let* (G1)–(G2) *hold. Then* $(\Theta^*(\cdot), v^*(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ *is a closed-loop Nash equilibrium of Problem* (SDG) *on* $[t,T]$ *if and only if the following hold:*

(i) *For* $i=1,2$, *the solution* $P_i(\cdot)\in C([t,T];\mathbb{S}^n)$ *to the Lyapunov type equation*

$$\begin{cases} \dot P_i + P_iA + A^\top P_i + C^\top P_iC + Q^i + (\Theta^*)^\top(R^i + D^\top P_iD)\Theta^* \\ \quad + \big[P_iB + C^\top P_iD + (S^i)^\top\big]\Theta^* + (\Theta^*)^\top\big[B^\top P_i + D^\top P_iC + S^i\big] = 0, \\ P_i(T) = G^i, \end{cases} \tag{5.7}$$

satisfies the following two conditions:

$$R_{ii}^i + D_i^\top P_iD_i \geqslant 0, \qquad \text{a.e. } s\in[t,T], \tag{5.8}$$

$$B_i^\top P_i + D_i^\top P_iC + S_i^i + (R_i^i + D_i^\top P_iD)\Theta^* = 0, \qquad \text{a.e. } s\in[t,T]. \tag{5.9}$$

(ii) *For* $i=1,2$, *the adapted solution* $(\eta_i(\cdot), \zeta_i(\cdot))$ *to the BSDE on* $[t,T]$

$$\begin{cases} d\eta_i = -\Big\{A^\top\eta_i + C^\top\zeta_i + (\Theta^*)^\top\big[B^\top\eta_i + D^\top\zeta_i + D^\top P_i\sigma + \rho^i + (R^i + D^\top P_iD)v^*\big] \\ \qquad\qquad + \big[P_iB + C^\top P_iD + (S^i)^\top\big]v^* + C^\top P_i\sigma + P_ib + q^i\Big\}ds + \zeta_i dW, \\ \eta_i(T) = g^i, \end{cases} \tag{5.10}$$

satisfies

$$B_i^\top\eta_i + D_i^\top\zeta_i + D_i^\top P_i\sigma + \rho_i^i + (R_i^i + D_i^\top P_iD)v^* = 0, \qquad \text{a.e. } s\in[t,T], \text{ a.s.} \tag{5.11}$$

Proof. We first prove the necessity. Suppose that $(\Theta^*(\cdot), v^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (SDG) on $[t, T]$, where $\Theta^*(\cdot) \equiv (\Theta_1^*(\cdot)^\top, \Theta_2^*(\cdot)^\top)^\top$ and $v^*(\cdot) \equiv (v_1^*(\cdot)^\top, v_2^*(\cdot)^\top)^\top$. Let $\mathbb{X}(\cdot)$ and $\mathbb{Y}_i(\cdot)$ $(i = 1, 2)$ be the solutions of (5.1) and (5.2), respectively. Consider the following linear ordinary differential equation (ODE, for short) which is equivalent to (5.7):

$$\begin{cases} \dot{P}_i + P_i(A + B\Theta^*) + (A + B\Theta^*)^\top P_i + (C + D\Theta^*)^\top P_i(C + D\Theta^*) \\ \quad + Q^i + (\Theta^*)^\top S^i + (S^i)^\top \Theta^* + (\Theta^*)^\top R^i \Theta^* = 0, \qquad s \in [t, T], \\ P_i(T) = G^i. \end{cases} \tag{5.12}$$

Such an equation admits a unique solution $P_i(\cdot) \in C([t, T]; \mathbb{S}^n)$. By Itô's formula, we have

$$\begin{aligned} d(P_i\mathbb{X}) &= \dot{P}_i\mathbb{X}ds + P_i(A + B\Theta^*)\mathbb{X}ds + P_i(C + D\Theta^*)\mathbb{X}dW \\ &= -\big\{(A + B\Theta^*)^\top P_i\mathbb{X} + (C + D\Theta^*)^\top P_i(C + D\Theta^*)\mathbb{X} \\ &\quad + \big[Q^i + (\Theta^*)^\top S^i + (S^i)^\top \Theta^* + (\Theta^*)^\top R^i \Theta^*\big]\mathbb{X}\big\}ds \\ &\quad + P_i(C + D\Theta^*)\mathbb{X}dW. \end{aligned}$$

Comparing the above with (5.2), by the uniqueness of adapted solutions to BSDEs, one has

$$\mathbb{Y}_i = P_i\mathbb{X}, \quad \mathbb{Z}_i = P_i(C + D\Theta^*)\mathbb{X}; \qquad i = 1, 2.$$

From (5.1), we see that the process $\mathbb{X}(\cdot)$ is invertible almost surely. Then, the above together with (5.3) leads to (5.9). Now let $X^*(\cdot)$ be the solution to (5.5), and for $i = 1, 2$, let $(Y_i^*(\cdot), Z_i^*(\cdot))$ be the adapted solution to (5.6). Define

$$\begin{cases} \eta_i = Y_i^* - P_iX^*, \\ \zeta_i = Z_i^* - P_i(C + D\Theta^*)X^* - P_i(Dv^* + \sigma). \end{cases} \tag{5.13}$$

Then $\eta_i(T) = g^i$, and

$$\begin{aligned} d\eta_i &= dY_i^* - \dot{P}_iX^*ds - P_idX^* \\ &= -\Big\{(A + B\Theta^*)^\top Y_i^* + (C + D\Theta^*)^\top Z_i^* + (S^i + R^i\Theta^*)^\top v^* + P_i(Bv^* + b) \\ &\quad + q^i + (\Theta^*)^\top \rho^i \\ &\quad + \big[\dot{P}_i + P_i(A + B\Theta^*) + Q^i + (\Theta^*)^\top S^i + (S^i)^\top \Theta^* + (\Theta^*)^\top R^i\Theta^*\big]X^*\Big\}ds \\ &\quad + \Big\{Z_i^* - P_i\big[(C + D\Theta^*)X^* + Dv^* + \sigma\big]\Big\}dW \\ &= -\Big\{(A + B\Theta^*)^\top Y_i^* + (C + D\Theta^*)^\top Z_i^* + (S^i + R^i\Theta^*)^\top v^* + P_i(Bv^* + b) \\ &\quad + q^i + (\Theta^*)^\top \rho^i \\ &\quad - (A + B\Theta^*)^\top P_iX^* - (C + D\Theta^*)^\top P_i(C + D\Theta^*)X^*\Big\}ds + \zeta_idW \\ &= -\Big\{(A + B\Theta^*)^\top \eta_i + (C + D\Theta^*)^\top \zeta_i + (C + D\Theta^*)^\top P_i(Dv^* + \sigma) \\ &\quad + (S^i + R^i\Theta^*)^\top v^* + P_i(Bv^* + b) + q^i + (\Theta^*)^\top \rho^i\Big\}ds + \zeta_idW \\ &= -\Big\{A^\top \eta_i + C^\top \zeta_i + (\Theta^*)^\top\big[B^\top \eta_i + D^\top \zeta_i + D^\top P_i\sigma + \rho^i + (R^i + D^\top P_iD)v^*\big] \\ &\quad + \big[P_iB + C^\top P_iD + (S^i)^\top\big]v^* + C^\top P_i\sigma + P_ib + q^i\Big\}ds + \zeta_idW. \end{aligned}$$

Thus, (η_i, ζ_i) is the adapted solution to BSDE (5.10). Next, from the proof of Proposition 5.1 we know that (5.4) holds. Thus (noting (5.9) and (5.13)),

$$
\begin{aligned}
0 &= B_i^\top Y_i^* + D_i^\top Z_i^* + (S_i^i + R_i^i \Theta^*)X^* + R_i^i v^* + \rho_i^i \\
&= B_i^\top \eta_i + D_i^\top \zeta_i + D_i^\top P_i \sigma + \rho_i^i + (R_i^i + D_i^\top P_i D)v^* \\
&\quad + \big[B_i^\top P_i + D_i^\top P_i C + S_i^i + (R_i^i + D_i^\top P_i D)\Theta^*\big]X^* \\
&= B_i^\top \eta_i + D_i^\top \zeta_i + D_i^\top P_i \sigma + \rho_i^i + (R_i^i + D_i^\top P_i D)v^*,
\end{aligned}
$$

which is (5.11). The proof of (5.8) will be included in the proof of sufficiency.

To prove the sufficiency, we take any $v(\cdot) = (v_1(\cdot)^\top, v_2(\cdot)^\top)^\top \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$. Denote $w = (v_1^\top, (v_2^*)^\top)^\top$, and let

$$
X(\cdot) = X(\cdot\,; t, x, \Theta_1^*(\cdot), v_1(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot))
$$

be the state process corresponding to (t,x) and $(\Theta_1^*(\cdot), v_1(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot))$. By applying Itô's formula to $s \mapsto \langle P_1(s)X(s), X(s)\rangle + 2\langle \eta_1(s), X_1(s)\rangle$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\big\langle G^1 X(T), X(T)\big\rangle + 2\big\langle g^1, X(T)\big\rangle\Big] - \mathbb{E}\Big[\langle P_1(t)x, x\rangle + 2\langle \eta_1(t), x\rangle\Big] \\
&= \mathbb{E}\int_t^T \Big\{\big\langle \dot P_1 X, X\big\rangle + 2\langle P_1 X, (A + B\Theta^*)X + Bw + b\rangle \\
&\quad + \big\langle P_1\big[(C + D\Theta^*)X + Dw + \sigma\big], (C + D\Theta^*)X + Dw + \sigma\big\rangle \\
&\quad - 2\big\langle A^\top \eta_1 + C^\top \zeta_1 + (\Theta^*)^\top\big[B^\top \eta_1 + D^\top \zeta_1 + D^\top P_1 \sigma + \rho^1 \\
&\quad + (R^1 + D^\top P_1 D)v^*\big], X\big\rangle \\
&\quad - 2\big\langle\big[P_1 B + C^\top P_1 D + (S^1)^\top\big]v^* + C^\top P_1 \sigma + P_1 b + q^1, X\big\rangle \\
&\quad + 2\langle \eta_1, (A + B\Theta^*)X + Bw + b\rangle + 2\langle \zeta_1, (C + D\Theta^*)X + Dw + \sigma\rangle\Big\}ds \\
&= \mathbb{E}\int_t^T \Big\{\big\langle\big[\dot P_1 + P_1(A + B\Theta^*) + (A + B\Theta^*)^\top P_1 \\
&\quad + (C + D\Theta^*)^\top P_1 (C + D\Theta^*)\big]X, X\big\rangle \\
&\quad + 2\langle P_1 X, Bw + b\rangle + 2\langle P_1(C + D\Theta^*)X, Dw + \sigma\rangle + \langle P_1(Dw + \sigma), Dw + \sigma\rangle \\
&\quad - 2\big\langle(\Theta^*)^\top\big[D^\top P_1 \sigma + \rho^1 + (R^1 + D^\top P_1 D)v^*\big], X\big\rangle \\
&\quad - 2\big\langle\big[P_1 B + C^\top P_1 D + (S^1)^\top\big]v^* + C^\top P_1 \sigma + P_1 b + q^1, X\big\rangle \\
&\quad + 2\langle \eta_1, Bw + b\rangle + 2\langle \zeta_1, Dw + \sigma\rangle\Big\}ds \\
&= \mathbb{E}\int_t^T \Big\{\big\langle\big[\dot P_1 + P_1(A + B\Theta^*) + (A + B\Theta^*)^\top P_1 \\
&\quad + (C + D\Theta^*)^\top P_1 (C + D\Theta^*)\big]X, X\big\rangle \\
&\quad + 2\big\langle(P_1 B + C^\top P_1 D)w - \big[P_1 B + C^\top P_1 D + (S^1)^\top\big]v^* - q^1, X\big\rangle \\
&\quad + 2\big\langle D^\top P_1 D w - (R^1 + D^\top P_1 D)v^* - \rho^1, \Theta^* X\big\rangle + \big\langle D^\top P_1 D w, w\big\rangle \\
&\quad + 2\big\langle B^\top \eta_1 + D^\top \zeta_1 + D^\top P_1 \sigma, w\big\rangle + \langle P_1 \sigma, \sigma\rangle + 2\langle \eta_1, b\rangle + 2\langle \zeta_1, \sigma\rangle\Big\}ds.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&J^1(t,x;\,\Theta^*X(\cdot)+w(\cdot))-\mathbb{E}\Big[\big\langle G^1X(T),X(T)\big\rangle+2\big\langle g^1,X(T)\big\rangle\Big]\\
&=\mathbb{E}\int_t^T\Big[\Big\langle\begin{pmatrix}Q^1+(\Theta^*)^\top S^1+(S^1)^\top\Theta^*+(\Theta^*)^\top R^1\Theta^* & (S^1)^\top+(\Theta^*)^\top R^1\\ S^1+R^1\Theta^* & R^1\end{pmatrix}\\
&\quad\times\begin{pmatrix}X\\ w\end{pmatrix},\begin{pmatrix}X\\ w\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}q^1+(\Theta^*)^\top\rho^1\\ \rho^1\end{pmatrix},\begin{pmatrix}X\\ w\end{pmatrix}\Big\rangle\Big]ds\\
&=\mathbb{E}\int_t^T\Big\{\big\langle\big[Q^1+(\Theta^*)^\top S^1+(S^1)^\top\Theta^*+(\Theta^*)^\top R^1\Theta^*\big]X,X\big\rangle+2\big\langle(S^1)^\top w+q^1,X\big\rangle\\
&\qquad\qquad+\,2\big\langle R^1w+\rho^1,\Theta^*X\big\rangle+\big\langle R^1w,w\big\rangle+2\big\langle\rho^1,w\big\rangle\Big\}ds.
\end{aligned}
$$

Combining the above two equations, together with Eq. (5.12) (which is equivalent to (5.7)) and conditions (5.9) and (5.11), one obtains

$$
\begin{aligned}
&J^1(t,x;\,\Theta^*X(\cdot)+w(\cdot))-\mathbb{E}\Big[\langle P_1(t)x,x\rangle+2\langle\eta_1(t),x\rangle\Big]\\
&\quad=\mathbb{E}\int_t^T\Big\{2\big\langle\big[P_1B+C^\top P_1D+(S^1)^\top\big](w-v^*),X\big\rangle\\
&\quad+2\big\langle(R^1+D^\top P_1D)(w-v^*),\Theta^*X\big\rangle+\big\langle(R^1+D^\top P_1D)w,w\big\rangle\\
&\quad+2\big\langle B^\top\eta_1+D^\top\zeta_1+D^\top P_1\sigma+\rho^1,w\big\rangle+\langle P_1\sigma,\sigma\rangle+2\langle\eta_1,b\rangle+2\langle\zeta_1,\sigma\rangle\Big\}ds\\
&\quad=\mathbb{E}\int_t^T\Big\{2\big\langle\big[P_1B_1+C^\top P_1D_1+(S_1^1)^\top\big](v_1-v_1^*),X\big\rangle\\
&\quad+2\big\langle(R_1^1+D_1^\top P_1D)^\top(v_1-v_1^*),\Theta^*X\big\rangle\\
&\quad+\big\langle(R_{11}^1+D_1^\top P_1D_1)v_1,v_1\big\rangle+2\big\langle(R_{12}^1+D_1^\top P_1D_2)v_2^*,v_1\big\rangle+\big\langle(R_{22}^1+D_2^\top P_1D_2)v_2^*,v_2^*\big\rangle\\
&\quad+2\big\langle B_1^\top\eta_1+D_1^\top\zeta_1+D_1^\top P_1\sigma+\rho_1^1,v_1\big\rangle+2\big\langle B_2^\top\eta_1+D_2^\top\zeta_1+D_2^\top P_1\sigma+\rho_2^1,v_2^*\big\rangle\\
&\quad+\langle P_1\sigma,\sigma\rangle+2\langle\eta_1,b\rangle+2\langle\zeta_1,\sigma\rangle\Big\}ds\\
&\quad=\mathbb{E}\int_t^T\Big\{\big\langle(R_{11}^1+D_1^\top P_1D_1)v_1,v_1\big\rangle-2\big\langle(R_{11}^1+D_1^\top P_1D_1)v_1^*,v_1\big\rangle\\
&\quad+\big\langle(R_{22}^1+D_2^\top P_1D_2)v_2^*,v_2^*\big\rangle\\
&\quad+2\big\langle B_2^\top\eta_1+D_2^\top\zeta_1+D_2^\top P_1\sigma+\rho_2^1,v_2^*\big\rangle+\langle P_1\sigma,\sigma\rangle+2\langle\eta_1,b\rangle+2\langle\zeta_1,\sigma\rangle\Big\}ds\\
&\quad=\mathbb{E}\int_t^T\Big\{\big\langle(R_{11}^1+D_1^\top P_1D_1)(v_1-v_1^*),v_1-v_1^*\big\rangle-\big\langle(R_{11}^1+D_1^\top P_1D_1)v_1^*,v_1^*\big\rangle\\
&\quad+\big\langle(R_{22}^1+D_2^\top P_1D_2)v_2^*,v_2^*\big\rangle+2\big\langle B_2^\top\eta_1+D_2^\top\zeta_1+D_2^\top P_1\sigma+\rho_2^1,v_2^*\big\rangle\\
&\quad+\langle P_1\sigma,\sigma\rangle+2\langle\eta_1,b\rangle+2\langle\zeta_1,\sigma\rangle\Big\}ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&J^1(t,x;\,\Theta^*X(\cdot)+w(\cdot))-J^1(t,x;\,\Theta^*X^*(\cdot)+v^*(\cdot))\\
&\quad=\mathbb{E}\int_t^T\big\langle(R_{11}^1+D_1^\top P_1D_1)(v_1-v_1^*),v_1-v_1^*\big\rangle ds.
\end{aligned}
$$

It follows that for any $v_1(\cdot) \in \mathcal{U}_1[t, T]$,

$$J^1(t, x;\ \Theta_1^*(\cdot)X^*(\cdot) + v_1^*(\cdot),\ \Theta_2^*(\cdot)X(\cdot) + v_2^*(\cdot)) \leqslant J^1(t, x;\ \Theta_1^*(\cdot)X(\cdot) + v_1(\cdot),\ \Theta_2^*(\cdot)X(\cdot) + v_2^*(\cdot)),$$

if and only if

$$R_{11}^1 + D_1^\top P_1 D_1 \geqslant 0, \qquad \text{a.e. } s \in [t, T].$$

Similarly, for any $v_2(\cdot) \in \mathcal{U}_2[t, T]$,

$$J^2(t, x;\ \Theta_1^*(\cdot)X^*(\cdot) + v_1^*(\cdot),\ \Theta_2^*(\cdot)X(\cdot) + v_2^*(\cdot)) \leqslant J^2(t, x;\ \Theta_1^*(\cdot)X(\cdot) + v_1^*(\cdot),\ \Theta_2^*(\cdot)X(\cdot) + v_2(\cdot)),$$

if and only if

$$R_{22}^2 + D_2^\top P_2 D_2 \geqslant 0, \qquad \text{a.e. } s \in [t, T].$$

This proves the sufficiency, as well as the necessity of (5.8). □

Note that conditions (5.9) and (5.11) are, respectively, equivalent to:

$$\begin{pmatrix} B_1^\top P_1 + D_1^\top P_1 C + S_1^1 \\ B_2^\top P_2 + D_2^\top P_2 C + S_2^2 \end{pmatrix} + \begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix} \Theta^* = 0,$$

$$\begin{pmatrix} B_1^\top \eta_1 + D_1^\top \zeta_1 + D_1^\top P_1 \sigma + \rho_1^1 \\ B_2^\top \eta_2 + D_2^\top \zeta_2 + D_2^\top P_2 \sigma + \rho_2^2 \end{pmatrix} + \begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix} v^* = 0.$$

Therefore,

$$\begin{aligned} \Theta^* &= -\begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix}^{-1} \begin{pmatrix} B_1^\top P_1 + D_1^\top P_1 C + S_1^1 \\ B_2^\top P_2 + D_2^\top P_2 C + S_2^2 \end{pmatrix}, \\ v^* &= -\begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix}^{-1} \begin{pmatrix} B_1^\top \eta_1 + D_1^\top \zeta_1 + D_1^\top P_1 \sigma + \rho_1^1 \\ B_2^\top \eta_2 + D_2^\top \zeta_2 + D_2^\top P_2 \sigma + \rho_2^2 \end{pmatrix}, \end{aligned} \tag{5.14}$$

provided the involved inverse (which is an $\mathbb{R}^{m\times m}$-valued function) exists. By plugging such a $\Theta^*(\cdot)$ into (5.7), we see that the equations for $P_1(\cdot)$ and $P_2(\cdot)$ are coupled, symmetric, and of Riccati type.

Now, let us try to rewrite the Lyapunov type equations in a more compact form. Note that (recalling the notation we introduced in the previous section)

$$\begin{aligned} 0 &= \begin{pmatrix} B_1^\top P_1 + D_1^\top P_1 C + S_1^1 \\ B_2^\top P_2 + D_2^\top P_2 C + S_2^2 \end{pmatrix} + \begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix} \Theta^* \\ &= \begin{pmatrix} B_1^\top & 0 \\ 0 & B_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} + \begin{pmatrix} D_1^\top & 0 \\ 0 & D_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} \\ &\quad + \begin{pmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m_2} \end{pmatrix}\begin{pmatrix} S^1 & 0 \\ 0 & S^2 \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} \\ &\quad + \left[\begin{pmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m_2} \end{pmatrix}\begin{pmatrix} R^1 & 0 \\ 0 & R^2 \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} D_1^\top & 0 \\ 0 & D_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix}\right] \Theta^* \end{aligned}$$

$$\equiv \mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S}\big)\mathbf{I}_n+\big[\mathbf{J}^\top\big(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D}\big)\mathbf{I}_m\big]\Theta^*,$$

with

$$\mathbf{P}(\cdot)\equiv\begin{pmatrix}P_1(\cdot) & 0\\ 0 & P_2(\cdot)\end{pmatrix}.$$

Hence, in the case that $\big[\mathbf{J}^\top\big(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D}\big)\mathbf{I}_m\big]^{-1}\equiv\begin{pmatrix}R_1^1+D_1^\top P_1 D\\ R_2^2+D_2^\top P_2 D\end{pmatrix}^{-1}$ exists and is bounded, we have

$$\Theta^*=-\big[\mathbf{J}^\top\big(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D}\big)\mathbf{I}_m\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S}\big)\mathbf{I}_n, \tag{5.15}$$

which is the same as (5.14). On the other hand, (5.7) can be written as

$$\begin{aligned}
0=&\begin{pmatrix}\dot P_1 & 0\\ 0 & \dot P_2\end{pmatrix}+\begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix}+\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix}^\top\begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}\\
&+\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}^\top\begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}+\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}\\
&+\begin{pmatrix}\Theta^* & 0\\ 0 & \Theta^*\end{pmatrix}^\top\begin{pmatrix}R^1+D^\top P_1 D & 0\\ 0 & R^2+D^\top P_2 D\end{pmatrix}\begin{pmatrix}\Theta^* & 0\\ 0 & \Theta^*\end{pmatrix}\\
&+\begin{pmatrix}P_1B+C^\top P_1 D+(S^1)^\top & 0\\ 0 & P_2B+C^\top P_2D+(S^2)^\top\end{pmatrix}\begin{pmatrix}\Theta^* & 0\\ 0 & \Theta^*\end{pmatrix}\\
&+\begin{pmatrix}\Theta^* & 0\\ 0 & \Theta^*\end{pmatrix}^\top\begin{pmatrix}B^\top P_1+D^\top P_1C+S^1 & 0\\ 0 & B^\top P_2+D^\top P_2C+S^2\end{pmatrix}.
\end{aligned}$$

Consequently, one sees that the following holds:

$$\begin{cases}\dot{\mathbf{P}}+\mathbf{P}\mathbf{A}+\mathbf{A}^\top\mathbf{P}+\mathbf{C}^\top\mathbf{P}\mathbf{C}+\mathbf{Q}+\boldsymbol{\Theta}^\top\big(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D}\big)\boldsymbol{\Theta}\\ \quad+\big(\mathbf{P}\mathbf{B}+\mathbf{C}^\top\mathbf{P}\mathbf{D}+\mathbf{S}^\top\big)\boldsymbol{\Theta}+\boldsymbol{\Theta}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S}\big)=0, \qquad \text{a.e. } s\in[t,T],\\ \mathbf{P}(T)=\mathbf{G},\end{cases} \tag{5.16}$$

where

$$\boldsymbol{\Theta}(\cdot)=\begin{pmatrix}\Theta^*(\cdot) & 0\\ 0 & \Theta^*(\cdot)\end{pmatrix},$$

and Θ^* is given by (5.15). Clearly, (5.16) is symmetric.

6. Two examples

From the previous sections, we see that the existence of an open-loop Nash equilibrium is equivalent to the solvability of a coupled system of two FBSDEs, together with the convexity condition for the cost functionals (see (4.3)); and that the existence of a closed-loop Nash equilibrium is equivalent to the solvability of a coupled system of two symmetric Riccati equations satisfying certain type of non-negativity condition (see (5.8)). Then a natural question is: Are open-loop and closed-loop Nash equilibria really different? In this section, we will present two examples showing that they are indeed different.

The following example shows that Problem (SDG) may have only open-loop Nash equilibria.

Example 6.1. Consider the following Problem (SDG) with one-dimensional state equation

$$\begin{cases} dX(s) = \big[u_1(s) + u_2(s)\big]ds + \big[u_1(s) - u_2(s)\big]dW(s), \qquad s \in [t,1], \\ X(t) = x, \end{cases}$$

and cost functionals

$$J^1(t,x;u_1(\cdot),u_2(\cdot)) = J^2(t,x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}X(1)^2 \equiv J(t,x;u_1(\cdot),u_2(\cdot)).$$

Let $\beta \geqslant \frac{1}{1-t}$. We claim that

$$\Big(u_1^\beta(s), u_2^\beta(s)\Big) = -\left(\frac{\beta x}{2}\mathbf{1}_{[t,t+\frac{1}{\beta}]}(s), \frac{\beta x}{2}\mathbf{1}_{[t,t+\frac{1}{\beta}]}(s)\right), \qquad s \in [t,1],$$

is an open-loop Nash equilibrium of the problem for the initial pair (t,x). Indeed, it is clear that for any $u_1(\cdot) \in L^2_{\mathbb{F}}(t,1;\mathbb{R})$,

$$J(t,x;u_1(\cdot),u_2^\beta(\cdot)) \geqslant 0.$$

On the other hand, the state process $X^\beta(\cdot)$ corresponding to $\big(u_1^\beta(s), u_2^\beta(s)\big)$ and (t,x) satisfies $X^\beta(1) = 0$. Hence,

$$J(t,x;u_1^\beta(\cdot),u_2^\beta(\cdot)) = 0 \leqslant J(t,x;u_1(\cdot),u_2^\beta(\cdot)), \qquad \forall\, u_1(\cdot) \in L^2_{\mathbb{F}}(t,1;\mathbb{R}).$$

Likewise,

$$J(t,x;u_1^\beta(\cdot),u_2^\beta(\cdot)) = 0 \leqslant J(t,x;u_1^\beta(\cdot),u_2(\cdot)), \qquad \forall\, u_2(\cdot) \in L^2_{\mathbb{F}}(t,1;\mathbb{R}).$$

This establishes the claim.

However, this problem does *not* admit a closed-loop Nash equilibrium. We now show this by contradiction. Suppose $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is a closed-loop Nash equilibrium. Consider the corresponding ODEs in Theorem 5.2, which now become

$$\begin{cases} \dot{P}_i + P_i(\Theta_1^* - \Theta_2^*)^2 + 2P_i(\Theta_1^* + \Theta_2^*) = 0, \\ P_i(1) = 1, \end{cases} \qquad i = 1,2. \tag{6.1}$$

The corresponding constraints read

$$P_1, P_2 \geqslant 0, \qquad P_1 + P_1(\Theta_1^* - \Theta_2^*) = 0, \qquad P_2 - P_2(\Theta_1^* - \Theta_2^*) = 0. \tag{6.2}$$

Since $P_1(\cdot)$ and $P_2(\cdot)$ satisfy the same ODE (6.1), we have $P_1(\cdot) = P_2(\cdot)$. Then (6.2) implies $P_1(\cdot) = 0$, which contradicts the terminal condition $P_1(1) = 1$.

The following example shows that Problem (SDG) may have only closed-loop Nash equilibria.

Example 6.2. Consider the following Problem (SDG) with one-dimensional state equation

$$\begin{cases} dX(s) = u_1(s)ds + u_2(s)dW(s), \qquad s \in [t,1], \\ X(t) = x, \end{cases}$$

and cost functionals

$$J^1(t,x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}\Big\{|X(1)|^2 + \int_t^1 |u_1(s)|^2 ds\Big\},$$

$$J^2(t,x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}\Big\{-|X(1)|^2 + \int_t^1 \Big[-|X(s)|^2 + |u_2(s)|^2\Big]ds\Big\}.$$

We claim that the problem admits a closed-loop Nash equilibrium of form $(\Theta_1(\cdot), 0;\ \Theta_2(\cdot), 0)$. In fact, by Theorem 5.2, we need to solve the following Riccati equations for $P_1(\cdot)$ and $P_2(\cdot)$:

$$\begin{cases} \dot P_1(s) + P_1(s)\Theta_2(s)^2 + 2P_1(s)\Theta_1(s) + \Theta_1(s)^2 = 0, \\ P_1(1) = 1, \\ P_1(s) + \Theta_1(s) = 0, \end{cases} \tag{6.3}$$

$$\begin{cases} \dot P_2(s) + P_2(s)\Theta_2(s)^2 + 2P_2(s)\Theta_1(s) + \Theta_2(s)^2 - 1 = 0, \\ P_2(1) = -1, \\ 1 + P_2(s) \geqslant 0, \\ [1 + P_2(s)]\Theta_2(s) = 0. \end{cases} \tag{6.4}$$

By the fourth equation in (6.4), we may assume $\Theta_2(\cdot) = 0$. Then (6.3)–(6.4) become (taking into account $\Theta_1(\cdot) = -P_1(\cdot)$ from the third equation in (6.3))

$$\begin{cases} \dot P_1(s) = P_1(s)^2, \\ P_1(1) = 1, \end{cases} \qquad \begin{cases} \dot P_2(s) = 2P_1(s)P_2(s) + 1, \\ P_2(1) = -1, \quad 1 + P_2(s) \geqslant 0. \end{cases}$$

A straightforward calculation leads to

$$P_1(s) = \frac{1}{2-s}, \qquad P_2(s) = \frac{-(2-s)^3 - 2}{3(2-s)^2}.$$

Therefore, $((s-2)^{-1}, 0; 0, 0)$ is a closed-loop Nash equilibrium of the problem.

Next, we claim that the problem does *not* have open-loop Nash equilibria. Indeed, suppose $(u_1^*(\cdot), u_2^*(\cdot))$ is an open-loop Nash equilibrium for some initial pair (t, x). Then $u_2^*(\cdot)$ is an open-loop optimal control of the following Problem (SLQ) with state equation

$$\begin{cases} dX(s) = u_1^*(s)ds + u_2(s)dW(s), \qquad s \in [t, 1], \\ X(t) = x, \end{cases} \tag{6.5}$$

and cost functional

$$\widetilde J(t, x; u_2(\cdot)) = \mathbb{E}\Big\{ -|X(1)|^2 + \int_t^1 \Big[-|X(s)|^2 + |u_2(s)|^2 \Big] ds \Big\}. \tag{6.6}$$

For any $u_2(\cdot) \in L^2_{\mathbb{F}}(t, 1; \mathbb{R})$, the corresponding solution to (6.5) is given by

$$X(s) = x + \int_t^s u_1^*(r)dr + \int_t^s u_2(r)dW(r). \tag{6.7}$$

Let $\varepsilon > 0$ be undetermined. Substituting (6.7) into (6.6) and using the inequality $(a+b)^2 \geqslant (1 - \frac{1}{\varepsilon})a^2 + (1-\varepsilon)b^2$, we see

$$\begin{aligned} \widetilde J(t, x; u_2(\cdot)) &\leqslant \Big(\frac{1}{\varepsilon} - 1\Big)\mathbb{E}\Big(x + \int_t^1 u_1^*(s)ds\Big)^2 + (\varepsilon - 1)\mathbb{E}\Big(\int_t^1 u_2(s)dW(s)\Big)^2 \\ &\quad + \Big(\frac{1}{\varepsilon} - 1\Big)\mathbb{E}\int_t^1 \Big(x + \int_t^s u_1^*(r)dr\Big)^2 ds \\ &\quad + (\varepsilon - 1)\mathbb{E}\int_t^1 \Big(\int_t^s u_2(r)dW(r)\Big)^2 ds + \mathbb{E}\int_t^1 |u_2(s)|^2 ds \\ &= \Big(\frac{1}{\varepsilon} - 1\Big)\mathbb{E}\Big[\Big(x + \int_t^1 u_1^*(s)ds\Big)^2 + \int_t^1 \Big(x + \int_t^s u_1^*(r)dr\Big)^2 ds\Big] \\ &\quad + \varepsilon\mathbb{E}\int_t^1 |u_2(s)|^2 ds + (\varepsilon - 1)\mathbb{E}\int_t^1 \int_t^s |u_2(r)|^2 dr ds. \end{aligned}$$

Now, by taking $u_2(s) = \lambda$, $\lambda \in \mathbb{R}$, we have

$$\widetilde{J}(t,x;\lambda) \leqslant \Big(\frac{1}{\varepsilon}-1\Big)\mathbb{E}\Big[\Big(x+\int_t^1 u_1^*(s)ds\Big)^2+\int_t^1\Big(x+\int_t^s u_1^*(r)dr\Big)^2 ds\Big] + \frac{\lambda^2(1-t)}{2}\big[2\varepsilon+(\varepsilon-1)(1-t)\big].$$

Choosing $\varepsilon > 0$ small enough so that $2\varepsilon + (\varepsilon - 1)(1 - t) < 0$ and then letting $\lambda \to \infty$, we see that

$$\inf_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} \widetilde{J}(t,x;u_2(\cdot)) = -\infty,$$

which contradicts the fact that $u_2^*(\cdot)$ is an open-loop optimal control of the associated LQ problem.

7. Closed-loop representation of open-loop Nash equilibria

Inspired by the decoupling technique introduced in [13,14,25,26], we now look at the solvability of FBSDE (4.1)–(4.2). Recall that with the notation introduced in Section 4, (4.1) and (4.2) are equivalent to (4.8) and (4.9), respectively. To solve FBSDE (4.8)–(4.9), we make the ansatz that the adapted solution $(X(\cdot), \mathbf{Y}(\cdot), \mathbf{Z}(\cdot))$ to Eq. (4.8) has the form

$$\begin{aligned}&\mathbf{Y}(\cdot) = \begin{pmatrix}\Pi_1(\cdot)X(\cdot)+\eta_1(\cdot)\\ \Pi_2(\cdot)X(\cdot)+\eta_2(\cdot)\end{pmatrix} \equiv \mathbf{\Pi}(\cdot)X(\cdot)+\boldsymbol{\eta}(\cdot);\\ &\mathbf{\Pi}(\cdot) \triangleq \begin{pmatrix}\Pi_1(\cdot)\\ \Pi_2(\cdot)\end{pmatrix}, \qquad \boldsymbol{\eta}(\cdot) \triangleq \begin{pmatrix}\eta_1(\cdot)\\ \eta_2(\cdot)\end{pmatrix},\end{aligned} \tag{7.1}$$

where $\Pi_i : [t,T] \to \mathbb{R}^{n\times n}$; $i = 1, 2$, are differentiable maps to be determined, and $\boldsymbol{\eta}(\cdot)$ is a stochastic process satisfying a certain equation. To match the terminal condition $\mathbf{Y}(T) = \mathbf{G}\mathbf{I}_n X(T) + \mathbf{g}$, we impose the requirements

$$\mathbf{\Pi}(T) = \mathbf{G}\mathbf{I}_n, \qquad \boldsymbol{\eta}(T) = \mathbf{g}.$$

The second requirement suggests that the equation for $\boldsymbol{\eta}(\cdot)$ should be a BSDE:

$$\begin{cases} d\boldsymbol{\eta}(s) = \alpha(s)ds + \boldsymbol{\zeta}(s)dW(s), \qquad s\in[t,T],\\ \boldsymbol{\eta}(T) = \mathbf{g},\end{cases}$$

where $\alpha : [t,T]\times\Omega \to \mathbb{R}^{2n}$ is to be determined, and

$$\boldsymbol{\zeta}(\cdot) = \begin{pmatrix}\zeta_1(\cdot)\\ \zeta_2(\cdot)\end{pmatrix}.$$

Applying Itô's formula to $s \mapsto \mathbf{\Pi}(s)X(s) + \boldsymbol{\eta}(s)$, we have

$$\begin{aligned}&-\big(\mathbf{A}^\top\mathbf{Y}+\mathbf{C}^\top\mathbf{Z}+\mathbf{Q}\mathbf{I}_nX+\mathbf{S}^\top\mathbf{I}_mu+\mathbf{q}\big)ds+\mathbf{Z}dW(s) = d\mathbf{Y}\\ &\quad = \big[\dot{\mathbf{\Pi}}X+\mathbf{\Pi}(AX+Bu+b)+\alpha\big]ds+\big[\mathbf{\Pi}(CX+Du+\sigma)+\boldsymbol{\zeta}\big]dW(s)\\ &\quad = \big[(\dot{\mathbf{\Pi}}+\mathbf{\Pi}A)X+\mathbf{\Pi}Bu+\mathbf{\Pi}b+\alpha\big]ds+\big[\mathbf{\Pi}CX+\mathbf{\Pi}Du+\mathbf{\Pi}\sigma+\boldsymbol{\zeta}\big]dW(s).\end{aligned}$$

Hence, one should have

$$\mathbf{Z} = \mathbf{\Pi}CX+\mathbf{\Pi}Du+\mathbf{\Pi}\sigma+\boldsymbol{\zeta}. \tag{7.2}$$

The stationarity condition (4.9) then becomes

$$0 = \mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{Y}+\mathbf{D}^\top\mathbf{Z}+\mathbf{S}\mathbf{I}_nX+\mathbf{R}\mathbf{I}_mu+\boldsymbol{\rho}\big)$$

$$
\begin{aligned}
&= \mathbf{J}^\top\big[\mathbf{B}^\top(\boldsymbol{\Pi}X+\boldsymbol{\eta})+\mathbf{D}^\top(\boldsymbol{\Pi}CX+\boldsymbol{\Pi}Du+\boldsymbol{\Pi}\sigma+\boldsymbol{\zeta})+\mathbf{SI}_nX+\mathbf{RI}_mu+\boldsymbol{\rho}\big]\\
&= \mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{SI}_n\big)X+\mathbf{J}^\top\big(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D\big)u\\
&\quad +\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathbf{J}^\top\big(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D\big) &= \begin{pmatrix} I_{m_1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_2}\end{pmatrix}\begin{pmatrix} R^1+D^\top\Pi_1 D\\ R^2+D^\top\Pi_2 D\end{pmatrix}\\
&= \begin{pmatrix} R^1_{11}+D_1^\top\Pi_1D_1 & R^1_{12}+D_1^\top\Pi_1D_2\\ R^2_{21}+D_2^\top\Pi_2D_1 & R^2_{22}+D_2^\top\Pi_2D_2\end{pmatrix}.
\end{aligned}
$$

This is an $\mathbb{R}^{m\times m}$-valued function which is not symmetric in general, even Π_1 and Π_2 are symmetric. If the matrix $\mathbf{J}^\top\big(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D\big)$ is invertible, then

$$
\begin{aligned}
u &= -\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{SI}_n\big)X\\
&\quad -\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big),
\end{aligned}
\tag{7.3}
$$

and

$$
\begin{aligned}
0 &= \big(\dot{\boldsymbol{\Pi}}+\boldsymbol{\Pi}A\big)X+\boldsymbol{\Pi}Bu+\boldsymbol{\Pi}b+\alpha+\mathbf{A}^\top(\boldsymbol{\Pi}X+\boldsymbol{\eta})\\
&\qquad +\mathbf{C}^\top(\boldsymbol{\Pi}CX+\boldsymbol{\Pi}Du+\boldsymbol{\Pi}\sigma+\boldsymbol{\zeta})+\mathbf{QI}_nX+\mathbf{S}^\top\mathbf{I}_mu+\mathbf{q}\\
&= \big(\dot{\boldsymbol{\Pi}}+\boldsymbol{\Pi}A+\mathbf{A}^\top\boldsymbol{\Pi}+\mathbf{C}^\top\boldsymbol{\Pi}C+\mathbf{QI}_n\big)X+\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)u\\
&\qquad +\alpha+\mathbf{A}^\top\boldsymbol{\eta}+\mathbf{C}^\top\boldsymbol{\zeta}+\boldsymbol{\Pi}b+\mathbf{C}^\top\boldsymbol{\Pi}\sigma+\mathbf{q}\\
&= \big(\dot{\boldsymbol{\Pi}}+\boldsymbol{\Pi}A+\mathbf{A}^\top\boldsymbol{\Pi}+\mathbf{C}^\top\boldsymbol{\Pi}C+\mathbf{QI}_n\big)X\\
&\qquad -\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{SI}_n\big)X\\
&\qquad -\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\\
&\qquad \times\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big)\\
&\qquad +\alpha+\mathbf{A}^\top\boldsymbol{\eta}+\mathbf{C}^\top\boldsymbol{\zeta}+\boldsymbol{\Pi}b+\mathbf{C}^\top\boldsymbol{\Pi}\sigma+\mathbf{q}\\
&= \Big\{\dot{\boldsymbol{\Pi}}+\boldsymbol{\Pi}A+\mathbf{A}^\top\boldsymbol{\Pi}+\mathbf{C}^\top\boldsymbol{\Pi}C+\mathbf{QI}_n\\
&\qquad -\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{SI}_n\big)\Big\}X\\
&\qquad -\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\\
&\qquad \times\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big)\\
&\qquad +\alpha+\mathbf{A}^\top\boldsymbol{\eta}+\mathbf{C}^\top\boldsymbol{\zeta}+\boldsymbol{\Pi}b+\mathbf{C}^\top\boldsymbol{\Pi}\sigma+\mathbf{q}.
\end{aligned}
$$

This suggests that $\boldsymbol{\Pi}(\cdot)$ should be a solution to the Riccati equation on $[t,T]$:

$$
\begin{cases}
\dot{\boldsymbol{\Pi}}+\boldsymbol{\Pi}A+\mathbf{A}^\top\boldsymbol{\Pi}+\mathbf{C}^\top\boldsymbol{\Pi}C+\mathbf{QI}_n\\
\quad -\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{RI}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\\
\quad \times\,\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{SI}_n\big)=0,\\
\boldsymbol{\Pi}(T)=\mathbf{GI}_n,
\end{cases}
\tag{7.4}
$$

and that the BSDE for $(\boldsymbol{\eta}(\cdot),\boldsymbol{\zeta}(\cdot))$ is

$$\begin{cases} d\boldsymbol{\eta}=-\Big\{\Big(\mathbf{A}^\top-\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\mathbf{B}^\top\Big)\boldsymbol{\eta} \\ \qquad +\Big(\mathbf{C}^\top-\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\mathbf{D}^\top\Big)\boldsymbol{\zeta} \\ \qquad +\Big(\mathbf{C}^\top-\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\mathbf{D}^\top\Big)\boldsymbol{\Pi}\sigma \\ \qquad +\boldsymbol{\Pi}b+\mathbf{q}-\big(\boldsymbol{\Pi}B+\mathbf{C}^\top\boldsymbol{\Pi}D+\mathbf{S}^\top\mathbf{I}_m\big)\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\boldsymbol{\rho}\Big\}ds+\boldsymbol{\zeta}dW, \\ \boldsymbol{\eta}(T)=\mathbf{g}. \end{cases} \tag{7.5}$$

The above procedure implies that if the Riccati equation (7.4) indeed admits a solution $\boldsymbol{\Pi}(\cdot)$, then the triple $(X(\cdot),\mathbf{Y}(\cdot),\mathbf{Z}(\cdot))$, defined through the FSDE on $[t,T]$

$$\begin{cases} dX=\Big\{\Big(A-B\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{S}\mathbf{I}_n\big)\Big)X \\ \qquad\quad -B\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big)+b\Big\}ds \\ \qquad +\Big\{\Big(C-D\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\Pi}+\mathbf{D}^\top\boldsymbol{\Pi}C+\mathbf{S}\mathbf{I}_n\big)\Big)X \\ \qquad\quad -D\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\boldsymbol{\Pi}D)\big]^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta}+\mathbf{D}^\top\boldsymbol{\zeta}+\mathbf{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}\big)+\sigma\Big\}dW, \\ X(t)=x, \end{cases}$$

(7.1) and (7.2), is an adapted solution to FBSDE (4.8) with respect to the control $u(\cdot)$ defined by (7.3), and the stationarity condition (4.9) holds. If, in addition, the convexity condition (4.3) holds for $i=1,2$, then by Theorem 4.1, Problem (SDG) admits an open-loop Nash equilibrium for every initial state x, and the open-loop Nash equilibria take the form

$$u(\cdot)=\Theta(\cdot)X(\cdot)+v(\cdot), \tag{7.6}$$

for some $(\Theta(\cdot),v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ which is independent of x. (7.6) is called a *closed-loop representation* of the open-loop Nash equilibria of Problem (SDG). More precisely, we have the following definition.

Definition 7.1. We say that open-loop Nash equilibria of Problem (SDG) on $[t,T]$ admit a *closed-loop representation*, if there exists a pair $(\Theta(\cdot),v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ such that for any initial state $x\in\mathbb{R}^n$, the process

$$u(s)\triangleq\Theta(s)X(s)+v(s),\qquad s\in[t,T] \tag{7.7}$$

is an open-loop Nash equilibrium of Problem (SDG) for (t,x), where $X(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} dX(s)=\big\{[A(s)+B(s)\Theta(s)]X(s)+B(s)v(s)+b(s)\big\}ds \\ \qquad\qquad +\big\{[C(s)+D(s)\Theta(s)]X(s)+D(s)v(s)+\sigma(s)\big\}dW(s), \\ X(t)=x. \end{cases} \tag{7.8}$$

Comparing Definitions 3.2 and 7.1, it is natural to ask whether the closed-loop representation of open-loop Nash equilibria is the outcome of some closed-loop Nash equilibrium. The following example shows that this is not the case in general.

Example 7.2. Consider the following state equation:

$$\begin{cases} dX(s)=\big[u_1(s)+u_2(s)\big]ds+X(s)dW(s),\qquad s\in[t,T], \\ X(t)=x, \end{cases}$$

with cost functionals

$$J^1(t,x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}\Big[X(T)^2 + \int_t^T u_1(s)^2 ds\Big],$$
$$J^2(t,x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}\Big[X(T)^2 + \int_t^T u_2(s)^2 ds\Big].$$

For this case, we have

$$\begin{cases} A=0,\ C=1, & B_1=B_2=1, & D_1=D_2=0, & b=\sigma=0, \\ Q^1=Q^2=0, & S^1=S^2=0, & R^1=\begin{pmatrix}1&0\\0&0\end{pmatrix}, & R^2=\begin{pmatrix}0&0\\0&1\end{pmatrix}, \\ G^1=G^2=1, & q^1=q^2=0, & \rho^1=\rho^2=0, & g^1=g^2=0. \end{cases}$$

Clearly, the convexity condition (4.3) holds for $i = 1,2$. In this example, the Riccati equation (7.4) can be written componentwise as follows:

$$\begin{cases} \dot\Pi_1(s)+\Pi_1(s)-\Pi_1(s)\big[\Pi_1(s)+\Pi_2(s)\big]=0, \qquad s\in[t,T], \\ \Pi_1(T)=1, \end{cases} \tag{7.9}$$

$$\begin{cases} \dot\Pi_2(s)+\Pi_2(s)-\Pi_2(s)\big[\Pi_1(s)+\Pi_2(s)\big]=0, \qquad s\in[t,T], \\ \Pi_2(T)=1. \end{cases} \tag{7.10}$$

It is easy to see that

$$\Pi_1(s)=\Pi_2(s)=\frac{e^{T-s}}{2e^{T-s}-1}$$

are solutions to (7.9) and (7.10), respectively. Note that in this case the adapted solution $(\eta(\cdot),\zeta(\cdot))$ to BSDE (7.5) is $(0,0)$. Then by the preceding argument, the open-loop Nash equilibria of this Problem (SDG) on $[t,T]$ admit a closed-loop representation given by

$$u_1(s)=u_2(s)=-\frac{e^{T-s}}{2e^{T-s}-1}X(s), \qquad s\in[t,T]. \tag{7.11}$$

Next we verify that the problem admits a closed-loop Nash equilibrium of form $(\Theta_1(\cdot),0;\Theta_2(\cdot),0)$. In light of Theorem 5.2, we need to solve the following Riccati equations for $P_1(\cdot)$ and $P_2(\cdot)$:

$$\begin{cases} \dot P_1(s)+P_1(s)+\Theta_1(s)^2+2P_1(s)\big[\Theta_1(s)+\Theta_2(s)\big]=0, \\ P_1(T)=1, \\ P_1(s)+\Theta_1(s)=0, \end{cases} \tag{7.12}$$

$$\begin{cases} \dot P_2(s)+P_2(s)+\Theta_2(s)^2+2P_2(s)\big[\Theta_1(s)+\Theta_2(s)\big]=0, \\ P_2(T)=1, \\ P_2(s)+\Theta_2(s)=0. \end{cases} \tag{7.13}$$

Noting the third equations in (7.12) and (7.13), we can further write (7.12)–(7.13) as follows:

$$\begin{cases} \dot P_1(s)=P_1(s)^2+2P_1(s)P_2(s)-P_1(s), \\ P_1(T)=1, \end{cases} \tag{7.14}$$

$$\begin{cases} \dot P_2(s)=P_2(s)^2+2P_2(s)P_1(s)-P_2(s), \\ P_2(T)=1. \end{cases} \tag{7.15}$$

Now it is easily seen that

$$P_1(s)=P_2(s)=\frac{e^{T-s}}{3e^{T-s}-2}.$$

Hence,

$$\Theta_1(s) = \Theta_2(s) = -P_1(s) = -\frac{e^{T-s}}{3e^{T-s}-2}. \tag{7.16}$$

Comparing (7.11) with (7.16), we see that the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria.

Now we give a characterization of the closed-loop representation of open-loop Nash equilibria.

Theorem 7.3. *Let* (G1)–(G2) *hold and let* $(\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$. *Then open-loop Nash equilibria of Problem* (SDG) *on* $[t,T]$ *admit the closed-loop representation* (7.7) *if and only if the following hold:*

(i) *The convexity condition* (4.3) *holds for* $i = 1, 2$.

(ii) *The solution* $\mathbf{\Pi}(\cdot) \in C([t,T];\mathbb{R}^{n\times 2n})$ *to the ODE on* $[t,T]$

$$\begin{cases}\dot{\mathbf{\Pi}} + \mathbf{\Pi}A + \mathbf{A}^\top\mathbf{\Pi} + \mathbf{C}^\top\mathbf{\Pi}C + \mathbf{Q}\mathbf{I}_n + \big(\mathbf{\Pi}B + \mathbf{C}^\top\mathbf{\Pi}D + \mathbf{S}^\top\mathbf{I}_m\big)\Theta = 0,\\ \mathbf{\Pi}(T) = \mathbf{G}\mathbf{I}_n,\end{cases} \tag{7.17}$$

satisfies

$$\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m + \mathbf{D}^\top\mathbf{\Pi}D)\big]\Theta + \mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{\Pi} + \mathbf{D}^\top\mathbf{\Pi}C + \mathbf{S}\mathbf{I}_n\big) = 0, \tag{7.18}$$

and the adapted solution $(\boldsymbol{\eta}(\cdot), \boldsymbol{\zeta}(\cdot))$ *to the BSDE on* $[t,T]$

$$\begin{cases} d\boldsymbol{\eta} = -\big[\mathbf{A}^\top\boldsymbol{\eta} + \mathbf{C}^\top\boldsymbol{\zeta} + \big(\mathbf{\Pi}B + \mathbf{C}^\top\mathbf{\Pi}D + \mathbf{S}^\top\mathbf{I}_m\big)v\\ \qquad\qquad + \mathbf{C}^\top\mathbf{\Pi}\sigma + \mathbf{\Pi}b + \mathbf{q}\big]ds + \boldsymbol{\zeta}dW,\\ \boldsymbol{\eta}(T) = \mathbf{g},\end{cases} \tag{7.19}$$

satisfies

$$\big[\mathbf{J}^\top(\mathbf{R}\mathbf{I}_m + \mathbf{D}^\top\mathbf{\Pi}D)\big]v + \mathbf{J}^\top\big(\mathbf{B}^\top\boldsymbol{\eta} + \mathbf{D}^\top\boldsymbol{\zeta} + \mathbf{D}^\top\mathbf{\Pi}\sigma + \boldsymbol{\rho}\big) = 0. \tag{7.20}$$

Proof. For any $x \in \mathbb{R}^n$, let $X(\cdot)$, $\mathbf{\Pi}(\cdot)$, and $(\boldsymbol{\eta}(\cdot), \boldsymbol{\zeta}(\cdot))$ be the solutions to (7.8), (7.17), and (7.19), respectively. Let $u(\cdot)$ be defined by (7.7) and set

$$\mathbf{Y} = \mathbf{\Pi}X + \boldsymbol{\eta}, \qquad \mathbf{Z} = \mathbf{\Pi}(C + D\Theta)X + \mathbf{\Pi}Dv + \mathbf{\Pi}\sigma + \boldsymbol{\zeta}.$$

Then $\mathbf{Y}(T) = \mathbf{G}\mathbf{I}_n X(T) + \mathbf{g}$, and

$$\begin{aligned} d\mathbf{Y} &= \dot{\mathbf{\Pi}}Xds + \mathbf{\Pi}dX + d\boldsymbol{\eta}\\ &= \big[\dot{\mathbf{\Pi}}X + \mathbf{\Pi}(A + B\Theta)X + \mathbf{\Pi}Bv + \mathbf{\Pi}b - \mathbf{A}^\top\boldsymbol{\eta} - \mathbf{C}^\top\boldsymbol{\zeta}\\ &\quad - \big(\mathbf{\Pi}B + \mathbf{C}^\top\mathbf{\Pi}D + \mathbf{S}^\top\mathbf{I}_m\big)v - \mathbf{C}^\top\mathbf{\Pi}\sigma - \mathbf{\Pi}b - \mathbf{q}\big]ds\\ &\quad + \big[\mathbf{\Pi}(C + D\Theta)X + \mathbf{\Pi}Dv + \mathbf{\Pi}\sigma + \boldsymbol{\zeta}\big]dW\\ &= \big[-\big(\mathbf{A}^\top\mathbf{\Pi} + \mathbf{C}^\top\mathbf{\Pi}C + \mathbf{Q}\mathbf{I}_n + \mathbf{C}^\top\mathbf{\Pi}D\Theta + \mathbf{S}^\top\mathbf{I}_m\Theta\big)X - \mathbf{A}^\top\boldsymbol{\eta} - \mathbf{C}^\top\boldsymbol{\zeta}\\ &\quad - \big(\mathbf{C}^\top\mathbf{\Pi}D + \mathbf{S}^\top\mathbf{I}_m\big)v - \mathbf{C}^\top\mathbf{\Pi}\sigma - \mathbf{q}\big]ds + \mathbf{Z}dW\\ &= \big\{-\mathbf{A}^\top(\mathbf{\Pi}X + \boldsymbol{\eta}) - \mathbf{Q}\mathbf{I}_n X - \mathbf{C}^\top\big[\mathbf{\Pi}(C + D\Theta)X + \mathbf{\Pi}Dv + \mathbf{\Pi}\sigma + \boldsymbol{\zeta}\big]\\ &\quad - \mathbf{S}^\top\mathbf{I}_m(\Theta X + v) - \mathbf{q}\big\}ds + \mathbf{Z}dW \end{aligned}$$

$$= \big(-\mathbf{A}^\top\mathbf{Y} - \mathbf{Q}\mathbf{I}_n X - \mathbf{C}^\top\mathbf{Z} - \mathbf{S}^\top\mathbf{I}_m u - \mathbf{q}\big)ds + \mathbf{Z}dW.$$

This shows that $(X(\cdot), \mathbf{Y}(\cdot), \mathbf{Z}(\cdot), u(\cdot))$ satisfies the FBSDE (4.8). According to Theorem 4.1, the process $u(\cdot)$ defined by (7.7) is an open-loop Nash equilibrium for (t, x) if and only if (i) holds and

$$\begin{aligned}
0 &= \mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{Y} + \mathbf{D}^\top\mathbf{Z} + \mathbf{S}\mathbf{I}_n X + \mathbf{R}\mathbf{I}_m u + \boldsymbol{\rho}\big) \\
&= \mathbf{J}^\top\big\{\mathbf{B}^\top(\mathbf{\Pi}X + \boldsymbol{\eta}) + \mathbf{D}^\top[\mathbf{\Pi}(C + D\Theta)X + \mathbf{\Pi}Dv + \mathbf{\Pi}\sigma + \boldsymbol{\zeta}] \\
&\qquad + \mathbf{S}\mathbf{I}_n X + \mathbf{R}\mathbf{I}_m(\Theta X + v) + \boldsymbol{\rho}\big\} \\
&= \mathbf{J}^\top\big[\mathbf{B}^\top\mathbf{\Pi} + \mathbf{D}^\top\mathbf{\Pi}C + \mathbf{S}\mathbf{I}_n + (\mathbf{R}\mathbf{I}_m + \mathbf{D}^\top\mathbf{\Pi}D)\Theta\big]X \\
&\qquad + \mathbf{J}^\top\big[\mathbf{B}^\top\boldsymbol{\eta} + \mathbf{D}^\top\boldsymbol{\zeta} + \mathbf{D}^\top\mathbf{\Pi}\sigma + \boldsymbol{\rho} + (\mathbf{R}\mathbf{I}_m + \mathbf{D}^\top\mathbf{\Pi}D)v\big].
\end{aligned}$$

Since the initial state x is arbitrary and $\mathbf{J}^\top[\mathbf{B}^\top\boldsymbol{\eta} + \mathbf{D}^\top\boldsymbol{\zeta} + \mathbf{D}^\top\mathbf{\Pi}\sigma + \boldsymbol{\rho} + (\mathbf{R}\mathbf{I}_m + \mathbf{D}^\top\mathbf{\Pi}D)v]$ is independent of x, the above leads to (7.18) and (7.20). □

Let us write (7.17)–(7.20) componentwise as follows: For $i = 1, 2$,

$$\begin{cases} \dot{\Pi}_i + \Pi_i A + A^\top\Pi_i + C^\top\Pi_i C + Q^i + \big[\Pi_i B + C^\top\Pi_i D + (S^i)^\top\big]\Theta = 0, \\ \Pi_i(T) = G^i, \end{cases} \tag{7.21}$$

$$\begin{pmatrix} R_1^1 + D_1^\top\Pi_1 D \\ R_2^2 + D_2^\top\Pi_2 D \end{pmatrix}\Theta + \begin{pmatrix} B_1^\top\Pi_1 + D_1^\top\Pi_1 C + S_1^1 \\ B_2^\top\Pi_2 + D_2^\top\Pi_2 C + S_2^2 \end{pmatrix} = 0, \tag{7.22}$$

$$\begin{cases} d\eta_i = -\Big\{A^\top\eta_i + C^\top\zeta_i + \big[\Pi_i B + C^\top\Pi_i D + (S^i)^\top\big]v \\ \qquad\qquad + C^\top\Pi_i\sigma + \Pi_i b + q^i\Big\}ds + \zeta_i dW, \\ \eta_i(T) = g^i, \end{cases} \tag{7.23}$$

$$\begin{pmatrix} R_1^1 + D_1^\top\Pi_1 D \\ R_2^2 + D_2^\top\Pi_2 D \end{pmatrix} v + \begin{pmatrix} B_1^\top\eta_1 + D_1^\top\zeta_1 + D_1^\top\Pi_1\sigma + \rho_1^1 \\ B_2^\top\eta_2 + D_2^\top\zeta_2 + D_2^\top\Pi_2\sigma + \rho_2^2 \end{pmatrix} = 0. \tag{7.24}$$

Noting the relation (7.22), one sees the equations for $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are coupled and none of them is symmetric. Consequently, $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are not symmetric in general. Whereas the Riccati equations (5.7) for $P_i(\cdot)$ $(i = 1, 2)$ are symmetric. This is the main reason that the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria.

8. Zero-sum cases

In the previous section, we have seen that for Problem (SDG), the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria in general. Now we would like to take a look at the situation for LQ stochastic two-person zero-sum differential games. In this case, Nash equilibria are usually called saddle points. According to (1.4), we have

$$
\begin{aligned}
&G^1 = -G^2 \equiv G, \quad g^1 = -g^2 \equiv g, \quad Q^1(\cdot) = -Q^2(\cdot) \equiv Q(\cdot),\\
&q^1(\cdot) = -q^2(\cdot) \equiv q(\cdot),\\
&\begin{pmatrix} R^1_{11}(\cdot) & R^1_{12}(\cdot) \\ R^1_{21}(\cdot) & R^1_{22}(\cdot) \end{pmatrix} \equiv -\begin{pmatrix} R^2_{11}(\cdot) & R^2_{12}(\cdot) \\ R^2_{21}(\cdot) & R^2_{22}(\cdot) \end{pmatrix} \equiv \begin{pmatrix} R_{11}(\cdot) & R_{12}(\cdot) \\ R_{21}(\cdot) & R_{22}(\cdot) \end{pmatrix} \equiv \begin{pmatrix} R_1(\cdot) \\ R_2(\cdot) \end{pmatrix} \equiv R(\cdot),\\
&\begin{pmatrix} S^1_1(\cdot) \\ S^1_2(\cdot) \end{pmatrix} = -\begin{pmatrix} S^2_1(\cdot) \\ S^2_2(\cdot) \end{pmatrix} \equiv \begin{pmatrix} S_1(\cdot) \\ S_2(\cdot) \end{pmatrix} \equiv S(\cdot), \quad \begin{pmatrix} \rho^1_1(\cdot) \\ \rho^1_2(\cdot) \end{pmatrix} = -\begin{pmatrix} \rho^2_1(\cdot) \\ \rho^2_2(\cdot) \end{pmatrix} \equiv \begin{pmatrix} \rho_1(\cdot) \\ \rho_2(\cdot) \end{pmatrix} \equiv \rho(\cdot),
\end{aligned} \tag{8.1}
$$

and

$$
\begin{aligned}
&J^1(t,x;u_1(\cdot),u_2(\cdot)) = -J^2(t,x;u_1(\cdot),u_2(\cdot))\\
&\quad = \mathbb{E}\Bigg\{ \langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle\\
&\qquad + \int_t^T \Bigg[\Bigg\langle \begin{pmatrix} Q(s) & S_1(s)^\top & S_2(s)^\top \\ S_1(s) & R_{11}(s) & R_{12}(s) \\ S_2(s) & R_{21}(s) & R_{22}(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix} \Bigg\rangle\\
&\qquad + 2\Bigg\langle \begin{pmatrix} q(s) \\ \rho_1(s) \\ \rho_2(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix} \Bigg\rangle \Bigg] ds \Bigg\}\\
&\quad \equiv J(t,x;u_1(\cdot),u_2(\cdot)).
\end{aligned}
$$

Let $(\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ and assume the open-loop saddle points of Problem (SDG) on $[t,T]$ admit the closed-loop representation (7.7). Eqs. (7.21) ($i = 1, 2$) for $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ now become

$$
\begin{cases} \dot{\Pi}_1 + \Pi_1 A + A^\top \Pi_1 + C^\top \Pi_1 C + Q + \big(\Pi_1 B + C^\top \Pi_1 D + S^\top\big)\Theta = 0,\\ \Pi_1(T) = G, \end{cases}
$$

and

$$
\begin{cases} \dot{\Pi}_2 + \Pi_2 A + A^\top \Pi_2 + C^\top \Pi_2 C - Q + \big(\Pi_2 B + C^\top \Pi_2 D - S^\top\big)\Theta = 0,\\ \Pi_2(T) = -G, \end{cases}
$$

respectively. Obviously, both $\Pi_1(\cdot)$ and $-\Pi_2(\cdot)$ satisfy

$$
\begin{cases} \dot{\Pi} + \Pi A + A^\top \Pi + C^\top \Pi C + Q + \big(\Pi B + C^\top \Pi D + S^\top\big)\Theta = 0,\\ \Pi(T) = G. \end{cases} \tag{8.2}
$$

Thus, $\Pi_1(\cdot) = -\Pi_2(\cdot) \equiv \Pi(\cdot)$, and (7.22) becomes

$$
\begin{pmatrix} R_1 + D_1^\top \Pi D \\ -R_2 - D_2^\top \Pi D \end{pmatrix} \Theta + \begin{pmatrix} B_1^\top \Pi + D_1^\top \Pi C + S_1 \\ -B_2^\top \Pi - D_2^\top \Pi C - S_2 \end{pmatrix} = 0,
$$

or equivalently,

$$
(R + D^\top \Pi D)\Theta + B^\top \Pi + D^\top \Pi C + S = 0.
$$

This is also equivalent to

$$
\begin{cases} \mathscr{R}(B^\top \Pi + D^\top \Pi C + S) \subseteq \mathscr{R}(R + D^\top \Pi D), \qquad \text{a.e. } s \in [t,T],\\ (R + D^\top \Pi D)^\dagger (B^\top \Pi + D^\top \Pi C + S) \in L^2(t,T;\mathbb{R}^{m\times n}), \end{cases} \tag{8.3}
$$

and

$$\begin{aligned}\Theta &= -(R+D^\top \Pi D)^\dagger(B^\top \Pi + D^\top \Pi C + S)\\ &\quad + [I-(R+D^\top \Pi D)^\dagger(R+D^\top \Pi D)]\theta,\end{aligned} \tag{8.4}$$

for some $\theta(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$. Upon substitution of (8.4) into (8.2), the latter becomes

$$\begin{cases}\dot{\Pi} + \Pi A + A^\top \Pi + C^\top \Pi C + Q\\ \quad -(\Pi B + C^\top \Pi D + S^\top)(R+D^\top \Pi D)^\dagger(B^\top \Pi + D^\top \Pi C + S) = 0,\\ \Pi(T) = G,\end{cases} \tag{8.5}$$

with constraints (8.3). Note that Eq. (8.5) is symmetric. Likewise, we have $(\eta_1(\cdot),\zeta_1(\cdot)) = -(\eta_2(\cdot),\zeta_2(\cdot)) \equiv (\eta_\Pi(\cdot),\zeta_\Pi(\cdot))$ satisfying

$$\begin{cases} d\eta_\Pi = -\Big\{\big[A^\top - (\Pi B + C^\top \Pi D + S^\top)(R+D^\top \Pi D)^\dagger B^\top\big]\eta_\Pi\\ \qquad\qquad + \big[C^\top - (\Pi B + C^\top \Pi D + S^\top)(R+D^\top \Pi D)^\dagger D^\top\big]\zeta_\Pi\\ \qquad\qquad + \big[C^\top - (\Pi B + C^\top \Pi D + S^\top)(R+D^\top \Pi D)^\dagger D^\top\big]\Pi\sigma\\ \qquad\qquad - (\Pi B + C^\top \Pi D + S^\top)(R+D^\top \Pi D)^\dagger \rho + \Pi b + q\Big\}ds + \zeta_\Pi dW,\\ \eta_\Pi(T) = g,\end{cases} \tag{8.6}$$

with constraints

$$\begin{cases} B^\top \eta_\Pi + D^\top \zeta_\Pi + D^\top \Pi\sigma + \rho \in \mathscr{R}(R+D^\top \Pi D), \qquad \text{a.e. } s\in[t,T], \text{ a.s.}\\ (R+D^\top \Pi D)^\dagger(B^\top \eta_\Pi + D^\top \zeta_\Pi + D^\top \Pi\sigma + \rho) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m),\end{cases} \tag{8.7}$$

and in this case,

$$\begin{aligned} v &= -(R+D^\top \Pi D)^\dagger(B^\top \eta_\Pi + D^\top \zeta_\Pi + D^\top \Pi\sigma + \rho)\\ &\quad + \big[I-(R+D^\top \Pi D)^\dagger(R+D^\top \Pi D)\big]\nu,\end{aligned}$$

for some $\nu(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. To summarize, we have the following result for LQ stochastic two-person zero-sum differential games.

Theorem 8.1. *Let* (G1)–(G2) *and* (8.1) *hold. Then the open-loop saddle points of Problem* (SDG) *on* $[t,T]$ *admit a closed-loop representation if and only if the following hold:*

(i) *The following convexity–concavity condition holds: For* $i = 1, 2$,

$$\begin{aligned}(-1)^{i-1}\mathbb{E}\Big\{\int_t^T \Big[\langle Q(s)X_i(s),X_i(s)\rangle + 2\langle S_i(s)X_i(s),u_i(s)\rangle + \langle R_{ii}(s)u_i(s),u_i(s)\rangle\Big]ds\\ + \langle GX_i(T),X_i(T)\rangle\Big\} \geqslant 0, \qquad \forall\, u_i(\cdot)\in\mathcal{U}_i[t,T],\end{aligned} \tag{8.8}$$

where $X_i(\cdot)$ *is the solution to FSDE* (4.4).

(ii) *The Riccati equation* (8.5) *admits a solution* $\Pi(\cdot) \in C([t,T];\mathbb{S}^n)$ *such that* (8.3) *holds, and the adapted solution of* (8.6) *satisfies* (8.7).

In the above case, all the closed-loop representations of open-loop saddle points are given by

$$\begin{aligned} u &= \Big\{-(R+D^\top \Pi D)^\dagger(B^\top \Pi + D^\top \Pi C + S)\\ &\quad + \big[I-(R+D^\top \Pi D)^\dagger(R+D^\top \Pi D)\big]\theta\Big\}X\\ &\quad - (R+D^\top \Pi D)^\dagger(B^\top \eta_\Pi + D^\top \zeta_\Pi + D^\top \Pi\sigma + \rho)\\ &\quad + \big[I-(R+D^\top \Pi D)^\dagger(R+D^\top \Pi D)\big]\nu,\end{aligned}$$

where $\theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$.

Proof. The result can be proved by combining Theorem 7.3 and the previous argument. We leave the details to the interested reader. □

Now let us recall from [22] the characterization of closed-loop saddle points of LQ stochastic two-person zero-sum differential games.

Theorem 8.2. *Let* (G1)–(G2) *and* (8.1) *hold. Then Problem* (SDG) *admits a closed-loop saddle point on* $[t, T]$ *if and only if the following hold:*

(i) *The Riccati equation*

$$\begin{cases} \dot P + PA + A^\top P + C^\top PC + Q \\ \quad -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) = 0, \\ P(T) = G, \end{cases} \tag{8.9}$$

admits a solution $P(\cdot) \in C([t, T]; \mathbb{S}^n)$ *such that the following hold:*

$$\begin{cases} \mathscr{R}(B^\top P + D^\top PC + S) \subseteq \mathscr{R}(R + D^\top PD), \qquad a.e.\ s \in [t, T], \\ (R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) \in L^2(t, T; \mathbb{R}^{m\times n}), \end{cases} \tag{8.10}$$

$$R_{11} + D_1^\top PD_1 \geqslant 0, \qquad R_{22} + D_2^\top PD_2 \leqslant 0, \qquad a.e.\ s \in [t, T]. \tag{8.11}$$

(ii) *The adapted solution* $(\eta_P(\cdot), \zeta_P(\cdot))$ *of the BSDE on* $[t, T]$

$$\begin{cases} d\eta_P = -\Big\{\big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta_P \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta_P \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma \\ \qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger\rho + Pb + q\Big\}ds + \zeta_P dW, \\ \eta_P(T) = g, \end{cases} \tag{8.12}$$

satisfies

$$\begin{cases} B^\top\eta_P + D^\top\zeta_P + D^\top P\sigma + \rho \in \mathscr{R}(R + D^\top PD), \qquad a.e.\ s \in [t, T],\ a.s. \\ (R + D^\top PD)^\dagger(B^\top\eta_P + D^\top\zeta_P + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m). \end{cases} \tag{8.13}$$

In this case, the closed-loop saddle point $(\Theta^*(\cdot), v^*(\cdot))$ *admits the following representation:*

$$\begin{cases} \Theta^* = -(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) \\ \qquad + \big[I - (R + D^\top PD)^\dagger(R + D^\top PD)\big]\theta, \\ v^* = -(R + D^\top PD)^\dagger(B^\top\eta_P + D^\top\zeta_P + D^\top P\sigma + \rho) \\ \qquad + \big[I - (R + D^\top PD)^\dagger(R + D^\top PD)\big]\nu, \end{cases} \tag{8.14}$$

where $\theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$.

Comparing Theorems 8.1 and 8.2, one may ask: For LQ stochastic two-person zero-sum differential games, when both the closed-loop representation of open-loop saddle points and the closed-loop saddle point exist, does the closed-loop representation coincide with the outcome of the closed-loop saddle point? The answer to this question is affirmative, as shown by the following result.

Theorem 8.3. *Let* (G1)–(G2) *and* (8.1) *hold. If both the closed-loop representation of open-loop saddle points and the closed-loop saddle point exist on* $[t, T]$, *then the closed-loop representation coincides with the outcome of the closed-loop saddle point.*

Proof. The proof is immediate from Theorems 8.1 and 8.2, once we show that the solution $\Pi(\cdot)$ to the Riccati equation (8.5) with constraints (8.3) coincides with the solution $P(\cdot)$ to (8.9) with constraints (8.10)–(8.11).

First, we note that if the convexity–concavity condition (8.8) holds for initial time t, it also holds for any $t' \in [t, T]$. Indeed, for any $t' \in [t, T]$, and any $u_1(\cdot) \in \mathcal{U}_1[t', T]$, let $X_1(\cdot)$ be the solution to

$$\begin{cases} dX_1(s) = \big[A(s)X_1(s) + B_1(s)u_1(s)\big]ds \\ \qquad\qquad +\big[C(s)X_1(s) + D_1(s)u_1(s)\big]dW(s), \qquad s \in [t', T], \\ X_1(t') = 0, \end{cases}$$

and define the *zero-extension* of $u_1(\cdot)$ as follows:

$$[0I_{[t,t')} \oplus u_1](s) = \begin{cases} 0, & s \in [t, t'), \\ u_1(s), & s \in [t', T]. \end{cases}$$

Then $\tilde{u}_1(\cdot) \equiv [0I_{[t,t')} \oplus u_1](\cdot) \in \mathcal{U}_1[t, T]$, and due to the initial state being 0, the solution $\widetilde{X}_1(s)$ of

$$\begin{cases} d\widetilde{X}_1(s) = \big[A(s)\widetilde{X}_1(s) + B_1(s)\tilde{u}_1(s)\big]ds \\ \qquad\qquad +\big[C(s)\widetilde{X}_1(s) + D_1(s)\tilde{u}_1(s)\big]dW(s), \qquad s \in [t, T], \\ \widetilde{X}_1(t) = 0, \end{cases}$$

satisfies

$$\widetilde{X}_1(s) = \begin{cases} 0, & s \in [t, t'), \\ X_1(s), & s \in [t', T]. \end{cases}$$

Hence,

$$\begin{aligned} &\mathbb{E}\Big\{\int_{t'}^{T} \Big[\langle QX_1, X_1\rangle + 2\langle S_1X_1, u_1\rangle + \langle R_{11}u_1, u_1\rangle\Big]ds + \langle GX_1(T), X_1(T)\rangle\Big\} \\ &\quad = \mathbb{E}\Big\{\int_{t}^{T} \Big[\big\langle Q\widetilde{X}_1, \widetilde{X}_1\big\rangle + 2\big\langle S_1\widetilde{X}_1, \tilde{u}_1\big\rangle + \big\langle R_{11}\tilde{u}_1, \tilde{u}_1\big\rangle\Big]ds + \big\langle G\widetilde{X}_1(T), \widetilde{X}_1(T)\big\rangle\Big\} \geqslant 0. \end{aligned}$$

This proves the case $i = 1$. The case $i = 2$ can be treated similarly.

Now let $(\Theta^*(\cdot), v^*(\cdot))$ be a closed-loop saddle point of Problem (SDG) on $[t, T]$. Under the assumption of the theorem, it is clear from Theorem 8.1 that for any initial pair (t', x) with $t' \in [t, T]$, the outcome

$$u^*(s) = \Theta^*(s)X^*(s) + v^*(s), \qquad s \in [t', T]$$

of $(\Theta^*(\cdot), v^*(\cdot))$ is an open-loop saddle point for (t', x), where $X^*(\cdot)$ is the solution to

$$\begin{cases} dX^*(s) = \big\{[A(s) + B(s)\Theta^*(s)]X^*(s) + B(s)v^*(s) + b(s)\big\}ds \\ \qquad\qquad + \big\{[C(s) + D(s)\Theta^*(s)]X^*(s) + D(s)v^*(s) + \sigma(s)\big\}dW(s), \qquad s \in [t', T], \\ X^*(t') = x. \end{cases}$$

By Theorem 8.2, $(\Theta^*(\cdot), v^*(\cdot))$ admits the representation (8.14), and a straightforward calculation shows that

$$\dot{P} + P(A + B\Theta^*) + (A + B\Theta^*)^\top P + (C + D\Theta^*)^\top P(C + D\Theta^*)$$

$$+(\Theta^*)^\top R\Theta^* + S^\top\Theta^* + (\Theta^*)^\top S + Q = 0,$$

and that the adapted solution $(\eta_P(\cdot), \zeta_P(\cdot))$ of (8.12) satisfies

$$\begin{aligned} d\eta_P = &-\big[(A+B\Theta^*)^\top\eta_P + (C+D\Theta^*)^\top\zeta_P + (C+D\Theta^*)^\top P\sigma \\ &+(\Theta^*)^\top\rho + Pb + q\big]ds + \zeta_P dW. \end{aligned}$$

Then applying Itô's formula to $s \mapsto \langle P(s)X^*(s), X^*(s)\rangle + 2\langle\eta_P(s), X^*(s)\rangle$ and noting that

$$(R + D^\top PD)\Theta^* + B^\top P + D^\top PC + S = 0,$$

we have

$$\begin{aligned} &J(t', x; u^*(\cdot)) = J(t', x; \Theta^*(\cdot)X^*(\cdot) + v^*(\cdot)) \\ &= \mathbb{E}\Big\{\langle GX^*(T), X^*(T)\rangle + 2\langle g, X^*(T)\rangle + \int_{t'}^T \Big[\langle QX^*, X^*\rangle + 2\langle SX^*, \Theta^*X^* + v^*\rangle \\ &\qquad + \langle R(\Theta^*X^* + v^*), \Theta^*X^* + v^*\rangle + 2\langle q, X^*\rangle + 2\langle\rho, \Theta^*X^* + v^*\rangle\Big]ds\Big\} \\ &= \mathbb{E}\Big\{\langle P(t')x, x\rangle + 2\langle\eta_P(t'), x\rangle \\ &\qquad + \int_{t'}^T \Big[\langle\dot{P}X^*, X^*\rangle + 2\langle PX^*, (A+B\Theta^*)X^* + Bv^* + b\rangle \\ &\qquad + \langle P[(C+D\Theta^*)X^* + Dv^* + \sigma], (C+D\Theta^*)X^* + Dv^* + \sigma\rangle \\ &\qquad - 2\big\langle(A+B\Theta^*)^\top\eta_P + (C+D\Theta^*)^\top\zeta_P + (C+D\Theta^*)^\top P\sigma \\ &\qquad + (\Theta^*)^\top\rho + Pb + q, X^*\big\rangle \\ &\qquad + 2\langle\eta_P, (A+B\Theta^*)X^* + Bv^* + b\rangle + 2\langle\zeta_P, (C+D\Theta^*)X^* + Dv^* + \sigma\rangle \\ &\qquad + \big\langle\big[Q + S^\top\Theta^* + (\Theta^*)^\top S + (\Theta^*)^\top R\Theta^*\big]X^*, X^*\big\rangle + 2\langle(R\Theta^* + S)X^*, v^*\rangle \\ &\qquad + 2\big\langle q + (\Theta^*)^\top\rho, X^*\big\rangle + \langle Rv^*, v^*\rangle + 2\langle\rho, v^*\rangle\Big]ds\Big\} \\ &= \mathbb{E}\Big\{\langle P(t')x, x\rangle + 2\langle\eta_P(t'), x\rangle + \int_{t'}^T \Big[\langle P\sigma, \sigma\rangle + 2\langle\eta_P, b\rangle + 2\langle\zeta_P, \sigma\rangle \\ &\qquad + \big\langle(R + D^\top PD)v^*, v^*\big\rangle + 2\big\langle B^\top\eta_P + D^\top\zeta_P + D^\top P\sigma + \rho, v^*\big\rangle\Big]ds\Big\}. \end{aligned} \tag{8.15}$$

Next, let $\theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$, $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$ and denote

$$\begin{cases} \Theta = -(R + D^\top\Pi D)^\dagger(B^\top\Pi + D^\top\Pi C + S) \\ \qquad + [I - (R + D^\top\Pi D)^\dagger(R + D^\top\Pi D)]\theta, \\ v = -(R + D^\top\Pi D)^\dagger(B^\top\eta_\Pi + D^\top\zeta_\Pi + D^\top\Pi\sigma + \rho) \\ \qquad + [I - (R + D^\top\Pi D)^\dagger(R + D^\top\Pi D)]\nu. \end{cases}$$

For any initial pair (t', x) with $t' \in [t, T]$, define $u(\cdot) \in \mathcal{U}[t', T]$ by

$$u(s) = \Theta(s)X(s) + v(s), \qquad s \in [t', T],$$

with $X(\cdot)$ being the solution to

$$\begin{cases} dX(s) = \big\{[A(s) + B(s)\Theta(s)]X(s) + B(s)v(s) + b(s)\big\}ds \\ \qquad\qquad + \big\{[C(s) + D(s)\Theta(s)]X(s) + D(s)v(s) + \sigma(s)\big\}dW(s), \qquad s \in [t', T], \\ X(t') = x. \end{cases}$$

By Theorem 8.1, $u(\cdot)$ is an open-loop saddle point for (t', x), and by a computation similar to (8.15), we obtain

$$\begin{aligned} J(t', x; u(\cdot)) = \mathbb{E}\Big\{ & \langle \Pi(t')x, x\rangle + 2\langle \eta_\Pi(t'), x\rangle \\ & + \int_{t'}^{T} \Big[\langle \Pi\sigma, \sigma\rangle + 2\langle \eta_\Pi, b\rangle + 2\langle \zeta_\Pi, \sigma\rangle \\ & + \big\langle (R + D^\top \Pi D)v, v\big\rangle + 2\big\langle B^\top \eta_\Pi + D^\top \zeta_\Pi + D^\top \Pi\sigma + \rho, v\big\rangle \Big] ds \Big\}. \end{aligned} \tag{8.16}$$

Since both $u^*(\cdot) \equiv (u_1^*(\cdot)^\top, u_2^*(\cdot)^\top)^\top$ and $u(\cdot) \equiv (u_1(\cdot)^\top, u_2(\cdot)^\top)^\top$ are open-loop saddle points for (t', x), we have

$$\begin{aligned} J(t', x; u_1^*(\cdot), u_2^*(\cdot)) & \leqslant J(t', x; u_1(\cdot), u_2^*(\cdot)) \leqslant J(t', x; u_1(\cdot), u_2(\cdot)) \\ & \leqslant J(t', x; u_1^*(\cdot), u_2(\cdot)) \leqslant J(t', x; u_1^*(\cdot), u_2^*(\cdot)). \end{aligned}$$

Therefore, $J(t', x; u^*(\cdot)) = J(t', x; u(\cdot))$ for all (t', x) with $t' \in [t, T]$, which, together with (8.15) and (8.16), yields $\Pi(\cdot) = P(\cdot)$. □

Remark 8.4. Theorem 8.3 is based on the assumption that both the closed-loop representation of open-loop saddle points and the closed-loop saddle points exist on $[t, T]$. This assumption is necessary because, in general, neither of these two kinds of existence implies the other (see Sun–Yong [22]). It is different from Problem (SLQ), in which closed-loop solvability always implies open-loop solvability. Recall from Theorem 2.5 that for Problem (SLQ), when a closed-loop optimal strategy exists, the solution P to the corresponding Riccati equation satisfies $R + D^\top PD \geqslant 0$. This positivity condition actually implies the convexity condition (ii) of Theorem 2.2. However, in the case of Problem (SDG), one cannot deduce the convexity–concavity condition (8.8) from the counterpart (8.11) of $R + D^\top PD \geqslant 0$, nor (8.11) from (8.8).

Finally, we have the following corollary for Problem (SLQ), which should be but has not been stated in [21].

Corollary 8.5. *For Problem* (SLQ)*, if the open-loop optimal controls admit a closed-loop representation, then every open-loop optimal control must be an outcome of a closed-loop optimal strategy.*

9. Concluding remarks

We have investigated a linear–quadratic stochastic two-person nonzero-sum differential game and have come to the conclusion that the existence of an open-loop Nash equilibrium is equivalent to the solvability of a system of FBSDEs with constraints, together with a convexity condition of the cost functionals, and the existence of a closed-loop Nash equilibrium is equivalent to the solvability of a system of coupled *symmetric* Riccati differential equations whose solution is required to satisfy certain regularity as well as a condition on the solution of a BSDE. We have shown by some examples that the existence of open-loop Nash equilibria does not imply that of closed-loop Nash equilibria, and vice-versa, and that even if both open-loop and closed-loop Nash equilibria exist, the outcome of a closed-loop Nash equilibrium is not necessarily the closed-loop representation of open-loop Nash equilibria. Moreover, we find that the situation in the zero-sum case is totally different: The closed-loop representation of open-loop saddle points coincides with the outcome of the corresponding closed-loop saddle point, as long as both exist. In particular, for stochastic LQ optimal control problem, if an open-loop optimal control admits a closed-loop representation, then the problem must be close-loop solvable, and the representation is the outcome of the corresponding closed-loop optimal strategy.

As we have seen in the preceding sections, we can obtain an open-loop Nash equilibrium and a close-loop Nash equilibrium, respectively, in terms of the solutions to the linear FBSDE (4.1) with the constraint (4.2) and the symmetric Riccati equation (5.7) with the constraints (5.8) and (5.9). Concerning general FBSDEs, there are mainly three approaches in the literature: (i) Fixed point approach, which requires the time duration T is small enough [1,17]. This is a little too restrictive. (ii) The Four-Step Scheme originally introduced in [13] (see also [14]), which is basically a decoupling method. By using it, one reduces the FBSDE problem to a Riccati equation, which in our case is non-symmetric (see [25,26]). (iii) The monotone conditions [9,18] and the method of continuation [24,27], which, in the current case, can only essentially cover the case of (4.7). From this general picture, we can see that the theory of FBSDEs is still far away from mature. On the other hand, the solvability of Riccati equations also is still a very challenging problem. Even for the case of deterministic optimal control problems (i.e., $C(\cdot) = 0$ and $D(\cdot) = 0$), in which the Riccati equation becomes symmetric, besides the classical conditions (similar to (4.7)), there is no nice conditions that guarantee the existence and uniqueness of solutions. We mention here that some relevant results can be found in [15,21], and [28], but more complete results are still not available.

To conclude, we would like to point out that our study of stochastic linear–quadratic two-person non-zero sum differential games provides a nice motivation for further investigation of FBSDEs and Riccati equations.

Acknowledgments

The authors would like to thank the anonymous referees for their suggestive comments, which lead to this improved version of the paper.

References

[1] F. Antonelli, Backward-forward stochastic differential equations, Ann. Appl. Probab. 3 (1993) 777–793.
[2] A. Bensoussan, Points de Nash dans le cas de fonctionnelles quadratiques et jeux différentiels linéaires à N personnes, SIAM J. Control 12 (1974) 460–499.
[3] R. Buckdahn, P. Cardaliaguet, C. Rainer, Nash equilibirum payoffs for nonzero-sum stochastic differential games, SIAM J. Control Optim. 43 (2004) 624–642.
[4] N. El Karoui, S. Hamadène, BSDEs and risk-sensitive control, zero-sum and nonzero-sum game prolems of stochastic functional differential equations, Stochastic Process. Appl. 107 (2003) 145–169.
[5] A. Friedman, Stochastic differential games, J. Differential Equations 11 (1972) 79–108.
[6] S. Hamadène, Backward-forward SDE's and stochastic differential games, Stochastic Process. Appl. 77 (1998) 1–15.
[7] S. Hamadène, Nonzero sum linear-quadratic stochastic differential games and backward-forward equations, Stoch. Anal. Appl. 17 (1999) 117–130.
[8] S. Hamadène, R. Mu, Existence of Nash equilibrium points for Markovian non-zero-sum stochastic differential games with unbounded coefficients, Stoch. Int. J. Probab. Stoch. Process 87 (2015) 85–111.
[9] Y. Hu, S. Peng, Solutions of forward-backward stochastic differential equations, Probab. Theory Related Fields 103 (1995) 273–283.
[10] I. Karatzas, Q. Li, BSDE approach to non-zero-sum stochastic differential games of control and stopping, in: S.N. Cohen, et al. (Eds.), Stochastic Processes, Finance and Control. A Festschrift in Honor of R. J. Elliott. Advances in Statistics, Probability and Actuarial Science, Vol. 1, 2012, pp. 105–153.
[11] Q. Lin, A BSDE approach to Nash equilibrium payoffs for stochastic differential games with nonlinear cost functionals, Stochastic Process. Appl. 122 (2012) 357–385.
[12] D.L. Lukes, D.L. Russell, A global theory for linear-quadratic differential games, J. Math. Anal. Appl. 33 (1971) 96–123.

[13] J. Ma, P. Protter, J. Yong, Solving forward-backward stochastic differential equations explicitly — a four-step scheme, Probab. Theory Related Fields 98 (1994) 339–359.
[14] J. Ma, J. Yong, Forward-Backward Stochastic Diffrential Equations and their Applications, in: Lecture Notes in Math., vol. 1702, Springer-Verlag, New York, 1999.
[15] M. McAsey, L. Mou, Generalized Riccati equations arising in stochastic games, Linear Algebra Appl. 416 (2006) 710–723.
[16] J. Nash, Non-cooperative games, Ann. of Math. 54 (1951) 286–295.
[17] E. Pardoux, S. Tang, Forward-backward stochastic differential equations and quasilinear parabolic PDEs, Probab. Theory Related Fields 114 (1999) 123–150.
[18] S. Peng, Z. Wu, Fully coupled forward-backward stochastic differential equations and applications to optimal control, SIAM J. Control Optim. 37 (1999) 825–843.
[19] R. Penrose, A generalized inverse of matrices, Proc. Cambridge Philos. Soc. 52 (1955) 17–19.
[20] C. Rainer, Two different approaches to nonzero-sum stochastic differential games, Appl. Math. Optim. 56 (2007) 131–144.
[21] J. Sun, X. Li, J. Yong, Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems, SIAM J. Control Optim. 54 (2016) 2274–2308.
[22] J. Sun, J. Yong, Linear quadratic stocahastic differential games: open-loop and closed-loop saddle points, SIAM J. Control Optim. 52 (2014) 4082–4121.
[23] J. Sun, J. Yong, S. Zhang, Linear quadratic stochastic two-person zero-sum differential games in an infnite horizon, ESAIM Control Optim. Calc. Var. 22 (2016) 743–769.
[24] J. Yong, Finding adapted solutions of forward-backward stochastic differential equations — method of continuation, Probab. Theory Related Fields 107 (1997) 537–572.
[25] J. Yong, Linear forward-backward stochastic differential equations, Appl. Math. Optim. 39 (1999) 93–119.
[26] J. Yong, Linear forward-backward stochastic differential equations with random coefficients, Probab. Theory Related Fields 135 (2006) 53–83.
[27] J. Yong, Forward backward stochastic differential equations with mixed initial and terminal conditions, Trans. Amer. Math. Soc. 362 (2010) 1047–1096.
[28] J. Yong, X.Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999.