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# Bianalytic free maps between spectrahedra and spectraballs

J. William Helton [a,1], Igor Klep [b,2], Scott McCullough [c,*,3], Jurij Volčič [d,4]

[a] Department of Mathematics, University of California, San Diego, United States of America
[b] Department of Mathematics, University of Ljubljana, Slovenia
[c] Department of Mathematics, University of Florida, Gainesville, United States of America
[d] Department of Mathematics, Texas A& M University, College Station, United States of America

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ABSTRACT

Linear matrix inequalities (LMIs) are ubiquitous in real algebraic geometry, semidefinite programming, control theory and signal processing. LMIs with (dimension free) matrix unknowns are central to the theories of completely positive maps and operator algebras, operator systems and spaces, and serve as the paradigm for matrix convex sets. The matricial feasibility set of an LMI is called a free spectrahedron.

In this article, the bianalytic maps between a very general class of *ball-like* free spectrahedra (examples of which include row or column contractions, and tuples of contractions) and arbitrary free spectrahedra are characterized and seen to have an elegant algebraic form. They are all highly structured rational maps. In the case that both the domain and codomain are ball-like, these bianalytic maps are explicitly determined and the article gives necessary and sufficient conditions for the

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* Corresponding author.

*E-mail addresses:* helton@math.ucsd.edu (J.W. Helton), igor.klep@fmf.uni-lj.si (I. Klep), sam@math.ufl.edu (S. McCullough), volcic@post.bgu.ac.il (J. Volčič).

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existence of such a map with a specified value and derivative at a point. In particular, this result leads to a classification of automorphism groups of ball-like free spectrahedra. The proofs depend on a novel free Nullstellensatz, established only after new tools in free analysis are developed and applied to obtain fine detail, geometric in nature locally and algebraic in nature globally, about the boundary of ball-like free spectrahedra.

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**Contents**

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## 1. Introduction

Fix a positive integer $g$. For positive integers $n$, let $M_n(\mathbb{C})^g$ denote the set of $g$-tuples $X = (X_1, \dots, X_g)$ of $n \times n$ matrices with entries from $\mathbb{C}$. Given a tuple $E = (E_1, \dots, E_g)$ of $d \times e$ matrices, the sequence $\mathcal{B}_E = (\mathcal{B}_E(n))_n$ defined by

$$\mathcal{B}_E(n) = \{X \in M_n(\mathbb{C})^g : \|\sum E_j \otimes X_j\| \leq 1\}$$

is a **spectraball**. The spectraball at **level** one, $\mathcal{B}_E(1)$, is a rotationally invariant closed convex subset of $\mathbb{C}^g$. Conversely, a rotationally invariant closed convex subset of $\mathbb{C}^g$ can be approximated by sets of the form $\mathcal{B}_E(1)$. A spectraball $\mathcal{B}_E$ is not determined by $\mathcal{B}_E(1)$. For example, letting $F_1 = (1 \quad 0)$, $F_2 = (0 \quad 1)$, and $E_j = F_j^*$, we have $\mathcal{B}_E(1) = \mathcal{B}_F(1) = \mathbb{B}^2$, the unit ball in $\mathbb{C}^2$, but $\mathcal{B}_E(2) \neq \mathcal{B}_F(2)$. Indeed, $\mathcal{B}_F$ (resp. $\mathcal{B}_E$) is the two variable **row ball** (resp. **column ball**) equal the set of pairs $(X_1, X_2)$ such that $X_1X_1^* + X_2X_2^* \preceq I$ (resp. $X_1^*X_1 + X_2^*X_2 \preceq I$), where the inequality $T \succeq 0$ indicates the selfadjoint matrix $T$ is positive semidefinite. Another well-known example is the **free polydisc**. It is the spectraball $\mathcal{B}_E$ determined by the tuple $E = (e_1e_1^*, \dots, e_ge_g^*) \in M_g(\mathbb{C})^g$, where $\{e_1, \dots, e_g\}$ is the standard orthonormal basis for $\mathbb{C}^g$. Thus $\mathcal{B}_E(n)$ is the set of tuples $X \in M_n(\mathbb{C})^g$ such that $\|X_j\| \leq 1$ for each $j$.

For $A \in M_d(\mathbb{C})^g$, let $L_A(x, y)$ denote the **monic pencil**

$$L_A(x, y) = I + \sum A_jx_j + \sum A_j^*y_j,$$

and let

$$L_A^{\mathrm{re}}(x) = L_A(x, x^*) = I + \sum A_jx_j + \sum A_j^*x_j^*$$

denote the corresponding **hermitian monic pencil**. The set $\mathcal{D}_A(1)$ consisting of $x \in \mathbb{C}^g$ such that $L_A^{\mathrm{re}}(x) \succeq 0$ is a **spectrahedron**. Spectrahedra are basic objects in a number of areas of mathematics; e.g. semidefinite programming, convex optimization and in real algebraic geometry [10]. They also figure prominently in determinantal representations [12,22,47,53], in the solution of the Kadison-Singer paving conjecture [44], the solution of the Lax conjecture [34], and in systems engineering [11,52].

For $A \in M_{d\times e}(\mathbb{C})^g$, the **homogeneous linear pencil** $\Lambda_A(x) = \sum_j A_jx_j$ evaluates at $X \in M_n(\mathbb{C})^g$ as

$$\Lambda_A(X) = \sum A_j \otimes X_j \in M_{d\times e}(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

In the case $A$ is square ($d = e$), the hermitian monic pencil $L_A^{\mathrm{re}}$ evaluates at $X$ as

$$L_A^{\mathrm{re}}(X) = I + \Lambda_A(X) + \Lambda_A(X)^* = I + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^*.$$

Thus $L_A^{\mathrm{re}}(X)^* = L_A^{\mathrm{re}}(X)$. Similarly, if $Y \in M_n(\mathbb{C})^g$, then $L_A(X, Y) = I + \Lambda_A(X) + \Lambda_{A^*}(Y)$. In particular, $L_A^{\mathrm{re}}(X) = L_A(X, X^*)$.

The **free spectrahedron** determined by $A \in M_r(\mathbb{C})^g$ is the sequence of sets $\mathcal{D}_A = (\mathcal{D}_A(n))$, where

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A^{\mathrm{re}}(X) \succeq 0\}.$$

The spectraball $\mathcal{B}_E$ is a spectrahedron since $\mathcal{B}_E = \mathcal{D}_B$ for $B = \left(\begin{smallmatrix} 0 & E \\ 0 & 0 \end{smallmatrix}\right)$. Free spectrahedra arise naturally in applications such as systems engineering [16] and in the theories of matrix convex sets, operator algebras and operator spaces and completely positive maps [17,27,48,49]. They also provide tractable useful relaxations for spectrahedral inclusion problems that arise in semidefinite programming and control theory such as the matrix cube problem [8,15,28].

The **interior** of the free spectrahedron $\mathcal{D}_A$ is the sequence $\mathrm{int}(\mathcal{D}_A) = (\mathrm{int}(\mathcal{D}_A(n)))_n$, where

$$\mathrm{int}(\mathcal{D}_A(n)) = \{X \in M_n(\mathbb{C})^g : L_A^{\mathrm{re}}(X) \succ 0\}.$$

A **free mapping** $\varphi : \mathrm{int}(\mathcal{D}_B) \to \mathrm{int}(\mathcal{D}_A)$ is a sequence of maps $\varphi_n : \mathrm{int}(\mathcal{D}_B(n)) \to \mathrm{int}(\mathcal{D}_A(n))$ such that if $X \in \mathrm{int}(\mathcal{D}_B(n))$ and $Y \in \mathrm{int}(\mathcal{D}_B(m))$, then

$$\varphi_{n+m}\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right) = \begin{pmatrix} \varphi_n(X) & 0 \\ 0 & \varphi_m(Y) \end{pmatrix},$$

and if $X \in \mathrm{int}(\mathcal{D}_B(n))$ and $S$ is an invertible $n \times n$ matrix such that

$$S^{-1}XS = \left(S^{-1}X_1S, \dots, S^{-1}X_gS\right) \in \mathrm{int}(\mathcal{D}_B(n)),$$

then

$$\varphi_n(S^{-1}XS) = S^{-1}\varphi_n(X)S.$$

Often we omit the subscript $n$ and write only $\varphi(X)$. The free mapping $\varphi$ is **analytic** if each $\varphi_n$ is analytic.

The central result of this article, Theorem 1.1, explicitly characterizes the free bianalytic mappings $\varphi$ between $\mathrm{int}(\mathcal{B}_E)$ and $\mathrm{int}(\mathcal{D}_A)$. These maps are birational and highly structured. Up to affine linear change of variable, they are what we call **convexotonic** (see Subsection 1.1 below). In the special case that $\mathcal{D}_A = \mathcal{B}_C$ is also a spectraball, given $b \in \mathrm{int}(\mathcal{B}_C)$ and a $g \times g$ matrix $M$, Corollary 1.3 gives explicit necessary and sufficient algebraic relations between $E$ and $C$ for the existence of a free bianalytic mapping $\varphi : \mathrm{int}(\mathcal{B}_E) \to \mathrm{int}(\mathcal{B}_C)$ satisfying $\varphi(0) = b$ and $\varphi'(0) = M$. As an illustration of the result, this corollary classifies, from first principles, the free automorphisms of the matrix balls – the row and column balls are special cases – and of the free polydiscs. See Remark 1.2(d) and Subsubsections 5.3.1 and 5.3.2.

There are two other results we would like to highlight in this introduction. Theorem 1.6, establishes an equivalence between an algebraic *irreducibility* condition on the defining polynomial of a spectraball and a geometric property of its boundary critical in the study of bianalytic maps between free spectrahedra. Its proof requires detailed information, both local and global, about the boundary of a spectraball, collected in Section 4. As a consequence of Theorem 1.6, we obtain a version of the main result from

[3] characterizing bianalytic maps between free spectrahedra that send the origin to the origin with elegant irreducibility and minimality hypotheses on the free spectrahedra replacing our earlier cumbersome geometric conditions. See Theorem 1.5 in Subsection 1.3. Another consequence of Theorem 1.6, and an essential ingredient in the proof of Theorem 1.1, is an of independent interest Nullstellensatz. It is stated as Proposition 1.7 in Subsection 1.5. Roughly, it says that a matrix-valued analytic free polynomial, singular on the boundary of a spectraball, is 0.

### *1.1. Convexotonic maps*

A $g$-tuple of $g \times g$ matrices $(\Xi_1, \dots, \Xi_g) \in M_g(\mathbb{C})^g$ satisfying

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} \Xi_s,$$

for each $1 \le j, k \le g$, is a **convexotonic tuple**. The expressions $p = \begin{pmatrix} p^1 & \cdots & p^g \end{pmatrix}$ and $q = \begin{pmatrix} q^1 & \cdots & q^g \end{pmatrix}$ whose entries are

$$p^i(x) = \sum_j x_j e_j^* (I - \Lambda_\Xi(x))^{-1} e_i \qquad \text{and} \qquad q^i(x) = \sum x_j e_j^* (I + \Lambda_\Xi(x))^{-1} e_i,$$

that is, in row form,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \text{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1},$$

are **convexotonic maps**. Here $p$ evaluates at $X \in M_n(\mathbb{C})^g$ as

$$p(X) = \begin{pmatrix} X_1 & \cdots & X_g \end{pmatrix} \left( I_{gn} - \sum_{j=1}^{g} \Xi_j \otimes X_j \right)^{-1}$$

and the output $p(X) \in M_{n \times gn}(\mathbb{C}) = M_n(\mathbb{C})^g$ is interpreted as a $g$-tuple of $n \times n$ matrices. It turns out the mappings $p$ and $q$ are free rational maps (as explained in Section 2) and inverses of one another (see [3, Proposition 6.2]).

Convexotonic tuples arise naturally as the structure constants of a finite dimensional algebra. If $A \in M_r(\mathbb{C})^g$ is linearly independent (meaning the set $\{A_1, \dots, A_g\} \subseteq M_r(\mathbb{C})$ is linearly independent) and spans an algebra, then, e.g. by Lemma 2.7 below, there is a uniquely determined convexotonic tuple $\Xi = (\Xi_1, \dots, \Xi_g) \in M_g(\mathbb{C})^g$ such that

$$A_k A_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s. \tag{1.1}$$

### *1.2. Free bianalytic maps from a spectraball to a free spectrahedron*

A tuple $E \in M_{d\times e}(\mathbb{C})^g$ is **ball-minimal** (for $\mathcal{B}_E$) if there does not exist $E'$ of size $d' \times e'$ with $d' + e' < d + e$ such that $\mathcal{B}_E = \mathcal{B}_{E'}$. In fact, if $E$ is ball-minimal and $\mathcal{B}_{E'} = \mathcal{B}_E$, then $d \le d'$ and $e \le e'$, by Lemma 3.2(9)[5] and $E$ is unique in the following sense. Given another tuple $F \in M_{d\times e}(\mathbb{C})^g$, the tuples $E$ and $F$ are **ball-equivalent** if there exists unitaries $W$ and $V$ of sizes $d \times d$ and $e \times e$ respectively such that $F = WEV$. Evidently if $E$ and $F$ are ball-equivalent, then $\mathcal{B}_E = \mathcal{B}_F$. Conversely, if $E$ and $F$ are both ball-minimal and $\mathcal{B}_E = \mathcal{B}_F$, then $E$ and $F$ are ball-equivalent (see Lemma 3.2(9) and more generally [21]).

Given $A \in M_r(\mathbb{C})^g$, we say $L_A$ (or $L_A^{\mathrm{re}}$) is **minimal** for a free spectrahedron $\mathcal{D}$ if $\mathcal{D} = \mathcal{D}_A$ and if for any other $B \in M_{r'}(\mathbb{C})^g$ satisfying $\mathcal{D} = \mathcal{D}_B$ it follows that $r' \ge r$. A minimal $L_A$ for $\mathcal{D}_A$ exists and is unique up to unitary equivalence [26,57]. We can now state Theorem 1.1, our principal result on bianalytic mappings from a spectraball onto a free spectrahedron. Since the hypotheses of Theorem 1.1 are invariant under affine linear change of variables, the normalizations $f(0) = 0$ and $f'(0) = I$ are simply a matter of convenience. Given $B \in M_d(\mathbb{C})^g$, by a free bianalytic map $f : \mathrm{int}(\mathcal{D}_B) \to \mathrm{int}(\mathcal{D}_A)$, we mean $f$ is a free analytic map and there exists a free analytic map $g : \mathrm{int}(\mathcal{D}_A) \to \mathrm{int}(\mathcal{D}_B)$ such that $g_n(f_n(X)) = X$ and $f_n(g_n(Y)) = Y$ for each $n$, $X \in \mathrm{int}(\mathcal{D}_B(n))$ and $Y \in \mathrm{int}(\mathcal{D}_A(n))$.

**Theorem 1.1.** *Suppose $E \in M_{d\times e}(\mathbb{C})^g$ and $A \in M_r(\mathbb{C})^g$ are linearly independent. If $f : \mathrm{int}(\mathcal{B}_E) \to \mathrm{int}(\mathcal{D}_A)$ is a free bianalytic mapping with $f(0) = 0$ and $f'(0) = I_g$, then $f$ is convexotonic.*

*If, in addition, $A$ is minimal for $\mathcal{D}_A$, then there is convexotonic tuple $\Xi \in M_g(\mathbb{C})^g$ such that equation (1.1) holds, and $f$ is the corresponding convexotonic map, namely*

$$f(x) = x(I - \Lambda_\Xi(x))^{-1}. \tag{1.2}$$

*In particular, $\{A_1, \dots, A_g\}$ spans an algebra.*

*If $A$ is minimal for $\mathcal{D}_A$ and $E$ is ball-minimal, then $\max\{d, e\} \le r \le d + e$ and there is an $r \times r$ unitary matrix $U$ such that, up to unitary equivalence,*

$$A = U \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.3}$$

*Conversely, given a linearly independent $E \in M_{d\times e}(\mathbb{C})^g$, an integer $r \ge \max\{d, e\}$ and an $r \times r$ unitary matrix $U$, let $A$ be given by equation (1.3). If there is a tuple $\Xi$ such that equation (1.1) holds, then $f$ of equation (1.2) is a free bianalytic map $f : \mathrm{int}(\mathcal{B}_E) \to \mathrm{int}(\mathcal{D}_A)$.*

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[5] See also [23, Section 5 or Lemma 1.2].

**Proof.** See Corollary 2.5 and Section 5.2. □

**Remark 1.2.**

(a) The normalizations $f(0) = 0$ and $f'(0) = I_g$ can easily be enforced. Given a $g \times g$ matrix $\Delta$ and a tuple $C \in M_{d\times e}(\mathbb{C})^g$, let $\Delta \cdot C \in M_{d\times e}(\mathbb{C})^g$ denote the tuple

$$(\Delta \cdot C)_j = \sum_k \Delta_{j,k} C_k. \tag{1.4}$$

In the case $f : \mathrm{int}(\mathcal{B}_E) \to \mathrm{int}(\mathcal{D}_A)$ is bianalytic, but $f(0) = b \neq 0$ or $f'(0) = M \neq I$, let $\lambda : \mathcal{D}_A \to \mathcal{D}_F$ denote the affine linear map $\lambda(x) = x \cdot M + b$, where

$$F = M \cdot (\mathfrak{H} A \mathfrak{H}) \quad \text{and} \quad \mathfrak{H} = L_A^{\mathrm{re}}(b)^{-1/2}.$$

By Proposition 3.3, $h = \lambda^{-1} \circ f : \mathrm{int}(\mathcal{B}_E) \to \mathrm{int}(\mathcal{D}_F)$ is bianalytic with $h(0) = 0$ and $h'(0) = I_g$ and, if $A$ is minimal for $\mathcal{D}_A$, then $B$ is minimal for $\mathcal{D}_F$. In particular, $f$ is, up to affine linear equivalence, convexotonic.

Further, with a bit of bookkeeping the algebraic conditions of equations (1.3) and (1.1) can be expressed intrinsically in terms of $E$ and $A$. In the case $\mathcal{D}_A$ is a spectraball, these conditions are spelled out in Corollary 1.3 below.

(b) In the context of Theorem 1.1 (and Remark 1.2), $f^{-1}$ extends analytically to an open set containing $\mathcal{D}_A$ and if $\mathcal{D}_A$ is bounded, then $f$ extends analytically to an open set containing $\mathcal{B}_E$. The precise result is stated as Theorem 2.1 below. Theorem 2.1 is an elaboration on [3, Theorem 1.1].

(c) Given $A$ as in equation (1.3) and writing $U = (U_{j,k})_{j,k=1}^2$ in the natural block form, equation (1.1) is equivalent to $E_k U_{11} E_j = \sum_s (\Xi_j)_{k,s} E_s$.

(d) Corollary 6.2 and Theorem 6.1 extend Theorem 1.1 to cases where the codomain is matrix convex,[6] but not, by assumption, the interior of a free spectrahedron assuming the inverse of the bianalytic map is rational.

(e) Here is an example of a free spectrahedron that is not a spectraball, but is bianalytically equivalent to a spectraball. Let

$$E = I_2,\ E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and set

$$A = U \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \in M_3(\mathbb{C})^2.$$

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[6] In the present setting, matrix convex is the same as the convexity at each level.

With $\Xi_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Xi_2 = 0$, the tuples $A$ and $\Xi$ satisfy equation (1.1) and the corresponding convexotonic map is given by $f(x_1, x_2) = (x_1, x_2 + x_1^2)$. It is thus bianalytic from $\operatorname{int}(\mathcal{B}_E)$ to $\operatorname{int}(\mathcal{D}_A)$. Moreover, $\mathcal{D}_A$ is not a spectraball since $\mathcal{D}_A(1)$ is not rotationally invariant. $\square$

For a matrix $T$ with $\|T\| \le 1$, let $D_T$ denote the positive square root of $I - T^*T$. Thus, if $T$ is $k \times \ell$, then $D_T$ is $\ell \times \ell$ and $D_{T^*}$ is $k \times k$.

**Corollary 1.3.** *Suppose $E \in M_{d\times e}(\mathbb{C})^g$ and $C \in M_{k\times \ell}(\mathbb{C})^g$ are linearly independent and ball-minimal, $b \in \operatorname{int}(\mathcal{B}_C)$ and $M \in M_g(\mathbb{C})$. There exists a free bianalytic mapping $\varphi : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{B}_C)$ such that $\varphi(0) = b$ and $M = \varphi'(0)$ if and only if $E$ and $C$ have the same size (that is, $k = d$ and $\ell = e$) and there exist $d \times d$ and $e \times e$ unitary matrices $\mathscr{W}$ and $\mathscr{V}$ respectively and a convexotonic $g$-tuple $\Xi \in M_g(\mathbb{C})^g$ such that*

(a) $-E_j \mathscr{V}^* \Lambda_C(b)^* \mathscr{W} E_k = \sum_s (\Xi_k)_{j,s} E_s = (\Xi_k \cdot E)_j$; *and*
(b) $D_{\Lambda_C(b)^*} \mathscr{W} E_j \mathscr{V}^* D_{\Lambda_C(b)} = \sum_s M_{js} C_s = (M \cdot C)_j$,

*for all $1 \le j, k \le g$. Moreover, in this case $\varphi = \psi \cdot M + b$, where $\psi$ is the convexotonic map associated to $\Xi$; i.e., $\psi(x) = x(I - \Lambda_\Xi(x))^{-1}$.*

The proof of Corollary 1.3 appears in Subsubsection 5.3.3.

**Remark 1.4.**

(a) If $\mathcal{B}_E$ and $\mathcal{B}_C$ are bounded (equivalently $E$ and $C$ are linearly independent [26, Proposition 2.6(2)]), then any free bianalytic map $\varphi : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{B}_C)$ is, up to an affine linear bijection, convexotonic without any further assumptions (e.g., $C$ and $E$ need not be ball-minimal). Indeed, simply replace $E$ and $C$ by ball-minimal $E'$ and $C'$ with $\mathcal{B}_{E'} = \mathcal{B}_E$ and $\mathcal{B}_{C'} = \mathcal{B}_C$ and apply Corollary 1.3. The ball-minimal hypothesis allows for an explicit description of $\varphi$.
(b) While $M$ is not assumed invertible, both the condition $M = \varphi'(0)$ (for a bianalytic $\varphi$) and the identity of Corollary 1.3(b) (since $E$ is assumed linearly independent) imply it is.
(c) Assuming $E$ and $C$ of Corollary 1.3 are ball-minimal, by using the relation between $E$ and $C$ from Corollary 1.3(b), item (a) can be expressed purely in terms of $C$ as

$$C_j D_{\Lambda_C(b)}^{-1} \Lambda_C(b)^* D_{\Lambda_C(b)^*}^{-1} C_k \in \operatorname{span}\{C_1, \dots, C_g\}. \tag{1.5}$$

In particular, given a ball-minimal tuple $C \in M_{d\times e}(\mathbb{C})^g$ and $b \in \operatorname{int}(\mathcal{B}_C)$, if equation (1.5) holds then, for any choice of $M, \mathscr{W}$ and $\mathscr{V}$ and solving equation (b) for $E$, there is a free bianalytic map $\varphi : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{B}_C)$ such that $\varphi(0) = b$ and $\varphi'(0) = M$.
(d) Among the results in [45] is a complete analysis of the free bianalytic maps between the free versions of matrix balls, antecedents and special cases of which appear

elsewhere in the literature such as [29] and [50]. The connection between the results in [45] on free matrix balls and Corollary 1.3 is worked out in Subsubsection 5.3.2. Subsubsection 5.3.1 gives a complete classification of free automorphisms of free polydiscs. □

### *1.3. Main result on maps between free spectrahedra*

The article [3] characterizes the triples $(p, A, B)$ such that $p : \mathcal{D}_A \to \mathcal{D}_B$ is bianalytic under unconventional geometric hypotheses (sketched in Subsection 1.4 below), cf. [3, §7]. Here we obtain Theorem 1.5 by converting those geometric hypotheses to algebraic *irreducibility* hypotheses that we now describe.

For a tuple of rectangular matrices $E = (E_1, \dots, E_g) \in M_{d\times e}(\mathbb{C})^g$ denote

$$Q_E(x,y) := I - \Lambda_{E^*}(y)\Lambda_E(x), \qquad \mathbb{L}_E(x,y) := \begin{pmatrix} I & \Lambda_E(x) \\ \Lambda_{E^*}(y) & I \end{pmatrix},$$

$$\ker(E) := \bigcap_{j=1}^{g} \ker(E_j) = \ker(\begin{pmatrix} E_1 \\ \vdots \\ E_g \end{pmatrix}), \quad \operatorname{ran}(E) = \operatorname{ran}((E_1 \quad \dots E_g)).$$

Thus $\mathbb{L}_E(x,y) = L_F(x,y)$ where

$$F = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

We also let $\mathbb{L}_E^{\mathrm{re}}$ denote the hermitian monic pencil,

$$\mathbb{L}_E^{\mathrm{re}}(x) := \mathbb{L}_E(x,x^*) = L_F(x,x^*) = L_F^{\mathrm{re}}(x)$$

and likewise

$$Q_E^{\mathrm{re}}(x) = Q_E(x,x^*).$$

Observe $\mathcal{B}_E = \mathcal{D}_{\mathbb{L}_E^{\mathrm{re}}} := \{X : \mathbb{L}_E(X,X^*) \succeq 0\} = \mathcal{D}_F$. Finally, for a monic pencil $L_A$, let

$$\mathcal{Z}_{L_A} = \{(X,Y) : \det(L_A(X,Y)) = 0\}, \quad \mathcal{Z}_{L_A}^{\mathrm{re}} = \{X : \det(L_A^{\mathrm{re}}(X)) = 0\}.$$

We also use the notation $\mathcal{Z}_{Q_E} = \mathcal{Z}_{\mathbb{L}_E}$.

Let $\mathbb{C}<x>$ denote the **free algebra** of noncommutative polynomials in the letters $x = \{x_1, \dots, x_g\}$. Thus elements of $\mathbb{C}<x>$ are finite $\mathbb{C}$-linear combinations of words in the letters $\{x_1, \dots, x_g\}$. For each positive integer $n$, an element $p$ of $\mathbb{C}<x>$ naturally induces a function, also denoted $p$, mapping $M_n(\mathbb{C})^g \to M_n(\mathbb{C})$ by replacing the letter $x_1, \dots, x_g$ by $n \times n$ matrices $X_1, \dots, X_g$. In this way, we view $p$ as a function on the disjoint union of the sets $M_n(\mathbb{C})^g$ (parameterized by $n$). When $e > 1$ there are non-constant $F \in \mathbb{C}<x>^{e\times e}$ that are invertible, and the appropriate analog of irreducible elements of $\mathbb{C}<x>^{e\times e}$ reads as follows. An $F \in \mathbb{C}<x>^{e\times e}$ with $\det f(0) \neq 0$ is an **atom** [13, Chapter 3] if $F$ does not factor; i.e., $F$ cannot be written as $F = F_1F_2$ for some non-invertible $F_1, F_2 \in \mathbb{C}<x>^{e\times e}$. As a consequence of Lemma 3.2(8) below, we will see that if $Q_E$ is an atom, $\ker(E) = \{0\}$ and $\ker(E^*) = \{0\}$, then $E$ is ball-minimal.

**Theorem 1.5.** *Suppose $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^g$ and*

(a) *$\mathcal{D}_A$ is bounded;*
(b) *$Q_A$ and $Q_B$ are atoms, $\ker(B) = \{0\}$ and $A^*$ is ball-minimal;*
(c) *$t > 1$ and $p : \mathrm{int}(t\mathcal{D}_A) \to M(\mathbb{C})^g$ and $q : \mathrm{int}(t\mathcal{D}_B) \to M(\mathbb{C})^g$ are free bianalytic mappings;*
(d) *$p(0) = 0$, $p'(0) = I$, $q(0) = 0$ and $q'(0) = I$.*

*If $q(p(X)) = X$ and $p(q(Y)) = Y$ for $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$ respectively, then $p$ is convexotonic, $A$ and $B$ are of the same size $d = e$, and there exist $d \times d$ unitary matrices $Z$ and $M$ and a convexotonic $g$-tuple $\Xi$ such that*

(1) *$p$ is the convexotonic map $p = x(I - \Lambda_\Xi(x))^{-1}$, where for each $1 \leq j, k \leq g$,*

$$A_k(Z - I)A_j = \sum_s (\Xi_j)_{k,s} A_s; \tag{1.6}$$

*in particular, the tuple $R = (Z - I)A$ spans an algebra with multiplication table $\Xi$,*

$$R_kR_j = \sum_s (\Xi_j)_{k,s} R_s;$$

(2) *$B_j = M^*ZA_jM$ for $1 \leq j \leq g$.*

**Proof.** See Section 4.4. □

### *1.4. Geometry of the boundary vs irreducibility*

At the core of the proofs of our main theorems in this paper is a richness of the geometry of the boundary, $\partial\mathcal{B}_E$, of a spectraball, $\mathcal{B}_E$. We shall show that a (rather ungainly) key geometric property of the boundary of $\mathcal{B}_E$ is equivalent to the defining polynomial $Q_E$ of $\mathcal{B}_E$ being an atom and $\ker(E) = \{0\}$.

To describe the geometric structure involved, fix $E \in M_{d\times e}(\mathbb{C})^g$. The **detailed boundary** $\widehat{\partial\mathcal{B}_E}$ of $\mathcal{B}_E$ is the sequence of sets

$$\widehat{\partial\mathcal{B}_E}(n) := \{(X, v) \in M_n(\mathbb{C})^g \times [\mathbb{C}^e \otimes \mathbb{C}^n] : X \in \partial\mathcal{B}_E,\ v \neq 0,\ Q_E^{\mathrm{re}}(X, X^*)v = 0\}.$$

For $n \in \mathbb{N}$, let $\widehat{\partial^1 \mathcal{B}_E}(n)$ denote the points $(X, v)$ in $\widehat{\partial \mathcal{B}_E}(n)$ such that $\dim \ker Q_E^{\mathrm{re}}(X, X^*) = 1$. For a vector $v \in \mathbb{C}^e \otimes \mathbb{C}^n = \mathbb{C}^{en}$, partitioned as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

for $v_k \in \mathbb{C}^e$, define $\pi(v) = v_1$. The geometric property important to mapping studies is that $\pi(\widehat{\partial^1 \mathcal{B}_E})$ contain enough vectors to span $\mathbb{C}^e$ or better yet to hyperspan $\mathbb{C}^e$. Here a set $\{u^1, \dots, u^{e+1}\}$ of vectors in $\mathbb{C}^e$ **hyperspans** $\mathbb{C}^e$ provided each $e$ element subset spans; i.e., is a basis of $\mathbb{C}^e$.

**Theorem 1.6.** *Let* $E \in M_{d\times e}(\mathbb{C})^g$. *Then*

(1) *$E$ is ball-minimal if and only if $\pi(\widehat{\partial^1 \mathcal{B}_E})$ spans $\mathbb{C}^e$.*
(2) *$Q_E$ is an atom and* $\ker(E) = \{0\}$ *if and only if $\pi(\widehat{\partial^1 \mathcal{B}_E})$ contains a hyperspanning set for $\mathbb{C}^e$.*

**Proof.** Part (1) is established in Proposition 4.2, while (2) is Proposition 4.4. □

### 1.5. A Nullstellensatz

Theorem 1.1 uses the following Nullstellensatz whose proof depends upon Theorem 1.6.

**Proposition 1.7.** *Suppose* $E = (E_1, \dots, E_g) \in M_{d\times e}(\mathbb{C})^g$ *is ball-minimal and* $V \in \mathbb{C}<x>^{\ell\times e}$ *is a (rectangular) matrix polynomial. If $V$ vanishes on $\widehat{\partial \mathcal{B}_E}$; that is $V(X)\gamma = 0$ whenever $(X, \gamma) \in \widehat{\partial \mathcal{B}_E}$, then $V = 0$.*

**Proof.** See Subsection 5.1. □

### 1.6. An overview of the proof of Theorem 1.1

We are now in a position to convey, in broad strokes, an outline of the proof of Theorem 1.1. The conversely direction is an immediate consequence of Proposition 2.2 (see Corollary 2.5) of Section 2. Its proof reflects the fact that convexotonic maps are bianalytic between certain special spectrahedral pairs. Proposition (2.2) is also the starting point for the proof of the more challenging converse. Given the tuple $A$, let $J = (J_1, \dots, J_h)$ denote a basis for the algebra spanned by $A$ with $J_j = A_j$, for $1 \le j \le g$. Proposition 2.2 says that $\mathcal{D}_J$ and $\mathcal{B}_J$ are bianalytic via the convexotonic map associated to the convexotonic $h$-tuple $\Xi$ determined by the tuple $J$ via equation

(1.1) (with $J$ in place of $A$). Starting with the free bianalytic map $f : \mathcal{B}_E \to \mathcal{D}_A$, observe that $G = \varphi \circ \iota \circ f : \mathcal{B}_E \to \mathcal{B}_J$ is a free proper map satisfying $G(0) = 0$ and $G'(0) = \begin{pmatrix} I_g & 0_{g\times(h-g)} \end{pmatrix}$, where $\iota : \mathcal{D}_A \to \mathcal{D}_J$ is the inclusion, since $\varphi(0) = 0$ and $\varphi'(0) = I_h$. An argument that uses Proposition 1.7 produces a representation for $G$ that can be thought of as an analog of the Schwarz Lemma (see equation (5.8)). In simple cases,

$$G(x) = \begin{pmatrix} x & 0 \end{pmatrix} \tag{1.7}$$

from which it follows that the $g$-tuple $\widehat{\Xi} \in M_g(\mathbb{C})^g$ defined by

$$(\widehat{\Xi}_j)_{s,t} = (\Xi_j)_{s,t}, \quad 1 \le j, s, t \le g$$

is convexotonic and thus $A$ spans an algebra. Hence $h = g$, the map $\varphi$ (and hence $\varphi^{-1}$) is convexotonic and $f = \varphi^{-1}$. In general only a weaker version of equation (1.7) holds, an inconvenience that does not conceptually alter the argument, but one that does make the proof more technical.

## 2. Free rational maps and convexotonic maps

In this section we review the notions of a free set and free rational function and provide further background on free functions and mappings. In particular, convexotonic maps are seen to be free rational mappings. In Subsection 2.3 we show how algebras of matrices give rise to convexotonic bianalytic maps between free spectrahedra. See Theorem 2.1.

### *2.1. Free sets, free analytic functions and mappings*

Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A **subset** $\Gamma$ of $M(\mathbb{C})^g$ is a sequence $(\Gamma_n)_n$ where $\Gamma_n \subseteq M_n(\mathbb{C})^g$. (Sometimes we write $\Gamma(n)$ in place of $\Gamma_n$.) The subset $\Gamma$ is a **free set** if it is closed under direct sums and simultaneous unitary similarity. Examples of such sets include spectraballs and free spectrahedra introduced above. We say the free set $\Gamma = (\Gamma_n)_n$ is **open** if each $\Gamma_n$ is open. Generally adjectives are applied level-wise to free sets unless noted otherwise.

A **free function** $f : \Gamma \to M(\mathbb{C})$ is a sequence of functions $f_n : \Gamma_n \to M_n(\mathbb{C})$ that **respects intertwining**; that is, if $X \in \Gamma_n$, $Y \in \Gamma_m$, $T : \mathbb{C}^m \to \mathbb{C}^n$, and

$$XT = (X_1T, \dots, X_gT) = (TY_1, \dots, TY_g) = TY,$$

then $f_n(X)T = Tf_m(Y)$. In the case $\Gamma$ is open, $f$ is **free analytic** if each $f_n$ is analytic in the ordinary sense. We refer the reader to [1,2,23,25,38,54] for a fuller discussion of free sets and functions. For further results, not already cited, on free bianalytic and proper free analytic maps see [24,29,40,46,50,51] and the references therein.

A **free mapping** $p : \Gamma \to M(\mathbb{C})^h$ is a tuple $p = \begin{pmatrix} p^1 & p^2 & \cdots & p^h \end{pmatrix}$ where each $p^j : \Gamma \to M(\mathbb{C})$ is a free function. The free mapping $p$ is **free analytic** if each $p^j$ is a free analytic function. If $h = g$ and $\Delta \subseteq M(\mathbb{C})^g$ is a free set, then $p : \Gamma \to \Delta$ is **bianalytic** if $p$ is analytic and $p$ has an inverse, that is necessarily free and analytic, $q : \Delta \to \Gamma$.

### *2.2. Free rational functions and mappings*

Based on the results of [37, Theorem 3.1] and [55, Theorem 3.5] a **free rational function regular at** 0 can, for the purposes of this article, be defined with minimal overhead as an expression of the form

$$r(x) = c^* \big(I - \Lambda_S(x)\big)^{-1} b, \tag{2.1}$$

where, for some positive integer $s$, we have $S \in M_s(\mathbb{C})^g$ and $b, c \in \mathbb{C}^s$. The expression $r$ is known as a **realization**. Realizations are easy to manipulate and a powerful tool as developed in the series of papers [5–7] of Ball-Groenewald-Malakorn; see also [9,13]. The realization $r$ is evaluated in the obvious fashion on a tuple $X \in M_n(\mathbb{C})^g$ as long as $I - \Lambda_S(X)$ is invertible. Importantly, free rational functions are free analytic.

Given a tuple $T \in M_k(\mathbb{C})^g$, let

$$\mathscr{I}_T = \{X \in M(\mathbb{C})^g : \det(I - \Lambda_T(X)) \neq 0\}. \tag{2.2}$$

A realization $\tilde{r}(x) = \tilde{c}^*(I - \Lambda_{\tilde{S}})^{-1}\tilde{b}$ is **equivalent** to the realization $r$ as in (2.1) if $r(X) = \tilde{r}(X)$ for $X \in \mathscr{I}_S \cap \mathscr{I}_{\tilde{S}}$. A free rational function is an equivalence class of realizations and we identify $r$ with its equivalence class and refer to it as a free rational function. The realization (2.1) is **minimal** if $s$ is the minimum size among all realizations equivalent to $r$. By [37,55], if $r$ is minimal and $\tilde{r}$ is equivalent to $r$, then $\mathscr{I}_S \supseteq \mathscr{I}_{\tilde{S}}$. Moreover, the results in [55] explain precisely, in terms of evaluations, the sense in which $\mathscr{I}_S$ deserves to be called the **domain of the free rational function** $r$, denoted $\operatorname{dom}(r)$.

A free polynomial $p$ is a free rational function regular at 0 and, as is well known, its domain is $M(\mathbb{C})^g$. If $f$ and $g$ are free rational functions regular at 0, then so are $f + g$ and $fg$. Moreover, $\operatorname{dom}(f + g)$ and $\operatorname{dom}(fg)$ both contain $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ as a consequence of [56, Theorem 3.10]. Free rational functions regular at 0 are determined by their evaluations near 0; that is if $f(X) = g(X)$ in some neighborhood of 0 in $\operatorname{dom}(f) \cap \operatorname{dom}(g)$, then $f = g$. In what follows, we often omit *regular at* 0 when it is understood from context. We refer the reader to [37,55] for a fuller discussion of the domain of a free rational function.

A **free rational mapping** $p$ is a tuple of rational functions $p = \begin{pmatrix} p^1 & \cdots & p^g \end{pmatrix}$. The domain of $p$ is the intersection of the domains of the $p^j$. By [3, Proposition 1.11], if $r$ is a free rational mapping with no singularities on a bounded free spectrahedron $\mathcal{D}_A$, then there is a $t > 1$ such that $r$ has no singularities on $t\mathcal{D}_A$.

### *2.3. Algebras and convexotonic maps*

Theorem 2.1 below is an expanded version of [3, Theorem 1.1]. To begin we discuss a sufficient condition for a tuple $X \in M_n(\mathbb{C})^g$ to lie in $\operatorname{dom}(p)$, the domain of a convexotonic mapping

$$p = \begin{pmatrix} p^1 & \cdots & p^g \end{pmatrix} = x(I - \Lambda_\Xi(x))^{-1}.$$

Since

$$p^j = \sum_{k=1}^{g} x_k \left[ e_k^*(I - \Lambda_\Xi(x))^{-1} e_j \right],$$

it follows that $\mathscr{I}_\Xi \subseteq \cap \operatorname{dom}(p^j) = \operatorname{dom}(p)$. Now suppose $R \in M_N(\mathbb{C})^g$ and $f_{k,s,a,b}, g_{k,s,a,b}, h_k \in \mathbb{C} <x>$ and let $r^k$ denote the free rational function

$$r^k(x) = \sum_{s,a,b} f_{k,s,a,b}(x)[e_a^* \left(I - \Lambda_R(x)\right)^{-1} e_b]\, g_{k,s,a,b}(x) + h_k.$$

If $r^j = p^j$ in some neighborhood of 0 lying in $\mathscr{I}_\Xi \cap \mathscr{I}_R$, then $r^j$ and $p^j$ represent the same free rational function. In particular, $\mathscr{I}_R \subseteq \operatorname{dom}(p^j)$ and therefore $\mathscr{I}_R \subseteq \operatorname{dom}(p)$.

Let $\operatorname{ext}(\mathcal{D}_B)$ denote the sequence $(\operatorname{ext}(\mathcal{D}_B(n)))_n$ where $\operatorname{ext}(\mathcal{D}_B(n))$ is the complement of $\mathcal{D}_B(n)$. Likewise let $\partial\mathcal{D}_B(n)$ denote the boundary of $\mathcal{D}_B(n)$ and let $\partial\mathcal{D}_B$ denote the sequence $(\partial\mathcal{D}_B(n))_n$.

**Theorem 2.1.** *Suppose $\mathfrak{A}, \mathfrak{B} \in M_r(\mathbb{C})^g$ are linearly independent, $U \in M_r(\mathbb{C})^g$ is unitary and $\mathfrak{B} = U\mathfrak{A}$. If there exists a tuple $\Xi \in M_g(\mathbb{C})^g$ such that*

$$\mathfrak{A}_\ell(U - I)\mathfrak{A}_j = \sum_{s=1}^{g} (\Xi_j)_{\ell,s} \mathfrak{A}_s,$$

*then $\Xi$ is convexotonic and the convexotonic maps $p$ and $q$ associated to $\Xi$ are bianalytic maps between $\mathcal{D}_\mathfrak{A}$ and $\mathcal{D}_\mathfrak{B}$ in the following sense.*

(a) $\operatorname{int}(\mathcal{D}_\mathfrak{A}) \subseteq \operatorname{dom}(p)$, $\operatorname{int}(\mathcal{D}_\mathfrak{B}) \subseteq \operatorname{dom}(q)$; *and* $p : \operatorname{int}(\mathcal{D}_\mathfrak{A}) \to \operatorname{int}(\mathcal{D}_\mathfrak{B})$ *is bianalytic.*
(b) *If* $X \in \operatorname{ext}(\mathcal{D}_\mathfrak{A}) \cap \operatorname{dom}(p)$, *then* $p(X) \in \operatorname{ext}(\mathcal{D}_\mathfrak{B})$.
(c) *If* $X \in \partial\mathcal{D}_\mathfrak{A} \cap \operatorname{dom}(p)$, *then* $p(X) \in \partial\mathcal{D}_\mathfrak{B}$.
(d) *If* $\mathcal{D}_\mathfrak{B}(1)$ *is bounded, then* $\mathcal{D}_\mathfrak{A} \subseteq \operatorname{dom}(p)$.

Before taking up the proof of Theorem 2.1, we prove the following proposition and collect a few of its consequences that will be used in the sequel.

**Proposition 2.2** *([4, Proposition 1.3]). Suppose $J \in M_d(\mathbb{C})^g$ is linearly independent and spans an algebra with convexotonic tuple $\Xi$ (as in equation (1.1) with $J$ in place of $A$).*

*Let* $p = x(I - \Lambda_\Xi(x))^{-1}$ *and* $q = x(I + \Lambda_\Xi(x))^{-1}$ *denote the corresponding convexotonic maps.*

(i) $\operatorname{int}(\mathcal{B}_J) \subseteq \operatorname{dom}(p)$ *and* $p : \operatorname{int}(\mathcal{B}_J) \to \operatorname{int}(\mathcal{D}_J)$.
(ii) $\mathcal{D}_J \subseteq \operatorname{dom}(q)$ *and* $q : \operatorname{int}(\mathcal{D}_J) \to \operatorname{int}(\mathcal{B}_J)$ *and* $q(\partial\mathcal{D}_J) \subseteq \partial\mathcal{B}_J$.
(iii) $p : \operatorname{int}(\mathcal{B}_J) \to \operatorname{int}(\mathcal{D}_J)$ *and* $q : \operatorname{int}(\mathcal{D}_J) \to \operatorname{int}(\mathcal{B}_J)$ *are birational inverses of one another.*
(iv) *If* $X \in \operatorname{dom}(p)$, *but* $X \notin \operatorname{int}(\mathcal{B}_J)$, *then* $p(X) \notin \operatorname{int}(\mathcal{D}_J)$.
(v) *If* $\mathcal{D}_J$ *is bounded, then the domain of* $p$ *contains* $\mathcal{B}_J$ *and* $p(\partial\mathcal{B}_J) \subseteq \partial\mathcal{D}_J$.
(vi) *If* $Y \in \operatorname{dom}(q)$, *but* $Y \notin \mathcal{D}_J$, *then* $q(Y) \notin \mathcal{B}_J$.

**Lemma 2.3.** *Suppose* $F \in M_d(\mathbb{C})^g$. *If* $I + \Lambda_F(X) + \Lambda_F(X)^* \succeq 0$, *then* $I + \Lambda_F(X)$ *is invertible.*

**Proof.** Arguing the contrapositive, suppose $I+\Lambda_F(X)$ is not invertible. In this case there is a unit vector $\gamma$ such that

$$\Lambda_F(X)\gamma = -\gamma.$$

Hence,

$$\langle (I + \Lambda_F(X) + \Lambda_F(X)^*)\gamma, \gamma\rangle = \langle \Lambda_F(X)^*\gamma, \gamma\rangle = \langle \gamma, \Lambda_F(X)\gamma\rangle = -1. \quad \square$$

**Lemma 2.4.** *Let* $T \in M_d(\mathbb{C})$. *Then*

(a) $I + T + T^* \succeq 0$ *if and only if* $I + T$ *is invertible and* $\|(I + T)^{-1}T\| \leq 1$;
(b) $I + T + T^* \succ 0$ *if and only if* $I + T$ *is invertible and* $\|(I + T)^{-1}T\| < 1$.
(c) *If* $\|T\| < 1$, *then* $I - T$ *is invertible and* $I + (I - T)^{-1}T + \big((I - T)^{-1}T\big)^* \succ 0$.
(d) *If* $\|T\| = 1$ *and* $I - T$ *is invertible, then* $I + (I - T)^{-1}T + \big((I - T)^{-1}T\big)^*$ *is positive semidefinite and singular.*

**Proof.** Item (a) follows from the chain of equivalences,

$$\begin{aligned}
\|(I + T)^{-1}T\| \leq 1 \quad &\iff \quad I - \big((I + T)^{-1}T\big)\big((I + T)^{-1}T\big)^* \succeq 0 \\
&\iff \quad I - (I + T)^{-1}TT^*(I + T)^{-*} \succeq 0 \\
&\iff \quad (I + T)(I + T)^* - TT^* \succeq 0 \\
&\iff \quad I + T + T^* \succeq 0.
\end{aligned}$$

The proof of item (b) is the same.

The proof of (c) is routine. Indeed, it is immediate that $I - T$ is invertible and

$$I + (I - T)^{-1}T + \big((I - T)^{-1}T\big)^* = (I - T)^{-1}\ (I - TT^*)\ (I - T)^{-*} \succ 0.$$

The proof of item (d) is similar. □

**Proof of Proposition 2.2.** Compute

$$
\begin{aligned}
\Lambda_J(q(x))\,\Lambda_J(x) &= \sum_{s,k=1}^{g} q^s(x) x_k J_s J_k = \sum_{j=1}^{g}\sum_{s=1}^{g} q^s(x)\left[\sum_{k=1}^{g} x_k(\Xi_k)_{s,j}\right] J_j \\
&= \sum_{j=1}^{g}\sum_{s=1}^{g} q^s(x)(\Lambda_\Xi(x))_{s,j} J_j \\
&= \sum_{j=1}^{g}\sum_{t=1}^{g} x_t \left[\sum_{s=1}^{g}(I+\Lambda_\Xi(x))^{-1}_{t,s}(\Lambda_\Xi(x))_{s,j}\right] J_j \\
&= \sum_{j=1}^{g}\sum_{t=1}^{g} x_t[(I+\Lambda_\Xi(x))^{-1}\Lambda_\Xi(x)]_{t,j} J_j.
\end{aligned}
$$

Hence,

$$
\Lambda_J(q(x))\,(I+\Lambda_J(x)) = \sum_{j=1}^{g}\sum_{t=1}^{g} x_t[(I+\Lambda_\Xi(x))^{-1}(I+\Lambda_\Xi(x))]_{t,j} J_j = \Lambda_J(x).
$$

Thus, as free (matrix-valued) rational functions regular at 0,

$$
\Lambda_J(q(x)) = (I+\Lambda_J(x))^{-1}\,\Lambda_J(x) =: F(x). \tag{2.3}
$$

Since $J$ is linearly independent, given $1 \le k \le g$, there is a linear functional $\lambda$ such that $\lambda(J_j) = 0$ for $j \neq k$ and $\lambda(J_k) = 1$. Applying $\lambda$ to equation (2.3), gives

$$
q^k(x) = \lambda(F(x)). \tag{2.4}
$$

Since $\lambda(F(x))$ is a free rational function whose domain contains

$$
\mathscr{D} = \{X : I + \Lambda_J(X) \text{ is invertible}\},
$$

the same is true for $q^k$. (As a technical matter, each side of equation (2.4) is a rational expression. Since they are defined and agree on a neighborhood of 0, they determine the same free rational function. It is the domain of this rational function that contains $\mathscr{D}$. See [55], and also [37], for full details.) By Lemma 2.3, $\mathscr{D}$ contains $\mathcal{D}_J$, (as $X \in \mathcal{D}_J$ implies $I + \Lambda_J(X)$ is invertible). Hence the domain of the free rational mapping $q$ contains $\mathcal{D}_J$. By Lemma 2.4 and equation (2.3), $q$ maps the interior of $\mathcal{D}_J$ into the interior of $\mathcal{B}_J$ and the boundary of $\mathcal{D}_J$ into the boundary of $\mathcal{B}_J$. Thus item (ii) is proved.

Similarly,

$$
(I - \Lambda_J(x))^{-1}\,\Lambda_J(x) = \Lambda_J(p(x)). \tag{2.5}
$$

Arguing as above shows the domain of $p$ contains the set

$$\mathscr{E} = \{X : I - \Lambda_J(X) \text{ is invertible}\},$$

which in turn contains $\text{int}(\mathcal{B}_J)$ (since $\|\Lambda_J(X)\| < 1$ allows for an application of Lemma 2.4). By Lemma 2.4 and equation (2.5), $p$ maps the interior of $\mathcal{B}_J$ into the interior of $\mathcal{D}_J$, proving item (i). Since $p$ and $q$ are formal rational inverses of one another, it follows from items (i) and (ii) that they are inverses of one another as maps between $\mathcal{D}_J$ and $\mathcal{B}_J$, proving item (iii). Further, if $X$ is in the boundary of $\mathcal{B}_J$, then for $t \in \mathbb{C}$ and $|t| < 1$, we have $p(tX) \in \text{int}(\mathcal{D}_J)$ and

$$\Lambda_J(p(tX)) = (I - \Lambda_J(tX))^{-1}\,\Lambda_J(tX).$$

Assuming $\mathcal{D}_J$ is bounded, it follows that $I - \Lambda_J(X)$ is invertible and thus, by Lemma 2.4, $X$ is in the domain of $p$ and $p(X)$ is in the boundary of $\mathcal{D}_J$, proving item (v). Finally, to prove item (iv), suppose $X \notin \text{int}(\mathcal{B}_J)$, but $p(X) \in \text{int}(\mathcal{D}_J)$. By item (i), there is a $Y \in \text{int}(\mathcal{B}_J)$ such that $p(Y) = p(X)$. By item (ii), $p(Y) = p(X) \in \text{dom}(q)$ and therefore, $Y = q(p(Y)) = q(p(X)) = X$, a contradiction. The proof of (vi) is similar. □

The converse portion of Theorem 1.1 is an immediate consequence of Proposition 2.2, stated below as Corollary 2.5.

**Corollary 2.5.** *Suppose* $E \in M_{d\times e}(\mathbb{C})^g$ *is linearly independent,* $r \geq \max\{d, e\}$*, the* $r \times r$ *matrix* $U$ *is unitary and*

$$A = U \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

*If there exists a tuple* $\Xi \in M_g(\mathbb{C})^g$ *such that equation* (1.1) *holds, then* $\Xi$ *is convexotonic and the associated convexotonic map* $p$ *is a bianalytic mapping* $\text{int}(\mathcal{B}_E) = \text{int}(\mathcal{B}_A) \to \text{int}(\mathcal{D}_A)$*. Moreover,* $\mathcal{D}_A \subseteq \text{dom}(q)$ *and* $q(\partial\mathcal{D}_A) \subseteq \partial\mathcal{B}_A$*, where* $q = x(I + \Lambda_\Xi(x))^{-1}$ *is the inverse of* $p$*.*

**Proof.** By the definition of $A$ we have $\mathcal{B}_A = \mathcal{B}_E$. The rest follows by Proposition 2.2. □

In the case $J$ does not span an algebra, we have the following variant of Proposition 2.2. It says that each free spectrahedron can be mapped properly to a bounded spectraball and is used in the proof of Theorem 1.1. Recall a mapping between topological spaces is **proper** if the inverse image of each compact sets is compact. Thus, for free open sets $\mathcal{U} \subseteq M(\mathbb{C})^g$ and $\mathcal{V} \subseteq M(\mathbb{C})^h$, a free mapping $f : \mathcal{U} \to \mathcal{V}$ is proper if each $f_n : \mathcal{U}_n \to \mathcal{V}_n$ is proper. For perspective, given subsets $\Omega \subseteq \mathbb{C}^g$ and $\Delta \subseteq \mathbb{C}^h$ (that are not necessarily closed), and a proper analytic map $\psi : \Omega \to \Delta$, if $\Omega \ni z^j \to \partial\Omega$, then $\psi(z^j) \to \partial\Delta$. [42, page 429].

**Corollary 2.6.** *Let $A \in M_d(\mathbb{C})^g$ and assume $A$ is linearly independent. Let $C_{g+1}, \dots, C_h \in M_d(\mathbb{C})$ be any matrices such that the tuple $J = (J_1, \dots, J_h) = (A_1, \dots, A_g, C_{g+1}, \dots, C_h)$ is a basis for the algebra generated by the tuple $A$. Let $\Xi \in M_h(\mathbb{C})^h$ denote the convexotonic tuple associated to $J$, let $p : \operatorname{int}(\mathcal{B}_J) \to \operatorname{int}(\mathcal{D}_J)$ denote the corresponding convexotonic map, let $q$ denote the inverse of $p$, and let $\iota : \operatorname{int}(\mathcal{D}_A) \to \operatorname{int}(\mathcal{D}_J)$ denote the inclusion. Then we have the commutative diagram*

$$\begin{tikzcd}
 & \operatorname{int}(\mathcal{B}_J) \arrow[d, "p"', "\cong"] \\
\operatorname{int}(\mathcal{D}_A) \arrow[ur, dashed, "f"] \arrow[r, hook, "\iota"'] & \operatorname{int}(\mathcal{D}_J)
\end{tikzcd}$$

*and the mapping*

$$f(x) = q \circ \iota(x) = \begin{pmatrix} x_1 & \cdots & x_g & 0 & \cdots & 0 \end{pmatrix} \Big( I + \sum_{j=1}^{g} \Xi_j x_j \Big)^{-1} \tag{2.6}$$

*is (injective) proper and extends analytically to a neighborhood of $\mathcal{D}_A$.*

**Proof.** By Proposition 2.2, $p : \operatorname{int}(\mathcal{B}_J) \to \operatorname{int}(\mathcal{D}_J)$ is birational and the domain of its inverse $q$ contains $\mathcal{D}_J$ and maps $\partial \mathcal{D}_J$ into $\partial \mathcal{B}_J$. In particular $q$ is proper.

Given $X \in M(\mathbb{C})^g$, letting $Y = \begin{pmatrix} X & 0 \end{pmatrix}$,

$$\Lambda_J(Y) = \sum_{j=1}^{h} J_j \otimes Y_j = \sum_{j=1}^{g} A_j \otimes X_j.$$

Hence $L_J^{\mathrm{re}}(\begin{pmatrix} X & 0 \end{pmatrix}) = L_A^{\mathrm{re}}(X)$ and it follows that $X \in \operatorname{int}(\mathcal{D}_A)$ if and only if $Y \in \operatorname{int}(\mathcal{D}_J)$. Hence, we obtain a mapping $\iota : \operatorname{int}(\mathcal{D}_A) \to \operatorname{int}(\mathcal{D}_J)$ defined by $\iota(X) = Y$.

Fix $m \in \mathbb{N}$ and suppose $K \subseteq \operatorname{int}(\mathcal{D}_J(m))$ is compact and let $K_* = \iota^{-1}(K) \subseteq \mathcal{D}_A(m)$. If $(X^n)$ is a sequence from $K_*$, then $Y^n = \begin{pmatrix} X^n & 0 \end{pmatrix}$ is a sequence from $K$. Since $K$ is compact, $(Y^n)_n$ has a subsequence $(Y^{n_j})_j$ that converges to some $Y \in K$. It follows that $Y = \begin{pmatrix} X & 0 \end{pmatrix} \in K \subseteq \operatorname{int}(\mathcal{D}_J)$ for some $X \in K_*$. Hence $(X^{n_j})_j$ converges to $X$ and we conclude that $K_*$ is compact. Thus $\iota$ is proper. Since $q$ is also proper, $f = q \circ \iota$ is too. Letting $z = (z_1, \dots, z_h)$ denote an $h$ tuple of freely noncommuting indeterminates,

$$q(z) = z(I + \Lambda_\Xi(z))^{-1}$$

and thus $f$ takes the form of equation (2.6). □

### 2.4. *Proof of Theorem 2.1*

**Lemma 2.7.** *Suppose* $G \in M_{d\times e}(\mathbb{C})^g$ *is linearly independent,* $C \in M_{e\times d}(\mathbb{C})$ *and* $\Psi \in M_g(\mathbb{C})^g$. *If*

$$G_\ell C G_j = \sum_{s=1}^{g} (\Psi_j)_{\ell,s} G_s,$$

*then the tuple* $\Psi$ *is convexotonic. Moreover, letting* $T = CG \in M_e(\mathbb{C})^g$,

$$G_\ell T^\alpha = \sum_{s=1}^{g} (\Psi^\alpha)_{\ell,s} G_s. \tag{2.7}$$

*In particular, if* $A \in M_d(\mathbb{C})^g$ *is linearly independent and spans an algebra, then the tuple* $\Psi$ *uniquely determined by equation* (1.1) *is convexotonic.*

Note that the hypothesis implies $T$ spans an algebra (but not that $T$ is linearly independent).

**Proof.** Routine calculations give

$$(G_\ell T_j)T_k = \sum_{t=1}^{g} (\Psi_j)_{\ell,t} G_t\, T_k = \sum_{s,t=1} (\Psi_j)_{\ell,t} (\Psi_k)_{t,s} G_s = \sum_{s} (\Psi_j\, \Psi_k)_{\ell,s} G_s.$$

On the other hand

$$G_\ell(T_j T_k) = G_\ell C(G_j T_k) = \sum_t G_\ell (\Psi_k)_{j,t} T_t = \sum_{s,t} (\Psi_t)_{\ell,s} (\Psi_k)_{j,t} G_s.$$

By independence of $G$,

$$(\Psi_j \Psi_k)_{\ell,s} = \sum_t (\Psi_k)_{j,t} (\Psi_t)_{\ell,s}$$

and therefore

$$\Psi_j \Psi_k = \sum_t (\Psi_k)_{j,t} \Psi_t.$$

Hence $\Psi$ is convexotonic.

A straightforward induction argument establishes the identity (2.7). □

**Proposition 2.8.** *Suppose* $A, B \in M_t(\mathbb{C})^g$ *are linearly independent,* $U \in M_t(\mathbb{C})^g$ *is unitary,* $B = UA$ *and there exists a convexotonic tuple* $\Xi \in M_g(\mathbb{C})^g$ *such that*

$$A_\ell(U-I)A_j = \sum_{s=1}^{g} (\Xi_j)_{\ell,s} A_s.$$

*Letting $p$ denote the associated convexotonic map, $R$ the tuple $(U-I)A = B-A$ and*

$$Q(x) = I - \Lambda_R(x),$$

(a) *we have*

$$\big(I + \Lambda_B(p(x))\big)Q(x) = I + \Lambda_A(x);$$

(b) *if $Z \in \mathrm{dom}(p)$, then*

$$\big(I + \Lambda_B(p(Z))\big)Q(Z) = I + \Lambda_A(Z), \tag{2.8}$$

*and*

$$Q(Z)^* L_B^{\mathrm{re}}(p(Z))Q(Z) = L_A^{\mathrm{re}}(Z); \tag{2.9}$$

(c) *if $Z \in M(\mathbb{C})^g$ and $Q(Z)$ is invertible, then $Z \in \mathrm{dom}(p)$ and equation (2.9) holds.*

**Proof.** Item (a) is straightforward, so we merely outline a proof. From Lemma 2.7, for words $\alpha$ and $1 \le j \le g$,

$$A_j R^\alpha = \sum_{s=1}^{g} (\Xi^\alpha)_{j,s} A_s.$$

Hence

$$B_j R^\alpha = \sum_{s=1}^{g} (\Xi^\alpha)_{j,s} B_s,$$

from which it follows that, letting $\{e_1, \dots, e_g\}$ denote the standard basis for $\mathbb{C}^g$,

$$\begin{aligned}
\Lambda_B(p(x)) &= \sum_s B_s p^s(x) = \sum_{s=1}^{g} \sum_{j=1}^{g} x_j \, [e_j^*(I - \Lambda_\Xi(x))^{-1} e_s] \\
&= \sum_{n=0}^{\infty} \sum_{j,s=1}^{g} x_j \, [e_j^* \Lambda_\Xi(x)^n e_s] = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} [\sum_{j,s=1}^{g} (\Xi^\alpha)_{j,s} B_s] x_j \alpha \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{g} B_j x_j \sum_{|\alpha|=n} R^\alpha \alpha \\
&= \sum_{j=1}^{g} B_j x_j \sum_{n=0}^{\infty} \Lambda_R(x)^n = \Lambda_B(x)(I - \Lambda_R(x))^{-1}.
\end{aligned}$$

In particular,

$$\begin{aligned}\big(I+\Lambda_B(p(x))\big)Q(x) &= \big(I+\Lambda_B(p(x))\big)(I-\Lambda_R(x))\\ &= I-\Lambda_R(x)+\Lambda_B(x)=I+\Lambda_A(x),\end{aligned}$$

since $R=B-A$. This computation also shows if both $\|\Lambda_\Xi(Z)\|<1$ and $\|\Lambda_R(Z)\|<1$, then equation (2.8) holds. Since both sides of equation (2.8) are rational functions, equation (2.8) holds whenever $Z\in\operatorname{dom}(p)$. Finally, using $\Lambda_B(p(x))Q(x)=\Lambda_B(x)$ as well as $R=B-A$ and $B=UA$,

$$\begin{aligned}Q(Z)^*L_B^{\mathrm{re}}(p(Z))Q(Z) &= Q^*(Z)Q(Z)+Q(Z)^*\Lambda_B(Z)+\Lambda_B(X)^*Q(Z)\\ &= I+\Lambda_A(Z)+\Lambda_A(Z)+\Lambda_B(Z)^*\Lambda_B(Z)-\Lambda_A(Z)^*\Lambda_A(Z)\\ &= L_A^{\mathrm{re}}(Z),\end{aligned}$$

a routine calculation shows that equation (2.8) implies equation (2.9).

Since $B\in M_t(\mathbb{C})^g$ is linearly independent, for each $1\le k\le g$ there exists a linear functional $\lambda_k : M_t(\mathbb{C})\to\mathbb{C}$ such that $\lambda_k(B_k)=1$ and $\lambda_k(B_j)=0$ if $j\neq k$. For each $k$, there is a matrix $\Psi_k\in M_t(\mathbb{C})$ such that $\lambda_k(T)=\operatorname{trace}(T\Psi_k)$. Writing $\Psi_k=\sum_s v_{k,s}u_{k,s}^*$ for vectors $u_{k,s},v_{k,s}\in\mathbb{C}^t$,

$$\lambda_k(T)=\sum_s u_{k,s}^*Tv_{k,s}.$$

Let

$$r^k(x)=\sum_{\ell,s}(u_{k,s}^*+u_{k,s}^*A_\ell x_\ell)(I-\Lambda_R(x))^{-1}v_{k,s}-\lambda_k(I).$$

Hence, for $X\in M_n(\mathbb{C})^g$ sufficiently close to 0, and with $W=Q^{-1}$ and $\Phi_k=\lambda_k\otimes I_n$,

$$\begin{aligned}p^k(X) &= \Phi_k\left(\Lambda_B(p(X))\right)=\Phi_k\left([I_t\otimes I_n+\Lambda_A(X)]\,W(X)-I_t\otimes I_n\right)\\ &= \sum_{\ell,s}[u_{k,s}^*\otimes I+(u_{k,s}^*A_j\otimes I_n)(I_t\otimes X_j)](I_t\otimes I_n-\Lambda_R(X))^{-1}[v_{k,s}\otimes I_n]\\ &\quad -\lambda_k(I)\otimes I_n=r^k(X).\end{aligned}$$

Thus, in the notation of equation (2.2), $\mathscr{I}_R\subseteq\operatorname{dom}(p)$; that is, if $Q(Z)=I-\Lambda_R(Z)$ is invertible, then $Z\in\operatorname{dom}(p)$, proving item (c). □

**Proof of Theorem 2.1.** That $\Xi$ is convexotonic follows from Lemma 2.7. Let $p$ denote the resulting convexotonic map. Let $R=\mathfrak{B}-\mathfrak{A}=(U-I)\mathfrak{A}$ and $Q(x)=I-\Lambda_R(x)$. From Proposition 2.8,

$$Q(X)^*L_{\mathfrak{B}}^{\mathrm{re}}(p(X))Q(X)=L_{\mathfrak{A}}^{\mathrm{re}}(X), \tag{2.10}$$

holds whenever $Q(X)$ is invertible.

Let $X \in \operatorname{int}(\mathcal{D}_{\mathfrak{A}}(n))$ be given. The function $F_X(z) = \Lambda_{\mathfrak{B}}(p((1-z)X))$ is a $M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$-valued rational function (of the single complex variable $z$ that is regular at $z = 1$). Suppose $\lim_{z\to 0} F_X(z)$ exists and let $T$ denote the limit. In that case,

$$\begin{aligned} Q(X)^*(I + T + T^*)Q(X) &= \lim_{z\to 0} Q((1-z)X)^*(I + F_X(z) + F_X(z)^*)Q((1-z)X) \\ &= L^{\mathrm{re}}_{\mathfrak{A}}(X) \succ 0 \end{aligned}$$

and therefore $Q(X)$ is invertible (and $I + T + T^* \succ 0$). Hence, if $\lim_{z\to 0} F_X(z)$ exists, then $Q(X)$ is invertible.

We now show the limit $\lim_{z\to 0} F_X(z)$ must exist, arguing by contradiction. Accordingly, suppose this limit fails to exist. Equivalently, $F_X(z)$ has a pole at 0. In this case there exists a $M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$ matrix-valued function $\Psi(z)$ analytic and never 0 in a neighborhood of 0 and a positive integer $m$ such that $F_X(z) = z^{-m}\Psi(z)$. Since $\Psi(0) \neq 0$, there is a vector $\gamma$ such that $\langle \Psi(0)\gamma, \gamma\rangle \neq 0$ (since the scalar field is $\mathbb{C}$). Choose a real number $\theta$ such that $\kappa := e^{-im\theta}\langle \Psi(0)\gamma, \gamma\rangle < 0$. Hence, for $t$ real and positive,

$$\begin{aligned} &\langle (F_X(te^{i\theta}) + F_X(te^{i\theta})^*)\gamma, \gamma\rangle \\ &\quad = t^{-m}\langle [e^{-im\theta}\Psi(te^{i\theta}) + e^{im\theta}\Psi(te^{i\theta})^*]\gamma, \gamma\rangle \\ &\quad = t^{-m}\big[2\langle e^{-im\theta}\Psi(0)\gamma, \gamma\rangle \\ &\qquad + \langle [e^{-im\theta}[\Psi(te^{-i\theta}) - \Psi(0)]\gamma, \gamma\rangle + e^{im\theta}\langle [\Psi(te^{-i\theta})^* - \Psi(0)^*]\gamma, \gamma\rangle\big] \\ &\quad \leq 2t^{-m}[\kappa + \delta_t], \end{aligned}$$

where $\delta_t$ tends to 0 as $t$ tends to 0. Hence, for $0 < t$ sufficiently small,

$$\langle L^{\mathrm{re}}_{\mathfrak{B}}(p((1 - te^{-im\theta})X))\gamma, \gamma\rangle = \langle (I + F_X(te^{i\theta}) + F_X(te^{i\theta})^*)\gamma, \gamma\rangle < 0,$$

contradicting the fact that $(1 - te^{-im\theta})X \in \operatorname{int}(\mathcal{D}_{\mathfrak{A}}) \cap \operatorname{dom}(p)$ for all $0 < t$ sufficiently small. At this point we have shown if $X \in \operatorname{int}(\mathcal{D}_{\mathfrak{A}})$, then $Q(X)$ is invertible and therefore, by Proposition 2.8, $X \in \operatorname{dom}(p)$. Further, if $X \in \operatorname{int}(\mathcal{D}_{\mathfrak{A}})$, then, by equation (2.10),

$$Q(X)^* L^{\mathrm{re}}_{\mathfrak{B}}(p(X))Q(X) = L^{\mathrm{re}}_{\mathfrak{A}}(X) \succ 0$$

and thus $L^{\mathrm{re}}_{\mathfrak{B}}(p(X)) \succ 0$; that is $p(X) \in \operatorname{int}(\mathcal{D}_{\mathfrak{B}})$. By symmetry, the same is true for $q$. Consequently, $p : \operatorname{int}(\mathcal{D}_{\mathfrak{A}}) \to \operatorname{int}(\mathcal{D}_{\mathfrak{B}})$ is bianalytic with inverse $q : \operatorname{int}(\mathcal{D}_{\mathfrak{B}}) \to \operatorname{int}(\mathcal{D}_{\mathfrak{A}})$, proving item (a).

If $X \in \operatorname{ext}(\mathcal{D}_{\mathfrak{A}}) \cap \operatorname{dom}(p)$, then $L^{\mathrm{re}}_{\mathfrak{B}}(p(X)) \not\succeq 0$ by Proposition 2.8(b) and equation (2.9), proving item (b).

Now suppose $\mathcal{D}_{\mathfrak{B}}(1)$ is bounded and $Z \in \partial\mathcal{D}_{\mathfrak{A}}(n)$. By [26, Proposition 2.4], $\mathcal{D}_{\mathfrak{B}}(n)$ is also bounded. For $0 < t < 1$, we have $tZ \in \operatorname{dom}(p)$ (by item (a)) and hence $\varphi$, defined on $(0, 1)$ by $\varphi_Z(t) := p(tZ)$, maps into $\operatorname{int}(\mathcal{D}_{\mathfrak{B}}(n))$ and is thus bounded. It follows that

$G_Z(t) = \Lambda_{\mathfrak{B}}(\varphi_Z(t))$ is also a bounded function on $(0,1)$. Arguing by contradiction, suppose $Q(Z) = I - \Lambda_R(Z)$ is not invertible. Thus there is a unit vector $\gamma$ such that $Q(zZ)\gamma = (1-z)\gamma$. For $0 < t < 1$, equation (2.10) gives,

$$(1-t)^2 \langle L^{\mathrm{re}}_{\mathfrak{B}}(\varphi_Z(t))\gamma, \gamma\rangle = 1 - t[-\langle[\Lambda_{\mathfrak{A}}(Z) + \Lambda_{\mathfrak{A}}(Z)^*]\gamma, \gamma\rangle].$$

Since the left hand side converges to 0 as $t$ approaches 1 from below, the right hand equals $1-t$. Hence

$$(1-t)\langle L^{\mathrm{re}}_{\mathfrak{B}}(\varphi_Z(t))\gamma, \gamma\rangle = 1,$$

and we have arrived at a contradiction, as the left hand side converges to 0 as $t$ tends to 1 from below. Hence $Q(Z)$ is invertible. By Proposition 2.8 (c), if $\mathcal{D}_{\mathfrak{B}}$ is bounded, then $\mathcal{D}_{\mathfrak{A}} \subseteq \mathrm{dom}(p)$, proving item (d).

Suppose $X \in \mathrm{dom}(p) \cap \partial\mathcal{D}_{\mathfrak{A}}$. Since $\mathrm{dom}(p)$ is open, $tX \in \mathrm{dom}(p)$ for $t \in \mathbb{R}$ sufficiently close to 1. Further $p(tX) \in \mathrm{int}(\mathcal{D}_A)$ for $t < 1$ and $p(tX) \in \mathrm{ext}(\mathcal{D}_{\mathfrak{B}})$ for $t > 1$. By continuity, $p(X) \in \partial\mathcal{D}_{\mathfrak{B}}$, proving item (c). □

## 3. Minimality and indecomposability

A monic pencil $L_A = L_A(x,y)$ of size $e$ is **indecomposable** if its coefficients $\{A_1, \dots, A_g, A_1^*, \dots, A_g^*\}$ generate $M_e(\mathbb{C})$ as a $\mathbb{C}$-algebra.[7] A collection of sets $\{S_1, \dots, S_k\}$ is **irredundant** if $\bigcap_{j\neq\ell} S_j \nsubseteq S_\ell$ for all $\ell$. A collection $\{L_{A^1}, \dots, L_{A^k}\}$ of monic pencils is **irredundant** if $\{\mathcal{D}_{A^j} : 1 \le j \le k\}$ is irredundant.

**Lemma 3.1.** *Given $B \in M_r(\mathbb{C})^g$, there exists a reducing subspace $\mathscr{M}$ for $\{B_1, \dots, B_g\}$ such that, with $A = B|_{\mathscr{M}}$, the monic pencil $L_A$ is minimal for $\mathcal{D}_B = \mathcal{D}_A$.*

*If $L_A$ and $L_B$ are both minimal and $\mathcal{D}_A = \mathcal{D}_B$, then $A$ and $B$ are unitarily equivalent. In particular $A$ and $B$ have the same size.*

*Given a monic pencil $L_A(x,y) = I + \sum A_j x_j + \sum A_j^* y$, there is a $k$ and indecomposable monic pencils $L_{A^j}$ such that*

$$L_A = \bigoplus_{j=1}^{k} L_{A^j} = L_{\bigoplus_{j=1}^k A^j},$$

*where the direct sum is in the sense of an orthogonal direct sum decomposition of the space that $A$ acts upon. Moreover, $L_A$ is minimal if and only if $\{L_{A^j} : 1 \le j \le \ell\}$ is irredundant.*

**Proof.** Zalar [57] (see also [26]) establishes this result over the reals, but the proofs work (and are easier) over $\mathbb{C}$; it can also be deduced from the results in [41] and [30]. □

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[7] Previously, in [41] such pencils were called irreducible.

Note if $E$ is ball-minimal then $\ker(E) = \{0\}$ and $\ker(E^*) = \{0\}$, an observation that will be used repeatedly in the sequel.

**Lemma 3.2.** *Let $E$ be a $g$-tuple of $d \times e$ matrices and assume* $\ker(E^*) = \{0\}$ *and* $\ker(E) = \{0\}$.

(1) *We have*

$$\begin{pmatrix} I & 0 \\ \Lambda_{E^*} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_E \end{pmatrix} \begin{pmatrix} I & \Lambda_E \\ 0 & I \end{pmatrix} = \mathbb{L}_E. \tag{3.1}$$

(2) *The monic pencil $\mathbb{L}_E$ is indecomposable if and only if $Q_E$ is an atom.*
(3) *$E$ is ball-minimal if and only if $\mathbb{L}_E^{\mathrm{re}}$ is minimal.*
(4) *If $A \in M_N(\mathbb{C})^g$ and $A_m A_j = 0$ for all $1 \le j, m \le g$ then,* $\dim \operatorname{rg} A + \dim \operatorname{rg} A^* \le N$ *and for any $s \ge$* $\dim \operatorname{rg} A$ *and $t \ge$* $\dim \operatorname{rg} A^*$ *with $s + t = N$, there exists a tuple $F \in M_{s \times t}(\mathbb{C})^g$ such that $A$ is unitarily equivalent to*

$$\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

(5) *If $L_A$ is minimal and $\mathcal{D}_A$ is a spectraball, then there exist ball-minimal tuples $F^1, \dots, F^k$ such that each $\mathbb{L}_{F^j}$ is an indecomposable monic pencil, $\{\mathcal{B}_{F^1}, \dots, \mathcal{B}_{F^k}\}$ is irredundant and $L_A$ is unitarily equivalent to $\mathbb{L}_{F^1} \oplus \cdots \oplus \mathbb{L}_{F^k}$.*
(6) *If $A$ is ball-minimal, then $L_A$ is minimal.*
(7) *If $E$ is ball-minimal, then, up to unitary equivalence, $Q_E = Q_{E^1} \oplus \cdots \oplus Q_{E^k}$, where the $Q_{E^j} \in \mathbb{C}\!<\!x, y\!>^{e_j \times e_j}$ are atoms,* $\ker(E^j) = \{0\}$ *for all $j$, and the spectraballs $\mathcal{B}_{E^j}$ are irredundant.*
(8) *If $Q_E$ is an atom, then $E$ is ball-minimal.*
(9) *If $E$ ball-minimal, $F \in M_{k \times \ell}(\mathbb{C})^g$ and $\mathcal{B}_E = \mathcal{B}_F$, then there is a tuple $R \in M_{(k-d)\times(\ell-e)}(\mathbb{C})^g$ and unitaries $U, V$ of sizes $k \times k$ and $\ell \times \ell$ respectively such that $\mathcal{B}_E \subseteq \mathcal{B}_R$ and*

$$F = U \begin{pmatrix} E & 0 \\ 0 & R \end{pmatrix} V. \tag{3.2}$$

*In particular,*

(a) *$d \le k$ and $e \le \ell$;*
(b) *if $F \in M_{d \times e}(\mathbb{C})^g$ is ball-minimal too, then $E$ and $F$ are ball-equivalent.*

Item (9) can be interpreted in terms of completely contractive maps and as special cases of the rectangular operator spaces of [21]. Indeed, letting $\mathscr{E}$ and $\mathscr{F}$ denote the spans of $\{E_1, \dots, E_g\}$ and $\{F_1, \dots, F_g\}$ respectively, the inclusion $\mathcal{B}_E \subseteq \mathcal{B}_F$ is equivalent to the mapping $\Phi : \mathscr{E} \to \mathscr{F}$ defined by $\Phi(E_j) = F_j$ being completely contractive. Hence $\mathcal{B}_E = \mathcal{B}_F$ if and only if $\Phi$ is completely isometric.

**Proof.** (1) Straightforward.

(2) By (3.1), $Q_E$ and $\mathbb{L}_E$ are stably associated, cf. [30, Section 4]. Hence $\mathbb{L}_E$ does not factor in $\mathbb{C}<x,y>^{(d+e)\times(d+e)}$ if and only if $Q_E$ does not factor in $\mathbb{C}<x,y>^{e\times e}$ by [30, Section 4]. Next, $\mathbb{L}_E$ is indecomposable if and only if it does not factor and

$$\ker(\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}) \cap \ker(\begin{pmatrix} 0 & 0 \\ E^* & 0 \end{pmatrix}) = \{0\}$$

([30, Section 2.1 and Theorem 3.4]). Thus $\mathbb{L}_E$ is indecomposable if and only if $Q_E$ does not factor.

(3) Let $L_B$ be minimal for $\mathcal{D}_B = \mathcal{B}_E$ and let $N$ denote the size of $B$. By [18, Theorem 1.1(2)] there exists positive integers $s,t$ such that $s+t=N$ and a tuple $F \in M_{s\times t}(\mathbb{C})^g$ such that

$$B = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{B}_E = \mathcal{B}_F$. On the other hand, with

$$A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix},$$

$\mathcal{D}_A = \mathcal{B}_E$ too. By minimality of $B$, $s+t \le d+e$. If $E$ is ball-minimal, then, since $\mathcal{B}_E = \mathcal{B}_F$, we have $s+t \ge d+e$ and hence $L_A^{\mathrm{re}} = \mathbb{L}_E^{\mathrm{re}}$ is minimal. On the other hand, if $\mathbb{L}_E^{\mathrm{re}}$ is minimal, then $\mathbb{L}_E^{\mathrm{re}}$ and $L_B$ have the same size, $N = s+t = d+e$ and thus $E$ is ball-minimal.

(4) Let $\mathscr{R} = \operatorname{rg} A$ and $\mathscr{R}_* = \operatorname{rg} A^*$. Since $A_m A_j = 0$ it follows that $\mathscr{R}$ and $\mathscr{R}_*$ are orthogonal and also that $A_m \mathscr{R} = 0$ and $A_m^* \mathscr{R}_* = 0$ for $1 \le m \le g$. In particular, $\dim \mathscr{R} + \dim \mathscr{R}_* \le N$. Letting $V$ and $V_*$ denote the inclusions of $\mathscr{R}$ and $\mathscr{R}_*$ into $\mathbb{C}^N$ respectively,

$$A = \begin{pmatrix} 0 & 0 & V^* A V_* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

with respect to the decomposition $\mathbb{C}^N = (\mathscr{R} \oplus \mathscr{R}_*)^\perp \oplus \mathscr{R} \oplus \mathscr{R}_*$. Now any choice of $s \ge \dim \mathscr{R}$ and $t \ge \dim \mathscr{R}_*$ with $s+t=N$ applied to (3.3) gives the desired decomposition.

(5) Since $L_A$ is minimal, by Lemma 3.1, $L_A$ is unitarily equivalent to $L_{A^1} \oplus \cdots \oplus L_{A^k}$ for some indecomposable irredundant monic pencils $L_{A^1}, \dots, L_{A^k}$. Let $N_j$ denote the size of $A^j$. Now suppose $\mathcal{D}_A$ is a spectraball. Thus, there exists $m, \ell$ and a ball-minimal tuple $G \in M_{m\times\ell}(\mathbb{C})^g$ such that $\mathcal{D}_A = \mathcal{B}_G$. By item (3) $\mathbb{L}_G^{\mathrm{re}}$ is minimal for $\mathcal{D}_A$. Thus

$$B := \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in M_{m+\ell}(\mathbb{C})^g$$

is unitarily equivalent to $A^1 \oplus \cdots \oplus A^k$ by Lemma 3.1. Since $B_m B_j = 0$ for $1 \leq j, m \leq g$, it follows that $A^\ell_m A^\ell_j = 0$ for all $j, m, \ell$. By item (4), there exists $s_j, t_j$ such that $s_j + t_j = N_j$ and tuples $F^j \in M_{s_j \times t_j}(\mathbb{C})^g$ such that, up to unitary equivalence,

$$A^j = \begin{pmatrix} 0 & F^j \\ 0 & 0 \end{pmatrix} \in M_{N_j}(\mathbb{C})^g.$$

Moreover, since $L_A$ is minimal and $\mathcal{D}_A = \cap_{j=1}^k \mathcal{B}_{F^j}$, each $F^j$ is ball-minimal.

(6) Given a tuple $A \in M_d(\mathbb{C})^g$, observe that $X \in \mathcal{B}_A$ if and only if $S \otimes X \in \mathcal{D}_A$, where

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, if $B \in M_r(\mathbb{C})^d$ and $\mathcal{D}_B = \mathcal{D}_A$, then $\mathcal{B}_B = \mathcal{B}_A$ and by ball-minimality, $r \geq d$. Hence $L_A$ is minimal.

(7) Combine items (3), (5) and (2) in that order.

(8) By item (2), $\mathbb{L}_E$ is indecomposable. For a pencil $L$, indecomposability of $L$ implies minimality of $L^{\mathrm{re}}$ by Lemma 3.1. Thus $\mathbb{L}_E^{\mathrm{re}}$ is minimal and hence $E$ is ball-minimal by item (3).

(9) Let

$$A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \in M_{d+e}(\mathbb{C})^g.$$

By item (3), $L_A^{\mathrm{re}} = \mathbb{L}_E^{\mathrm{re}}$ is minimal. Since $\mathbb{L}_F^{\mathrm{re}}$ defines $\mathcal{B}_E$, there is a reducing subspace $\mathscr{M}$ for

$$B = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \in M_{k+\ell}(\mathbb{C})^g$$

such that the restriction of $B$ to $\mathscr{M}$ is unitarily equivalent to $A$ by Lemma 3.1. Thus, there is unitary $Z \in M_{k+\ell}(\mathbb{C})$ and a tuple $C \in M_{(k+\ell)-(d+e)}(\mathbb{C})^g$ such that, with respect to the decomposition $\mathscr{M} \oplus \mathscr{M}^\perp$,

$$B = Z^* \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} Z.$$

Since $B_m B_j = 0$ for all $j, m$, we have $C_m C_j = 0$ too. Further, using ball-minimality of $E$, $\ell \geq \operatorname{rk} F^* F = \operatorname{rk} E^* E + \operatorname{rk} C^* C = e + \operatorname{rk} C^* C$. Thus $\dim \operatorname{rg} C \leq \ell - e$. Likewise, $\dim \operatorname{rg} C^* \leq k - d$. By item (4), there exists a tuple $R \in M_{(k-d)\times(\ell-e)}(\mathbb{C})^g$ such that, up to unitary equivalence,

$$C = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}.$$

Thus, letting $G = \left(\begin{smallmatrix} E & 0 \\ 0 & R \end{smallmatrix}\right) \in M_{k\times\ell}(\mathbb{C})^g$,

$$\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} X = X \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}$$

for some unitary matrix $X$. Writing $X = (X_{j,k})_{j,k=1}^2$ with respect to the decomposition $\mathbb{C}^k \oplus \mathbb{C}^\ell$, it follows that

$$X_{11}G = FX_{22}, \quad X_{21}G = 0, \quad FX_{21} = 0.$$

Hence $FX_{22}X_{22}^* = F$ and $X_{11}^*X_{11}G = G$. Thus $X_{11}$ is isometric on $\operatorname{rg} G$ and therefore $X_{11}$ extends to a unitary mapping $U$ on all of $\mathbb{C}^k$ such that $UG = X_{11}G$. Similarly, $X_{22}^*$ is isometric on $\operatorname{rg} F^*$ and hence $X_{22}^*$ extends to a unitary $V$ on all of $\mathbb{C}^\ell$ such that $VF^* = X_{21}^*F^*$. Finally, $UG = X_{11}G = FX_{22} = FV^*$. Hence equation (3.2) holds, which implies $\mathcal{B}_E = \mathcal{B}_F = \mathcal{B}_E \cap \mathcal{B}_R$. Thus $\mathcal{B}_E \subseteq \mathcal{B}_R$ and the remainder of item (9) follows. □

Minimality and indecomposability of monic pencils are preserved under an affine linear change of variables.

**Proposition 3.3.** *Consider a hermitian monic pencil $L_A^{\mathrm{re}}$ and an affine linear change of variables $\lambda : x \mapsto xM + b$ for some invertible $g \times g$ matrix $M$ and vector $b \in \mathbb{C}^g$. If $L_A^{\mathrm{re}}(b) \succ 0$, then $\lambda^{-1}(\mathcal{D}_A) = \mathcal{D}_F$, where*

$$F = M \cdot (\mathfrak{H}A\mathfrak{H}) \quad \text{and} \quad \mathfrak{H} = L_A^{\mathrm{re}}(b)^{-1/2}. \tag{3.4}$$

*Further,*

1. *$L_A$ is indecomposable if and only if $L_F$ indecomposable;*
2. *$L_A$ is minimal if and only if $L_F$ is minimal.*

**Proof.** Equation (3.4) is proved in [3, §8.2].

Turning to item (1), let us first settle the special case $M = I$. If $L_A$ is not indecomposable, then there is a common non-trivial reducing subspace $\mathscr{M}$ for $A$. It follows that $\mathscr{M}$ is reducing for $L_A^{\mathrm{re}}(b)$ and hence for $F = \mathfrak{H}A\mathfrak{H}$.

Now suppose $L_F$ is not indecomposable; that is, there is a non-trivial reducing subspace $\mathscr{N}$ for $F = \mathfrak{H}A\mathfrak{H}$. Since

$$\mathfrak{H}(L_A^{\mathrm{re}}(b) - I)\mathfrak{H} = \mathfrak{H}(\Lambda_A(b) + \Lambda_A(b)^*)\mathfrak{H} = \Lambda_F(b) + \Lambda_F(b)^*,$$

we conclude that

$$\big(I - L_A^{\mathrm{re}}(b)^{-1}\big)\mathscr{N} = \mathfrak{H}(L_A^{\mathrm{re}}(b) - I)\mathfrak{H}\mathscr{N} \subseteq \mathscr{N}.$$

Hence $\mathscr{N}$ is invariant for $L_A^{\mathrm{re}}(b)^{-1}$. Since $\mathscr{N}$ is finite dimensional and $L_A^{\mathrm{re}}(b)^{-1}$ is invertible, $L_A^{\mathrm{re}}(b)^{-1}\mathscr{N} = \mathscr{N}$ and consequently $\mathfrak{H}\mathscr{N} = \mathscr{N}$. Because $F = \mathfrak{H}A\mathfrak{H}$ it is now evident that $\mathscr{N}$ is reducing for $A$.

Now consider the special case $b = 0$. A subspace $\mathscr{M}$ reduces $A$ if and only if it reduces $M \cdot A$. Combining these two special cases proves item (1).

Finally we prove item (2). By Lemma 3.1, $L_A$ is unitarily equivalent to $\bigoplus_{j=1}^{\ell} L_{A^j}$, where the $L_{A^j}$ are indecomposable monic pencils. Now $L_F$ is unitarily equivalent to $\bigoplus_{j=1}^{\ell} L_{F^j}$, where $F^j = M \cdot (\mathfrak{H}A^j\mathfrak{H})$. By item (1), each of these summands $L_{F^j}$ is indecomposable. Furthermore, since $\Psi$ is bijective it is clear that $\bigcap_{k\neq i} \mathcal{D}_{A^k} \subseteq \mathcal{D}_{A^i}$ if and only if $\bigcap_{k\neq j} \mathcal{D}_{F^k} \subseteq \mathcal{D}_{F^j}$. Therefore $\{L_{A^j} : 1 \le j \le \ell\}$ is irredundant if and only if $\{L_{F^j} : 1 \le j \le \ell\}$ is irredundant. Hence $L_A$ is minimal for $\mathcal{D}_A$ if and only if $L_F$ is minimal for $\mathcal{D}_F$, again by Lemma 3.1. □

**Example 3.4.** Even with $M = I$, the property (1) of Proposition 3.3 fails for a general positive definite $\mathfrak{H}$ and $F$ as in (3.4). For example, let

$$A = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad \mathfrak{H} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}^{-1}.$$

Then $L_A$ is indecomposable, but since

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the monic pencil $L_F$ is clearly not. □

**Remark 3.5.** Suppose $E \in M_{d\times e}(\mathbb{C})^g$ and $C \in M_g(\mathbb{C})$ is invertible. If $E$ is ball-minimal, then $C \cdot E$ (see equation (1.4)) is ball-minimal. □

## 4. Characterizing bianalytic maps between spectrahedra

In this section we prove Theorem 1.5 and Proposition 1.6, stated as Propositions 4.2 and 4.4 below. A major accomplishment, exposited in Subsection 4.3, is the reduction of the eig-generic type hypotheses of [3] to various natural and cleaner algebraic conditions on the corresponding pencils defining spectrahedra.

**Lemma 4.1.** *Let $L_A$ be a monic pencil. The set $\{(X, X^*) : X \in \mathcal{Z}_{L_A}^{\mathrm{re}}(n)\}$ is Zariski dense in the set $\mathcal{Z}_{L_A}(n)$ for every $n$. Likewise, $\{(X, X^*) : X \in \mathcal{Z}_{Q_A}^{\mathrm{re}}(n)\}$ is Zariski dense in $\mathcal{Z}_{Q_A}(n) = \{(X, Y) \in M_n(\mathbb{C})^{2g} : \det Q_A(X, Y) = 0\}$.*

**Proof.** The first statement holds by [41, Proposition 5.2]. The second follows immediately from the first. □

### 4.1. The detailed boundary

Let $\rho$ be a **hermitian** $d \times d$ **free matrix polynomial** with $\rho(0) = I_d$. Thus $\rho \in \mathbb{C} <x, y>^{d\times d}$ and $\rho(X, X^*)^* = \rho(X, X^*)$ for all $X \in M(\mathbb{C})^g$. The **detailed boundary** of $\mathcal{D}_\rho$ is the sequence of sets

$$\widehat{\partial \mathcal{D}_\rho}(n) := \big\{(X, v) \in M_n(\mathbb{C})^g \times (\mathbb{C}^{dn} \setminus \{0\}) : X \in \partial \mathcal{D}_\rho,\ \rho(X, X^*)v = 0\big\}$$

over $n \in \mathbb{N}$. The nomenclature and notation are somewhat misleading in that $\widehat{\partial D_\rho}$ is not determined by the set $\mathcal{D}_\rho$ but by its defining polynomial $\rho$. Denote also

$$\widehat{\partial^1 \mathcal{D}_\rho}(n) := \Big\{(X, v) \in \widehat{\partial \mathcal{D}_\rho}(n) : \dim \ker(\rho(X, X^*)) = 1\Big\}.$$

For $(X, v) \in \widehat{\partial^1 \mathcal{D}_\rho(n)}$, we call $v$ the **hair** at $X$. Letting

$$\pi_1 : M_n(\mathbb{C})^g \times \mathbb{C}^{dn} \to M_n(\mathbb{C})^g \quad \text{and} \quad \pi_2 : M_n(\mathbb{C})^g \times \mathbb{C}^{dn} \to \mathbb{C}^{dn}$$

denote the canonical projections, set

$$\partial^1 \mathcal{D}_\rho(n) = \pi_1 \left(\widehat{\partial^1 \mathcal{D}_\rho}(n)\right), \qquad \operatorname{hair} \mathcal{D}_\rho(n) = \pi_2 \left(\widehat{\partial^1 \mathcal{D}_\rho}(n)\right).$$

Observe $\widehat{\partial \mathcal{B}_E}(n) := \widehat{\partial \mathcal{D}_{Q_E}}(n)$, etc.

#### 4.1.1. Boundary hair spans

In this subsection we connect the notion of boundary hair to ball-minimality. Given a tuple $E \in M_{d\times e}(\mathbb{C})^g$, a subset $\mathscr{S} \subseteq \widehat{\partial^1 \mathcal{B}_E}$ is closed under unitary similarity if for each $n$, each $(X, v) \in \widehat{\partial^1 \mathcal{B}_E}(n)$ and each $n \times n$ unitary $U$, we have $(UXU^*, (I_e \otimes U)v) \in \mathscr{S}(n)$. Assuming $\mathscr{S} \subseteq \widehat{\partial^1 \mathcal{B}_E}$ is closed under unitary similarity, let

$$\pi(\operatorname{hair} \mathscr{S}) = \Big\{u \in \mathbb{C}^e : \exists n \in \mathbb{N},\ \exists v \in \mathscr{S}(n) \cap \operatorname{hair} \mathcal{B}_E(n) :\ v = u \otimes e_1 + \sum_{j=2}^{n} u_j \otimes e_j\Big\},$$

where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Because $\mathscr{S}$ is invariant under unitary similarity, the definition of $\pi(\operatorname{hair} \mathscr{S})$ does not actually depend on the choice of orthonormal basis for $\mathbb{C}^n$. Thus, for instance, $\pi(\operatorname{hair} \partial^1 \mathcal{B}_E)$ is the set of those vectors $u \in \mathbb{C}^e$ such that there exists an $n$, a pair $(X, v) \in M_n(\mathbb{C})^g \oplus [\mathbb{C}^e \otimes \mathbb{C}^n]$ and a unit vector $h \in \mathbb{C}^n$ such that $Q_E^{\mathrm{re}}(X) \succeq 0$, $\dim \ker(Q_E^{\mathrm{re}}(X)) = 1$, $Q_E^{\mathrm{re}}(X)v = 0$ and $u = (I_e \otimes h^*)v$. For notational convenience we write $\pi(\operatorname{hair} \mathcal{B}_E)$ as shorthand for $\pi(\operatorname{hair} \widehat{\partial^1 \mathcal{B}_E})$.

**Proposition 4.2.** *A tuple $E \in M_{d\times e}(\mathbb{C})^g$ is ball-minimal if and only if $\pi(\operatorname{hair}\mathcal{B}_E)$ spans $\mathbb{C}^e$ and $\ker(E^*) = \{0\}$. Moreover, if $\pi(\operatorname{hair}\mathcal{B}_E)$ spans $\mathbb{C}^e$, then there exists a positive integer $r$*[8] *and pairs $(\alpha^a, \gamma^a) \in \widehat{\partial^1\mathcal{B}_E(r)}$ for $1 \le a \le e$ such that, writing $\gamma^a = \sum_{t=1}^r \delta_t^a \otimes e_t \in \mathbb{C}^e \otimes \mathbb{C}^r$ the set $\{\delta_1^a : 1 \le a \le e\}$ spans $\mathbb{C}^e$.*

**Proof.** Suppose $E$ is ball-minimal and let $e' \le e$ denote the dimension of the span of $\pi(\operatorname{hair}\mathcal{B}_E)$. Let

$$\mathcal{T}_E = \{(X, X^*)\colon X \in \partial^1\mathcal{D}_{Q_E} = \partial^1\mathcal{B}_E\}.$$

Let $W$ denote the inclusion of $\operatorname{span}\pi(\operatorname{hair}\mathcal{B}_E)$ into $\mathbb{C}^e$. Observe that

$$W^*Q_E(x,y)W = W^*W - W^*\Lambda_{E^*}(y)\Lambda_E(x)W = Q_{EW}(x,y).$$

Thus $\mathcal{B}_E \subseteq \mathcal{B}_{EW}$ and moreover $(X, v) \in \widehat{\partial^1\mathcal{D}_{Q_E} = \partial^1\mathcal{B}_E}$ implies

$$Q_{EW}^{\mathrm{re}}(X)(W^* \otimes I)v = (W^* \otimes I)Q_E^{\mathrm{re}}(X)v = 0,$$

so $\mathcal{T}_E \subseteq \mathcal{Z}_{\mathbb{L}_{EW}}$. Since $\partial^1\mathcal{D}_{\mathbb{L}_E} = \partial^1\mathcal{D}_{Q_E} = \partial^1\mathcal{B}_E$ by equation (3.1), $\mathbb{L}_E^{\mathrm{re}}$ (equivalently $\mathbb{L}_E$) is minimal by Lemma 3.2(3), and $\mathcal{T}_E$ is Zariski dense in $\mathcal{Z}_{\mathbb{L}_E}$ by [30, Corollary 8.5], it follows that $\mathcal{Z}_{\mathbb{L}_E} \subseteq \mathcal{Z}_{\mathbb{L}_{EW}}$. Since are convex sets containing 0 in their interiors, and their boundaries are contained in $\mathcal{Z}_{\mathbb{L}_E}$ and $\mathcal{Z}_{\mathbb{L}_{EW}}$ respectively, the inclusion $\mathcal{Z}_{\mathbb{L}_E} \subseteq \mathcal{Z}_{\mathbb{L}_{EW}}$ implies $\mathcal{B}_{EW} \subseteq \mathcal{B}_E$. Indeed, if $X \in \mathcal{B}_{EW}$ but $X \notin \mathcal{B}_E$, then there is a $0 < t < 1$ such that $tX \in \partial\mathcal{B}_E \cap \mathcal{B}_{EW}$. Thus $(tX, tX^*) \in \mathcal{Z}_{\mathbb{L}_E} \subseteq \mathcal{Z}_{\mathbb{L}_{EW}}$. Consequently $Q_{EW}^{\mathrm{re}}(tX)$ has a kernel and finally $Q_E^{\mathrm{re}}(X) \not\succeq 0$, contradicting $X \in \mathcal{B}_{EW}$. Hence $E$ and $EW$ define the same spectraball. Since $EW$ is a $d \times e'$-tuple and $E$ is ball-minimal and $d \times e$, Lemma 3.2(9) implies $e' \ge e$. Thus $e' = e$ and $\pi(\operatorname{hair}\mathcal{B}_E)$ spans $\mathbb{C}^e$. If $\ker(E^*) \ne \{0\}$ then $E$ is not ball-minimal. Hence we have shown, if $E$ is ball-minimal, then $\pi(\operatorname{hair}\mathcal{B}_E)$ spans and $\ker(E^*) = \{0\}$.

To prove the converse, suppose $F \in M_{k\times\ell}(\mathbb{C})^g$ is not ball-minimal, but $\ker(F^*) = \{0\}$. Let $\mathscr{H}_F \subseteq \mathbb{C}^\ell$ denote the span of $\pi(\operatorname{hair}\mathcal{B}_F)$. It suffices to show $\mathscr{H}_F \ne \mathbb{C}^\ell$. Let $E \in M_{d\times e}(\mathbb{C})^g$ be ball-minimal with $\mathcal{B}_F = \mathcal{B}_E$. By Lemma 3.2(9), $d \le k$ and $e \le \ell$ and, letting $d' = k - d$ and $e' = \ell - e$, there is a tuple $R \in M_{d'\times e'}(\mathbb{C})^g$ and $k \times k$ and $\ell \times \ell$ unitary matrices $U$ and $V$ respectively so that equation (3.2) holds and $\mathcal{B}_E \subseteq \mathcal{B}_R$. Note that $e' \ne 0$ since $\ker(F^*) = \{0\}$ and further

$$Q_F = V^*\begin{pmatrix} Q_E & 0 \\ 0 & Q_R \end{pmatrix} V = V^*(Q_E \oplus Q_R)V.$$

Without loss of generality, we may assume $V = I$.

---

[8] While it is not needed here, $r$ can be chosen at most $e$.

Suppose $X \in \partial^1\mathcal{B}_F(n)$ and $0 \neq v \in \mathbb{C}^\ell \otimes \mathbb{C}^n$ is in the kernel of $Q_F^{\mathrm{re}}(X)$. With respect to the decomposition of $\mathbb{C}^\ell \otimes \mathbb{C}^n = [\mathbb{C}^e \otimes \mathbb{C}^n] \oplus [\mathbb{C}^{e'} \otimes \mathbb{C}^n]$, decompose $v = u \oplus u'$. It follows that $0 = Q_F^{\mathrm{re}}(X)v = Q_E^{\mathrm{re}}(X)u \oplus Q_R^{\mathrm{re}}(X)u'$ and hence both $Q_E^{\mathrm{re}}(X)u = 0$ and $Q_R^{\mathrm{re}}(X)u' = 0$. Therefore, $\left(\begin{smallmatrix} 0 \\ u' \end{smallmatrix}\right)$ is in the kernel of $Q_F^{\mathrm{re}}(X)$. On the other hand, $X \in \partial\mathcal{B}_E(n)$. Hence there is a $0 \neq w \in \mathbb{C}^e \otimes \mathbb{C}^n$ such that $Q^{\mathrm{re}}(X)w = 0$. Thus $0 \neq \left(\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right)$ is in the kernel of $Q_F^{\mathrm{re}}(X)$. Since the dimension of the kernel of $Q_F^{\mathrm{re}}(X)$ is one, $u' = 0$ and therefore $\mathscr{H}_F \subseteq \mathbb{C}^e \oplus \{0\} \subsetneq \mathbb{C}^e \oplus \mathbb{C}^{e'} = \mathbb{C}^\ell$.

To prove the moreover portion of the proposition, note that the assumption that the $\pi(\mathrm{hair}\,\mathcal{B}_E)$ spans implies the existence of $n_1, \dots, n_e \in \mathbb{N}$ and pairs $(\alpha^a, \gamma^a) \in M_{n_a}(\mathbb{C})^g \times [\mathbb{C}^e \otimes \mathbb{C}^{n_a}]$ such that, writing $\gamma^a = \sum_{t=1}^{n_a} \delta_t^a \otimes e_t$, the set $\{\delta_1^a : 1 \leq a \leq e\}$ spans $\mathbb{C}^e$. By choosing $r = \max\{n_a : 1 \leq a \leq e\}$ and padding $\delta^a$ and $\gamma^a$ by zeros as needed, it can be assumed that $n_a = r$ for all $a$. $\square$

## *4.2. From basis to hyperbasis*

Call an $e+1$-element subset $\mathcal{U} = \{u^1, \dots, u^{e+1}\}$ of $\mathbb{C}^e$ a **hyperbasis** if each $e$-element subset of $\mathcal{U}$ is a basis. This notion critically enters the genericity conditions considered in [3].

**Lemma 4.3.** *Given $E \in M_{d\times e}(\mathbb{C})^g$ and $n \in \mathbb{N}$, if $\mathcal{Z}_{Q_E}(n)$ is an irreducible hypersurface in $M_n(\mathbb{C})^{2g}$,*

$$\{(X, X^*) \colon X \in \partial^1\mathcal{B}_E(n)\}$$

*is Zariski dense in $\mathcal{Z}_{Q_E}(n)$, and $\pi(\mathrm{hair}\,\mathcal{B}_E)$ spans $\mathbb{C}^e$, then $\pi(\mathrm{hair}\,\mathcal{B}_E)$ contains a hyperbasis for $\mathbb{C}^e$.*

**Proof.** By Proposition 4.2 there exist a positive integer $r$, tuples $X^1, \dots, X^e \in \partial^1\mathcal{B}_E(r)$ and vectors $\gamma^j = \sum_{t=1}^r \delta_t^j \otimes e_t \in \ker(Q_E^{\mathrm{re}}(X^j)) \subseteq \mathbb{C}^e \otimes \mathbb{C}^r$, such that $\{\delta_1^j : 1 \leq j \leq e\}$ spans $\mathbb{C}^e$. Note too that $\delta_1^j = (I \otimes \varrho_1^*)\delta^j$, where $\{\varrho_1, \dots \varrho_r\}$ is the standard orthonormal basis for $\mathbb{C}^r$.

If $X \in \partial^1\mathcal{B}_E(n)$, then the adjugate matrix, $\mathrm{adj}(Q_E^{\mathrm{re}}(X))$, is of rank one and its range is $\ker(Q_E^{\mathrm{re}}(X))$. Let $M_{(i)}$ denote the $i$-th column of a matrix $M$ and suppose $\gamma = \sum_{t=1}^r \delta_t \otimes e_t$ spans $\ker(Q_E^{\mathrm{re}}(X))$. It follows that $(I \otimes \varrho_1^*)\,\mathrm{adj}(Q_E^{\mathrm{re}}(X^k))_{(i)} = \mu\delta_1$ for some $\mu \in \mathbb{C}$. Moreover, for every $k = 1, \dots, e$ there exists $1 \leq i_k \leq er$ such that $\ker(Q_E^{\mathrm{re}}(X^k)) = \mathrm{span}(\mathrm{adj}\,Q_E^{\mathrm{re}}(X^k))_{(i_k)}$, and hence $(I \otimes e_1^*)\,\mathrm{adj}(Q_E^{\mathrm{re}}(X^k))_{(i_k)} = \mu_k\delta_1^k$ for some $\mu_k \neq 0$. Now consider

$$v(t, X, Y) := \sum_{k=1}^{e} t_k\,(I \otimes \varrho_1^*)\,\mathrm{adj}(Q_E(X, Y))_{(i_k)} \in \mathbb{C}^e \tag{4.1}$$

as a vector of polynomials in indeterminates $t = (t_1, \dots, t_e)$ and entries of $(X, Y)$ (i.e., coordinates of $M_r(\mathbb{C})^{2g}$). Let $\{\varepsilon_1, \dots, \varepsilon_e\}$ denote the standard basis for $\mathbb{C}^e$. For every $k$

we have $v(\varepsilon_k, X^k, X^{k*}) = (I \otimes \varrho_1^*)\operatorname{adj}(Q^{\mathrm{re}}(X^k))_{(i_k)} = \mu_k \delta_1^k \neq 0$. Since the complements of zero sets are Zariski open and dense in the affine space, for each $k$ the set $U_k = \{t \in \mathbb{C}^g : v(t, X^k X^{k*}) \neq 0\} \subseteq \mathbb{C}^g$ is open and dense and thus so is $\bigcap_{k=1}^e U_k$. Hence there exists $\lambda \in \mathbb{C}^e$ such that $v(\lambda, X^k, X^{k*}) \neq 0$ for every $k$. Now define the map

$$u : \mathcal{Z}_{Q_E}(n) \to \mathbb{C}^e, \qquad u(X, Y) := v(\lambda, X, Y).$$

Note that $u$ is a polynomial map by (4.1) and, for $X \in \partial^1 \mathcal{B}_E(r)$ and $0 \neq \delta = \sum_{t=1}^r \delta_t \otimes \varrho_t \in \ker(Q_E^{\mathrm{re}}(X))$,

$$u(X, X^*) = \sum_{s=1}^e \lambda_s (I \otimes \varrho_1^*)\operatorname{adj}(Q_E^{\mathrm{re}}(X))_{(i_s)} = \sum_{s=1}^e \lambda_s \nu_s \delta^1 = \nu \delta_1,$$

for some $\nu \in \mathbb{C}$. In particular, if $U(X, X^*) \neq 0$, then $u(X, X^*) \in \pi(\operatorname{hair} \mathcal{B}_E)$.

$$0 \neq u(X^k, X^{k*}) = \nu_k \delta_k^1,$$

for each $k$ and hence $u(X^1, X^{1*}), \dots, u(X^e, X^{e*})$ form a basis of $\mathbb{C}^e$. Therefore,

$$u(X, Y) = \sum_{k=1}^e r_k(X, Y) u(X^k, X^{k*})$$

for $(X, Y) \in \mathcal{Z}_{Q_E}(n)$, where $r_k$ are polynomial functions on $M_r(\mathbb{C})^{2g}$. In particular, $r_k(X^j, X^{j*}) = \delta_{j,k}$, where $\delta$ is the Kronecker delta function.

Suppose that the product $r_1 \cdots r_e \equiv 0$ on

$$\{(X, X^*) \colon X \in \partial^1 \mathcal{B}_E(n)\} \subseteq \mathcal{Z}_{Q_E}.$$

Then $r_1 \cdots r_e \equiv 0$ on $\mathcal{Z}_{Q_E}(n)$ by the Zariski denseness hypothesis. Therefore $r_k \equiv 0$ on $\mathcal{Z}_{Q_E}(n)$ for some $k$ by the irreducibility hypothesis, contradicting $r_k(X^k, X^{k*}) = 1$. Consequently there exists $X^0 \in \partial^1 \mathcal{B}_E(n)$ such that $r_1(X^0, X^{0*}) \cdots r_e(X^0, X^{0*}) \neq 0$. By the construction it follows that $\{u(X^0, X^{0*}), u(X^1, X^{1*}), \dots, u(X^e, X^{e*})\} \subseteq \pi(\operatorname{hair} \mathcal{B}_E)$ forms a hyperbasis of $\mathbb{C}^e$. □

**Proposition 4.4.** *Let $E \in M_{d\times e}(\mathbb{C})^g$. Then $Q_E$ is an atom and* $\ker(E) = \{0\}$ *if and only if $\pi(\operatorname{hair} \mathcal{B}_E)$ contains a hyperbasis of $\mathbb{C}^e$.*

**Proof.** Let $\iota$ denote the inclusion of $\operatorname{rg}(E)$ into $\mathbb{C}^d$ and let $\widehat{E} = \iota^* E$. Note that $\mathcal{B}_E = \mathcal{B}_{\widehat{E}}$ and thus $\pi(\operatorname{hair} \mathcal{B}_E) = \pi(\operatorname{hair} \mathcal{B}_{\widehat{E}})$. Further $Q_E = Q_{\widehat{E}}$ and $\ker(\widehat{E}) = \ker(E)$ and $\ker(\widehat{E}^*) = \{0\}$. It follows that $Q_E$ is an atom if and only if $Q_{\widehat{E}}$ is an atom; $\ker(E) = \{0\}$ if and only if $\ker(\widehat{E}) = \{0\}$; and $\pi(\operatorname{hair} \mathcal{B}_E)$ contains a hyperbasis of $\mathbb{C}^e$ if and only if $\pi(\operatorname{hair} \mathcal{B}_{\widehat{E}})$ does. Thus, by replacing $E$ with $\widehat{E}$ we may assume that $\ker(E^*) = \{0\}$.

($\Rightarrow$) Suppose $Q_E$ is an atom and $\ker(E) = \{0\}$ and $\ker(E^*) = \{0\}$. By Lemma 3.2(2), $\mathbb{L}_E$ (equivalently $\mathbb{L}_E^{\mathrm{re}}$) is indecomposable. By [41, Proposition 3.12],[9] $\mathcal{Z}_{\mathbb{L}_E}$ is an irreducible free locus. By [30, Corollary 3.6], $\mathcal{Z}_{\mathbb{L}_E}(n)$ is an irreducible hypersurface for large enough $n$. Thus, by [30, Corollary 8.5], $\partial^1\mathcal{B}_E(n) = \partial^1 Q_E^{\mathrm{re}}(n)$ is Zariski dense in $\mathcal{Z}_{\mathbb{L}_E^{\mathrm{re}}}^{\mathrm{re}}(n)$ for large enough $n$. Thus $\{(X, X^*) : X \in \partial^1\mathcal{B}_E(n)\}$ is Zariski dense in $\{(X, X^*) : X \in M_n(\mathbb{C})^g, \det \mathbb{L}_E^{\mathrm{re}}(X) = 0\}$ for large enough $n$. By Lemma 4.1 it now follows that $\{(X, X^*) : X \in \partial^1\mathcal{B}_E\}$ is Zariski dense in $\mathcal{Z}_{\mathbb{L}_E} = \mathcal{Z}_{Q_E} = \{(X, Y) : \det Q_E(X, Y) = 0\}$. Thus the assumptions of Lemma 4.3 are satisfied for some $n \in \mathbb{N}$, so $\pi(\operatorname{hair}\mathcal{B}_E)$ contains a hyperbasis for $\mathbb{C}^e$.

($\Leftarrow$) Suppose $Q_E$ is not an atom. If $E$ is not ball-minimal, then $\pi(\operatorname{hair}\mathcal{B}_E)$ does not span $\mathbb{C}^e$ by Proposition 4.2, since $\ker(E^*) = \{0\}$. If $E$ is ball-minimal, then $\mathbb{L}_E^{\mathrm{re}}$ is minimal but not indecomposable by Lemma 3.2 items (2) and (3). Thus $\mathbb{L}_E^{\mathrm{re}}$ decomposes non-trivially as $\mathbb{L}_{E^1}^{\mathrm{re}} \oplus \mathbb{L}_{E^2}^{\mathrm{re}}$ by Lemma 3.2(5). Hence $Q_E$ decomposes as $Q_{E^1} \oplus Q_{E^2}$. Letting $e_i \geq 1$ denote the size of $Q_{E^i}$,

$$\pi(\operatorname{hair}\mathcal{B}_E) \subseteq (\mathbb{C}^{e_1} \oplus \{0\}^{e_2}) \cup (\{0\}^{e_1} \oplus \mathbb{C}^{e_2}).$$

Thus $\pi(\operatorname{hair}\mathcal{B}_E)$ cannot contain a hyperbasis for $\mathbb{C}^e = \mathbb{C}^{e_1} \oplus \mathbb{C}^{e_2}$. □

**Remark 4.5.**

(1) Note that $Q_E$ is an atom, $\ker(E) = \{0\}$ and $\ker(E^*) = \{0\}$ (or equivalently, $\mathbb{L}_E$ is indecomposable) if and only if the centralizer of

$$\begin{pmatrix} 0 & E_1 \\ 0 & 0 \end{pmatrix}, \dots \begin{pmatrix} 0 & E_g \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ E_1^* & 0 \end{pmatrix}, \dots \begin{pmatrix} 0 & 0 \\ E_g^* & 0 \end{pmatrix},$$

is trivial. Verification of this fact amounts to checking whether a system of linear equations has a solution.

(2) If $\mathbb{L}_E$ is indecomposable, then so is $L_E$. Indeed, if $L_E = L_{E^1} \oplus L_{E^2}$, then $\mathbb{L}_E$ equals $\mathbb{L}_{E^1} \oplus \mathbb{L}_{E^2}$ up to a canonical shuffle.

However, the converse is not true. For example, with $\Lambda(x) = \begin{pmatrix} 0 & x_2 \\ x_1 & 0 \end{pmatrix}$,

$$I + \Lambda(x) + \Lambda^*(y) = \begin{pmatrix} 1 & x_2 + y_1 \\ x_1 + y_2 & 1 \end{pmatrix}$$

is an indecomposable monic pencil, but

$$I - \Lambda\Lambda^* = \begin{pmatrix} 1 - x_1 y_1 & 0 \\ 0 & 1 - x_2 y_2 \end{pmatrix}$$

factors. □

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[9] Irreducible in [41] is indecomposable here.

### *4.3. The eig-generic conditions*

In this subsection we connect the various genericity assumptions on tuples in $M_d(\mathbb{C})^g$ used in [3] to clean, purely algebraic conditions of the corresponding hermitian monic pencils, see Proposition 4.8. We begin by recalling these assumptions precisely.

**Definition 4.6** *([3, §7.1.2]).* A tuple $A \in M_d(\mathbb{C})^g$ is **weakly eig-generic** if there exists an $\ell \le d+1$ and, for $1 \le j \le \ell$, positive integers $n_j$ and tuples $\alpha^j \in M_{n_j}(\mathbb{C})^g$ such that

(a) for each $1 \le j \le \ell$, the eigenspace corresponding to the largest eigenvalue of $\Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ has dimension one and hence is spanned by a vector $u^j = \sum_{a=1}^{n_j} u_a^j \otimes e_a$; and

(b) the set $\mathscr{U} = \{u_a^j : 1 \le j \le \ell, 1 \le a \le n_j\}$ contains a hyperbasis for $\ker(A)^\perp = \operatorname{rg}(A^*)$.

The tuple is **eig-generic** if it is weakly eig-generic and $\ker(A) = \{0\}$ (equivalently, $\operatorname{rg}(A^*) = \mathbb{C}^d$).

Finally, a tuple $A$ is **$*$-generic** (resp. **weakly $*$-generic**) if there exists an $\ell \le d$ and tuples $\beta^j \in M_{n_j}(\mathbb{C})^g$ such that the kernels of $I - \Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ have dimension one and are spanned by vectors $\mu^j = \sum \mu_a^j \otimes e_a$ for which the set $\{\mu_a^j : 1 \le j \le \ell, 1 \le a \le n_j\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(A) = \ker(A^*)^\perp$).

**Remark 4.7.** One can replace $n_j$ with $\sum_{j=1}^{\ell} n_j$ in Definition 4.6, so we can without loss of generality assume $n_1 = \cdots = n_g$. $\square$

Mixtures of these generic conditions were critical assumptions in the main theorems of [3]. The next proposition gives elegant and much more familiar replacements for them.

**Proposition 4.8.** *Let $A \in M_d(\mathbb{C})^g$.*

(1) *$A$ is eig-generic if and only if $Q_A$ is an atom and* $\ker(A) = \{0\}$.
(2) *$A$ is $*$-generic and* $\ker(A) = \{0\}$ *if and only if $A^*$ is ball-minimal.*
(3) *Let $\iota$ denote the inclusion of* $\operatorname{rg}(A^*)$ *into $\mathbb{C}^d$. Then $A$ is weakly eig-generic if and only if $Q_{A\iota}$ is an atom and* $\ker(A\iota) = \{0\}$.
(4) *Let $\iota$ denote the inclusion of* $\operatorname{rg}(A)$ *into $\mathbb{C}^d$. Then $A$ is weakly $*$-generic and* $\ker(A) = \{0\}$ *if and only if $A^*\iota$ is ball-minimal.*

**Proof.** It is immediate from the definitions that if $\pi(\widehat{\operatorname{hair}\mathcal{B}_A})$ contains a hyperbasis, then $A$ is eig-generic. On the other hand, if $(\alpha, u) \in \widehat{\partial^1\mathcal{B}_E}$, then $u$ is an eigenvector of $\Lambda_A(\alpha)^*\Lambda_A(\alpha)$ corresponding to its largest eigenvalue 1. Writing $u = \sum_{a=1}^{n} u_a \otimes e_a \neq 0$, each $u_a \in \pi(\operatorname{hair}\mathcal{B}_E)$ because if $U$ is a unitary matrix, then $(U\alpha U^*, Uu) \in \widehat{\partial^1\mathcal{B}_E}$. Hence

$\pi(\operatorname{hair}\mathcal{B}_A)$ contains a hyperbasis if and only if $A$ is eig-generic and therefore item (1) Follows from Proposition 4.4 and Remark 4.7.

A similar argument to that above shows $\pi(\operatorname{hair}\mathcal{B}_{A^*})$ spans if and only if $A$ is $*$-generic. Thus item (2) follows from the Proposition 4.2 and Remark 4.7.

Item (3) follows from (1) since $A\iota$ is eig-generic and $\ker(A\iota) = \{0\}$.

Item (4) follows from (2) since $\iota^* A$ is weakly $*$-generic and $\ker(\iota^* A) = \ker(A)$. $\square$

### 4.4. *Proof of Theorem 1.5*

We use Proposition 4.8. In the terminology of [3], assumptions (a) and (b) imply that $A$ is eig-generic and $*$-generic, and $B$ is eig-generic, since the ball-minimal hypothesis on $A^*$ implies $\ker(A) = \{0\}$. Theorem 1.5 thus follows from [3, Corollary 7.11] once it is verified that the assumptions imply $\mathcal{D}_B$ is bounded, $p(\partial\mathcal{D}_A) \subseteq \partial\mathcal{D}_B$ and $q(\partial\mathcal{D}_B) \subseteq \partial\mathcal{D}_A$. For instance, if $X \in \partial\mathcal{D}_A$, but $p(X) \in \operatorname{int}(\mathcal{D}_B)$, then there is a $Z \notin \mathcal{D}_A$ such that $p(Z) \in \mathcal{D}_B$. But then, $Z = q(p(Z)) \in \mathcal{D}_A$, a contradiction. $\square$

## 5. Bianalytic maps between spectraballs and free spectrahedra

In this section we prove the rest of our main results, Proposition 1.7, and then Theorem 1.1 and its Corollary 1.3.

### 5.1. *The proof of Proposition 1.7*

Throughout this subsection, we fix a tuple $E \in M_{d\times e}(\mathbb{C})^g$, a positive integer $M$ and an $F \in \mathbb{C}<x>^{1\times e}$ of degree at most $M$. Write $F = \begin{pmatrix} F^1 & \cdots & F^e \end{pmatrix}$ and

$$F^s = \sum_{|w|\le M} F^s_w w,$$

where $|w|$ denotes the **length of the word** $w$ and $F^s_w \in \mathbb{C}$.

Let $S$ denote the tuple of shifts on the truncated Fock space $\mathscr{F}_M$ with orthonormal basis the words of length at most $M$ in the freely noncommuting variables $\{x_1, \dots, x_g\}$. When viewing a word $w$ as an element of the finite dimensional Hilbert space $\mathscr{F}_M$ we will write $\underline{w}$. Thus $S_\ell \underline{w} = \underline{x_\ell w}$ if $|w| < M$ and $S_\ell \underline{w} = 0$ if $|w| = M$. Let $P$ denote the projection of $\mathscr{F}_M$ onto the subspace $\mathscr{F}_{M-1}$ and note that $S_k^* S_\ell = P$ if $k = \ell$ and $S_k^* S_\ell = 0$ if $k \ne \ell$.

Given a matrix $\beta = (\beta_{j,k})_{j,k=1}^g \in M_g(M_r(\mathbb{C}))$ and words $u, w$ of the same length $N$,

$$u = x_{j_1} x_{j_2} \cdots x_{j_N}, \quad w = x_{k_1} x_{k_2} \cdots x_{k_N},$$

let

$$\widehat{\beta}_{u,w} = \beta_{k_1,j_1} \beta_{k_2,j_2} \cdots \beta_{k_N,j_N}.$$

In particular, $\beta_{j,k} = \widehat{\beta}_{x_k,x_j}$

$$\widehat{\beta}_{u,w}\widehat{\beta}_{x_j,x_k} = \beta_{k_1,j_1}\beta_{k_2,j_2}\cdots\beta_{k_N,j_N}\beta_{k,j} = \widehat{\beta}_{ux_j,wx_k}. \tag{5.1}$$

Let

$$(\beta \cdot S)_j = \sum_{k=1}^{g} \beta_{j,k} \otimes S_k$$

and $\beta \cdot S = ((\beta \cdot S)_1, \ldots, (\beta \cdot S)_g)$.

**Lemma 5.1.** *Given $1 \le N \le M$ and a word $w$ of length $N$,*

$$(\beta \cdot S)^w = \sum_{|u|=N} \widehat{\beta}_{u,w} \otimes S^u.$$

**Proof.** We induct on $N$. For $N = 1$ and $w = x_t$,

$$(\beta \cdot S)^w = \sum_{k=1}^{g} \beta_{t,k} \otimes S_k = \sum_{k} \widehat{\beta}_{x_k,x_t} S_k = \sum_{|u|=1} \widehat{\beta}_{u,x_t} S^u$$

Now suppose the result holds for $N$. Let $v$ be a word of length $N$ and consider the word $w = vx_t$ of length $N + 1$. Using the induction hypothesis and equation (5.1),

$$\begin{aligned}(\beta \cdot S)^w &= (\beta \cdot S)^v (\beta \cdot S)^{x_t} = [\sum_{|u|=N} \widehat{\beta}_{u,v} \otimes S^u]\,[\sum_{k} \beta_{t,k} \otimes S_k] \\ &= \sum_{|u|=N} \sum_{k=1}^{g} \widehat{\beta}_{u,v}\beta_{t,k} \otimes S^u S_k = \sum_{|u|=N} \sum_{k=1}^{g} \widehat{\beta}_{ux_k,vx_t} \otimes S^{ux_k} \\ &= \sum_{|z|=N+1} \widehat{\beta}_{z,w} \otimes S^z. \quad \square\end{aligned}$$

Given $N$, let $\mathscr{G}_N$ denote the subspace of $\mathscr{F}_M$ spanned by words of length $N$. Thus the words of length $N$ form an orthonormal basis for $\mathscr{G}_N$. Given words $u, w \in \mathscr{G}_N$, let $\underline{u}\,\underline{w}^*$ denote the linear mapping on $\mathscr{G}_N$ determined by $\underline{u}\,\underline{w}^*\underline{v} = \langle \underline{v}, \underline{w} \rangle \underline{u}$, for words $v \in \mathscr{G}_N$. Let

$$B(\beta, N) = \sum_{|u|=N=|w|} \widehat{\beta}_{u,w} \otimes \underline{u}\,\underline{w}^* = \left(\widehat{\beta}_{u,w}\right)_{|u|=N=|w|} \in M_r(\mathbb{C}) \otimes M_{g^N}(C),$$

where the second equality is understood in the sense of unitary equivalence. In particular, $B(\beta, 1) = (\beta_{k,j})_{j,k=1}^{g}$.

**Lemma 5.2.** *For each positive integer $N$ the set of $\beta \in M_g(M_r(\mathbb{C}))$ such that $B(\beta, N)$ is invertible is open and dense.*

**Proof.** For the second statement, observe that $B(I, N)$ is the identity matrix since, with $\beta_{j,k} = \delta_{j,k} I_r$, we have $\widehat{\beta}_{u,w} = \delta_{u,w} I_r$. Hence the mapping $\psi : M_g(M_r(\mathbb{C})) \to \mathbb{C}$ defined by $\psi(\beta) = \det B(\beta, N)$ is a polynomial in the entries of $\beta$ that is not identically zero. Thus $\psi$ is nonzero on an open dense set and the result follows. $\square$

For notational purposes, let 1 denote the emptyword $\underline{\emptyset} \in \mathscr{F}_M$. Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_e\}$ denote the standard orthonormal basis for $\mathbb{C}^e$.

**Lemma 5.3.** *Suppose $\beta \in M_g(M_r(\mathbb{C}))$ and $\gamma = \sum_{s=1}^{e} \varepsilon_s \otimes \gamma_s \in \mathbb{C}^e \otimes \mathbb{C}^r$. If*

$$\sum_{s=1}^{e} F^s(\beta \cdot S)[\gamma_s \otimes 1] = 0,$$

*then, for $1 \le N \le M$ and each word $u$ of length $N$,*

$$\sum_{|w|=N} \widehat{\beta}_{u,w}[\sum_{s=1}^{e} F^s_w \gamma_s] = 0.$$

*Moreover, if $B(\beta, N)$ is invertible, then*

$$\sum_{s=1}^{e} F^s_w \gamma_s = 0$$

*for each word $|w| = N$.*

**Proof.** Since $F^s_w \in \mathbb{C}$, by Lemma 5.1,

$$\sum_{N=0}^{M} \sum_{|w|=N} F^s_w (\beta \cdot S)^w = \sum_{N=0}^{M} \sum_{|u|=N} \left[ \sum_{|w|=N} F^s_w \widehat{\beta}_{u,w} \right] \otimes S^u.$$

Thus,

$$0 = \sum_{s=1}^{e} F^s(\beta \cdot S)[\gamma_s \otimes 1] = \sum_{N=0}^{M} \sum_{|u|=N} \left( \sum_{|w|=N} \widehat{\beta}_{u,w}[\sum_{s=1}^{e} F^s_w \gamma_s] \right) \otimes \underline{u}$$

and the first part of the result follows.

To prove the second part, let

$$y = \sum_{|v|=N} y_v \otimes \underline{v} \in \mathbb{C}^r \otimes \mathscr{G}_N,$$

where $y_v = \sum_{s=1}^{e} F^s_v \gamma_s \in \mathbb{C}^r$. Thus

$$\begin{aligned} B(\beta, N)y &= \sum_{|u|=N=|w|} \widehat{\beta}_{u,w} \otimes \underline{u}\,\underline{w}^* \sum_{|v|=N} y_v \otimes v \\ &= \sum_{|u|=N=|w|} \widehat{\beta}_{u,w} y_w \otimes \underline{u} = \sum_{|u|=N} [\sum_{|w|=N} \widehat{\beta}_{u,w} y_w] \otimes \underline{u} = 0. \end{aligned}$$

Hence if $B(\beta, N)$ is invertible, then $y = 0$ and therefore $\sum_{s=1}^{e} F_w^s \gamma_s = 0$ for each $|w| = N$. $\square$

We continue to let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_e\}$ denote the standard basis for $\mathbb{C}^e$. Let $\{\varrho_1, \ldots, \varrho_r\}$ denote the standard orthonormal basis for $\mathbb{C}^r$.

**Proposition 5.4.** *Fix $1 \leq N \leq M$. If there exist a positive integer $r$ and $(\beta^a, \gamma^a) \in M_g(M_r(\mathbb{C})) \times [\mathbb{C}^e \otimes \mathbb{C}^r]$ for $1 \leq a \leq e$ such that,*

(a) *writing*

$$\gamma^a = \sum_{t=1}^{r} \delta_t^a \otimes \varrho_t$$

*the vectors $\{\delta_1^a : 1 \leq a \leq e\}$ span $\mathbb{C}^e$;*

(b) *$B(\beta^a, N)$ is invertible for each $1 \leq a \leq e$;*

(c) *$F(\beta^a \cdot S)[\gamma^a \otimes 1] = 0$ for each $1 \leq a \leq e$,*

*then $F_w^s = 0$ for each $1 \leq s \leq e$ and $|w| = N$.*

**Proof.** Note that

$$0 = F(\beta^a \cdot S)[\gamma^a \otimes 1] = \sum_{s=1}^{e} F^s(\beta^a \cdot S)[\gamma_s^a \otimes 1].$$

Thus items (b) and (c) validate the hypotheses of Lemma 5.3, and hence $\sum_s F_w^s \gamma_s^a = 0$ for each $|w| = N$ and $1 \leq a \leq e$. Writing $\gamma^a = \sum_{s=1}^{e} \varepsilon_s \otimes \gamma_s^a$, it follows that

$$\sum_{s=1}^{e} [\varrho_1^* \gamma_s^a] \varepsilon_s = (I \otimes \varrho_1^*) \sum_{s=1}^{e} \varepsilon_s \otimes \gamma_s^a = \delta_1^a = \sum_{s=1}^{e} [\varepsilon_s^* \delta_1^a] \varepsilon_s.$$

Therefore $\varrho_1^* \gamma_s^a = \varepsilon_s^* \delta_1^a$ and consequently, for $|w| = N$,

$$0 = \sum_{s=1}^{e} F_w^s [\varrho_1^* \gamma_s^a] = \sum_{s=1}^{e} F_w^s [\varepsilon_s^* \delta_1^a] = F_w \delta_1^a,$$

where $F_w = \begin{pmatrix} F_w^1 & \ldots & F_w^e \end{pmatrix} \in \mathbb{C}^{1 \times e}$. Since, by hypothesis, $\{\delta_1^a : 1 \leq a \leq e\}$ spans $\mathbb{C}^e$ it follows that $F_w = 0$ whenever $|w| = N$. Thus $F_w^s = 0$ for $1 \leq s \leq e$ and $|w| = N$. $\square$

Given $\beta = (\beta_{j,k}) \in M_g(M_r(\mathbb{C}))$, let

$$(E \cdot \beta)_k = \sum_{j=1}^{g} E_j \otimes \beta_{j,k}$$

**Lemma 5.5.** *For $\beta \in M_g(M_r(\mathbb{C}))$,*

$$\Lambda_E(\beta \cdot S) = \sum_k (E \cdot \beta)_k \otimes S_k$$

$$Q_E^{\mathrm{re}}(\beta \cdot S) = [I - \sum_k (E \cdot \beta)_k^* (E \cdot \beta)_k] \otimes P + I \otimes (I - P),$$

*where $P$ is the projection of $\mathscr{F}_M$ onto $\mathscr{F}_{M-1}$.*

**Proof.** Compute,

$$\Lambda_E(\beta \cdot S) = \sum_{j=1}^{g} E_j \otimes (\sum_{k=1}^{g} \beta_{j,k} \otimes S_k) = \sum_{k=1}^{g}[\sum_{j=1}^{g} E_j \otimes \beta_{j,k}] \otimes S_k = \sum_{k=1}^{g}(E \cdot \beta)_k \otimes S_k,$$

and thus

$$\Lambda_E(\beta \cdot S)^* \Lambda_E(\beta \cdot S) = [\sum_{k=1}^{g} (E \cdot \beta)_k^* (E \cdot \beta)_k] \otimes P$$

and the result follows. □

### 5.1.1. *The hair spanning condition*

A subset $\{(\alpha^a, \gamma^a) : 1 \leq a \leq e\} \subseteq M_r(\mathbb{C})^g \times [\mathbb{C}^e \otimes \mathbb{C}^r]$ is a **boundary spanning set** for $\mathcal{B}_E$ if each $(\alpha^a, \gamma^a) \in \widehat{\partial \mathcal{B}_E}$ and, writing $\gamma^a = \sum_{t=1}^r \delta_t^a \otimes \varrho_t$, the set $\{\delta_1^a : 1 \leq a \leq e\}$ spans $\mathbb{C}^e$. This set is a **boundary hair spanning set** for $\mathcal{B}_E$ if moreover $(\alpha^a, \gamma^a) \in \widehat{\partial^1 \mathcal{B}_E}$ for each $a$. By Proposition 4.2, if $E$ is ball-minimal, then there exists a boundary hair spanning set for $\mathcal{B}_E$.

**Proposition 5.6.** *Fix $1 \leq N \leq M$. If $E \in M_{d \times e}(\mathbb{C})^g$ is ball-minimal, then there exists a positive integer $r$ and a subset $\{(\beta^a, \gamma^a) : 1 \leq a \leq e\}$ of $M_g(M_r(\mathbb{C})) \otimes [\mathbb{C}^e \otimes \mathbb{C}^r]$ such that $B(\beta^a, N)$ is invertible for each $1 \leq a \leq e$ and $\{(\beta^a \cdot S, \gamma^a \otimes 1) : 1 \leq a \leq e\}$ is a boundary spanning set for $\mathcal{B}_E$.*

The proof of Proposition 5.6 uses the following special case of a standard result from the theory of perturbation of matrices [39, Chapter 2, Section 4].

**Lemma 5.7.** *Suppose $R \in M_d(\mathbb{C})$, $I - R \succeq 0$ and $\ker(I - R)$ is one-dimensional and spanned $v \in \mathbb{C}^d$. For each $\epsilon > 0$, there is a $\mu > 0$ such that if $Q \in M_d(\mathbb{C})$ is self-*

*adjoint and* $\|Q\| < \mu$*, then there is a* $c > 0$ *and* $w \in \mathbb{C}^d$ *such that* $I - c(R+Q) \succeq 0$*,* $\ker(I - c(R+Q))$ *is spanned by* $w$ *and* $\|v - w\| < \epsilon$*.*

**Proof of Proposition 5.6.** Since $E$ is ball-minimal, there is an $r$ and a boundary hair spanning set $\{(\alpha^a, \zeta^a) : 1 \le a \le e\} \subseteq M_r(\mathbb{C})^g \times [\mathbb{C}^e \otimes \mathbb{C}^r]$ for $\mathcal{B}_E$ by Proposition 4.2. In particular, writing $\zeta^a = \sum_{t=1}^r \chi_t^a \otimes \varrho_t$, the set $\{\chi_1^a : 1 \le a \le e\}$ spans $\mathbb{C}^e$. There is an $\epsilon > 0$ such that, if $\tau^a = \sum_{t=1}^g \tau_t^a \otimes \rho_t$ and $\|\zeta^a - \tau^a\| < \epsilon$ for each $1 \le a \le e$, then the set $\{\tau_1^a : 1 \le a \le e\}$ spans $\mathbb{C}^e$.

Fix $1 \le a \le e$ and let, for $1 \le j, k \le g$,

$$\widetilde{\beta}_{j,k}^a = \begin{cases} \alpha_j^a & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

Thus

$$I - [\sum_{j=1}^g E_j \otimes \widetilde{\beta}_{j,1}^a]^* \, [\sum_{j=1}^g E_j \otimes \widetilde{\beta}_{j,1}^a] = Q_E^{\mathrm{re}}(\alpha^a)$$

is positive semidefinite with kernel spanned by $\zeta^a$. By Lemmas 5.2 and 5.7, there exists a $\beta^a \in M_g(M_r(\mathbb{C}))$ such that $B(\beta^a, N)$ is invertible and

$$R(\beta^a) := I - \sum_{k=1}^g \left( [\sum_{j=1}^g E_j \otimes \beta_{j,k}]^* \, [\sum_{j=1}^g E_j \otimes \beta_{j,k}] \right) = I - \sum_{k=1}^g (E \cdot \beta)_k^* \, (E \cdot \beta)_k \tag{5.2}$$

is positive semidefinite and has kernel spanned by a vector $\gamma^a$ such that $\|\zeta^a - \gamma^a\| < \epsilon$. In particular, writing $\gamma^a = \sum_{t=1}^r \delta_t^a \otimes \varrho_t$, from the first paragraph of the proof, the set $\{\delta_1^a : 1 \le a \le e\}$ spans $\mathbb{C}^e$.

To complete the proof, observe, using $R(\beta^a)$ defined in equation (5.2) and Lemma 5.5, that

$$Q_E^{\mathrm{re}}(\beta^a \cdot S) = R(\beta^a) \otimes P \, + \, I \otimes (I - P).$$

It follows that $\{(\beta^a \cdot S, \gamma^a \otimes 1) : 1 \le a \le e\}$ is a boundary spanning set for $\mathcal{B}_E$. $\square$

### *5.1.2. Proof of Proposition 1.7*

Suppose $E$ is ball-minimal[10] and $F \in \mathbb{C} <x>^{1 \times e}$ vanishes on $\widehat{\partial \mathcal{B}_E}$ and has degree at most $M$.

Fix $1 \le N \le M$. By Proposition 5.6, there exists an $r > 0$ and $(\beta^a, \gamma^a) \in M_g(M_r(\mathbb{C})) \times [\mathbb{C}^e \otimes \mathbb{C}^r]$ such that $\{(\beta^a \cdot S, \gamma^a \otimes 1) : 1 \le a \le e\}$ is a boundary spanning set for $\mathcal{B}_E$ and $B(\beta^a, N)$ is invertible for each $1 \le a \le e$. Since $(\beta^a \cdot S, \gamma^a) \in \widehat{\partial \mathcal{B}_E}$, it

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[10] It is enough to assume that $PE$ is ball-minimal, where $P$ is the projection of $\mathbb{C}^d$ onto $\mathrm{rg}(E)$.

follows that $0 = F(\beta \cdot S)\gamma^a$. An application of Proposition 5.4 implies $F_w^s = 0$ for all $1 \le s \le e$ and $|w| = N$. Hence $F_w^s = 0$ for all $1 \le s \le e$ and $|w| \le M$ and therefore $F = 0$. To complete the proof, given $V \in \mathbb{C} <x>^{\ell \times e}$ that vanishes on $\widehat{\partial \mathcal{B}_E}$, apply what has already been proved to each row of $V$ to conclude $V = 0$. □

### *5.2. Theorem 1.1*

In this subsection we prove the first part Theorem 1.1. (The conversely portion was already proved as Corollary 2.5.)

A free analytic mapping $f$ into $M(\mathbb{C})^h$ defined in a neighborhood of 0 of $M(\mathbb{C})^g$ has a power series expansion ([25,38,54]),

$$f(x) = \sum_{j=0}^{\infty} G_j(x) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} f_\alpha x^\alpha, \tag{5.3}$$

where $f_\alpha \in \mathbb{C}^{1\times h}$. The term $G_j$ is the **homogeneous of degree** $j$ part of $f$. It is a polynomial mapping $M(\mathbb{C})^g \to M(\mathbb{C})^h$.

**Lemma 5.8.** *Let $E \in M_{d\times e}(\mathbb{C})^g$ and $B \in M_r(\mathbb{C})^h$. Suppose $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_B)$ is proper. For each positive integer $N$ there exists a free polynomial mapping $p = p_N$ of degree at most $N$ such that if $X \in \mathcal{B}_E$ is nilpotent of order $N$, then $f_X(z) := f(zX) = p(zX)$ for $z \in \mathbb{C}$ with $|z| < 1$. Further, if $X \in \partial\mathcal{B}_E$ (equivalently $\|\Lambda_E(X)\| = 1$), then $p(X) \in \partial\mathcal{D}_B$.*

**Proof.** Fix a positive integer $N$. The series expansion of equation (5.3) converges as written on $\mathcal{N}_\epsilon = \{X \in M(\mathbb{C})^g : \sum X_j X_j^* \prec \epsilon^2\}$ for any $\epsilon > 0$ such that $N_\epsilon \subseteq \operatorname{int}(\mathcal{B}_E)$ [25, Proposition 2.24]. In particular, if $X \in \mathcal{B}_E$ is nilpotent of order $N$ and $|z|$ is small, then

$$f_X(z) := f(zX) = \sum_{j=1}^{N} G_j(zX) = \sum_{j=1}^{N} \left[ \sum_{|\alpha|=j} f_\alpha \otimes X^\alpha \right] z^j =: p(zX).$$

It now follows that $f_X(z) = p(zX)$ for $|z| < 1$ (since $zX \in \operatorname{int}(\mathcal{B}_E)$ for such $z$ and both sides are analytic in $z$ and agree on a neighborhood of 0).

Now suppose $X \in \partial\mathcal{B}_E(n)$ (still nilpotent of order $N$). Since $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_B)$, it follows that $L_B^{\mathrm{re}}(p(tX)) \succ 0$ for $0 < t < 1$. Thus $L_B^{\mathrm{re}}(p(X)) \succeq 0$. Arguing by contradiction, suppose $L_B^{\mathrm{re}}(p(X)) \succ 0$; that is $p(X) \in \operatorname{int}(\mathcal{D}_B(n))$. Hence there is an $\eta$ such that

$$\overline{B}_\eta(p(X)) := \{Y \in M_n(\mathbb{C})^g : \|Y - p(X)\| \le \eta\} \subseteq \operatorname{int}(\mathcal{D}_B(n)).$$

Since $K = \overline{B}_\eta(p(X))$ is compact, $L = f_n^{-1}(K) \subseteq \operatorname{int}(\mathcal{B}_E)$ is also compact by the proper hypothesis on $f$ (and hence on each $f_n : \operatorname{int}(\mathcal{B}_E(n)) \to \operatorname{int}(\mathcal{D}_B(n))$). On the other hand, for $t < 1$ sufficiently large, $tX \in L$, but $X \notin \operatorname{int}(\mathcal{B}_E(n))$, and we have arrived at the contradiction that $L$ cannot be compact. $\square$

**Remark 5.9.** In view of Lemma 5.8, for $X \in \partial\mathcal{B}_E$ nilpotent we let $f(X)$ denote $f_X(1)$. Observe also, if $g = h$, $f(0) = 0$, $f'(0) = I_g$ and $X \in \mathcal{B}_E$ is nilpotent of order two, then $f(X) = X$. $\square$

**Lemma 5.10.** *Suppose $B \in M_r(\mathbb{C})^g$ and $\mathfrak{V} \in M_{r\times u}(\mathbb{C})$ and let $\mathscr{B}$ denote the algebra generated by $B$. Let $h$ denote the dimension of $\mathscr{B}$ as a vector space. If $\{B_1\mathfrak{V}, \dots, B_g\mathfrak{V}\}$ is linearly independent, then there exists a $g \le t \le h$ and a basis $\{J_1, \dots, J_h\}$ of $\mathscr{B}$ such that*

(1) *$J_j = B_j$ for $1 \le j \le g$;*
(2) *$\{J_1\mathfrak{V}, \dots, J_t\mathfrak{V}\}$ is linearly independent; and*
(3) *$J_j\mathfrak{V} = 0$ for $t < j \le h$.*

*Letting $\Xi \in M_h(\mathbb{C})^h$ denote the convexotonic tuple associated to $J$,*

$$(\Xi_j)_{\ell,k} = 0 \quad \text{for } j > t,\ k \le t \text{ and } 1 \le \ell \le h.$$

**Proof.** The set $\mathscr{N} = \{T \in \mathscr{B} : T\mathfrak{V} = 0\} \subseteq \mathscr{B}$ is a subspace (in fact a left ideal). Since $\{B_1\mathfrak{V}, \dots, B_g\mathfrak{V}\}$ is linearly independent, the subspace $\mathscr{M}$ of $\mathscr{B}$ spanned by $\{B_1, \dots, B_g\}$ has dimension $g$ and satisfies $\mathscr{M} \cap \mathscr{N} = \{0\}$. Thus there is a $g \le t \le h$ such that $h - t$ is the dimension of $\mathscr{N}$. Choose a basis $\{J_{t+1}, \dots, J_h\}$ for $\mathscr{N}$. Thus the set $\{B_1, \dots, B_g, J_{t+1}, \dots, J_h\}$ is linearly independent and $g \le t \le h$. Extend it to a basis $\{J_1, \dots, J_h\}$. To see that item (2) holds, we argue by contradiction. If $\{J_1\mathfrak{V}, \dots, J_t\mathfrak{V}\}$ is linearly dependent, then some linear combination of $\{J_1, \dots, J_t\}$ lies in $\mathscr{N}$.

The last statement is a consequence of the fact that $\mathcal{N}$ is a left ideal. Indeed, since the tuple $\Xi$ satisfies,

$$J_\ell J_j = \sum_{k=1}^{h} (\Xi_j)_{\ell,k} J_k$$

for $1 \le j, \ell \le h$ we have, for $j > t$ and $1 \le \ell \le h$,

$$0 = J_\ell J_j \mathfrak{V} = \sum_{k=1}^{h} (\Xi_j)_{\ell,k} J_k \mathfrak{V} = \sum_{k=1}^{t} (\Xi_j)_{\ell,k} J_k \mathfrak{V}.$$

By independence of $\{J_k\mathfrak{V} : 1 \le k \le t\}$, it follows that $(\Xi_j)_{\ell,k} = 0$ for $k \le t$. $\square$

**Lemma 5.11.** *Let $E \in M_{d\times e}(\mathbb{C})^g$ and $A \in M_r(\mathbb{C})^g$. If there is a proper free analytic mapping $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_A)$ such that $f(0) = 0$ and $f'(0) = I$, then $\mathcal{B}_E = \mathcal{B}_A$.*

**Proof.** We perform the off diagonal trick. Given a tuple $X$, let

$$S_X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Suppose $X \in M_n(\mathbb{C})^g$ and $\|\Lambda_E(X)\| = 1$. It follows that $\|\Lambda_E(S_X)\| = 1$. Thus $S_X \in \partial\mathcal{B}_E$. Since $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_A)$ is proper with $f(0) = 0$ and $f'(0) = I$ (and $S_X$ is nilpotent), $f(S_X) = S_X$ (see Remark 5.9), and $S_X \in \partial\mathcal{D}_A$. Thus $I + \Lambda_A(S_X) + \Lambda_A(S_X)^*$ is positive semidefinite and has a (non-trivial) kernel. Equivalently,

$$1 = \|\Lambda_A(S_X)\| = \|\Lambda_A(X)\|.$$

Hence, by homogeneity, $\|\Lambda_E(X)\| = \|\Lambda_A(X)\|$ for all $n$ and $X \in M_n(\mathbb{C})^g$. Thus $\mathcal{B}_E = \mathcal{B}_A$. $\square$

*5.2.1. Proof of Theorem 1.1*

We assume, without loss of generality, that $E$ is ball-minimal. We will now show $f$ is convexotonic.

Lemma 5.11 applied to the proper free analytic mapping $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_A)$ gives $\mathcal{B}_E = \mathcal{B}_A$. Applying Lemma 3.2(9) there exist $r \times r$ unitary matrices $\mathtt{W}$ and $\mathtt{V}$ such that $A = \mathtt{W}\left(\begin{smallmatrix} E & 0 \\ 0 & R \end{smallmatrix}\right)\mathtt{V}^*$, where $R \in M_{(r-d)\times(e-d)}(\mathbb{C})^g$ and $\mathcal{B}_E \subseteq \mathcal{B}_R$. Replacing $A$ with the unitarily equivalent tuple $\mathtt{V}^* A \mathtt{V}$, we assume

$$A = U \begin{array}{c c} \begin{array}{cc} e & r{-}e \end{array} & \\ \begin{pmatrix} E & 0 \\ 0 & R \end{pmatrix} & \begin{array}{l} d \\ r{-}d \end{array} \end{array} \tag{5.4}$$

where

$$U = \mathtt{V}^*\,\mathtt{W} = \begin{array}{c c} \begin{array}{cc} d & r{-}d \end{array} & \\ \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} & \begin{array}{l} e \\ r{-}e \end{array} \end{array} . \tag{5.5}$$

With respect to the orthogonal decomposition in equation (5.4), let

$$\mathfrak{V} = \begin{pmatrix} I_e \\ 0_{r-e,e} \end{pmatrix} \in M_{r\times e}(\mathbb{C}).$$

We will use later the fact that if $Q_E^{\mathrm{re}}(X) \succeq 0$ and $Q_E^{\mathrm{re}}(X)\gamma = 0$, then $Q_A^{\mathrm{re}}(X)\mathfrak{V}\gamma = 0$. For now observe

$$A_j\mathfrak{V} = U\begin{pmatrix} E_j \\ 0 \end{pmatrix}. \tag{5.6}$$

Thus, since $\{E_1, \dots, E_g\}$ is linearly independent, the set $\{A_1\mathfrak{V}, \dots, A_g\mathfrak{V}\}$ is linearly independent.

We now apply Lemma 5.10 to $A$ in place of $B$ and obtain a basis $\{J_1,\dots,J_h\}$ for $\mathscr{A}$, the algebra generated by $\{A_1,\dots,A_g\}$, and a $g \le t \le h$ such that $J_j = A_j$ for $1 \le j \le g$, the set $\{J_j\mathfrak{V} : 1 \le j \le t\}$ is linearly independent and $J_j\mathfrak{V} = 0$ for $t < j \le h$. Let $\xi \in M_h(\mathbb{C})^h$ denote the convexotonic tuple associated to $J$ and let $\Xi = -\xi$. Thus $(\Xi_j)_{\ell,k} = 0$ for $j > t$, $k \le t$, and all $\ell$ and

$$J_\ell J_j = -\sum_{s=1}^{h} (\Xi_j)_{\ell,s} J_s.$$

Let $\varphi : \operatorname{int}(\mathcal{D}_J) \to \operatorname{int}(\mathcal{B}_J)$ denote the convexotonic map

$$\varphi(x) = x(I - \Lambda_\Xi(x))^{-1}$$

from Proposition 2.2. Let $\iota : \mathcal{D}_A \to \mathcal{D}_J$ denote the inclusion. By Corollary 2.6 the composition $\varphi \circ \iota$ is proper from $\operatorname{int}(\mathcal{D}_A)$ to $\operatorname{int}(\mathcal{B}_J)$. Hence, $\mathscr{F} = \varphi \circ \iota \circ f$ is proper from $\operatorname{int}(\mathcal{B}_E)$ to $\operatorname{int}(\mathcal{B}_J)$. Further $\mathscr{F}(0) = 0$ and $\mathscr{F}'(0) = \begin{pmatrix} I_g & 0 \end{pmatrix}$ because essentially the same is true for each of the components $f, \iota, \varphi$. Thus $\mathscr{F}(x) = \begin{pmatrix} x & 0 \end{pmatrix} + \rho(x)$, where $\rho(0) = 0$ and $\rho'(0) = 0$.

Write

$$\mathscr{F} = \begin{pmatrix} \mathscr{F}^1 & \dots & \mathscr{F}^h \end{pmatrix}.$$

Expand $\mathscr{F}$ as a power series,

$$\mathscr{F} = \sum H_j = \sum_{j=1}^{\infty} \sum_{|\alpha|=j} \mathscr{F}_\alpha\, \alpha,$$

where $H_j$ is the homogeneous of degree $j$ part of $\mathscr{F}$. Thus,

$$H_j = \begin{pmatrix} H_j^1 & \dots & H_j^h \end{pmatrix}$$

and $H_1(x) = \begin{pmatrix} x & 0 \end{pmatrix}$. Likewise,

$$\mathscr{F}_{x_j}(x) = \begin{pmatrix} 0 & \dots & 0 & x_j & 0 & \dots & 0 \end{pmatrix}$$

for $1 \le j \le g$ and $\mathscr{F}_{x_j} = 0$ for $j > g$.

The next objective is to show $H_m^s = 0$ for $m \ge 2$ and $s \le t$. Given a positive integer $m$, let $S$ denote the $(m+1)\times(m+1)$ matrix, indexed by $j,k = 0,1,\dots,m$, with $S_{a,a+1} = 1$ and $S_{a,b} = 0$ otherwise. Thus $S$ has ones on the first super diagonal and 0 everywhere else and $S^{m+1} = 0$. Let $Y \in \mathcal{B}_E$ be given. Since $S \otimes Y$ is nilpotent with $(S \otimes Y)^\alpha = 0$ if $\alpha$ is a word with $|\alpha| > m$, Lemma 5.8 (and Remark 5.9) imply $\mathscr{F}(S \otimes Y) \in \mathcal{B}_J$; that is if $\|\Lambda_E(Y)\| \le 1$, then $\|\Lambda_J(\mathscr{F}(S \otimes Y))\| \le 1$. Let $\mathscr{X}^j = \mathscr{F}^j(S \otimes Y) = \sum_{\mu=1}^{m} S^\mu \otimes H_\mu^j(Y)$. With respect to the natural block matrix decomposition, $\mathscr{X}^j_{0,m} = H_m^j(Y)$ and $\mathscr{X}^j_{m-1,m} =$

$H_1^j(Y)$. Thus $\mathscr{X}_{m-1,m}^j = Y_j$ for $1 \leq j \leq g$ and $\mathscr{X}_{m-1,m}^j = H_1^j(Y) = 0$ for $j > g$. Now $\|\Lambda_J(\mathscr{X})\| \leq 1$ is equivalent to $I - \Lambda_J(\mathscr{X})^*\Lambda_J(\mathscr{X}) \succeq 0$. Thus,

$$I - \Lambda_A(Y)^*\Lambda_A(Y) - \Lambda_J(H_m(Y))^*\Lambda_J(H_m(Y)) \succeq 0.$$

Multiplying on the right by $\mathfrak{V} \otimes I$ and on the left by $\mathfrak{V}^* \otimes I$,

$$I - \Lambda_{A\mathfrak{V}}(Y)^*\Lambda_{A\mathfrak{V}}(Y) - \Lambda_{J\mathfrak{V}}(H_m(Y))^*\Lambda_{J\mathfrak{V}}(H_m(Y)) \succeq 0.$$

By equation (5.6) $\Lambda_{A\mathfrak{V}}(Y)^*\Lambda_{A\mathfrak{V}}(Y) = \Lambda_E(Y)^*\Lambda_E(Y)$, and hence,

$$\begin{aligned} Q_E^{\mathrm{re}}(Y) - &\Lambda_{J\mathfrak{V}}(H_m(Y))^*\Lambda_{J\mathfrak{V}}(H_m(Y)) \\ &= I - \Lambda_E(Y)^*\Lambda_E(Y) - \Lambda_{J\mathfrak{V}}(H_m(Y))^*\Lambda_{J\mathfrak{V}}(H_m(Y)) \succeq 0. \end{aligned} \tag{5.7}$$

Let $V(y) = \Lambda_{J\mathfrak{V}}(H_m(y))$. If $(Y, \gamma) \in \widehat{\partial\mathcal{B}_E}$, then $Q_E^{\mathrm{re}}(Y)\gamma = 0$ and hence, by equation (5.7), $V(Y)\gamma = 0$. Thus $V$ vanishes on $\widehat{\partial\mathcal{B}_E}$ and hence $V = 0$ by Proposition 1.7; that is

$$0 = V(y) = \Lambda_{J\mathfrak{V}}(H_m(y)) = \sum_{j=1}^{h} J_j\mathfrak{V}\, H_m^j(y) = \sum_{j=1}^{t} J_j\mathfrak{V}\, H_m^j(y).$$

Since $\{J_1\mathfrak{V}, \dots, J_t\mathfrak{V}\}$ is linearly independent, it follows that $H_m^j(y) = 0$ for all $1 \leq j \leq t$ and all $m \geq 2$. Hence,

$$\mathscr{F}(x) = \begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix}$$

where the 0 has length $t - g$ and $\Psi$ has length $h - t$ and moreover, $\Psi(0) = 0$ and $\Psi'(0) = 0$.

Let $\psi$ denote the inverse of $\varphi$,

$$\psi(x) = x(I + \Lambda_\Xi(x))^{-1}.$$

Thus, $\psi \circ \mathscr{F} = \iota \circ f = \begin{pmatrix} f(x) & 0 & 0 \end{pmatrix}$ and consequently,

$$\begin{pmatrix} f(x) & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix} \left( (I + \Lambda_\Xi(\begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix})) \right)^{-1}. \tag{5.8}$$

Rearranging gives,

$$\begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix} = \begin{pmatrix} f(x) & 0 & 0 \end{pmatrix} (I + \Lambda_\Xi(\begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix})). \tag{5.9}$$

We now examine the $k$-th entry on the right hand side of equation (5.9). First,

$$\begin{aligned} \big(I + \Lambda_\Xi(\begin{pmatrix} x & 0 & \Psi(x) \end{pmatrix})\big)_{\ell,k} &= \big(I + \sum_{j=1}^{g} \Xi_j x_j + \sum_{j=t+1}^{h} \Xi_j \Psi_{j-t}\big)_{\ell,k} \\ &= I_{\ell,k} + \sum_{j=1}^{g} (\Xi_j)_{\ell,k} x_j + \sum_{j=t+1}^{h} (\Xi_j)_{\ell,k} \Psi_{j-t}. \end{aligned}$$

Since $(\Xi_j)_{\ell,k} = 0$ for $j > t$ and $k \leq t$ (see Lemma 5.10), if $k \leq t$, then

$$\big(I + \Lambda_\Xi((x \quad 0 \quad \Psi(x)))\big)_{\ell,k} = I_{\ell,k} + \sum_{j=1}^{g} (\Xi_j)_{\ell,k} x_j$$

for all $\ell$. Hence, the right hand side of equation (5.9), for $g < k \leq t$ (so that $I_{\ell,k} = 0$ for $\ell \leq g$) is,

$$\sum_{\ell=1}^{g} f^\ell(x) \big(I + \Lambda_\Xi((x \quad 0 \quad \Psi(x)))\big)_{\ell,k} = \sum_{j,\ell=1}^{g} (\Xi_j)_{\ell,k} f^\ell(x)\, x_j \tag{5.10}$$

and similarly, for $1 \leq k \leq g$,

$$\sum_{\ell=1}^{g} f^\ell(x)\, (I + \sum_{j=1}^{g} \Xi_j x_j + \sum_{j=t+1}^{h} \Xi_j \Psi_{j-t})_{\ell,k} = f^k(x) + \sum_{j,\ell=1}^{g} (\Xi)_{\ell,k} f^\ell(x)\, x_j. \tag{5.11}$$

Combining equations (5.10) and (5.9), for $g < k \leq t$,

$$\sum_{j=1}^{g} \left[ \sum_{\ell=1}^{g} (\Xi_j)_{\ell,k} f^\ell(x) \right] x_j = 0.$$

Hence, for each $1 \leq j \leq g$ and $g < k \leq t$,

$$\sum_{\ell=1}^{g} (\Xi_j)_{\ell,k} f^\ell(x) = 0.$$

Since $\{f^1, \ldots, f^g\}$ is linearly independent, it follows that

$$(\Xi_j)_{\ell,k} = 0, \quad 1 \leq j, \ell \leq g,\ g < k \leq t. \tag{5.12}$$

We next show $\widehat{\Xi} \in M_g(\mathbb{C})^g$ defined by

$$(\widehat{\Xi}_j)_{\ell,k} = (\Xi_j)_{\ell,k}, \quad 1 \leq j, \ell, k \leq g$$

is convexotonic. Using equation (5.12), for $1 \leq j, \ell \leq g$,

$$\begin{aligned} A_\ell A_j \mathfrak{V} = J_\ell J_j \mathfrak{V} &= -\sum_{s=1}^{h} (\Xi_j)_{\ell,s} J_s \mathfrak{V} = -\sum_{s=1}^{t} (\Xi_j)_{\ell,s} J_s \mathfrak{V} \\ &= -\sum_{s=1}^{g} (\Xi_j)_{\ell,s} J_s \mathfrak{V} = -\sum_{s=1}^{g} (\Xi_j)_{\ell,s} A_s \mathfrak{V}. \end{aligned} \tag{5.13}$$

Multiplying equation (5.13) on the left by $U^*$ and using equation (5.6) gives

$$\begin{pmatrix} E_\ell & 0 \\ 0 & R_\ell \end{pmatrix} (-U) \begin{pmatrix} E_j \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{s=1}^g (\Xi_j)_{\ell,s} E_s \\ 0 \end{pmatrix}.$$

Using equation (5.5), it follows that

$$E_\ell(-U_{11})E_j = \sum_{s=1}^g (\Xi_j)_{\ell,s} E_s = \sum_{s=1}^g (\widehat{\Xi}_j)_{\ell,s} E_s. \tag{5.14}$$

By Lemma 2.7, the tuple $\widehat{\Xi}$ is convexotonic.

Combining equation (5.9) and equation (5.11), if $1 \le k \le g$, then

$$\begin{aligned} x_k &= \sum_{\ell=1}^g f^\ell(x) \left(I + \Lambda_\Xi((x \quad 0 \quad \Psi(x)))\right)_{\ell,k} \\ &= f^k(x) + \sum_{j,\ell=1}^g (\Xi_j)_{\ell,k} f^\ell(x)\, x_j = f^k(x) + \sum_{j,\ell=1}^g (\widehat{\Xi}_j)_{\ell,k} f^\ell(x)\, x_j. \end{aligned}$$

Thus,

$$x = f(x)(I + \Lambda_{\widehat{\Xi}}(x))$$

and consequently

$$f(x) = x(I + \Lambda_{\widehat{\Xi}}(x))^{-1} \tag{5.15}$$

is convexotonic.

We now complete the proof by showing, if $A$ is minimal for $\mathcal{D}_A$ (we continue to assume $E$ is ball-minimal), then $A$ is unitarily equivalent to

$$B = U \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U_{11}E & 0 \\ U_{21}E & 0 \end{pmatrix} \in M_r(\mathbb{C})^g \tag{5.16}$$

and $B$ spans an algebra. To this end, using equations (5.16) and (5.14), observe

$$B_\ell B_j = \begin{pmatrix} U_{11}E_\ell U_{11}E_j & 0 \\ U_{21}E_\ell U_{11}E_j & 0 \end{pmatrix} = \sum_{s=1}^g (-\widehat{\Xi}_j)_{\ell,s} \begin{pmatrix} U_{11}E_s & 0 \\ U_{21}E_s & 0 \end{pmatrix} = \sum_{s=1}^g (-\widehat{\Xi}_j)_{\ell,s} B_s.$$

Thus $B$ spans an algebra and, by Proposition 2.2, the convexotonic map $f$ of equation (5.15) is a bianalytic map $f : \operatorname{int}(\mathcal{B}_B) \to \operatorname{int}(\mathcal{D}_B)$. On the other hand, $\mathcal{B}_B = \mathcal{B}_E = \mathcal{B}_A$. Thus, as $f : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_A)$ is bianalytic, $\mathcal{D}_B = \mathcal{D}_A$. Since $A$ is minimal defining for $\mathcal{D}_A$ and $A$ and $B$ have the same size, $B$ is minimal for $\mathcal{D}_A$. Hence $A$ and $B$ are unitarily equivalent by Lemma 3.1. From the form of $B$, it is evident that $r \ge \max\{d, e\}$. On the other hand, if $r > d + e$, then $B$ must have 0 as a direct summand and so is not minimal. Thus $r \le d + e$. □

### *5.3. Corollary 1.3*

This subsection begins by illustrating Corollary 1.3 in the case of free automorphism of free matrix balls and free polydiscs before turning to the proof of the corollary.

#### *5.3.1. Automorphisms of free polydiscs*

Let $\{e_1, \dots, e_g\}$ denote the usual orthonormal basis for $\mathbb{C}^g$ and let $E_j = e_j e_j^*$. The spectraball $\mathcal{B}_E$ is then the **free polydisc** with

$$\operatorname{int}(\mathcal{B}_E(n)) = \{X \in M_n(\mathbb{C})^g : \|X_j\| < 1\}.$$

Let $b \in \operatorname{int}(\mathcal{B}_E(1)) = \mathbb{D}^g$ be given.

In the setting of Corollary 1.3, we choose $C = E$. If $\mathscr{V}, \mathscr{W}$ are $g \times g$ unitary matrices such that equation Corollary 1.3(b) holds, then there exists a $g \times g$ permutation matrix $\Pi$ and unitary diagonal matrices $\rho$ and $\mu$ such that $\mathscr{W} = \Pi\rho$ and $\mathscr{V} = \mu\Pi$. We can in fact assume $\mu = I_g$. It is now evident that item (a) of Corollary 1.3 holds and determines $\Xi$. Conversely, given a triple $(b, \Pi, \rho)$, where $b \in \mathbb{D}^g$, $\Pi$ is a $g \times g$ permutation matrix and $\rho$ is a diagonal unitary matrix, the equations (b) and (a) of Corollary 1.3 hold with $\mathscr{W} = \Pi\rho$ and $\mathscr{V} = \Pi$. Hence the automorphisms of $\mathcal{B}_E$ are determined by triples $(b, \Pi, \rho)$.

By pre (or post) composing with a permutation, we may assume $\Pi = I_g$. In this case $M$ is the $g \times g$ diagonal matrix with diagonal entries $M_{jj} = \rho_j(1 - |b_j|^2)$ and $\Xi_k = -\rho_j b_j^* E_k$. The corresponding convexotonic map $\psi(x) = x(I - \Lambda_\Xi(x))^{-1}$ has entries

$$\psi^j(x) = x_j(1 + c_j^* x_j)^{-1},$$

where $c_j = \rho_j b_j^*$. Thus the mapping $\varphi(x) = \psi(x) \cdot M + b$ has entries,

$$\varphi^j(x) = \rho_j x_j (1 + c_j^* x_j)^{-1}(1 - |b_j|^2) + b_j = \rho_j(x_j + c_j)(1 + c_j^* x_j)^{-1},$$

where $c_j = \rho_j b_j^*$. Hence, the automorphisms of the free polydisc are given by

$$\varphi(x) = \Big(\rho_{\pi(1)}(x_{\pi(1)} + c_{\pi(1)})(1 + c_{\pi(1)}^* x_{\pi(1)})^{-1}, \dots,$$
$$\rho_{\pi(g)}(x_{\pi(g)} + c_{\pi(g)})(1 + c_{\pi(g)}^* x_{\pi(g)})^{-1}, \Big)$$

for $c = (c_1, \dots, c_g) \in \mathbb{D}^g$, unimodular $\rho_j$ and a permutation $\pi$ of $\{1, \dots, g\}$.

#### *5.3.2. Automorphisms of free matrix balls*

Let $(E_{ij})_{i,j=1}^{d,e}$ denote the matrix units in $M_{d\times e}(\mathbb{C})$ and view $E \in M_{d\times e}(\mathbb{C})^{de}$. We consider automorphisms of $\mathcal{B}_E$, the **free** $d \times e$ **matrix ball**.

Before proceeding further, note, since $\{E_{ij} : 1 \le i \le d, 1 \le j \le e\}$ spans all of $M_{d\times e}(\mathbb{C})$, by the reverse implication in Corollary 1.3, any choice of $b$ in the unit ball of $M_{d\times e}(\mathbb{C})$ and $d \times d$ and $e \times e$ unitary matrices $\mathscr{W}$ and $\mathscr{V}$ determines uniquely a $g \times g$ invertible matrix $M$ satisfying the identity of item (b) of Corollary 1.3. Likewise a convexotonic tuple is uniquely determined by the identity of item (a). The resulting bianalytic automorphism $\varphi$ of $\mathcal{B}_E$ satisfying $\varphi(0) = b$ and $\varphi'(0) = M$ is then given by the formula in Corollary 1.3. Our objective in the remainder of this example is to show this formula for $\varphi$ agrees with that of [45, Theorem 13]. Doing so requires passing back and forth between row vectors of length $de$ and matrices of size $d \times e$.

First note that

$$\Lambda_E(b) = b.$$

From item (b) of Corollary 1.3 (which defines $M$ in terms of $b$, $\mathscr{V}$ and $\mathscr{W}$),

$$\begin{aligned}\sum_{u,v} M_{(i,j),(u,v)} E_{u,v} &= (M \cdot E)_{i,j} \\ &= D_{\Lambda_E(b)^*} \mathscr{W} E_{i,j} \mathscr{V}^* D_{\Lambda_E(b)} \\ &= \sum_{u,v} [e_u^* D_{\Lambda_E(b)^*} \mathscr{W} e_i]\, [e_j^* \mathscr{V}^* D_{\Lambda_E(b)} e_v]\, e_u e_v^*.\end{aligned}$$

Hence,

$$M_{(i,j),(u,v)} = [e_u^* D_{\Lambda_E(b)^*} \mathscr{W} e_i]\, [e_j^* \mathscr{V}^* D_{\Lambda_E(b)} e_v].$$

Next observe that,

$$-E_{ij} \mathscr{V}^* \Lambda_E(b)^* \mathscr{W} E_{st} = -e_i e_j^* \mathscr{V}^* b^* \mathscr{W} e_s e_t^* = -(e_j^* \mathscr{V}^* b^* \mathscr{W} e_s) E_{it}.$$

Hence, letting $\beta_{js} = -(e_j^* \mathscr{V}^* b^* \mathscr{W} e_s)$ for $1 \le j \le e$ and $1 \le s \le d$, the tuple $\Xi \in M_{de}(\mathbb{C})^{de}$ defined by (for $1 \le i, u \le d$ and $1 \le v \le e$)

$$(\Xi_{st})_{(i,j),(u,v)} = \begin{cases} \beta_{js} & v = t,\ u = i \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the identity of equation item (a) of Corollary 1.3. Hence the free bianalytic automorphism of $\mathcal{B}_E$ determined by $b$, $\mathscr{W}$ and $\mathscr{V}$ is

$$\varphi(x) = \psi(x) \cdot M + b \tag{5.17}$$

where $\psi = x(I - \Lambda_\Xi(x))^{-1}$ is the convexotonic map determined by $\Xi$.

We next express formula for $\varphi$ in equation (5.17) in terms of the canonical matrix structure on $\mathcal{B}_E$. Given a matrix $y = (y_{ij})_{i,j=1}^{d,e}$, let

$$\operatorname{row}(y) = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1e} & y_{21} & \cdots & y_{de} \end{pmatrix}.$$

Similarly, given $z = (z_j)_{j=1}^{de}$, let

$$\operatorname{mat}_{d\times e}(z) = \begin{pmatrix} z_1 & z_2 & \dots & z_e \\ z_{e+1} & z_{e+2} & \dots & z_{2e} \\ \vdots & \vdots & \dots & \vdots \\ z_{(d-1)e} & z_{(d-1)e+1} & \dots & z_{de} \end{pmatrix}.$$

Since $d$ and $e$ are fixed in this example, it is safe to abbreviate $\operatorname{mat}_{d\times e}$ to simply mat. For a tuple $y = (y_{s,t})_{s,t=1}^{d,e}$ of indeterminates,

$$\begin{aligned}(y \cdot M)_{u,v} &= \sum_{i,j} M_{(i,j),(u,v)} y_{i,j} \\ &= \sum_{i,j} [e_u^* D_{\Lambda_E(b)^*} \mathscr{W}]\, y_{i,j} e_i e_j^* \,[\mathscr{V}^* D_{\Lambda_E(b)} e_v] \\ &= e_u^* \,[D_{\Lambda_E(b)^*} \mathscr{W}] \operatorname{mat}(y)\, [\mathscr{V}^* D_{\Lambda_E(b)}]\, e_v.\end{aligned}$$

Thus,

$$\operatorname{mat}(y \cdot M) = D_{\Lambda_E(b)^*} \mathscr{W} \operatorname{mat}(y)\, \mathscr{V}^* D_{\Lambda_E(b)}. \tag{5.18}$$

Let

$$\Gamma_{(i,j),(u,v)}(x) := \Big( \sum_{s,t=1}^{d,e} \Xi_{st} x_{st} \Big)_{(i,j),(u,v)} = \begin{cases} \sum_{s=1}^d \beta_{js} x_{sv} & u = i \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Gamma$ is a $de \times de$ linear matrix polynomial of the form,

$$\Gamma = I_d \otimes \beta \operatorname{mat}(x)$$

and $(I - \Gamma)^{-1} = I_d \otimes (I - \beta \operatorname{mat}(x))^{-1}$. In the formula for the convexotonic map $\psi$ determined by $\Xi$, the indeterminates $x = (x_{st})_{s,t}$ are arranged in a row and we find,

$$\begin{aligned}\operatorname{row}(\psi(x)) &= \operatorname{row}(x)(I - \Lambda_\Xi(x))^{-1} \\ &= (x_{11} \quad x_{12} \quad \dots x_{1e} \quad x_{21} \quad \dots x_{de}) \Big( I \otimes (I - \beta \operatorname{mat}(x))^{-1} \Big) \\ &= \big( \hat{x}_1 (I - \beta \operatorname{mat}(x))^{-1} \quad \dots \quad \hat{x}_d (I - \beta \operatorname{mat}(x))^{-1} \big),\end{aligned}$$

where $\hat{x}_j = (x_{j1} \quad \dots x_{je})$. Thus,

$$\begin{aligned}&\operatorname{row}(x)(I - \Lambda_\Xi(x))^{-1} \\ &\quad = \Big( (\operatorname{mat}(x)[I - \beta \operatorname{mat}(x)]^{-1})_{11} \quad (\operatorname{mat}(x)[I - \beta \operatorname{mat}(x)]^{-1})_{12} \quad \dots \\ &\qquad (\operatorname{mat}(x)[I - \beta \operatorname{mat}(x)]^{-1})_{de} \Big).\end{aligned}$$

Hence, in matrix form,

$$\operatorname{mat}(\psi(x)) = \operatorname{mat}(x)(I - \beta \operatorname{mat}(x))^{-1} = \operatorname{mat}(x)(I + (\mathscr{V}^* b^* \mathscr{W}) \operatorname{mat}(x))^{-1}.$$

Let $c = \mathscr{W}^* b \mathscr{V}$ and note

$$I - \Lambda_E(b)\Lambda_E(b)^* = I - bb^* = I - \mathscr{W} cc^* \mathscr{W}^* = \mathscr{W}(I - cc^*)\mathscr{W}^* = \mathscr{W}(I - \Lambda_E(c)\Lambda_E(c)^*)\mathscr{W}^*.$$

Thus,

$$D_{\Lambda_E(b)^*} \mathscr{W} = \mathscr{W} D_{\Lambda_E(c)^*} \tag{5.19}$$

and similarly $\mathscr{V}^* D_{\Lambda_E(b)} = D_{\Lambda_E(c)} \mathscr{V}^*$. Consequently, using, in order, equations (5.17), (5.18), and (5.19) together with the definition of $c$ in the first three equalities followed by some algebra,

$$\begin{aligned}
\operatorname{mat}(\varphi(x)) &= \operatorname{mat}(\psi(x) \cdot M) + b \\
&= D_{\Lambda_E(b)^*} \mathscr{W} \operatorname{mat}(\psi) \mathscr{V}^* D_{\Lambda_E(b)} + b \\
&= \mathscr{W}[D^*_{\Lambda_E(c)} \operatorname{mat}(\psi) D_{\Lambda_E(c)} + c]\mathscr{V}^* \\
&= \mathscr{W} D_{\Lambda_E(c)^*}[\operatorname{mat}(\psi) + D^{-2}_{\Lambda_{E(c)^*}} c] D_{\Lambda_E(c)} \mathscr{V}^* \\
&= \mathscr{W} D_{\Lambda_E(c)^*}[\operatorname{mat}(x)(I + c^* \operatorname{mat}(x))^{-1} + D^{-2}_{\Lambda_{E(c)^*}} c] D_{\Lambda_E(c)} \mathscr{V}^* \\
&= \mathscr{W} D^{-1}_{\Lambda_E(c)^*}[D^2_{\Lambda_{E(c)^*}} \operatorname{mat}(x) + c(I + c^* \operatorname{mat}(x))]\, [I + c^* \operatorname{mat}(x)]^{-1} D_{\Lambda_E(c)} \mathscr{V}^* \\
&= \mathscr{W}(I - cc^*)^{-\frac{1}{2}}[(1 - cc^*) \operatorname{mat}(x) + c \\
&\quad + cc^* \operatorname{mat}(x)][I + c^* \operatorname{mat}(x)]^{-1} D_{\Lambda_E(c)} \mathscr{V}^* \\
&= \mathscr{W}(I - cc^*)^{-\frac{1}{2}}[\operatorname{mat}(x) + c][I + c^* \operatorname{mat}(x)]^{-1}(I - c^* c)^{\frac{1}{2}} \mathscr{V}^*,
\end{aligned}$$

giving the standard formula for the automorphisms of $\mathcal{B}_E$ that send 0 to $b$. (See, for example, [45].)

*5.3.3. Proof of Corollary 1.3*

Suppose $E = (E_1, \dots, E_g) \in M_{d\times e}(\mathbb{C})^g$ and $C = (C_1, \dots, C_g) \in M_{k\times \ell}(\mathbb{C})^g$ are linearly independent and ball-minimal and $\varphi : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{B}_C)$ is bianalytic.

Let $\widehat{C}$ denote the tuple

$$\widehat{C}_j = \begin{pmatrix} 0_{k,k} & C_j \\ 0_{\ell,k} & 0_{\ell,\ell} \end{pmatrix} \in M_r(\mathbb{C}),$$

where $r = k + \ell$. Thus $\mathcal{B}_C = \mathcal{D}_{\widehat{C}}$ and, since $C$ is ball-minimal, $\widehat{C}$ is minimal for $\mathcal{D}_{\widehat{C}}$ by Lemma 3.2(3).

Let $b = \varphi(0)$ and for notational convenience, let $\Lambda = \Lambda_C(b) \in M_{k\times\ell}(\mathbb{C})$. Set

$$\mathscr{G} = \begin{pmatrix} I_k & \Lambda \\ 0 & D_\Lambda \end{pmatrix}^{-1} = \begin{pmatrix} I_k & -\Lambda D_\Lambda^{-1} \\ 0 & D_\Lambda^{-1} \end{pmatrix} \in M_r(\mathbb{C}), \tag{5.20}$$

and observe that $\mathscr{G}^*\mathbb{L}_C(b)\mathscr{G} = I$ and therefore $\mathbb{L}_C(b)^{-1} = \mathscr{G}\mathscr{G}^*$. Hence there is a unitary matrix $T$ such that $\mathscr{G} = \mathbb{L}_E(b)^{-\frac{1}{2}}T$. It follows from Proposition 3.3, letting $A \in M_{r\times r}(\mathbb{C})^g$ denote the $g$-tuple with entries

$$A_j = \mathscr{G}^* \begin{pmatrix} 0 & (M\cdot C)_j \\ 0 & 0 \end{pmatrix} \mathscr{G} \in M_r(\mathbb{C})^g, \tag{5.21}$$

and $M = \varphi'(0)$, that the inverse of the mapping $\lambda(x) = x \cdot M + b$ is an affine linear bijection from $\mathcal{B}_C = \mathcal{D}_{\widehat{C}}$ to $\mathcal{D}_A$ and $A$ is minimal for $\mathcal{D}_A$.

The mapping

$$f := \lambda^{-1} \circ \varphi : \operatorname{int}(\mathcal{B}_E) \to \operatorname{int}(\mathcal{D}_A)$$

is a free bianalytic mapping with $f(0) = 0$ and $f'(0) = I$, where $E$ is ball-minimal and $A$ is minimal for $\mathcal{D}_A$. An application of Theorem 1.1 now implies that there is a convexotonic tuple $\Xi$ such that equation (1.1) holds, $f$ is the corresponding convexotonic map and there are unitaries $V$ and $W$ of size $r$ such that

$$A = W \begin{pmatrix} 0_{d,r-e} & E \\ 0_{r-d,r-e} & 0_{r-d,e} \end{pmatrix} V^*. \tag{5.22}$$

In particular, $\varphi(x) = f(x) \cdot M + b$.

From equation (5.22),

$$\sum A_j^* A_j = V \begin{pmatrix} 0 & 0 \\ 0 & \sum_j E_j^* E_j \end{pmatrix} V^*$$

and consequently $\operatorname{rk} \sum A_j^* A_j = \operatorname{rk} \sum E_j^* E_j$. Since $E$ is ball-minimal, $\ker(E) = \{0\}$. Equivalently, $\operatorname{rk} \sum E_j^* E_j = e$. On the other hand, from equation (5.21),

$$\sum A_j^* A_j = \mathscr{G}^* \begin{pmatrix} 0 & 0 \\ 0 & (M\cdot C)_j^* \Gamma (M\cdot C)_j \end{pmatrix} \mathscr{G},$$

where $\Gamma$ is the $(1,1)$ block entry of $\mathscr{G}\mathscr{G}^*$. Observe that $\Gamma$ is positive definite and, since $C$ is ball-minimal, $\ker(M \cdot C) = \{0\}$. Hence $\operatorname{rk} \sum A_j^* A_j = \ell$. Thus $e = \ell$. Computing $\sum A_j A_j^*$ using equation (5.22) shows $\operatorname{rk} \sum A_j A_j^* = d$. On the other hand, using equation (5.21),

$$\sum_{j=1}^g A_j A_j^* = \mathscr{G} \begin{pmatrix} \sum_{j=1}^g (M\cdot C)_j D_\Lambda^{-2} (M\cdot C)_j^* & 0 \\ 0 & 0 \end{pmatrix} \mathscr{G}^*.$$

Since $C$ is $k \times \ell$ and ball-minimal, $\ker((M \cdot C)^*) = \{0\}$ and $D_\Lambda^{-2}$ is positive definite, $\operatorname{rk} \sum_{j=1}^g (M \cdot C)_j D_\Lambda^{-2} (M \cdot C)_j^* = k$. Hence $d = \operatorname{rk} \sum_{j=1}^g A_j A_j^* = k$. Thus $E$ and $C$ have the same size $d \times e$.

Since $E$ and $C$ are both $d \times e$ and $r = d + e$, the matrices $V$ and $W$ decompose as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \qquad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}^r = \mathbb{C}^d \oplus \mathbb{C}^e$. In particular, $V_{jj}$ and $W_{jj}$ are all square. Comparing equation (5.22) and equation (5.21) gives

$$\begin{pmatrix} W_{11} E_j V_{12}^* & W_{11} E_j V_{22}^* \\ W_{21} E_j V_{12}^* & W_{21} E_j V_{22}^* \end{pmatrix} = \begin{pmatrix} 0 & (M \cdot C)_j D_\Lambda^{-1} \\ 0 & -D_\Lambda^{-1} \Lambda^* (M \cdot C)_j D_\Lambda^{-1} \end{pmatrix}. \tag{5.23}$$

Multiplying both sides of equation (5.23) by $\begin{pmatrix} W_{11}^* & W_{21}^* \end{pmatrix}$ and using the fact that $W$ is unitary shows,

$$E_j V_{12}^* = 0.$$

Since $E$ is ball-minimal and $\sum E_j^* E_j V_{12}^* = 0$ we conclude that $V_{12} = 0$. Since $V$ is unitary, $V_{22}$ is isometric and since $V_{22}$ is square ($e \times e$) it is unitary (and thus $V_{21} = 0$). Further,

$$\begin{aligned} W_{11} E_j V_{22}^* &= (M \cdot C)_j D_\Lambda^{-1} \\ W_{21} E_j V_{22}^* &= -D_\Lambda^{-1} \Lambda^* (M \cdot C)_j D_\Lambda^{-1}. \end{aligned} \tag{5.24}$$

Thus, $W_{21} E_j V_{22}^* = -D_\Lambda^{-1} \Lambda^* W_{11} E_j V_{22}^*$ and hence $W_{21} E_j = -D_\Lambda^{-1} \Lambda^* W_{11} E_j$. It follows that

$$W_{21} \sum E_j E_j^* = -D_\Lambda^{-1} \Lambda^* W_{11} \sum E_j E_j^*.$$

Thus, again using that $E$ is ball-minimal (so that $\ker(E^*) = \{0\}$),

$$W_{21} = -D_\Lambda^{-1} \Lambda^* W_{11}.$$

Hence,

$$I = W_{11}^* W_{11} + W_{21}^* W_{21} = W_{11}^* [I + \Lambda D_\Lambda^{-2} \Lambda^*] W_{11} = W_{11}^* D_{\Lambda^*}^{-2} W_{11}$$

and, since $W_{11}$ is $d \times d$, we conclude that it is invertible and

$$W_{11} W_{11}^* = D_{\Lambda^*}^2.$$

Consequently there is a $d \times d$ unitary $\mathscr{W}$ such that

$$\begin{aligned} W_{11} &= D_{\Lambda^*}\mathscr{W} \\ W_{21} &= -D_\Lambda^{-1}\Lambda^* D_{\Lambda^*}\mathscr{W} = -\Lambda^*\mathscr{W}. \end{aligned} \tag{5.25}$$

Combining the first bits of each of equations (5.24) and (5.25) and setting $\mathscr{V} = V_{22}$ gives Corollary 1.3(b). Namely,

$$(M \cdot C)_j = D_{\Lambda^*}\mathscr{W} E_j \mathscr{V}^* D_\Lambda.$$

Observe (using $E$ and $C$ have the same size) that,

$$A = W \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0_{e\times d} & I_e \\ I_d & 0_{d\times e} \end{pmatrix} V^*.$$

The tuple $A$ is, up to unitary equivalence, of the form of equation (1.3) where

$$U = \begin{pmatrix} 0 & V_{22}^* \\ V_{11}^* & 0 \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} \mathscr{V}^* W_{21} & * \\ * & * \end{pmatrix}.$$

Thus, $U_{11} = \mathscr{V}^* W_{21} = -\mathscr{V}^*\Lambda^*\mathscr{W}$. Since the pair $(A, \Xi)$ satisfies equation (1.1),

$$\begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix} U \begin{pmatrix} E_j & 0 \\ 0 & 0 \end{pmatrix} = \sum_s (\Xi_j)_{k,s} \begin{pmatrix} E_s & 0 \\ 0 & 0 \end{pmatrix},$$

item (a) holds.

To prove the converse, suppose $E, C \in M_{d\times e}(\mathbb{C})^g$ and $b \in \mathcal{B}_C(1)$ are given and there exists an invertible $M \in M_g(\mathbb{C})$, a convexotonic tuple $\Xi \in M_g(\mathbb{C})^g$ and unitaries $\mathscr{W}$ and $\mathscr{V}$ such that items (a) and (b) of Corollary 1.3 hold. Let $\Lambda = \Lambda_C(b)$ and define $\mathscr{G}$ and $A$ as in equations (5.20) and (5.21) respectively. The map $\lambda(x) = x \cdot M + b$ is again an affine linear bijection from $\mathcal{D}_A$ to $\mathcal{B}_C$.

Define $W_{11}$ and $W_{21}$ by equation (5.25). It follows that $W_{11}W_{11}^* + W_{21}W_{21}^* = I$. Choose $W_{12}$ and $W_{22}$ such that $W = (W_{ij})_{i,j=1}^2$ is a (block) unitary matrix. Let $V_{22} = \mathscr{V}$ and take any unitary $V_{11}$ (of the appropriate size) and set

$$V = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix}.$$

Next, using item (b), the definitions of $W_{11}$ and $W_{12}$ and $D_\Lambda^{-1}\Lambda^* D_{\Lambda^*} = \Lambda^*$,

$$\begin{aligned} A_k &= \mathscr{G}^* \begin{pmatrix} 0 & (M\cdot C)_k \\ 0 & 0 \end{pmatrix} \mathscr{G} = \begin{pmatrix} 0 & (M\cdot C)_k D_\Lambda^{-1} \\ 0 & -D_\Lambda^{-1}\Lambda^*(M\cdot C)_k D_\Lambda^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & D_{\Lambda^*}\mathscr{W} E_k \mathscr{V}^* \\ 0 & -\Lambda^*\mathscr{W} E_k \mathscr{V}^* \end{pmatrix} = \begin{pmatrix} 0 & W_{11} E_k \mathscr{V}^* \\ 0 & W_{21} E_k \mathscr{V}^* \end{pmatrix}. \end{aligned}$$

Thus, using item (a),

$$A_jA_k = \begin{pmatrix} 0 & W_{11}E_j\mathscr{V}^*W_{21}E_k\mathscr{V}^* \\ 0 & W_{21}E_j\mathscr{V}^*W_{21}E_k\mathscr{V}^* \end{pmatrix} = \sum_s (\Xi_k)_{j,s} \begin{pmatrix} 0 & W_{11}E_s\mathscr{V}^* \\ 0 & W_{21}E_s\mathscr{V}^* \end{pmatrix} = \sum_s (\Xi_k)_{j,s} A_s.$$

Thus $A$ spans an algebra with multiplication table given by $\Xi$. Consequently $f(x) = x(I - \Lambda_\Xi(x))^{-1}$ is convexotonic from $\operatorname{int}(\mathcal{B}_A)$ to $\operatorname{int}(\mathcal{D}_A)$ by Proposition 2.2. On the other hand, $\mathcal{B}_A = \mathcal{B}_E$, since

$$A_j^*A_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{V}E_j^*E_k\mathscr{V}^* \end{pmatrix}$$

(because $W_{11}^*W_{11} + W_{21}^*W_{21} = I$). Thus $f$ is convexotonic from $\operatorname{int}(\mathcal{B}_E)$ to $\operatorname{int}(\mathcal{D}_A)$. Finally, $\varphi = \lambda \circ f$ is convexotonic from $\operatorname{int}(\mathcal{B}_E)$ to $\operatorname{int}(\mathcal{B}_C)$ with $\varphi(0) = b$ and $\varphi'(0) = M$.

The uniqueness is well known. Indeed, if $\varphi$ and $\zeta$ are both bianalytic from $\mathcal{B}_E \to \mathcal{B}_C$, send 0 to $b$ and have the same derivative at 0, then $f = \varphi \circ \zeta^{-1}$ is an analytic automorphism of $\mathcal{B}_C$ sending 0 to 0 and having derivative the identity at 0. Since $\mathcal{B}_C$ is *circular*, the free version of Cartan's Theorem [24] says $f(x) = x$ and hence $\zeta = \varphi$. □

## 6. Convex sets defined by rational functions

In this section we employ a variant of the main result of [32] to extend Theorem 1.1 to cover birational maps from a matrix convex set to a spectraball. A free set is **matrix convex** if it is closed with respect to isometric conjugation. We refer the reader to [17,21,27,43,48] for the theory of matrix convex sets. For expository convenience, by free rational mapping $p : M(\mathbb{C})^g \to M(\mathbb{C})^g$ we mean $p = \begin{pmatrix} p^1 & p^2 & \dots p^g \end{pmatrix}$ where each $p^j = p^j(x)$ is a free rational function (in the $g$-variables $x = (x_1, \dots, x_g)$) regular at 0. Theorem 6.1 immediately below is the main result of this section. It is followed up by two corollaries.

**Theorem 6.1.** *Suppose $q : M(\mathbb{C})^g \to M(\mathbb{C})^g$ is a free rational mapping, $\mathscr{C} \subseteq M(\mathbb{C})^g$ is a bounded open matrix convex set containing the origin and $E \in M_{d\times e}(\mathbb{C})^g$. If $E$ is linearly independent, $\mathscr{C} \subseteq \operatorname{dom}(q)$ and $q : \mathscr{C} \to \operatorname{int}(\mathcal{B}_E)$ is bianalytic, then there exists an $r \le d+e$ and a tuple $A \in M_r(\mathbb{C})^g$ such that $\mathscr{C} = \operatorname{int}(\mathcal{D}_A)$ and $q$ is, up to affine linear equivalence, convexotonic.*

**Corollary 6.2.** *Suppose $p : M(\mathbb{C})^g \to M(\mathbb{C})^g$ is a free rational mapping, $E \in M_{d\times e}(\mathbb{C})^g$ is linearly independent and let*

$$\mathscr{C} := \{X : X \in \operatorname{dom}(p),\ \|\Lambda_E(p(X))\| < 1\}.$$

*Assume $\mathscr{C}$ is bounded, convex and contains 0. If $X_k \in \mathscr{C}(n)$ and the sequence $(X_k)_k$ converges to $X \in \partial\mathscr{C}$ implies $\lim_{k\to\infty} \|\Lambda_E(p(X_k))\| = 1$, then there exists an $r \le d+e$ and a tuple $A \in M_r(\mathbb{C})^g$ such that $\mathscr{C} = \operatorname{int}(\mathcal{D}_A)$ and $p : \operatorname{int}(\mathcal{D}_A) \to \operatorname{int}(\mathcal{B}_E)$ is bianalytic and, up to affine linear equivalence, convexotonic.*

**Proof.** By assumption $p : \mathscr{C} \to \operatorname{int}(\mathcal{B}_E)$ is a proper map. By [24, Theorem 3.1], $p$ is bianalytic. Hence Corollary 6.2 follows from Theorem 6.1. □

**Corollary 6.3.** *Suppose* $p : M(\mathbb{C})^g \to M(\mathbb{C})^g$ *is a free polynomial mapping,* $E \in M_{d\times e}(\mathbb{C})^g$ *is linearly independent and let*

$$\mathscr{C} := \{X : \|\Lambda_E(p(X))\| < 1\}.$$

*If* $\mathscr{C}$ *is bounded, convex and contains* 0*, then there exists an* $r \le d + e$ *and a tuple* $A \in M_r(\mathbb{C})^g$ *such that* $\mathscr{C} = \operatorname{int}(\mathcal{D}_A)$ *and* $p : \operatorname{int}(\mathcal{D}_A) \to \operatorname{int}(\mathcal{B}_E)$ *is bianalytic and, up to affine linear equivalence, convexotonic.*

**Proof.** By hypothesis $p : \mathscr{C} \to \operatorname{int}(\mathcal{B}_E)$. Let $X \in \partial\mathscr{C}$ be given. By convexity and continuity $p(tX) \in \operatorname{int}(\mathcal{B}_E)$ for $0 \le t < 1$ and $p(X) \in \mathcal{B}_E$. If $p(X) \in \operatorname{int}(\mathcal{B}_E)$, then there exists $t_* > 1$ such $p(t_*X) \in \operatorname{int}(\mathcal{B}_E)$. But then $0$, $t_*X \in \mathscr{C}$ and $X \notin \mathscr{C}$, violating convexity of $\mathscr{C}$. Hence $p(X) \in \partial\mathcal{B}_E$ and consequently $p$ is a proper map. Thus Corollary 6.3 follows from Corollary 6.2. □

The proof of Theorem 6.1 given here depends on two preliminary results. Let $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle$ denote the **skew field of free rational functions** in the freely noncommuting variables $(x, y) = (x_1, \dots, x_g, y_1, \dots, y_g)$. There is an involution $\check{}$ on $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle$ determined by $\check{x}_j = y_j$. A $p \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle$ is **symmetric** if $\check{p} = p$. An important feature of the involution is the fact that, if $p \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle$ and $(X, X^*) \in \operatorname{dom}(p)$, then $\check{p}(X, X^*) = p(X, X^*)^*$ and $p$ is symmetric if and only if $\check{p}(X, X^*) = p(X, X^*)$ for all $(X, X^*) \in \operatorname{dom}(p) \cap \operatorname{dom}(\check{p})$. These notions naturally extend to matrices over $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle$.

Proposition 6.4 below is a variant of the main result of [32]. Taking advantage of recent advances in our understanding of the singularities of free rational functions (e.g., [55]), the proof given here is rather short, compared to that of the similar result in [32].

**Proposition 6.4.** *Suppose* $s(x, y)$ *is a* $\mu \times \mu$ *symmetric matrix-valued free rational function in the* $2g$*-variables* $(x_1, \dots, x_g, y_1, \dots, y_g)$ *that is regular at* 0*. Let*

$$S = \{X \in M(\mathbb{C})^g : (X, X^*) \in \operatorname{dom}(s),\ s(X, X^*) \succ 0\},$$

*let* $S^0$ *denote the (level-wise) connected component of* 0 *of* $S$*, and assume* $S^0(1) \neq \emptyset$*. If each* $S^0(n)$ *is convex, then there is a positive integer* $N$ *and a tuple* $A \in M_N(\mathbb{C})^g$ *such that* $S^0 = \operatorname{int}(\mathcal{D}_A)$*.*

**Proof.** From [37,55] the free rational function $s$ has an observable and controllable realization. By [33], since $s$ is symmetric, this realization can be symmetrized. Hence, there exists a positive integer $t$, a tuple $T \in M_t(\mathbb{C})^g$, a signature matrix $J \in M_t(\mathbb{C})$ (thus $J = J^*$ and $J^2 = I$) and matrices $D$ and $C$ of sizes $\mu \times \mu$ and $t \times \mu$ respectively such that

$$s(x,y) = D + C^* L_{J,T}(x,y)^{-1} C$$

and $\operatorname{dom}(s) = \{(X,Y) : \det(L_{J,T}(X,Y)) \neq 0\}$, where

$$L_{J,T}(x,y) = J - \Lambda_T(x) - \Lambda_{T^*}(y) = J - \sum T_j x_j - \sum T_j^* y_j.$$

Let $\tilde{s}(x,y) = s(x,y)^{-1}$. Thus $\tilde{s}(x,y)$ is also a $\mu \times \mu$ symmetric matrix-valued free rational function. It has a representation,

$$\tilde{s}(x,y) = \tilde{D} + \tilde{C}^* L_{\tilde{J},\tilde{T}}(x,y)^{-1} \tilde{C},$$

with $\operatorname{dom}(\tilde{s}) = \{(X,Y) : \det(L_{\tilde{J},\tilde{T}}(X,Y)) \neq 0\}$. Let

$$Q(x) = \Big(\frac{J}{2} - \Lambda_T(x)\Big) \oplus \Big(\frac{\tilde{J}}{2} - \Lambda_{\tilde{T}}(x)\Big),$$

let $P(x,x^*) = Q(x) + Q(x)^*$, let $\mathscr{I} = \{X : \det(P(X)) \neq 0\}$ and let $\mathscr{I}^0$ denote its connected component of 0. Observe that $\{(X,X^*) : X \in \mathscr{I}\} = \{X : (X,X^*) \in \operatorname{dom}(s) \cap \operatorname{dom}(\tilde{s})\}$. In particular, if $X \in \mathscr{I}^0$, then $(X,X^*) \in \operatorname{dom}(s) \cap \operatorname{dom}(\tilde{s})$. On the other hand, if $(X,X^*) \in \operatorname{dom}(s)$ and $s(X,X^*) \succ 0$, then $s(X,X^*)$ is invertible and hence $(X,X^*) \in \operatorname{dom}(\tilde{s})$. Hence, if $X \in S^0$, then $(X,X^*) \in \operatorname{dom}(s) \cap \operatorname{dom}(\tilde{s})$ too.

Suppose $X \in S^0$. Thus $tX \in S^0$ for $0 \le t \le 1$ by convexity. It follows that $t(X,X^*) \in \operatorname{dom}(s) \cap \operatorname{dom}(\tilde{s})$. Hence $tX \in \mathscr{I}$ for $0 \le t \le 1$. Thus $X \in \mathscr{I}^0$ and $S^0 \subseteq \mathscr{I}^0$.

Arguing by contradiction, suppose there exists $X \in \mathscr{I}^0 \setminus S^0$. It follows that there is a (continuous) path $F$ in $\mathscr{I}^0$ such that $F(0) = 0$ and $F(1) = X$. There is a smallest $0 < \alpha \le 1$ with the property $Y = F(\alpha)$ is in the boundary of $S^0$. Since $Y \in \mathscr{I}^0$, $(Y,Y^*) \in \operatorname{dom}(s)$. Since $Y \notin S^0$, $s(Y,Y^*) \succeq 0$ is not invertible. It follows that $Y \in \mathscr{I}^0$, but $(Y,Y^*) \notin \operatorname{dom}(\tilde{s})$, a contradiction. Hence $\mathscr{I}^0 = S^0$ is the component of the origin of the set of $X \in M(\mathbb{C})^g$ such that $P(X)$ is invertible. By a variant of the main result in [31], $S^0$ is the interior of a free spectrahedron. □

**Lemma 6.5.** *If $q : M(\mathbb{C})^g \to M(\mathbb{C})^g$ is a free rational mapping and $E \in M_{d\times e}(\mathbb{C})^g$ is linearly independent, then*

1. *the domains of $q$ and $Q(x) := \Lambda_E(q(x))$ coincide;*
2. $\operatorname{dom}(\check{q}) = \operatorname{dom}(q)^* := \{X : X^* \in \operatorname{dom}(q)\}$*; and*
3. *the domain of*

$$r(x,y) := \begin{pmatrix} I_{d\times d} & Q(x) \\ \check{Q}(y) & I_{e\times e} \end{pmatrix} \tag{6.1}$$

*is* $\operatorname{dom}(q) \times \operatorname{dom}(q)^* = \{(X,Y) : X, Y^* \in \operatorname{dom}(q)\}$.

**Proof.** The inclusion $\operatorname{dom}(q) \subseteq \operatorname{dom}(Q)$ is evident. To prove the converse, let $1 \leq k \leq g$ be given. Using the linear independence of $\{E_1, \dots, E_g\}$, choose a linear functional $\lambda_k$ on the span of $\{E_1, \dots, E_g\}$ such that $\lambda_k(E_j) = 1$ if $j = k$ and 0 otherwise. It follows that the domain of $\lambda_k(Q(x)) = q^k(x)$ contains $\operatorname{dom}(Q)$. Hence $\operatorname{dom}(Q) \subseteq \operatorname{dom}(q)$, proving item (1).

Item (2) is evident as is the inclusion $\operatorname{dom}(r) \supseteq \operatorname{dom}(q) \times \operatorname{dom}(q)^*$ of (3). For $1 \leq j \leq g$, let

$$F_j = \begin{pmatrix} 0 & E_j \\ 0 & 0 \end{pmatrix}$$

and let $F_j = F_{j-g}^*$ for $g < j \leq 2g$. Observe that $r(x, y) = \Lambda_F(q(x), \breve{q}(y))$. It follows from item (1) applied to $(q(x), \breve{q}(y))$ and $F$ that

$$\operatorname{dom}(r) = [\operatorname{dom}(q) \times M(\mathbb{C})^g] \cap [M(\mathbb{C})^g \times \operatorname{dom}(\breve{q})] = \operatorname{dom}(q) \times \operatorname{dom}(q)^*,$$

proving item (3) and the lemma. □

**Proof of Theorem 6.1.** It is immediate that

$$\mathscr{C} \subseteq S := \{X : X \in \operatorname{dom}(q), \quad \|\Lambda_E(q(X))\| < 1\}.$$

Let $S^0$ denote the connected component of $S$ containing 0. Since $\mathscr{C}$ is open, connected and contains the origin, $\mathscr{C} \subseteq S^0$.

Let $Q = \Lambda_E \circ p$ and let $r$ denote the ($(d+e) \times (d+e)$ symmetric matrix-valued) free rational function defined in equation (6.1). By Lemma 6.5, $\{X : (X, X^*) \in \operatorname{dom}(r)\} = \operatorname{dom}(q)$ and moreover, for $X \in \operatorname{dom}(q)$, we have $q(X) \in \operatorname{int}(\mathcal{B}_E)$ if and only if $r(X, X^*) \succ 0$. Thus,

$$S = \{X : (X, X^*) \in \operatorname{dom}(r),\ r(X) \succ 0\}.$$

Arguing by contradiction, suppose $Y \in S^0$, but $Y \notin \mathscr{C}$. By connectedness, there is a continuous path $F$ in $S^0$ such that $F(0) = 0$ and $F(1) = Y$. Let $0 < \alpha \leq 1$ be the smallest number such that $X = F(\alpha) \in \partial\mathscr{C}$. Since $q : \mathscr{C} \to \operatorname{int}(\mathcal{B}_E)$ is bianalytic, it is proper. Hence, if $X \in \operatorname{dom}(q)$, then $q(X) \in \partial\mathcal{B}_E$ and consequently $X \notin S$. On the other hand, if $X \notin \operatorname{dom}(q)$, then $X \notin S$. In either case we obtain a contradiction. Hence $S^0 \subseteq \mathscr{C}$.

Since $\mathscr{C} = S^0$ is convex (and so connected), Proposition 6.4 implies there is a positive integer $N$ and tuple $A \in M_N(\mathbb{C})^g$ such that $\mathscr{C} = \operatorname{int}(\mathcal{D}_A)$. Since $\operatorname{int}(\mathcal{D}_A)$ is bounded, the tuple $A$ is linearly independent. Without loss of generality, we may assume that $A$ is minimal for $\mathcal{D}_A$. Since $p^{-1} : \operatorname{int}(\mathcal{D}_A) \to \operatorname{int}(\mathcal{B}_E)$ is bianalytic and $A$ and $E$ are linearly independent, Theorem 1.1 and Remark 1.2(a) together imply $p^{-1}$, and hence $p$, is, up to affine linear equivalence, convexotonic and $r \leq d + e$ by Theorem 1.1. □

## Appendix A. Context and motivation

The main development over the past two decades in convex programming has been the advent of linear matrix inequalities (LMIs); with the subject generally going under the heading of semidefinite programming (SDP). SDP is a generalization of linear programming and many branches of science have a collection of paradigm problems that reduce to SDPs, but not to linear programs. There is highly developed software for solving optimization problems presented as LMIs. In $\mathbb{R}^g$ sets defined by LMIs are very special cases of convex sets known as spectrahedra. However, as to be discussed, in the noncommutative case convexity is closely tied to free spectrahedra.

The study of free spectrahedra and their bianalytic equivalence derives motivation from systems engineering and connections to other areas of mathematics. Indeed the paradigm problems in linear systems engineering textbooks are *dimension free* in that what is given is a signal flow diagram and the algorithms and resulting software toolboxes handle *any* system having this signal flow diagram. Such a problem leads to a matrix inequality whose solution (feasible) sets $D$ is *free semialgebraic* [16]. Hence $D$ is closed under direct sums and simultaneous unitary conjugation, i.e., it is a free sets. In this dimension free setting, if $D$ is convex, then it is a free spectrahedron [31,43]. For optimization and design purposes, it is hoped that $D$ is convex (and hence a spectrahedron), and algorithm designers put great effort into converting (say by change of variables) the problem they face to one that is convex.

If the domain $D$ is not convex one might attempt to map it bianalytically to a free spectrahedron. The classical problems of linear control that reduce to convex problems all require a change of variables, see [52]. One bianalytic map composed with the inverse of another leads to a bianalytic map between free spectrahedra; thus maps between free spectrahedra characterize the non-uniqueness of bianalytic mappings from the solution set $D$ of a system of matrix inequalities to a free spectrahedron.

Studying bianalytic maps between free spectrahedra is a free analog of rigidity problems in several complex variables [14,19,20,35,36,42]. Indeed, there is a large literature on bianalytic maps on convex sets. For example, Forstnerič [20] showed that any proper map between balls with sufficient regularity at the boundary must be rational. The conclusions we see here in Theorems 1.1, 1.3 and 2.1 are vastly more rigid than mere birationality.

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