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Time and Energy-Optimal Lane Change Maneuvers for Cooperating Connected and Automated Vehicles*

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Abstract— **We derive optimal control policies for a Connected and Automated Vehicle (CAV) cooperating with neighboring CAVs to implement a highway lane change maneuver. We optimize the maneuver time and subsequently minimize the associated energy consumption of all cooperating vehicles in this maneuver. We prove structural properties of the optimal policies which simplify the solution derivations and lead to analytical optimal control expressions. The solutions, when they exist, are guaranteed to satisfy safety constraints for all vehicles involved in the maneuver. Simulation results show the effectiveness of the proposed solution and significant performance improvements compared to maneuvers performed by human-driven vehicles.**

I. Introduction

Advances in next generation transportation system technologies and the emergence of Connected and Automated Vehicles (CAVs), also known as "autonomous vehicles", have the potential to drastically improve a transportation network's performance in terms of safety, comfort, congestion reduction and energy efficiency. In highway driving, an overview of automated intelligent vehicle-highway systems was provided in [1] with more recent developments mostly focusing on autonomous car-following control [2],[3],[4]. Automating a lane change maneuver remains a challenging problem which has attracted increasing attention in recent years [5],[6],[7],[8].

The basic architecture of an automated lane-change maneuver can be divided into the strategy level and the control level [9]. The strategy level generates a feasible (possibly optimal in some sense) trajectory for a lane-change maneuver. The control level is responsible for determining how vehicles track the aforementioned trajectory. For example, [7] adopts such an architecture for an automated lane-change maneuver, but does not provide an analytical solution and assumes that there are no other vehicles in the left lane (the lane in which the controllable vehicle ends up after completing the maneuver). In [10], background vehicles are included in the left lane and the goal is to check whether there exists a lane-change trajectory or not; if one exists, the controllable vehicle will then track this trajectory. A similar approach is taken in [11] with the trajectory being updated during the maneuver based on the latest surrounding information. In

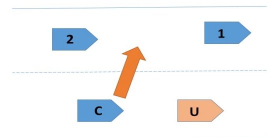

Fig. 1. The basic lane changing maneuver process.

these papers, only one vehicle can be controlled during the maneuver and no analytical solutions are provided.

The emergence of CAVs brings up the opportunity for cooperation among vehicles traveling in both left and right lanes in carrying out an automated lane-change maneuver [9],[12],[13]. Such cooperation presents several advantages relative to the two-level architecture mentioned above. In particular, when controlling a single vehicle and checking on the feasibility of a maneuver depending on the state of the surrounding traffic, as in [14],[15], the maneuver may be infeasible without the cooperation of other vehicles, especially under heavier traffic conditions. In contrast, a cooperative architecture can allow multiple interacting vehicles to implement controllers enabling a larger set of maneuvers. Feasible, but not necessarily optimal, vehicle trajectories for cooperative multi-agent lane-changing maneuvers are derived in [16]. The case of multiple cooperating vehicles simultaneously changing lanes is considered in [17] with the requirement that all vehicles are controllable and their velocities prior to the lane change are all the same. First, vehicles with a lower priority must adjust their positions in their current lane and give way to those with a higher priority so as to avoid collisions. Then, a lane changing optimal control problem is solved for each vehicle without considering the usual safe distance constraints between vehicles. This "progressively constrained dynamic optimization" method facilitates a numerical solution to the underlying optimal control problem at the expense of some loss in performance.

Our goal is to provide an optimal solution for the maneuver in Fig. 1, in which the controlled vehicle C attempts to overtake an uncontrollable vehicle U by using the left lane to pass. In this case, the initial velocities of all vehicles can be different and arbitrary. The overall lane changing and passing maneuver consists of three steps: (i) The target vehicle C moves to the left lane, (ii) C moves faster than U (and possibly other vehicles ahead of it) while on the left lane, (iii) C moves back to the right lane. The first step is further subdivided into two parts. First, vehicle C adjusts its position in the current lane to prepare for a lane shift, while vehicles 1 and 2 in Fig. 1 cooperate to create space for C in the left lane. Next, the latitudinal lane shift of C takes place. In this paper, we limit ourselves to the

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first part of this step. Our objective is to minimize both the maneuver time and the energy consumption of vehicles C, 1 and 2 which are all assumed to share their state information. We also impose a hard safe distance constraint between all adjacent vehicles located in the same lane, as well as constraints due to speed and acceleration limits imposed on all vehicles. We first determine a minimum feasible time for the maneuver (if one exists) and associated terminal positions for vehicles C, 1 and 2. We then solve a fixed terminal time decentralized optimal control problem for each of the three vehicles. We derive several properties of the optimal solution which facilitate obtaining explicit analytical solutions. Our approach applies to a wider range of scenarios relative to those in [10],[11],[14],[15] and incorporates the safety distance constraint not included in [16] and [17].

II. Problem Formulation

We define $x_i(t)$ to be the longitudinal position of vehicle i along its current lane measured with respect to a given origin, where we use $i = 1,2,C,U$. Similarly, $v_i(t)$ and $u_i(t)$ are vehicle i's velocity and (controllable) acceleration. The dynamics of vehicle i are

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t) \tag{1}$$

The maneuvers carried out by vehicles $1,2,C$ are initiated at time t_0 and end at time t_f. We define $d_i(v_i(t))$ to be the minimal safe distance between vehicle i and the one that precedes it in its lane, which in general depends on the vehicle's current speed. The control input and speed are constrained as follows for all $t \in [t_0, t_f]$:

$$u_{i\min} \le u_i(t) \le u_{i\max}, \quad v_{i\min} \le v_i(t) \le v_{i\max} \tag{2}$$

where $u_{i\max}$, $u_{i\min}$, $v_{i\max}$, $v_{i\min}$ are the maximal and minimal acceleration (respectively speed) limits. In Fig.1, we control vehicles 1, 2 and C to complete a lane change maneuver while minimizing the corresponding energy consumption and the maneuver time. For each vehicle $i = 1,2,C$ we formulate the following optimization problem assuming that $x_i(0)$ and $v_i(0)$ are given:

$$\begin{aligned} & J(t_f; u_i(t)) = \\ & \min_{u_i(t)} \int_0^{t_f} [w_t + [w_{1,u}u_1^2(t) + w_{2,u}u_2^2(t) + w_{C,u}u_C^2(t)]]dt \\ & \qquad \text{s.t. } (1), (2) \text{ and} \\ & x_1(t) - x_2(t) > d_2(v_2(t)), \quad t \in [0, t_f] \\ & x_U(t) - x_C(t) > d_C(v_C(t)), \quad t \in [0, t_f] \\ & x_1(t_f) - x_C(t_f) > d_C(v_C(t_f)), \quad x_C(t_f) - x_2(t_f) > d_2(v_2(t_f)) \end{aligned} \tag{3}$$

where w_t, $w_{i,u}$, $i = 1,2,C$, are weights associated with the maneuver time t_f and with a measure of the total energy expended. The two terms in $J(t_f; u_i(t))$ need to be properly normalized and we set $w_t = \frac{\rho}{T_{\max}}$ and $w_{i,u} = \frac{1-\rho}{\max\{u_{i\max}^2, u_{i\min}^2\}}$, where $\rho \in [0,1]$ and $T_{\max}$ is a prespecified upper bound on the maneuver time (e.g., $T_{\max} = l/\min\{v_{imin}\}$, $i = 1,2,C,U$, where l is the distance to the next highway exit). The safe distance is defined as $d_i(v_i(t)) = \phi v_i(t) + \delta$ where ϕ is the headway time (the general rule $\phi = 1.8$ is usually adopted as in [18]). As stated, the problem allows for a free terminal time t_f and terminal state constraints $x_i(t_f)$, $v_i(t_f)$. We will next specify the terminal time t_f as the solution of a minimization problem which allows each vehicle to specify a desired "aggressiveness level" relative to the shortest possible maneuver time subject to (2). Then, we will also specify $x_i(t_f)$, $i = 1,2,C$.

III. Optimal Control Solution

Terminal time specification. We begin by formulating the following minimization problem:

$$\min_{t_f > 0} \; t_f \tag{4}$$

$$\text{s.t. } x_1(0) + v_1(0)t_f + 0.5\alpha_1 u_{1\max} t_f^2 - x_C(0) - v_C(0)t_f - 0.5\alpha_C u_{C\max} t_f^2 > d_C(v_C(t_f)) \tag{4a}$$

$$x_U(t_f) - x_C(0) - v_C(0)t_f - 0.5\alpha_C u_{C\min} t_f^2 > d_C(v_C(t_f)) \tag{4b}$$

$$x_C(0) + v_C(0)t_f + 0.5\alpha_C u_{c\min} t_f^2 - x_2(0) - v_2(0)t_f - 0.5\alpha_2 u_{2\min} t_f^2 > d_2(v_2(t_f)) \tag{4c}$$

where $\alpha_i \in [0,1), i = 1,2,C$ is an "aggressiveness coefficient" for vehicle i which can be preset by the driver. Observe that $[x_i(t_0) + v_i(t_0)t_f + 0.5\alpha_i u_{i\max} t_f^2]$ is the terminal position of i under control $\alpha_i u_{i\max}$. To minimize t_f, vehicle 1 should accelerate and vehicle 2 decelerate so as to increase the gap between them in Fig. 1. If C accelerates, then (4a) ensures the safety constraint is still satisfied. If C has to decelerate because it is constrained by U, then (4b) ensures that the safety constraint between U and C is satisfied and (4c) ensures that the safety constraint between 2 and C is also satisfied. As we will subsequently show, the optimal control of C is either always non-positive or always non-negative throughout $[0, t_f]$ so that either the first or the last two constraints are relevant to it. Naturally, a solution to (4) may not exist, in which case we must iterate on the values of α_i and possibly abort the maneuver.

Terminal position specifications. Assuming a solution t_f is determined, we next seek to specify terminal vehicle positions $x_i(t_f)$, $i = 1,2,C$, to be associated with problem (3). To do so, we define $\Delta x_i(t_f) = x_i(t_f) - x_i(t_0) - v_i(t_0)t_f$ which is the difference between the actual terminal position of i and its ideal terminal position under constant speed $v_i(t_0)$; this is ideal from the energy point of view in (5), since the energy component is minimized when $u_i(t) = 0$. Thus, the energy-optimal value is $\Delta x_i(t_f) = 0$. We then seek terminal positions that minimize a measure of deviating form these energy-optimal values over all three vehicles:

$$\begin{aligned} \min_{x_i(t_f) > x_i(0), i=1,2,C} \; & \Delta x_C^2(t_f) + \Delta x_1^2(t_f) + \Delta x_2^2(t_f) \\ \text{s.t. } & \Delta x_i(t_f) = x_i(t_f) - x_i(0) - v_i(0)t_f \\ & x_1(t_f) - x_C(t_f) > \max\{d_C(v_C(t))\} \\ & x_C(t_f) - x_2(t_f) > \max\{d_2(v_2(t))\} \\ & x_U(t_f) - x_C(t_f) > \max\{d_C(v_C(t))\} \end{aligned} \tag{5}$$

The max values in (5) are assumed to be given by a prespecified maximum inter-vehicle safe distance. However, as subsequently shown in Theorem 1, they actually turn out to be the known initial or terminal values of $d_2(v_2(t))$ and

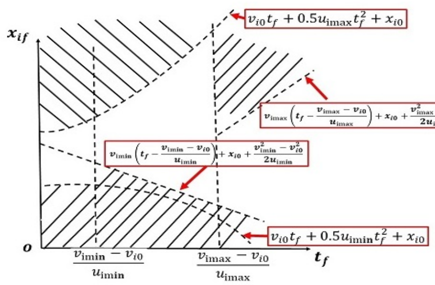

Fig. 2. The feasible state set of controllable vehicles in the left lane.

$d_C(v_C(t))$. For example, $\max\{d_2(v_2(t))\} = d_2(v_2(t_0))$ and $\max\{d_C(v_C(t))\} = d_C(v_C(t_0) + u_{C\max}t_f)$.

Lemma 1: The solution $x_i^*(t_f)$, $i = 1,2,C$, to (5) satisfies $\Delta x_1^*(t_f) \geq 0$ and $\Delta x_2^*(t_f) \leq 0$.

*Proof***:** All proofs of lemmas and theorems are omitted and may be found in [19]. ■

A. Optimal Control of Vehicles 1 *and* 2

With the terminal time t_f and longitudinal position $x_i(t_f)$, $i = 1,2$, set through (4) and (5) respectively, the optimal control problems of vehicles $i = 1,2$ in (3) become:

$$\min_{u_1(t)} \int_0^{t_f} \frac{1}{2}u_1^2(t)dt \quad \text{s.t. (1), (2), } x_1(t_f) = x_{1,f} \tag{6}$$

$$\min_{u_2(t)} \int_0^{t_f} \frac{1}{2}u_2^2(t)dt \quad \text{s.t. (1), (2), } x_2(t_f) \leq x_{2,f}, \tag{7}$$
$$x_1(t) - x_2(t) > d_2(v_2(t)), \quad t \in [0, t_f]$$

where $x_{1,f}$ and $x_{2,f}$ are given as above. In (7), we use an inequality $x_2(t_f) \leq x_{2,f}$ to describe the terminal position constraint instead of the equality since it suffices for the distance between the two vehicles to accommodate vehicle C while at the same time allowing for the cost under a control with $x_2(t_f) < x_{2,f}$ to be smaller than under a control with $x_2(t_f) = x_{2,f}$. In (6), there is no need to consider the case that $x_1(t_f) > x_{1,f}$ since it is clear that the optimal cost when $x_1(t_f) = x_{1,f}$ is always smaller compared to $x_1(t_f) > x_{1,f}$. The next result establishes the fact that the solution of these two problems involves vehicle 1 never decelerating and vehicle 2 never accelerating.

Theorem 1 The optimal control in (6) is $u_1^*(t) \geq 0$ and the optimal control in (7) is $u_2^*(t) \leq 0$.

Based on Theorem 1, in addition to showing that vehicle 1 never decelerates and vehicle 2 never accelerates, we also eliminate the safe distance constraint in (7) since the distance between the vehicles will increase in the course of the maneuver and the last two safety constraints in (3) ensure that this distance is eventually large enough to accommodate the length of vehicle C. Thus, (7) becomes

$$\min_{u_2(t)} \int_0^{t_f} \frac{1}{2}u_2^2(t)dt \quad \text{s.t. (1), (2), } x_2(t_f) = x_{2,f} \tag{8}$$

Feasible terminal state set. The constraints in (2) limit the sets of feasible terminal conditions $(x_{i,f}, t_f)$, $i = 1,2,C$ as shown in Fig. 2 where the feasible set is the unshaded area defined as follows for each $i = 1,2,C$: (i) Vehicle i cannot reach $x_{i,f}$ under its maximal acceleration if $u_{i\max}t_f + v_{i,0} \leq v_{i\max}$ and $v_{i,0}t_f + 0.5u_{i\max}t_f^2 < x_{i,f} - x_{i,0}$. (ii) Vehicle i cannot reach $x_{i,f}$ under its maximal acceleration after attaining its maximal velocity if $u_{i\max}t_f + v_{i,0} > v_{i\max}$ and $v_{i\max}(t_f - \frac{v_{i\max} - v_{i,0}}{u_{i\max}}) < x_{i,f} - x_{i,0} - \frac{v_{i\max}^2 - v_{i,0}^2}{2u_{i\max}}$. (iii) Vehicle i exceeds $x_{i,f}$ under the minimal acceleration if $u_{i\min}t_f + v_{i,0} \geq v_{i\min}$ and $v_{i,0}t_f + 0.5u_{i\min}t_f^2 > x_{i,f} - x_{i,0}$. (iv) Vehicle i exceeds $x_{i,f}$ under the minimal acceleration after attaining its minimal velocity if $u_{i\min}t_f + v_{i,0} < v_{i\min}$ and $v_{i\min}(t_f - \frac{v_{i\min} - v_{i,0}}{u_{i\min}}) > x_{i,f} - x_{i,0} - \frac{v_{i\min}^2 - v_{i,0}^2}{2u_{i\min}}$. In addition, vehicle C must also satisfy a safety distance constraint with respect to vehicle U, hence if $x_{C,f} > x_U(0) + v_U t_f - d_C(v_2(t_f))$, there is no feasible solution.

Note that if an optimal t_f is determined in (4) and the solution of (5) guarantees that $x_i(t_f)$, $i = 1,2,C$, do not violate the safety constraints, $(x_{i,f}, t_f)$ is expected to be feasible. However, if $(x_{i,f}, t_f)$ is infeasible for vehicle i, then the following algorithm is used to find a feasible such pair:

Algorithm 1:

(1) t_f is updated using $t_f = \beta t_f$, $\beta > 1$.

(2) With updated t_f, (5) is re-solved to obtain new $x_{i,f}$.

(3) If $(x_{i,f}, t_f)$ is feasible in Fig. 2, stop; else return to step (1) with a higher value of β.

In the above, the coefficient β is used to relax the maneuver time t_f so as to accommodate one or more of the constraints in Fig. 2 until a feasible $(x_{i,f}, t_f)$ is identified.

Solution of problem (6). We begin by adjoining the constraints in (6) to obtain the Hamiltonian, so as to obtain the Lagrangian functions:

$$\begin{aligned} L(v_1, u_1, \lambda, \eta) = & \frac{1}{2}u_1^2(t) + \lambda_x(t)v_1(t) + \lambda_v(t)u_1(t) + \\ & \eta_2(t)(u_1(t) - u_{1\max}) + \eta_3(t)(v_{1\min} - v_1(t)) \\ & + \eta_4(t)(v_1(t) - v_{1\max}) \end{aligned} \tag{9}$$

where $\lambda(t) = [\lambda_v(t), \lambda_x(t)]^T$ and $\eta = [\eta_1(t), ..., \eta_4(t)]^T$. The explicit solution of (6) is given next.

Theorem 2 Let $x_1^*(t)$, $v_1^*(t)$, $u_1^*(t)$ be a solution of (6). Then, $u_1^*(t) = \arg\min_{0 \leq u_1 \leq u_{1\max}} \frac{1}{2}u_1^2 + \frac{u_1^*(t_0)^2(t - \tau)u_1}{v_{1,0} - v_1^*(t_f) + (\tau - t_0)u_1^*(t_0)}$, where τ is the first time that $v_1^*(\tau) = v_{1\max}$ and $\tau = t_f$ if $v_{1\max}$ is never reached.

Furthermore, following a derivation similar to that in [20] we can obtain the optimal cost $J_1^*(t_f)$ in (6) based on several cases depending on the initial acceleration $u_{1,0}^*$ and the terminal velocity $v_1^*(t_f)$ which can be explicitly evaluated as in [20]. The final optimal cost is the minimal among all possible values obtained.

Case I: $u_{1,0}^* = u_{1\max}$ and $\dot{u}_1^*(t) = 0$. If $t_f < \frac{v_{1\max} - v_{1,0}}{u_{1\max}}$, then $u_1^*(t) = u_{1\max}$ for all $t \in [0, t_f]$. Otherwise, when $v_1(t) = v_{1\max}$, the control switches to $u_1^*(t) = 0$. Therefore,

$$J_1^*(t_f) = \begin{cases} \frac{1}{2}u_{1\max}(v_{1\max} - v_{1,0}) & \text{if } t_f \geq \frac{v_{1\max} - v_{1,0}}{u_{1\max}} \\ \frac{1}{2}u_{1\max}^2(t_f - t_0) & \text{otherwise} \end{cases} \tag{10}$$

Case II: $u_{1,0}^* = u_{1\max}$ and $v_1^*(t_f) = v_{1\max}$. We define t_1 as the time that $u_1^*(t)$ begins to decrease and τ as the first time that $u_1^*(\tau) = 0$. Thus, $u_1^*(t)$ is a piecewise linear function of time t and (following calculations similar to those in [20]):

$$J_1^*(t_f) = \frac{1}{2}(t_1 - t_0)u_{1\max}^2 + \frac{1}{24}\frac{u_{1\max}^4(\tau - t_1)^3}{[v_{1,0} - v_{1\max} + (\tau - t_0)u_{1\max}]^2} \tag{11}$$

Using similar calculations, we summarize below the remaining three cases:

Case III: $u^*_{1,0} = u_{1\max}$, $v^*_1(t_f) < v_{1\max}$ $\quad J^*_1(t_f) = \frac{1}{2}\frac{u^2_{1\max}(t_f + 2t_1 - 3t_0)}{3}$

Case IV: $u^*_{1,0} < u_{1\max}$, $v^*_1(t_f) = v_{1\max}$ $\quad J^*_1(t_f) = \frac{2}{3}\frac{(v_{1\max} - v_{1,0})^2}{\tau - t_0}$

Case V: $u^*_{1,0} < u_{1\max}$, $v^*_1(t_f) < v_{1\max}$ $\quad J^*_1(t_f) = \frac{3}{2}\frac{[x_{1,f} - v_{1,0}(t_f - t_0)]^2}{(t_f - t_0)^3}$

Solution of problem (8). Similar to the solution of (6), we can derive an explicit solution for (8), as follows.

Theorem 3 Let $x^*_2(t)$, $v^*_2(t)$, $u^*_2(t)$ be a solution of (8). Then, $u^*_2(t) = \arg\min_{u_{2\min}\le u_2 \le 0} \frac{1}{2}u_2^2 + \frac{u^*_2(t_0)^2(\tau - t)u_2}{v^*_2(t_f) - v_{2,0} - (\tau - t_0)u^*_2(t_0)}$, where τ is the first time that $v^*_1(\tau) = v_{2\min}$ and $\tau = t_f$ if $v_{2\min}$ is never reached.

We can also obtain the optimal cost $J^*_2(t_f)$ in (7) based on several cases depending on the initial acceleration $u^*_{2,0}$ and the terminal velocity $v^*_2(t_f)$ which can be explicitly evaluated as in [20]. In what follows, we define t_1 as the time that $u^*_2(t)$ begins to increase and τ as the first time that $u^*_2(\tau) = 0$.

Case I: $u^*_{2,0} = u_{2\min}$, $v^*_2(t_f) = v_{2\min}$ $\quad J^*_2(t_f) = \frac{1}{2}(t_1 - t_0)u^2_{2\min} + \frac{1}{24}\frac{u^4_{2\min}(\tau - t_1)^3}{[v_{2,0} - v_{2\min} + (\tau - t_0)u_{2\min}]^2}$

Case II: $u^*_{2,0} = u_{2\min}$, $v^*_2(t_f) > v_{2\min}$ $\quad J^*_2(t_f) = \frac{u^2_{2\min}(t_f + 2t_1 - 3t_0)}{6}$

Case III: $u^*_{2,0} > u_{2\min}$, $v^*_2(t_f) = v_{2\min}$ $\quad J^*_2(t_f) = \frac{2}{3}\frac{(v_{2\min} - v_{2,0})^2}{\tau - t_0}$

Case IV: $u^*_{2,0} > u_{2\min}$, $v^*_2(t_f) = v_{2\min}$ $\quad J^*_2(t_f) = \frac{3}{2}\frac{[x_{2,f} - v_{2,0}(t_f - t_0)]^2}{(t_f - t_0)^3}$

B. Optimal Control of Vehicle C

Unlike (6) and (8), deriving the optimal control of vehicle C as in Fig. 1 is more challenging. First, since we need to keep a safe distance between vehicles C and U, a constraint $x_U(0) + v_U t - x_C(t) > d_C(v_C(t))$ must hold for all $t \in [0, t_f]$. The resulting problem formulation is:

$$\begin{aligned} &\min_{u_C(t)} \int_0^{t_f} \frac{1}{2}u^2_C(t)dt \\ &\text{s.t. } (1),\ (2),\ x_C(t_f) = x_{C,f},\ t \in [0, t_f] \\ &x_U(0) + v_U t - x_C(t) > d_C(v_C(t)) \end{aligned} \tag{12}$$

in which $d_C(v_C(t))$ is time-varying. To simplify (12), we use $d_C \equiv \max\{d_C(v_C(t))\}$ instead of $d_C(v_C(t))$, which is a more conservative constraint still ensuring that the original one is not violated (the problem with $d_C(v_C(t)) = \phi v_C(t) + \delta$ can still be solved at the expense of added complexity and is the subject of ongoing research). The Lagrangian is now

$$\begin{aligned} L(x_C, v_C, u_C, \lambda, \eta) &= \frac{1}{2}u^2_C(t) + \lambda_x(t)v_C(t) + \lambda_v(t)u_C(t) \\ &+ \eta_1(t)(u_C(t) - u_{C\max}) + \eta_2(t)(u_{C\min} - u_C(t)) \\ &+ \eta_3(t)(v_C(t) - v_{C\max}) + \eta_4(t)(v_{C\min} - v_C(t)) \\ &+ \eta_5(t)(x_C(t) - x_U(0) - v_U(0)t + d_C) \end{aligned} \tag{13}$$

Based on Pontryagin's principle, we have

$$u^*_C(t) = \begin{cases} -\lambda_v(t) & \text{if } u_{C\min} \le -\lambda_v(t) \le u_{C\max} \\ u_{C\min} & \text{if } -\lambda_v(t) < u_{C\min} \\ u_{C\max} & \text{if } -\lambda_v(t) > u_{C\max} \end{cases} \tag{14}$$

when none of the constraints is active along an optimal trajectory. In order to account for the constraints becoming active, we identify several cases depending on the terminal states of vehicles U and C. Let us define $\bar{x}_C(t_f)$ to be the terminal position of C if $u_C(t) = 0$ for all $t \in [0, t_f]$. If $\bar{x}_C(t_f) < x_C(t_f)$, vehicle C must accelerate in order satisfy the terminal position constraint. Otherwise, C must decelerate. Also critical is the value of $x_U(t_f) - d_C$, i.e., the upper bound of the safe terminal position of C. In addition, during the entire maneuver process, we require that $x_C(t) \le x_U(t) - d_C$.

We begin with the 3! cases for ordering $x_C(t_f)$, $\bar{x}_C(t_f)$ and $x_U(t_f) - d_C$. Fortunately, we can exclude several cases as infeasible because $x_C(t_f) \le x_U(t_f) - d_C$ is a necessary condition to have feasible solutions. This leaves three remaining cases as follows.

Case 1: $\bar{x}_C(t_f) < x_C(t_f) < x_U(t_f) - d_C$.
Case 2: $x_C(t_f) < \bar{x}_C(t_f) < x_U(t_f) - d_C$.
Case 3: $x_C(t_f) < x_U(t_f) - d_C < \bar{x}_C(t_f)$.

These are visualized in Fig. 3. The following results provide structural properties of the optimal solution (14) depending on which case applies.

Lemma 2: If $x_U(0) + v_U(0)t - x_C(t) = d_C$, then $v_C(t) = v_U(0)$, $t \in [0, t_f]$.

Theorem 4 [*Case 1* in Fig. 3]: If $\bar{x}_C(t_f) < x_C(t_f) < x_U(t_f) - d_C$, then $u^*_C(t) \ge 0$ and $\eta^*_5(t) = 0$.

Theorem 5 [Case 2 in Fig. 3]: If $x_C(t_f) < \bar{x}_C(t_f) < x_U(t_f) - d_C$, then $u^*_C(t) \le 0$ and $\eta^*_5(t) = 0$.

Theorem 6 [Case 3 in Fig. 3] If $x_C(t_f) < x_U(t_f) - d_C < \bar{x}_C(t_f)$, then $u^*_C(t) \le 0$.

Based on Theorems 4,5, *Cases 1,2* in Fig. 3 can be solved without the safety constraint in (12) since we have shown that $\eta^*_5(t) = 0$. Therefore, the optimal control is the same as that derived for vehicles 1 and 2 in Theorems 2,3. This leaves only *Case 3* to analyze. We proceed by first solving (12) without the safety constraint, so it reduces to the solution in Theorem 3, since we know that $u^*_C(t) \le 0$. If a feasible optimal solution exists, then the problem is solved. Otherwise, we need to re-solve the problem in order to determine an optimal trajectory that includes at least one arc in which $x_U(0) + v_U(0)t - x^*_C(t) - d_C = 0$.

Based on Lemma 2, there exists a time $\tau_1 \in (0, t_f)$ that satisfies $v_C(\tau_1) = v_U(0)$ and $x_C(\tau_1) = x_U(0) + v_U(0)\tau_1 - d_C \equiv a$ (it is easy to see that there is at most one such constrained arc, since $v_C(t) = v_U(0)$ as soon as this arc is entered.) We then split problem (12) into two subproblems as follows:

$$\begin{aligned} &\min_{u_C(t)} \int_0^{\tau_1} \frac{1}{2}u^2_C(t)dt \\ &\text{s.t. } (1),\ (2),\ x_C(\tau_1) = a,\ v_C(\tau_1) = v_U(\tau_1),\ t \in [0, \tau_1] \end{aligned} \tag{15}$$

$$\begin{aligned} &\min_{u(t)} \int_{\tau_1}^{t_f} \frac{1}{2}u^2_C(t)dt \text{ s.t. } (1),\ (2), \\ &x_C(\tau_1) = a,\ v_C(\tau_1) = v_U(\tau_1),\ x_C(t_f) = x_{cf},\ t \in [\tau_1, t_f] \end{aligned} \tag{16}$$

where (15) has a fixed terminal time τ_1 (to be determined), position a, and speed $v_U(0)$, while (16) has a fixed terminal time t_f and position $x_{C,f}$ with given $x_C(\tau_1) = a$.

Let us first solve (15). Since $u^*_C(t) \le 0$ and the terminal speed is $v_U(0)$, only the acceleration constraint $u_{C\min} - u_C \le$

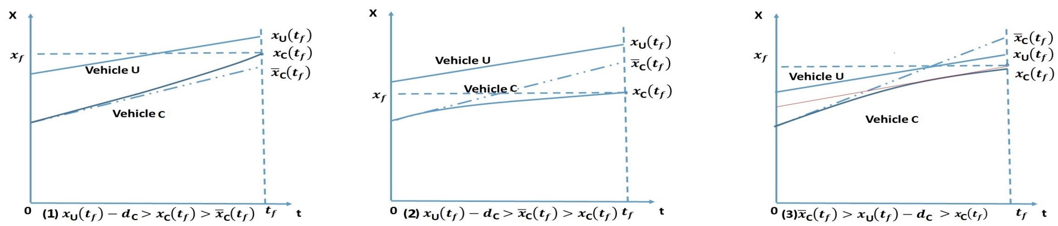

Fig. 3. The three feasible cases for the optimal maneuver of vehicle C.

0 can be active in $[0,\tau_1]$. Suppose that this constraint becomes active at time $\tau_2 < \tau_1$. Since $u_{C\min} - u_C$ is independent of t, $x_C(t)$, and $v_C(t)$, it follows (see [21]) that there are no discontinuities in the Hamiltonian or the costates, i.e., $\lambda_x(\tau_2^-) = \lambda_x(\tau_2^+)$, $\lambda_v(\tau_2^-) = \lambda_v(\tau_2^+)$, $H(\tau_2^-) = H(\tau_2^+)$. It follows from $H(\tau_2^-) - H(\tau_2^+) = 0$ and (13) that

$$[u_C^*(\tau_2^-) - u_C^*(\tau_2^+)][\frac{1}{2}(u_C^*(\tau_2^-)) + \frac{1}{2}(u_C^*(\tau_2^+)) + \lambda_v(\tau_2^-)] = 0$$

Therefore, either $u_C^*(\tau_2^-) = u_C^*(\tau_2^+)$ or $u_C^*(\tau_2^-) = -\lambda_v(\tau_2^-)$ based on (14). Either condition used in the above equation leads to the conclusion that $u_C^*(\tau_2^-) = u_C^*(\tau_2^+)$, i.e., $u_C^*(t)$ is continuous at τ_2.

Let us now evaluate the objective function in (15) as a function of τ_1 and a, denoting it by $J_1(\tau_1,a)$, under optimal control. In view of (14), there are two cases.

(a) $u_C^*(t) = u_{C\min}$ for $t \in [0,\tau_2)$, $u_C^*(t) = -\lambda_v(t)$ for $t \in [\tau_2,\tau_1]$. As in the proof of Theorem 2, the costate equations are $\dot{\lambda}_v(t) = -\lambda_x(t)$ and $\dot{\lambda}_x(t) = 0$. Therefore, $\lambda_v(t) = ct - b$ where b, c are to be determined. It follows that

$$u_C^*(t) = c(t-\tau_2) + u_{C\min}, \quad t \in [\tau_2,\tau_1] \tag{17}$$

and the following boundary conditions hold:

$$\begin{aligned} v_C(\tau_2) &= v_C(0) + u_{C\min}\tau_2 \\ v_C(\tau_1) &= v_U(0) = v_C(\tau_2) + \int_{\tau_2}^{\tau_1}[c(t-\tau_2) + u_{C\min}]dt \\ x_C(\tau_2) &= x_C(0) + v_C(0)\tau_2 + \frac{1}{2}u_{C\min}\tau_2^2 \\ x_C(\tau_1) &= a = x_C(\tau_2) + \int_{\tau_2}^{\tau_1}[\frac{c}{2}t^2 + (u_{C\min} - c\tau_2)(t-\tau_2) \\ &\quad - \frac{c}{2}\tau_2^2 + v_C(0) + u_{C\min}\tau_2]dt \end{aligned} \tag{18}$$

Using (17) and (18) to eliminate c and τ_2 and then evaluate $J_1(\tau_1,a)$ in (15) after some algebra yields:

$$\begin{aligned} J_1(\tau_1,a) =& \frac{1}{2}u_{C\min}(2v_U(0) - 2v_C(0) - u_{C\min}\tau_1) \\ &+ \frac{2(v_U(0) - v_C(0) - u_{C\min}\tau_1)^3}{9(a - x_C(0) - v_C(0)\tau_1 - 0.5u_{C\min}\tau_1^2)} \end{aligned} \tag{19}$$

(b) $u_C^*(t) = -\lambda_v(t)$ for $t \in [0,\tau_2)$, $u_C^*(t) = u_{C\min}$ for $t \in [\tau_2,\tau_1]$. Proceeding as above, we get

$$\begin{aligned} v_C(\tau_2) &= v_C(0) + \int_0^{\tau_2}[c(\tau_2 - t) + u_{C\min}]dt \\ v_C(\tau_1) &= v_U(0) = v_C(\tau_2) + (\tau_1 - \tau_2)u_{C\min} \\ x_C(\tau_2) &= x_C(0) + \int_0^{\tau_2}[v_C(0) + \frac{c}{2}t^2 + u_{C\min}t]dt \\ x_C(\tau_1) &= a = x_C(\tau_2) + \int_{\tau_2}^{\tau_1}[v_C(\tau_2) + u_{C\min}(t-\tau_2)]dt \end{aligned} \tag{20}$$

and, after some calculations, we obtain $J_1(\tau_1,a)$ in (15):

$$\begin{aligned} J_1(\tau_1,a) =& \frac{1}{2}u_{C\min}(2v_U(0) - 2v_C(0) - u_{C\min}\tau_1) \\ &- \frac{2(v_U(0) - v_C(0) - u_{C\min}\tau_1)^3}{9(a - x_C(0) - v_U\tau_1 + 0.5u_{C\min}\tau_1^2)} \end{aligned} \tag{21}$$

Proceeding to the second subproblem (16), note that the control at the entry point of the constrained arc at time τ_1 is no longer guaranteed to be continuous. This problem is of the same form as the optimal control problem for vehicle 2 in (8) whose solution is given in Theorem 3, except that initial conditions now apply at time τ_1 as given in (16). Proceeding exactly as before, we can obtain the cost $J_2(\tau_1,a)$ under optimal control. Adding the two costs, we obtain $J_C(\tau_1,a) = J_1(\tau_1,a) + J_2(\tau_1,a)$ in (12). This results in a simple nonlinear programming problem whose solution (τ_1^*,a^*) results from $\frac{\partial J_C(\tau_1,a)}{\partial \tau_1} = 0$ and $\frac{\partial J_C(\tau_1,a)}{\partial a} = 0$. The optimal control is the one corresponding to (τ_1^*,a^*).

Based on our analysis, we find that *Case 3* is the only one where the safety constraint may become active. This provides an option to the vehicle C controller: if *Case 3* applies, the maneuver may either be implemented or delayed until the conditions change to either one of *Cases 1,2* so as avoid the situation that arises through (15),(16).

IV. Simulation Results

We provide simulation results illustrating the time and energy-optimal optimal maneuver controller we have derived and compare its performance to a baseline of human-driven vehicles. In what follows, we set the minimal and maximal vehicle speeds to $1m/s$ and $33m/s$ respectively and the maximal acceleration and deceleration to $3.3m/s^2$ and $-7m/s^2$ respectively. The aggressiveness coefficients $\alpha_i, i = 1,2,C$ in (4) are all set to $\alpha_i = 0.5$. Extensive simulation examples for all three cases may be found in [19]. Here, we limit ourselves to an example for the more interesting Case 3.

Case 3 **in Fig. 3.** We set $x_1(0) = 40m$, $v_1(0) = 11m/s$, $x_U(0) = 40m$, $v_U(0) = 8m/s$ $x_2(0) = 10m$, $v_2(0) = 23m/s$, $x_C(0) = 13m$, $v_C(0) = 19m/s$. Solving (4) and (5), we get $t_f = 14.49s$ and $x_1(t_f) = 199.37m$, $x_2(t_f) = 75m$, $x_C(t_f) = 105.9m$. The optimal trajectories of vehicles 1,2 are similar to Case 1,2 and omitted here. In this case, vehicle 1 accelerates and vehicle 2 decelerates in order to create space for vehicle C. For vehicle C, we first solve the optimal control problem (12) without considering the safety constraint and find that it actually becomes active. Therefore, we proceed with the two subproblems (15) and (16) to derive the true optimal trajectories. We obtained $a^* = 43m$ and $\tau_1^* = 3.2s$, and Fig.

TABLE I

INITIAL STATES OF VEHICLES

Cases \ States	$x_1(0)[m]$	$v_1(0)[m/s]$	$x_2(0)[m]$	$v_2(0)[m/s]$	$x_C(0)[m]$	$v_C(0)[m/s]$	$x_U(0)[m]$	$v_U(0)[m/s]$	$d_C[m]$
(1)	95	13	0	18	13	10	120	9	30
(2)	120	13	30	18	13	16	100	10	30
(3)	100	11	10	23	213	19	290	8	30

TABLE II

ENERGY COMPARISON: CAVS VS HUMAN-DRIVEN VEHICLES

Cases \ Energy Consumption	CAVs	Human-driven Vehicles	Improvement
(1)	6.8	16.4	59%
(2)	23.0	46.0	50%
(3)	59.5	103.5	43%

4 shows the optimal trajectory of vehicle C. Observe that C decelerates over the maneuver and the safety distance constraint is active at $\tau_1^* = 3.2s$ when there is a jump in the acceleration trajectory. Following that, vehicle C continues decelerating until it reaches its terminal position.

Comparison of optimal maneuver control and human-driven vehicles. We use standard car-following models in the commercial SUMO simulator to simulate a lane change maneuver implemented by human-driven vehicles with the requirement that vehicle C changes lanes between vehicles 1 and 2. We considered all cases in Fig. 3 with both CAVs and human-driven vehicles sharing the same initial states as shown in Table I. The associated energy consumption is shown in Table II and provides evidence of savings in the range $43-59\%$ over all three cases.

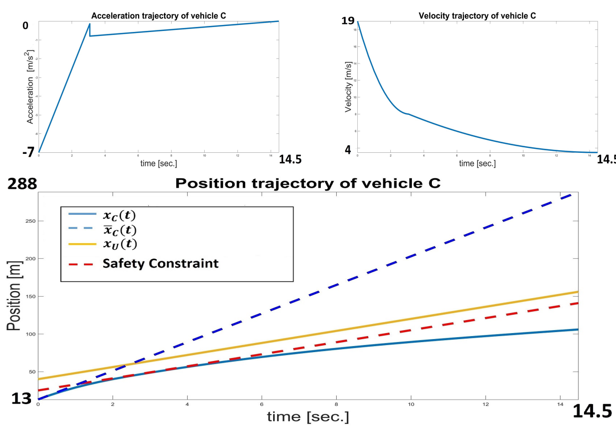

Fig. 4. Optimal trajectories of vehicle C in case (3) of Fig. 3.

V. CONCLUSION AND FUTURE WORK

We used an optimal control framework to derive time and energy-optimal policies for a CAV cooperating with neighboring CAVs to implement a highway lane change maneuver. Our solution is limited to the first step of the complete maneuver, i.e., all three cooperating vehicles adjust their positions before the lane-changing vehicle makes the lane shift. Our ongoing work aims to complete this step. In addition, we plan to incorporate a "comfort" factor in the problem by minimizing any resulting jerk and adopt a more general velocity-varying safety distance constraint.

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