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Efficiently Constructing Tangent Circles

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The Greek geometers of antiquity devised a game—we might call it geometrical solitaire—which . . . must surely stand at the very top of any list of games to be played alone. Over the ages it has attracted hosts of players, and though now well over 2000 years old, it seems not to have lost any of its singular charm or appeal. – Howard Eves

The Problem of Apollonius is to construct a circle tangent to three given ones in a plane. The three circles may also be limits of circles, that is, points or lines; and "construct" means using a straightedge and compass. Apollonius's own solution did not survive antiquity [**8**], and we only know of its existence through a "mathscinet review" by Pappus half a millennium later. Both Viéte and Gergonne rose to the challenge of devising their own constructions, and either may possibly have rediscovered the solution of Apollonius. In this note, we consider a special case of the problem of Apollonius, but from the point of view of *efficiency*. Our goal is to present, in what we believe is the most efficient way possible, a construction of four mutually tangent circles. Of course, five (generic) circles cannot be mutually tangent in the plane, for their tangency graph, the complete graph K_5, is non-planar.

Our measure of efficiency is the one used by Hartshorne [**7**, p. 20]. We aim to minimize the number of *moves*, where a move is the act of drawing either (1) a line through two points, or (2) a circle through a point with given center; like Euclid, our compass collapses when lifted. Marking a point does not count as a move, for no act of drawing is involved. There is another measure proposed by Lemoine in 1907 [**4**, p. 213], which seems to capture the likelihood of propagated error in a construction. We believe Hartshorne's measure is more in the spirit of Euclid and Plato, who thought of constructions as idealized.

The previous best construction appears to be Eppstein's [**2**], though one might argue that ours can be derived from Gergonne's more general construction. We discuss both in our closing remarks.

Baby cases: one and two circles

Constructing one circle costs one move: let A and Z be any distinct points in the plane and draw the circle $O_A(Z)$ with center A and passing through Z. (We will use the notation $O_P(Q)$ for the circle centered at P that goes through Q, or just O_P when there is no need to refer to Q.) Given O_A, constructing a second circle tangent to it costs two more moves: draw the line AZ, and put an arbitrary point B on this line (say, outside O_A). Now draw the circle $O_B(Z)$. Then O_A and O_B are obviously tangent at

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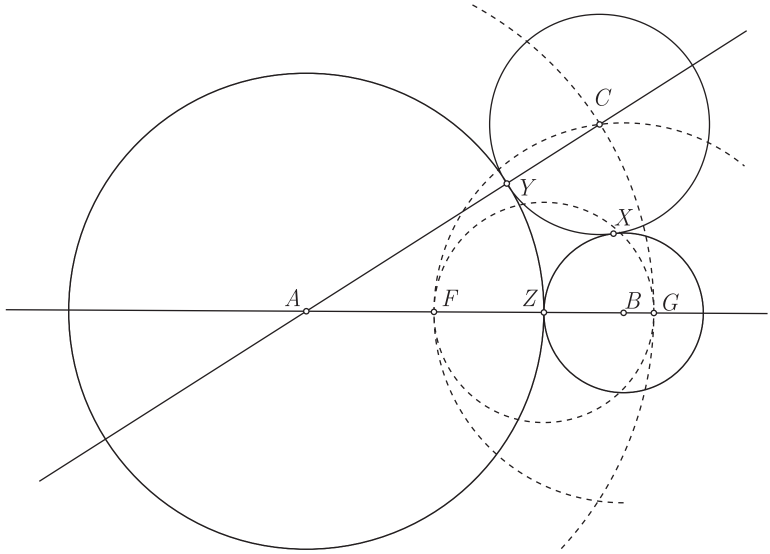

Figure 1 A third tangent circle. The solid lines/circles are the initial and final objects (or objects we wish to include in the next step), while the dotted figures are the intermediate constructions.

Z. In fact one cannot do better than two moves, for otherwise one could draw the circle O_B immediately, but this requires *a priori* knowledge of a point on O_B.

Warmup: three circles

Given two circles O_A and O_B, tangent at Z, and the line AB, how many moves does it take to construct a third circle tangent to both O_A and O_B? We encourage readers at this point to stop and try this problem themselves.

Proposition 1. *Given two mutually tangent circles O_A and O_B, and the line AB, a generic circle tangent to both O_A and O_B is constructible in five moves.*

We first give the construction, then the proof that it works.

The construction Draw a circle O_Z centered at Z and with arbitrary radius c, which will be the radius of our third circle (this is move 1). Let it intersect AB at F and G. Next draw the circles $O_A(G)$ (move 2) and $O_B(F)$ (move 3); see Figure 1. Let these two circles intersect at C. Construct the line AC (move 4) and let Y be its appropriate intersection with O_A. Finally, draw the circle $O_C(Y)$ (move 5). Then O_C is tangent to O_A and O_B.

Proof. It is clear that O_C is tangent to O_A. Since $|CY| = |GZ|$, the radius of O_C is c. Let X be the appropriate point where O_B intersects the line BC (the latter is not constructed). We wish to prove that X lies on O_C, as shown, and that O_C is therefore tangent to O_B at X. Note that $|CX| = |FZ| = c$, the radius of O_C, so X is indeed on O_C. Thus O_C is tangent to O_B at X, as claimed. ■

Remark. The locus of points C is a hyperbola with foci A and B. By choosing F to be on the same side of Z as A, as we did in Figure 1, we get one branch of the hyperbola. The other branch is obtained by choosing F on the other side of Z (and letting c be sufficiently large).

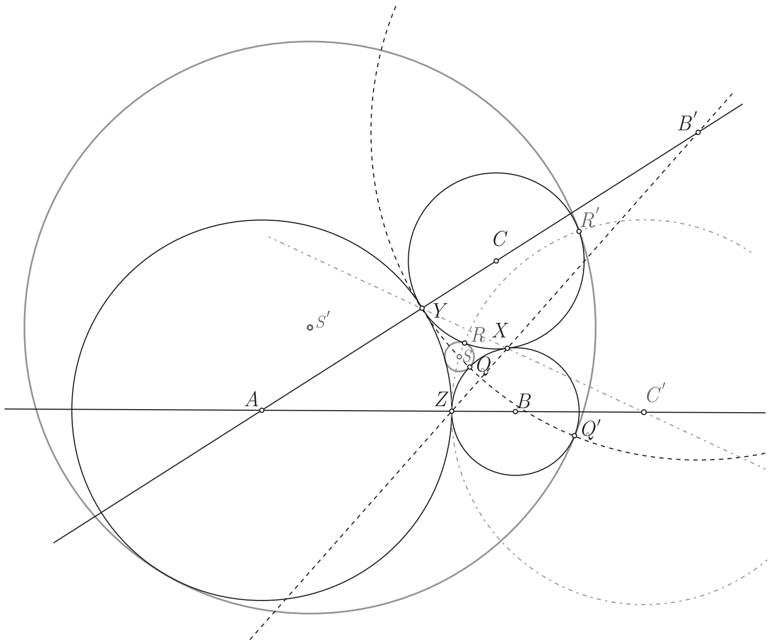

Figure 2 The circle $O_{B'}$ and points Q and Q'. The solid gray circles are the Apollonian circles O_S and $O_{S'}$ that we are in the process of constructing.

Main theorem: the fourth circle

Finally we come to the main event, the fourth tangent circle, which we call the Apollonian circle.* We are given three mutually tangent circles, O_A, O_B, and O_C, lines AB and AC, and the points of tangency X, Y, and Z; that is, we are given the already constructed objects in Figure 1.

Theorem 1. *An Apollonian circle tangent to O_A, O_B, and O_C in Figure 1 is constructible in seven moves.*

The construction Draw the line XZ (this is move 1) and let it intersect AC at B'. Draw the circle $O_{B'}(Y)$ (move 2). It intersects O_B at Q and Q', with Q, say, closer to A; see Figure 2. We repeat this procedure: draw the line XY, let it intersect AB at C', draw the circle $O_{C'}(Z)$ and let $O_{C'}$ intersect O_C at R and R', with R closer to A. This repetition used two more moves. Next draw the lines BQ and CR (now up to move 6) and let them meet at S. Finally, use the seventh move to draw the desired Apollonian circle $O_S(Q)$; see Figure 3.

Remark. If a pair of lines, e.g., AC and XZ, are parallel (so B' is at infinity), then use the line BY in lieu of $O_{B'}$ (the former is the limit of the latter as $B' \to \infty$).

Circle inversions Before the proof, we remind the reader about circle inversions. For any circle with center O and radius r, the inversion of a point P is, by definition, the point P' on the ray OP that satisfies

$$\frac{|OP|}{r} = \frac{r}{|OP'|}.$$

*Many objects in the literature are named after Apollonius though he had nothing to do with them, such as the Apollonian gasket and the Apollonian group (see, e.g., Kontorovich [9]). The fourth tangent circle really *is* due to him, though most authors refer to it as the "Soddy" circle.

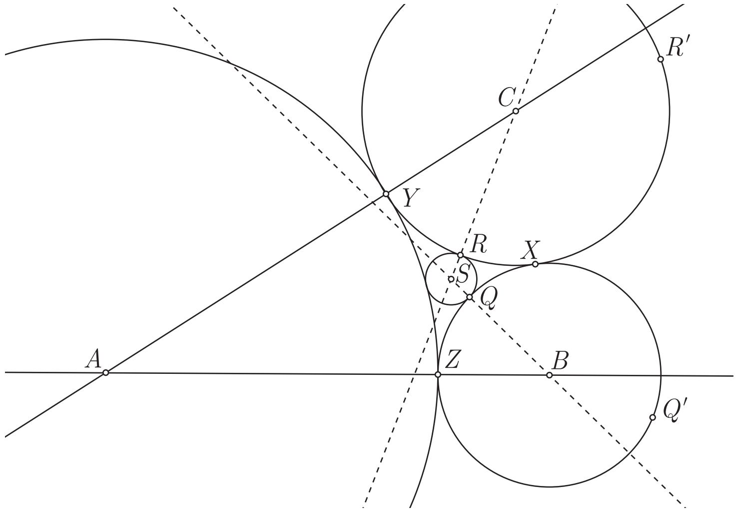

Figure 3 The construction of S and the Apollonian circle O_S.

Two well-known properties of inversions are: the image of a line or circle is itself a line or circle, and angles are preserved. See any good text for details (e.g., Eves [4, Section 3.4] or Hartshorne [7, Section 37]).

Proof. There is a unique circle such that inversion through it sends O_A to O_C. We claim that $O_{B'}$ is this circle. Indeed, such an inversion must fix O_B, and therefore sends X to Z. Hence, its center must lie on XZ. Its center also lies on the line perpendicular to O_A and O_C, which is the line AC. Thus, its center is $B' = XZ \cap AC$. Finally, the point Y is fixed by this inversion, giving the claim.

Next, we claim that the point of tangency of O_B and the Apollonian circle O_S must also lie on this inversion circle $O_{B'}$ (in which case this point must be Q, as constructed). Indeed, since the inversion preserves the initial configuration of three circles, it must also fix O_S, and hence also its point of tangency with O_B.

Finally, since O_B and O_S are tangent at Q, their centers are collinear with Q; that is, S lies on the line BQ. Similarly, S lies on CR. ∎

Remark. The second solution, $O_{S'}$, to the Apollonian problem can now be constructed in a further three moves. Indeed, the points of tangency Q' and R' are already on the page. Draw the lines BQ' and CR' (two more moves) and let them intersect at S'. Then draw $O_{S'}(Q')$ (our third move) to get our second Apollonian circle.

Remark. Let $A' = BC \cap YZ$ be constructed similarly to B' and C'. Note that the triangles ΔABC and ΔXYZ are perspective from the Gergonne point. By Desargues' theorem, they are therefore perspective from a line, which is the line $A'B'C'$, known as the *Gergonne line* of ΔABC; see Oldknow [10], who seems to have been just shy of discovering the construction presented here.

Closing remarks

Of the constructions in the literature, we highlight two.

Eppstein The previously simplest solution to our problem seems to be that of Eppstein [2,3,5], which uses eleven elementary moves to draw O_S. His construction finds the tangency point Q by first dropping the perpendicular to AC through B, and then connecting a second line from Y to one of the two points of intersection of this perpendicular with O_B. This second line intersects O_B at Q (or Q', depending on the choice of intersection point). Note that constructing a perpendicular line is not an elementary operation, costing 3 moves. The second line *is* elementary, so Eppstein can construct Q in 4 moves, then R in 4 more, then two more lines BQ and CR to get the center S, and finally the circle O_S in a total of 11 moves. To construct the other solution, $O_{S'}$, using his method, it would cost another five moves (as opposed to our three), since one needs to draw two more lines to produce Q' and R' (whereas our construction gives these as a byproduct).

Gergonne Gergonne's solution to the *general* Apollonian problem (i.e., when the given circles are not necessarily tangent) is perhaps closest to ours, but of course the problem he is solving is more complicated. He begins by constructing the radical circle O_I for the initial circles O_A, O_B, and O_C, and identifies the six points X, X', Y, Y', Z, and Z', where it intersects the three original circles. Those points are taken in order around O_I, with Y' and Z on O_A, Z' and X on O_B, and X' and Y on O_C. In our configuration, the radical circle is the incircle of triangle ABC and the six points are $X = X'$, $Y = Y'$, and $Z = Z'$.

Every pair of circles can be thought of as being similar to each other via a dilation through a point. In general, there are two such dilations. This gives us six points of similarity, which lie on four lines, the four *lines of similitude*. In Gergonne's construction, each line generates a pair of tangent circles. In our configuration, the point B' is the center of the dilation that sends O_A to O_C. Since O_A and O_C are tangent, there is only one dilation, so we get only one line of similitude, the Gergonne line.

The radical circle of O_B, O_I, and a pair of tangent circles is centered on the line of similitude, so is where XZ' intersects that line. In our configuration, that gives us B'. The radical circle is the one that intersects O_I perpendicularly, so in our configuration it goes through Y.

Efficiency or complexity Measures of efficiency or complexity come up in many branches of mathematics. Some, like height in number theory, are simple to define and quantify. At the other end of the spectrum, the *elegance* of proofs is a difficult notion to quantify, but one we nevertheless recognize and appreciate. Erdös would often refer to what he called *The Book*, a book in which God keeps the most elegant proofs. The invocation of an all-knowing is an acknowledgement that we can never know for sure (i.e., prove) whether a particular proof belongs in that book. With the measures of Hartshorne and Lemoine, we quantify elegance in constructions, and having done so, we now have the tools to prove that a construction is best possible or most elegant. This though appears to be a very difficult question to tackle for even modest constructions. Hartshorne refers to "par" scores and "doable in" scores, but shies away from calling anything best possible.

DeTemple, using Lemoine's measure, analyzes constructions of regular p-gons for $p = 5$, 17, and 257, and also muses about the complexity of showing a construction is best possible [1]. In earlier literature, there is a paper by Güntshe, who gives and analyzes (using Lemoine's measure) constructions for the regular 17-gon [6].

We leave the reader with a challenge: construct a (generic) configuration of four mutually tangent circles in the plane using fewer than 15 ($= 1 + 2 + 5 + 7$) moves. Or prove (as we suspect) that this is impossible!

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Summary. In this short note we present, by what we surmise is the most efficient method, a straight-edge and compass construction of four mutually tangent circles in a plane.

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