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4 Dimensional waypoint generation for conflict-free trajectory based operation

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ABSTRACT

This paper systematically investigates the safety implications of a new 4-dimensional air traffic management paradigm by quantitatively identifying conditions where conflict-free trajectories can be guaranteed by only using 4-dimensional waypoints. Towards this end, a concept called Trajectory Inclusion is first introduced based on geometry and physics-driven proofs and analysis. Strategies to achieve conflict-free trajectories are then developed and further explained with a leader–follower example. It is found out that the time-based waypoints alone can guarantee conflict-free trajectories if certain initial conditions are satisfied. In general, the results of this paper enhance the understanding of time-based management capabilities, and help formulate better time-based instructions to reduce unnecessary tactical maneuvers and improve the overall performance of the airspace, one of the main promises Trajectory Based Operations (TBO). Specifically, the algorithmic scheme created for the leader–follower example can be applied directly to Flight-deck Interval Management, a future concept of TBO.

1. Introduction

The Next Generation Air Transportation System (NextGen) envisions improving the safety and efficiency of airspace operations, while reducing the environmental impacts and increasing the capacity of the air transportation system [1]. At the heart of NextGen is Time Based Management for Trajectory Based Operations (TBO), an air traffic management system in which every aircraft is represented by a 4 dimensional trajectory (4DT) [2]. A 4DT includes a series of points from departure to arrival representing the aircraft's path in four dimensions: lateral (latitude and longitude), vertical (altitude), and time [3]. It plays an important role in NextGen's transition from traditional miles-in-trail [4] traffic management to (Extended) Time-based Metering, providing air traffic services to meet a scheduled time at which airborne aircraft should cross a metering point or arc instead of specifying a minimum spacing for flights.

As pointed out by [5], the goal of the transition to a time-based system is to provide *speed* or *time* control to keep flights on their optimal path. In TBO, controllers use the metering information, such as Scheduled Time of Arrival (STA) and Speed Advisories, to issue clearances to the aircraft in order to comply with the assigned STA [6]. **However, the decision of when to use speed or time control remains an open problem.** In fact, the "quality" of the 4DT waypoints affects the feasibility and effectiveness of the speed-based advisory for conflict resolution. On one hand, 4DT waypoints can decrease controller workload by allowing the aircraft to adjust speed autonomously to meet the crossing restriction, instead of issuing speed instructions to pilots to keep the aircraft on time [7]. However, on the other hand, inappropriate 4DT waypoints can cause unnecessary maneuvers and even render speed-based conflict resolution infeasible. Examples can be found in [8].

Therefore, this paper aims to understand, **to what extent, 4DT waypoints alone, *regardless of the speed profiles*, can guarantee conflict-free trajectories.** Previous work [8] shows *qualitatively* that the 4DT waypoints alone can guarantee conflict-free trajectories in some conditions but can also introduce hazards in other conditions. This paper tackles this problem by identifying the *quantitative* conditions, where no matter what speed profiles the pilots choose to fly, as long as the STAs are respected, conflict-free trajectories can be guaranteed.

This paper makes two contributions: (1) the novel introduction of Trajectory Inclusion and the geometry-based proof for conflict-free scenarios; (2) the finding that 4DT waypoints regardless of the speed profiles can guarantee conflict-free trajectories and the specific scheme to achieve them. Furthermore, the results from the leader–follower example can be directly applied to address Flight-deck Interval Management [9], where ADS-B [10] equipped aircraft pairs automatically achieve relative spacing by dynamically assigning waypoints to each airplane.

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The rest of the paper is organized as follows. In Section 2, related literature is reviewed. Section 3 defines the mathematical problem and assumptions. In Section 4, a key concept of this paper—Trajectory Inclusion—is proposed. Section 5 and Section 6 explain how to use 4DT waypoints to achieve conflict-free trajectories for single time interval and two consecutive time intervals respectively by applying Trajectory Inclusion. Section 7 shows how to achieve conflict-free trajectories for all time based on the findings in Section 5 and Section 6. Section 8 concludes this paper with some final remarks.

2. Literature review

2.1. Time based management

Time Based Management is at the heart of TBO [5]. Both time-based and speed-based instruction have their roles in Time Based Management, but the distinction between the roles are unclear. FAA's order JO 7210.3AA [11] prescribes that time-based metering allows the routine use of Performance Based Operations and only applies spacing **when needed**. It is consistent with FAA's Performance Based Navigation (PBN) strategy to continue transition from distance-based to time-based and speed-based air traffic management [12]. However, when is "when needed"? In fact, there is a general lack of understanding about what a NAS Time-based Scheduling and Management system means from the perspectives of both Air traffic and System adaptors [12], which coincides with the findings in [13] that there are general confusions about the usage and implementation of the time-based capabilities.

Based on [14–16], though the exact detailed algorithm is not open to the public and names of the modules where the functions reside in differ, it is certain that STA is calculated first and then airplanes are given speed advisories to catch each STA at the respective Constraint Satisfaction Point (CSP). [17] supports this claim by pointing out that 4D trajectory is generated to create control advisories, usually as speed and altitude profiles, on how to meet the STA at a waypoint. More specifically, [30] claims the TBFM system uses trajectory modeling functions to build a sequence and schedule of aircraft joining an arrival flow and provides a time schedule at meter reference points (MRPs). Its sub-function of speed advisories suggests airspeeds that ATC can provide to an aircraft to help meet its frozen scheduled time of arrival (STA) at an MRP. [31] touches upon the time-based and speed-based concepts by studying the difference and interaction between schedule-based management and spacing-based management at CMPs. [9] discusses the concept of operation for Interval Management, which is basically a speed-based decision-making tool to assist ATC to maintain STA and conduct relative spacing. More similar work can be found in [32,33].

In summary, many works describe using speed advisory to satisfy STA, but none of them considers to what extent the use of STAs affects the use of speed-based advisories, let alone a coordinated use of them both.

2.2. Conflict-free trajectory

There is a relevant body of literature that specifically relates to the generation of conflict-free trajectories. [18] proposes a stochastic linear hybrid system to describe the dynamics of an aircraft with changing flight modes and a computationally efficient algorithm is developed to estimate aircraft future trajectory. This approach makes better predictions because the aircraft's intent information is included in the model. However, this work does not mention how to generate appropriate, safe, or "good" intent in the first place. In [19], a probabilistic conflict detection algorithm is proposed to determine the evolution of uncertainty in the complex nonlinear dynamical systems with high computational efficiency and then an optimal control method is combined to solve the conflict resolution problem. [20] presents a way of obtaining a conflict-free solution for all planned trajectories during the strategic phase based on a data-driven conflict-resolution model and a multi-objective global optimization algorithm. In [21], a dynamic optimizer is proposed for complex (and realistic) lateral and vertical trajectories, producing vertical, lateral and speed profile. Direct collocation methods are used to convert the complex problem to a continuous multiphase optimal control problem that is solved with non-linear programming techniques to minimize fuel burn. [22] makes improvements in the areas of computationally efficient wind-optimal routing, aircraft conflict detection, and optimal conflict resolution. An algorithm is developed to compute a complete set of conflict-free optimal wind routes for double the current-day single flight level air traffic density in less than one minute on an average (450 MHz) workstation, which was an order-of-magnitude improvement over current state-of-the-art algorithms. More related work can be found in [23] and [24].

Despite a large variety of models and algorithms used to achieve conflict-free trajectories, most of them take STA as given and explicitly or implicitly assume to use speed to respect a 4DT waypoint and resolve a potential conflict. None of them considers using 4DT waypoints effectively to eliminate conflicts in the first place and use speed advisories only when time-based instructions alone cannot avoid conflict. The most similar work we can find is [25], but it still takes 4DT waypoints as given.

3. Problem definition

3.1. The leader–follower example

This paper develops methods to achieve conflict-free trajectories by only using 4DT waypoints regardless of the potential speed profiles. A leader–follower example (Fig. 1) is studied. Two airplanes, AC1 and AC2, fly from waypoints to waypoints following the same 3D trajectory (i.e. a 3D "flow") with any speed that the pilots choose to fly. Although vectoring and holding can also be used for safety clearance, they are not considered in this paper, for the following reasons. First, they are not the first choice for safety clearance due to their fuel inefficiency and attention-demanding nature. Second, and more importantly, they are only used "when a speed solution alone does not exist" [12]. This paper seeks to use 4DT waypoints to minimize the use of speed resolution, and hence can effectively eliminate the necessity to use the costly vectoring and holding in the first place.

Note that the "x" axis denotes positions along the 3D trajectory. Time ("t") is the fourth dimension and will be explicitly represented in Fig. 2 momentarily. In an earth-centered inertial coordinate system, the axis x (i.e. a 3D "flow") can be a straight line representing air corridors, or any curved path with altitude changes and turns such as in the standard routes for arrival and departure. As long as the aircraft fly in a flow, the results of this paper are applicable.

It is a simple but non-trivial example. In fact, it is an important enough example to be one of the three types of trajectory conflicts described in [26]. The authors called it "Trailing Conflict" and it is worth studying "as it is often the case on airways". Moreover, in 2014, NASA released "A Concept of Operations for Far-Term STBO" [27], which listed trailing conflicts as one of the operational scenarios called "Lead/Follow Conflict" to highlight the unique time-based capabilities of the far-term NextGen ConOps. More recently in 2017, [28] also studied trailing conflicts for future Trajectory Based Operations, where it is characterized as "Single-Altitude, On-Airway Airborne Spacing Operation".

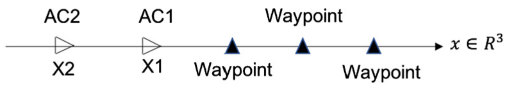

Fig. 1. The leader–follower example in a 3D flow.

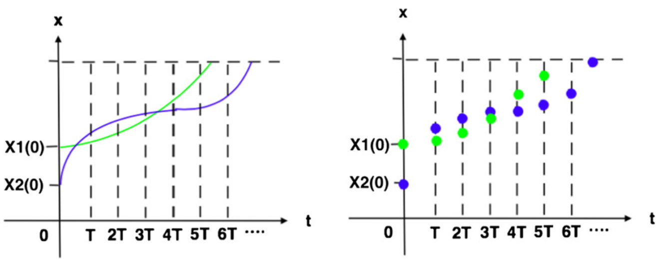

Fig. 2. The 4DT representation of the leader–follower example.

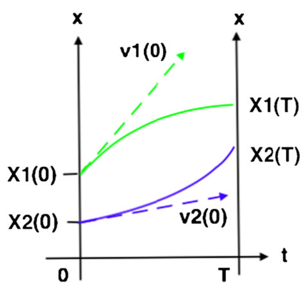

Fig. 3. Conflict-free trajectories for time interval $t \in [0, T]$ as an example.

3.2. Problem definition

Fig. 2 is the 4DT representation of the leader–follower example. The vertical axis denotes the 3D flow and the horizontal axis denotes the fourth dimension, time. The entire 4DT trajectory is broken down into time intervals T, and a 4DT trajectory is a set of 3D waypoints defined at all the time stamps, i.e., $\{x, t | t = 0, T, 2T, 3T, \dots\}$. If we can guarantee no conflict in each time interval, then we can guarantee conflict-free trajectories for all time. For the purpose of illustration, T is assumed the same among all the time intervals. In fact, as long as the length of the time intervals satisfy Assumption 3, they can be different and the results of this paper are still applicable.

Hence, the entire problem boils down to guaranteeing conflict-free trajectories for each single time interval. Taking time interval $t \in [0, T]$ for example (Fig. 3), the initial conditions (positions and speed) of the two airplanes are assumed known at $t = 0$, which are $X1(0)$, $v1(0)$ and $X2(0)$, $v2(0)$ respectively. The curved line represents the trajectory of the aircraft along $[0, T]$, and the straight dash line starting at $t = 0$ represents the initial speed of the aircraft, whose slope is the magnitude of the speed. The task is to select waypoints for both airplanes at $t = T$, i.e. $X1(T)$ and $X2(T)$, so that regardless of the potential speed profiles the pilots choose to fly, their trajectories, line $X1(0)X1(T)$ and line $X2(0)X2(T)$, do not intersect.

3.3. Definition and assumptions

Definition.

$x \in [\underline{x}, \overline{x}]$

- When x is a variable: the variables in this paper can be position, speed, acceleration and so on. $\overline{x}$ and $\underline{x}$ denote respectively the upper bound and lower bound of the quantity of the variable.
- When x is a point: $\overline{x}$ and $\underline{x}$ denote respectively the most advanced point and the least advanced point that Point x can be. The measurement of their positions are variables and denoted by $X_{\overline{x}}$ and $X_{\underline{x}}$.

Assumption 1.

$v \in [\underline{v}, \overline{v}]$ and $\overline{v} > 0, \underline{v} > 0$

The speed bound is uncertain, determined by both deterministic factors and non-deterministic factors such as altitude and wind. In fact, speed restrictions are already calculated in current Flight Management System in real time. Many factors are taken into consideration, such as wind model, flight phase, altitude, flight plan speed restrictions, flaps configuration and airframe speed envelope limitations [29]. Deciding the exact speed bound is out of the scope of this paper, but we assume here the speed bound is known in real time, $[\underline{v}, \overline{v}]$.

Assumption 2.

$a \in [-\underline{a}, \overline{a}]$ and $\underline{a} > 0, \overline{a} > 0$

Similar to speed, the acceleration/deceleration rate is determined by many factors, both deterministic and non-deterministic, such as engine thrust, altitude and wind. We assume an over-approximation of the acceleration and deceleration rate are known in real time, which are $[0, \overline{a}]$ and $[-\underline{a}, 0]$ respectively. The negative part of a means deceleration.

Assumption 3.

$$T > \frac{\overline{v} - \underline{v}}{\overline{a}} + \frac{\overline{v} - \underline{v}}{\underline{a}}$$

Obviously, the length of each time interval T is important. For simplicity, we assume T is long enough for the aircraft to speed up from the lowest speed to highest speed and then slow down to the lowest speed. A more comprehensive investigation about the effect of T should be conducted.

4. Trajectory inclusion

In this section, Trajectory Inclusion, the fundamental concept of our approach, is proposed. Intuitively, a Trajectory Inclusion is an over approximation of all the possible 4D trajectories that an airplane can take from one 4DT waypoint to another.

As shown in Fig. 4, an airplane flies along the 3D trajectory—the "x" axis. The current position at $t = 0$ is $X(0)$ and the waypoint to reach at $t = T$ is $X(T)$. Note that, on one hand, symbols like A and $\overline{D}$ are the geometry points, whose position measurement along the 3D trajectory are denoted in the fashion of X_A and $X_{\overline{D}}$. On the other hand, symbols without subscripts like $X(0)$, $X(T)$ and $\overline{X(T)}$ denote both geometry points and their position measurement.

As shown in Fig. 4(a), first start from Point $X(0)$ and draw two straight lines with slopes of $\overline{v}$ and $\underline{v}$ respectively. They intersect $t = T$ at Point A and B. Then start from Point $X(T)$ and draw two straight lines with slopes of $\overline{v}$ and $\underline{v}$ respectively. The intersect $t = 0$ at Point C and D. Line X(0)A intersects line X(T)C at Point N and line X(0)B intersects line X(T)D at Point M. Because the airplane speed is bounded with $[\underline{v}, \overline{v}]$, an airplane, starting from Point $X(0)$, can only reach points within the Triangle X(0)AB. Similarly, to reach $X(T)$ at $t = T$, an airplane has to be within the Triangle X(T)CD. Hence, an airplane, starting from $X(0)$ to reach $X(T)$, has to fly with the overlapping of Triangle X(0)AB and X(T)CD, thus the Trajectory Inclusion— diamond X(0)MX(T)N.

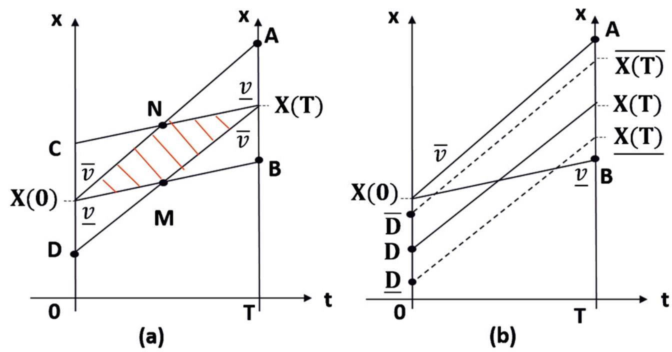

Fig. 4. (a) The red marked diamond is an inclusion of all the possible trajectories that an airplane can take to reach waypoint $X(T)$ at $t = T$ from position $X(0)$ at $t = 0$; (b) The feasible waypoint X(T) is bounded within $[\underline{X(T)}, \overline{X(T)}]$ when the initial speed v(0) is taken into consideration. (For interpretation of the colors in the figure(s), the reader is referred to the online version of this article.)

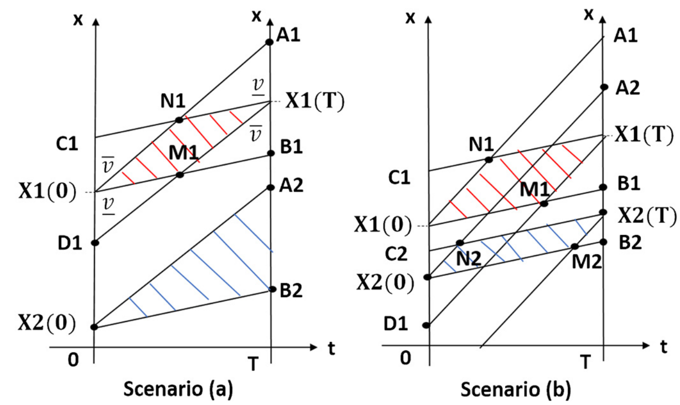

Fig. 5. Scenario (a): Conflict-free trajectories can be guaranteed if Point X2(0) is behind Point D1; Scenario (b): Conflict-free trajectories can be guaranteed as long as Point X2(T) is behind Point B1, even though Point X2(0) is before Point D1.

Furthermore, $X(T)$, the waypoint to reach, cannot be selected arbitrarily (Fig. 4(b)). First, it has to be apparently between Point A and B. Moreover, because it takes time for the airplane to transition from its initial speed $v(0)$ to the fastest/lowest speed, it can never reach the full range between A and B. We denote the resulting bound for $X(T)$ with $[\underline{X(T)}, \overline{X(T)}]$. Starting from Point $\underline{X(T)}$ and $\overline{X(T)}$, draw two straight lines with slope of $\overline{v}$ respectively. They intersect with $t = 0$ at Point $\underline{D}$ and $\overline{D}$.

The following are the quantitative representations of some of the points that are going to be extensively used in rest of the paper.

The position of Point B is,

$$X_B = X(0) + \underline{v}T \tag{1}$$

The position of Point D is,

$$X_D = X(T) - \overline{v}T \tag{2}$$

The calculation of $\underline{X(T)}$ and $\overline{X(T)}$ is straightforward. The idea is to accelerate/decelerate to the boundary speed $\overline{v}/\underline{v}$ with the highest rate $\overline{a}/\underline{a}$, and then maintain the speed $\overline{v}/\underline{v}$ to $t = T$.

$$X(T) \in [\underline{X(T)}, \overline{X(T)}] \tag{3}$$

where,

$$\overline{X(T)} = X(0) + \overline{v}T - \frac{(\overline{v} - v(0))^2}{2\overline{a}}$$
$$\underline{X(T)} = X(0) + \underline{v}T + \frac{(v(0) - \underline{v})^2}{2\underline{a}}$$

Applying (3) to (2), we have X_D:

$$X_D \in [X_{\underline{D}}, X_{\overline{D}}] \tag{4}$$

where,

$$X_{\overline{D}} = X(0) - \frac{(\overline{v} - v(0))^2}{2\overline{a}}$$
$$X_{\underline{D}} = X(0) + \underline{v}T + \frac{(v(0) - \underline{v})^2}{2\underline{a}} - \overline{v}T$$

5. Conflict-free trajectories for $[0, T]$

In this section, two **scenarios** are first (Section 5.1) proposed, where conflict-free trajectories are guaranteed based on the concept of Trajectory Inclusion, regardless of the potential speed profiles. **Strategies** (Section 5.2) to lead the airplanes into the conflict-free scenarios are then developed accordingly. However, not all situations can be led into the conflict-free scenarios with the strategies. Certain **conditions** (Section 5.3) have to be satisfied. They are called applicable conditions for the strategies.

5.1. Conflict-free scenarios based on trajectory inclusion

For the leader–follower problem, if there is no overlapping between the Trajectory Inclusions of the two airplanes during $[0, T]$, the conflict-free trajectories can be guaranteed during the time interval, regardless of the potential speed profiles. Note that, for the purpose of illustration, we assume the two airplanes have the same speed and acceleration/deceleration constraints in this paper. These results can be easily applied to situations where airplanes have different speed and acceleration/deceleration constraints.

As shown in Fig. 5, AC1, currently at Point $X1(0)$, leads AC2 at Point $X2(0)$ at $t = 0$. Points A1/A2, B1/B2, C1/C2 and D1/D2 corresponds to the points A, B, C and D in Fig. 4 for AC1/AC2 respectively.

Scenario (a)

When Point X2(0) is behind Point D1 (Fig. 5(a)), AC2's Trajectory Inclusion, Triangle X2(0)A2B2, has no overlap with AC1's Trajectory Inclusion, Diamond X1(0)M1X1(T)N1. Hence, the waypoints of both aircraft, X1(T) and X2(T), can be selected freely as long as they can be reached at $t = T$, i.e. $X1(T) \in [\underline{X1(T)}, \overline{X1(T)}]$ and $X2(T) \in [\underline{X2(T)}, \overline{X2(T)}]$.

It can be summarized as:

When $X2(0) < X_{D1}$:

$X1(T) \in [\underline{X1(T)}, \overline{X1(T)}]$, $X2(T) \in [\underline{X2(T)}, \overline{X2(T)}]$.

Scenario (b)

When Point X2(0) is before Point D1 (Fig. 5(a)), as long as X2(T) is behind Point B1, AC1's Trajectory Inclusion (Diamond X1(0)M1X1(T)N1) has no overlapping with AC2's Trajectory Inclusion (Diamond X2(0)M2X2(T)N2). Hence, $X1(T) \in [\underline{X1(T)}, \overline{X1(T)}]$ and $X2(T) \in [\underline{X2(T)}, X_{B1}]$.

It can be summarized as:

When $X2(0) > X_{D1}$:

$X1(T) \in [\underline{X1(T)}, \overline{X1(T)}]$ and $X2(T) \in [\underline{X2(T)}, X_{B1}]$.

5.2. Strategies to achieve conflict-free scenarios

In general, "strategy" in this paper means: given the initial condition $(X1(0), X2(0), v1(0), v2(0))$, if the 4DT waypoints $\{X1(T), X2(T)\}$ are selected following the strategies, then the conflict-free scenarios of Section 5.1 can be achieved.

Strategy (a):

To achieve Scenario (a), Strategy (a) is to select $X1(T)$ so that $X_{D1} > X2(0)$. After that $X2(T)$ can be selected freely. See the previous section for details.

Mathematically, Strategy (a) can be represented as follows. Note that $X_{D1}, \underline{X1(T)}, \overline{X1(T)}, \underline{X2(T)}$ and $\overline{X2(T)}$ have been expanded based on equations (2) and (4).

$$\begin{cases} X1(T) \in [max(X2(0) + \overline{v}T, X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}), \\ \qquad X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}}] \\ X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X2(0) + \overline{v}T - \frac{(\overline{v}-v2(0))^2}{2\overline{a}}] \end{cases}$$

Intuitively, it means given the initial condition $(X1(0), X2(0), v1(0), v2(0))$, as long as $X1(T)$ and $X2(T)$ are selected accordingly from the sets above, Scenario (a) can be achieved and hence conflict-free trajectories are guaranteed.

Strategy (b):

To achieve Scenario (b), Strategy (b) is to select $X1(T)$ and $X2(T)$, so that $X_{D1} < X2(0)$ and $X2(T) < X_{B1}$.

Mathematically, Strategy (b) can be represented as follows. Note that $X_{B1}, X_{D1}, \underline{X1(T)}, \overline{X1(T)}, \underline{X2(T)}$ and $\overline{X2(T)}$ have been expanded based on equations (1), (2) and (3),

$$\begin{cases} X1(T) \in [X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}, min(X2(0) + \overline{v}T, \\ \qquad X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}})] \\ X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \underline{v}T] \end{cases}$$

Intuitively, it means given the initial condition $(X1(0), X2(0), v1(0), v2(0))$, as long as $X1(T)$ and $X2(T)$ are selected accordingly from the sets above, Scenario (b) can be achieved and hence conflict-free trajectories are guaranteed.

Expanding the *max* and *min* above, Strategy (a) and (b) are further refined into the four sets. Given the initial condition $(X1(0), X2(0), v1(0), v2(0))$, as long as $X1(T)$ and $X2(T)$ are selected accordingly from the following sets, conflict-free trajectories can be guaranteed.

Strategy (a-1):

$$X1(T) \in [X2(0) + \overline{v}T, X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}}]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X2(0) + \overline{v}T - \frac{(\overline{v}-v2(0))^2}{2\overline{a}}]$$

Strategy (a-2):

$$X1(T) \in [X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}}]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X2(0) + \overline{v}T - \frac{(\overline{v}-v2(0))^2}{2\overline{a}}]$$

Strategy (b-1):

$$X1(T) \in [X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}, X2(0) + \overline{v}T]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \underline{v}T]$$

Strategy (b-2):

$$X1(T) \in [X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}}]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \underline{v}T]$$

5.3. Applicable conditions for the strategies

Strategies above are mappings from the set $\{X1(0), X2(0), v1(0), v2(0)\}$ to the set $\{X1(T), X2(T)\}$. However, not all elements in the former can be mapped to the latter. The upper bounds of $X1(T)$ and $X2(T)$ has to be greater than their lower bounds, so that $\{X1(T), X2(T)\}$ is not empty. The relationship among $X1(0), X2(0), v1(0)$ and $v2(0)$ that make $\{X1(T), X2(T)\}$ non-empty is called the "applicable condition" for the strategy.

Appendix B has the detailed mathematical proof and derivation of the applicable conditions. The results are summarized as follows:

Condition (a-1):

$$\frac{(\overline{v}-v1(0))^2}{2\overline{a}} < X1(0) - X2(0) < \overline{v}T - \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

Condition (a-2):

$$X1(0) - X2(0) > \overline{v}T - \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

Condition (b-1):

$$max(\frac{(\overline{v}-v1(0))^2}{2\overline{a}}, \frac{(v2(0)-\underline{v})^2}{2\underline{a}}) < X1(0) - X2(0)$$
$$< \overline{v}T - \underline{v}T - \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

Condition (b-2):

$$\frac{(v2(0)-\underline{v})^2}{2\underline{a}} < X1(0) - X2(0) < \frac{(\overline{v}-v1(0))^2}{2\overline{a}}$$

5.4. Observations

Observation 1. The relationship between scenario, strategy and condition can be summarized as: conflict-free *Scenarios* can be achieved by applying *Strategies*, which are only applicable when certain *Conditions* are satisfied. As shown in Fig. 6, the **initial condition** at the bottom determines whether there are strategies (and which strategy) to apply, so that 4DT waypoints exist to form the desired scenarios, and in turn secure the conflict-free trajectories during $[0, T]$.

Observation 2. Condition (a-1) and (a-2) can be merged into (5), which becomes the applicable condition for Strategy (a).

$$X1(0) - X2(0) > \frac{(\overline{v} - v1(0))^2}{2\overline{a}} \tag{5}$$

Similarly, Condition (b-1) and (b-2) can be merged. (6) is the resulting applicable condition for Strategy (b).

$$\frac{(v2(0) - \underline{v})^2}{2\underline{a}} < X1(0) - X2(0) < \overline{v}T - \underline{v}T - \frac{(v1(0) - \underline{v})^2}{2\underline{a}} \tag{6}$$

6. Conflict-free trajectories for $[0, 2T]$

The previous section concludes that $(X1(0), X2(0), v1(0), v2(0))$ determines whether conflict-free trajectories during $[0, T]$ **exist** regardless of the potential speed profiles. Similarly, $(X1(T), X2(T), v1(T), v2(T))$ determines the existence of conflict-free trajectories for $[T, 2T]$. However, because $v1(T)$ and $v2(T)$ are not supposed to be specified in the context of time-based control, there is no control of $(X1(T), X2(T), v1(T), v2(T))$. Therefore, there is no guarantee that conflict-free trajectories can exist during $[T, 2T]$ if $(X1(T), X2(T))$ are selected only for conflict-free trajectories in $[0, T]$. This limits the scalability of our approach and this section tackles it.

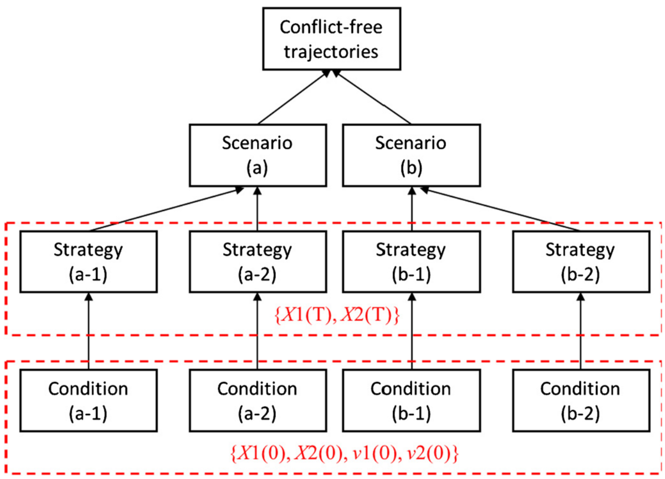

Fig. 6. Initial condition determines whether conflict-free trajectories exist.

6.1. *The existence of conflict-free trajectories during* $[T, 2T]$

This subsection explains, at $t = 0$, how to select waypoints $(X1(T), X2(T))$ so that conflict-free trajectories exist during $[T, 2T]$ regardless of the potential speed profiles during $[0, T]$.

First, it is easy to prove the following inequality by applying Property 1 (see Appendix A).

$$\overline{v}T - \underline{v}T - \frac{(v1(0) - \underline{v})^2}{2\underline{a}} > \frac{(\overline{v} - v1(0))^2}{2\overline{a}}$$

(5) and (6) are the conditions for the conflict-free trajectories to exist. They can be merged using inequality above and rewritten as (7) for $[T, 2T]$.

$$X1(T) - X2(T) > min(\frac{(\overline{v} - v1(T))^2}{2\overline{a}}, \frac{(v2(T) - \underline{v})^2}{2\underline{a}}) \tag{7}$$

(8) is a sufficient condition for (7). In other words, as long as (8) is true, (7) is true, and hence conflict-free trajectories during $[T, 2T]$ exist.

$$X1(T) - X2(T) > \frac{(v2(T) - \underline{v})^2}{2\underline{a}} \tag{8}$$

Second, as observed by Property 2 (see Appendix A), there is a quantitative relationship between $X(T)$ and $v(T)$: for each $X(T)$, a subset of $[\overline{v}, \underline{v}]$ exist for $v(T)$ to reach $X(T)$ on time. An extreme example is, if the most advanced waypoint is selected, i.e. $X(T) = \overline{X(T)}$, then the airplane has to accelerate immediately with $\overline{a}$ to reach the speed upper bound $\overline{v}$ and maintain $\overline{v}$ till $t = T$. In other words, only one value of $v(T)$ is possible for $X(T) = \overline{X(T)}$, which is $v(T) = \overline{v}$. Intuitively, this means even though the speed profiles are not explicitly specified, the selection of $(X1(T), X2(T))$ has an implication to the possible value of $(v1(T), v2(T))$.

Therefore, the problem becomes, at $t = 0$, how to select $(X1(T), X2(T))$ and use its implication to $(v1(T), v2(T))$, in order to ensure (8) is true. Appendix C is a full detailed address of this problem and the results are shown in Fig. 7. As long as $(X1(T), X2(T))$ is selected from the red marked area, (8) is always true, and hence the conflict-free trajectories always exist for $[T, 2T]$ without specifying $(v1(T), v2(T))$. Note that, L, M, N and K are all straight lines. For the sake of space, $\underline{X1(T)}$, $\overline{X1(T)}$, $\underline{X2(T)}$ and $\overline{X2(T)}$ are not fully expanded. The full expressions can be found in (3).

6.2. *Conflict-free trajectories for* $[0, 2T]$

"Conflict-free trajectories for $[0, 2T]$" means selecting $(X1(T), X2(T))$ at $t = 0$ so that, 1) conflict-free trajectories can be **guaranteed** for $[0, T]$ and 2) conflict-free trajectories **exist** for $[T, 2T]$.

For 1), $(X1(T), X2(T))$ has to be selected by following the strategies (a-1), (a-2), (b-1) or (b-2). In fact, each strategy represents a rectangle in a X1(T)X2(T)-plane, as of in Fig. 7. For 2), $(X1(T), X2(T))$ has to be selected from the red marked area as shown in Fig. 7.

Take Strategy (a-1) as an example. It is represented by all the points within the blue marked rectangle as shown in Fig. 8. It overlaps the red marked area at the blue-red marked area, within which all the points satisfy both 1) and 2). This area is called the "conflict-free waypoints area (a-1)" for Strategy (a-1). It is where the waypoints can be selected for conflict-free trajectories during $[0, 2T]$, if Strategy (a-1) is applicable. A precise expression of this "conflict-free waypoints area" can always be derived, but it is theoretically trivial, and thus no further discussion here. Similarly, the "conflict-free waypoints area" for (a-2), (b-1) or (b-2) can be derived by following the exactly same approach.

However, the "conflict-free waypoints area" can be empty. If that is the case, the corresponding strategy cannot lead to conflict-free trajectories for $[0, 2T]$. Taking Strategy (a-1) for example again (Fig. 8): $X1(T) = X1(0) + \overline{v}T - \frac{(\overline{v} - v1(0))^2}{2\overline{a}}$ has to be at the right of $X1(T) = \underline{X2(T)}$, so that "conflict-free waypoints area (a-1)" is not empty, and hence Strategy (a-1) can be used to select waypoints for conflict-free trajectories during $[0, 2T]$.

The rest of the section proves all the "conflict-free waypoints areas" are not empty. Note that $\overline{X2(T)}$ and $\underline{X2(T)}$ in Fig. 9 are expanded, in order to be consistent with the expressions used in the strategies.

Observe that, $X2(T)$'s lower bounds in all the strategies are the same, $X2(T) = X2(0) + \underline{v}T + \frac{(v2(0) - \underline{v})^2}{2\underline{a}}$. It corresponds to line K in Fig. 9. Therefore, as long as the upper bound of $X1(T)$ is at the right of $X1(T) = X2(0) + \underline{v}T + \frac{(v2(0) - \underline{v})^2}{2\underline{a}}$, the "conflict-free waypoints area" will not be empty.

The upper bound of $X1(T)$ in all the strategies are $X1(T) = X1(0) + \overline{v}T - \frac{(\overline{v} - v1(0))^2}{2\overline{a}}$ or $X1(T) = X2(0) + \overline{v}T$. It can be proved that both of them are greater than $X1(T) = X2(0) + \underline{v}T + \frac{(v2(0) - \underline{v})^2}{2\underline{a}}$. Hence, the "conflict-free waypoints areas" are never empty. See Appendix D for the detailed proof.

Therefore, to conclude this section: as long as there are waypoints $(X1(T), X2(T))$ to **guarantee** conflict-free trajectories for $[0, T]$, conflict-free trajectories for $[T, 2T]$ always **exist**.

7. Conflict-free trajectories for all time using 4D waypoints

Section 5 concludes that, as long as the applicable conditions (see Section 5.3) at $t = 0$ are satisfied, there are waypoints $(X1(T), X2(T))$ to guarantee conflict-free trajectories for $[0, T]$, regardless of the speed profiles.

Section 6 concludes that, as long as there are waypoints $(X1(T), X2(T))$ to guarantee conflict-free trajectories for $[0, T]$, conflict-free trajectories for $[T, 2T]$ always exist, regardless of instantaneous speed $(v1(T), v2(T))$ at $t = T$.

Combining those two conclusions, a new conclusion can be drawn: as long as the applicable conditions at $t = 0$ are satisfied, waypoints $(X1(T), X2(T))$ always exist to a) guarantee conflict-free trajectories for $[0, T]$ and b) ensure waypoints $(X1(2T), X2(2T))$ always exist to guarantee conflict-free trajectories for $[T, 2T]$.

Fig. 10 is a scheme to achieve conflict-free trajectories for all time using only 4D waypoints, regardless of the speed profiles.

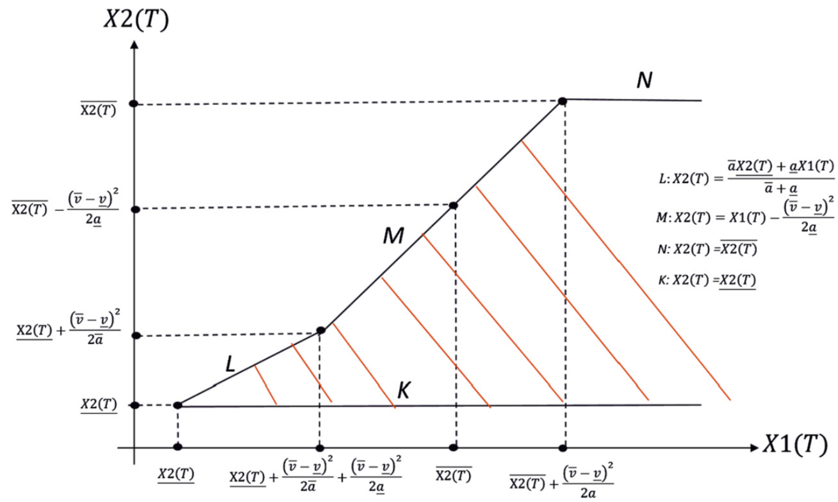

Fig. 7. Conflict-free trajectories always exist during $[T, 2T]$ if $X1(T)$ and $X2(T)$ are selected from the red marked area.

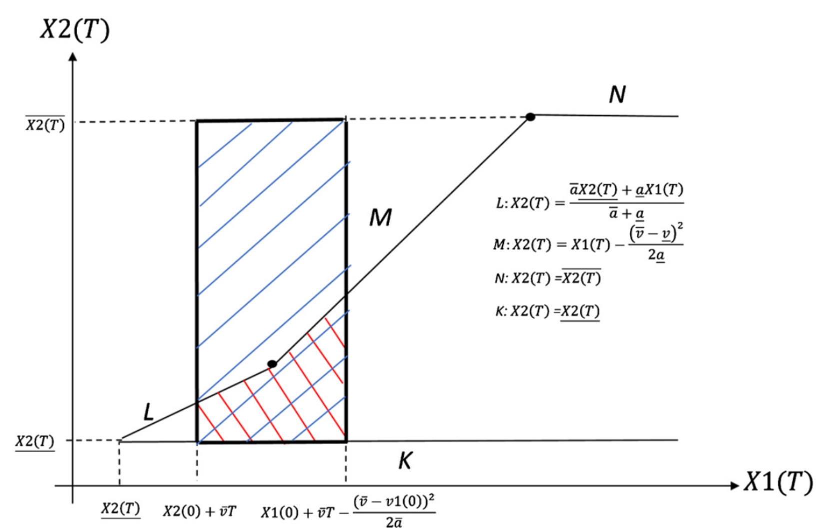

Fig. 8. When Strategy (a-1) is applicable, $(X1(T), X2(T))$ selected from the "conflict-free waypoints area (a-1)" (red-blue marked) can guarantee conflict-free trajectories during $[0, T]$ and the existence of conflict-free trajectories during $[T, 2T]$.

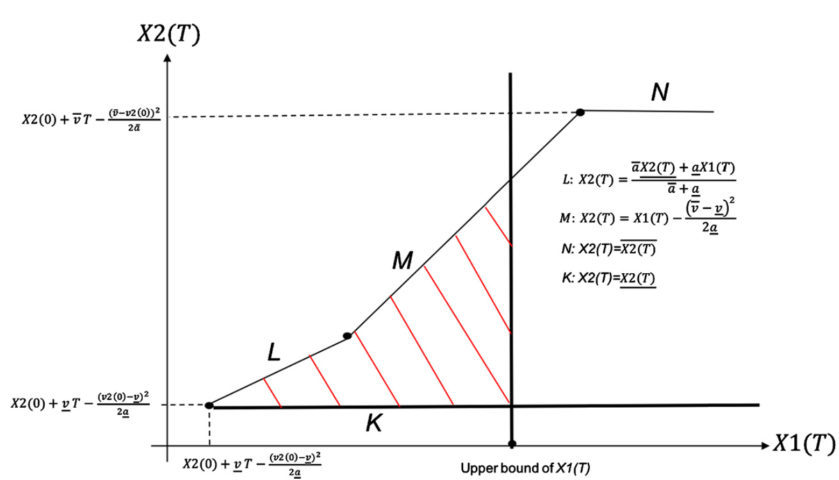

Fig. 9. When the upper bound of $X1(T)$ is at the right of $X1(T) = X2(0) + \underline{v}T + \frac{(v2(0) - \underline{v})^2}{2\underline{a}}$, the "conflict-free waypoints area" is not empty.

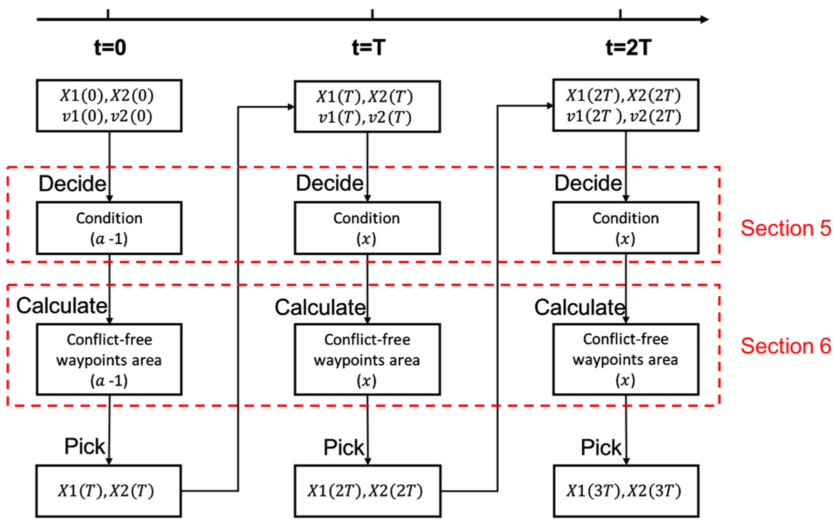

Fig. 10. The scheme for conflict-free trajectories using 4D waypoints, where x is a variable, whose value can be any of $\{a-1, a-2, b-1, b-2\}$.

(1) $t = 0$

First, assume initial condition $(X1(0), X2(0), v1(0), v2(0))$ satisfies the applicable conditions in Section 5.3, say Condition (a-1). Second, based on the conclusion of Section 6, the "conflict-free waypoints area (a-1)" always exist and can be calculated accordingly. Third, the waypoints $(X1(T), X2(T))$ is then picked from the area. $(X1(T), X2(T))$ can guarantee conflict-free trajectories during $[0, T]$ regardless of the speed profiles, and can also ensure $(X1(2T), X2(2T))$ exist for conflict-free trajectories during $[T, 2T]$.

(2) $t = T$

First, waypoints $(X1(T), X2(T))$ picked at $t = 0$ already ensured that $(X1(T), X2(T), v1(T), v2(T))$ satisfies the applicable conditions and $(X1(2T), X2(2T))$ always exists for conflict-free trajectories during $[T, 2T]$. It only has to decide which condition x is satisfied, in order to select the strategy accordingly. Similarly, based on the conclusion of Section 6, the corresponding "conflict-free waypoints area (x)" always exist and can be calculated. Then $(X1(2T), X2(2T))$ is picked from the "conflict-free waypoints area (x)" for $t = 2T$. Again, $(X1(2T), X2(2T))$ can guarantee conflict-free trajectories during $[T, 2T]$ regardless of the speed profiles, and can also ensure $(X1(3T), X2(3T))$ exist for conflict-free trajectories during $[2T, 3T]$.

(3) $t \geq 2T$

Follow the same process to select waypoints $(X1(3T), X2(3T))$ and conflict-free trajectories can be guaranteed for $[2T, 3T]$ and waypoints $(X1(4T), X2(4T))$ exist for conflict-free trajectories during $[3T, 4T]$. Repeat the same process for all the succeeding time stamps, and conflict-free trajectories can be guaranteed for all the time intervals.

In summary, as long as $(X1(T), X2(T))$ exists for conflict-free trajectories during $[0, T]$, 4DT waypoints can always be found to guarantee conflict-free trajectories for all the succeeding time intervals by following the scheme above. In other words, if the initial condition satisfies the applicable conditions defined in Section 5.3, conflict-free trajectories can be guaranteed **for all time** by using 4DT waypoints only.

8. Conclusion

Trajectory Based Operation in NextGen is a time-based air traffic management concept. Both time-based and speed-based instruction play important roles in TBO. However, the decision of when to use speed or time control remains an open problem. In this paper, we studied a leader–follower problem where two airplanes fly the same 3D trajectory and are regulated by their respective 4DT waypoints. It is a fundamental building block that supports a variety of important NextGen applications [26–28]. The core question we are asking is whether it is possible to achieve conflict-free trajectories by only prescribing the 4DT waypoints regardless of the speed profiles airplanes actually fly?

Toward this end, a concept called Trajectory Inclusion is first introduced based on geometry and physics-driven proofs and analysis. Conditions and strategies to achieve conflict-free trajectories are then developed and further explained with the leader–follower example. It is found that, regardless of the speed profiles, as long as certain initial condition is satisfied, the conflict-free trajectories can always be guaranteed for all time by 4DT waypoints alone.

The results of this paper can enhance the understanding of time-based management capabilities, and help formulate better time-based instructions to reduce unnecessary tactical maneuvers and improve the overall performance of the airspace, one of the main promises of TBO. Specifically, the algorithmic scheme created for the leader–follower example can be applied directly to Flight-deck Interval Management, a future concept of TBO.

Conflict of interest statement

The authors state that there is no conflict of interest.

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Appendix A. Properties

In this section, we provide two important properties that will be extensively used in the mathematical proof of this paper.

Property 1. Given Assumption 3, the following inequality is always true:

$$\overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{\underline{a}} > 0$$

Property 1 is a direct derivation from $T > \frac{\overline{v}-\underline{v}}{\overline{a}} + \frac{\overline{v}-\underline{v}}{\underline{a}}$ (Assumption 3), by multiplying $(\overline{v}-\underline{v})$ at both sides.

Property 2. Given initial condition $X(0)$ and $v(0)$ at $t = 0$ and the desired waypoint $X(T)$, all the possible $v(T)$ at $t = T$ can be summarized as follows:

1) When $X(T) \in [\underline{X(v(T))}_{min}, \underline{X(v(T))}_{max}]$

$$v(T) \in [\underline{v}, \underline{v} + \sqrt{2\overline{a}(X(T) - X(0) - \underline{v}T - \frac{(v(0)-\underline{v})^2}{2\underline{a}})}]$$

2) When $X(T) \in [\underline{X(v(T))}_{max}, \overline{X(v(T))}_{min}]$

$$v(T) \in [\underline{v}, \overline{v}]$$

3) When $X(T) \in [\overline{X(v(T))}_{min}, \overline{X(v(T))}_{max}]$

$$v(T) \in [\overline{v} - \sqrt{2\underline{a}(X(0) + \overline{v}T - \frac{(\overline{v}-v(0))^2}{2\overline{a}} - X(T))}, \overline{v}]$$

where

$$\underline{X(v(T))}_{max} = X(0) + \underline{v}T + \frac{(v(0)-\underline{v})^2}{2\underline{a}} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}},$$
$$\underline{X(v(T))}_{min} = X(0) + \underline{v}T + \frac{(v(0)-\underline{v})^2}{2\underline{a}},$$
$$\overline{X(v(T))}_{max} = X(0) + \overline{v}T - \frac{(\overline{v}-v(0))^2}{2\overline{a}},$$
$$\overline{X(v(T))}_{min} = X(0) + \overline{v}T - \frac{(\overline{v}-v(0))^2}{2\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}.$$

The intuition is that, not all $v(T) \in [\underline{v}, \overline{v}]$ is feasible for certain waypoint $X(T)$. An extreme example is, if the most advanced waypoint at $t = T$ is selected, meaning $X(T) = \overline{X(T)}$, then the airplane has to accelerate immediately with $\overline{a}$ to reach the speed of $\overline{v}$ and maintain $\overline{v}$ till $t = T$. In other words, only one value of $v(T)$, instead of the entire range of $[\underline{v}, \overline{v}]$, is possible for $X(T) = \overline{X(T)}$, which is $v(T) = \overline{v}$ in this case. Hence, there is a quantitative relationship between $X(T)$ and $v(T)$: for each $X(T)$, a subset of $[\overline{v}, \underline{v}]$ exist for $v(T)$ to reach $X(T)$ on time. Intuitively, this means even though the speed profiles are not explicitly specified, the selection of $X(T)$ has an implication to the possible value of $v(T)$.

Proof. For a specific $v(T)$, the airplane has to fly with $\overline{v}$ as long as possible to reach the furthest position. Similarly, to reach the closest position, it has to fly with $\underline{v}$ as long as possible. As shown in Fig. 11, the furthest position, denoted by $\overline{X(v(T))}$, is the area under the red lines plus $X(0)$ and the closest position, $\underline{X(v(T))}$, is the area under the blue lines plus $X(0)$.

$$\underline{X(v(T))} = X(0) + \underline{v}T + \frac{(v(0)-\underline{v})^2}{2\underline{a}} + \frac{(v(T)-\underline{v})^2}{2\overline{a}}$$
$$\overline{X(v(T))} = X(0) + \overline{v}T - \frac{(\overline{v}-v(0))^2}{2\overline{a}} - \frac{(\overline{v}-v(T))^2}{2\underline{a}}$$

$\underline{X(v(T))}$ and $\overline{X(v(T))}$ can be simplified by (2).

$$\underline{X(v(T))} = \underline{X(T)} + \frac{(v(T)-\underline{v})^2}{2\overline{a}}$$
$$\overline{X(v(T))} = \overline{X(T)} - \frac{(\overline{v}-v(T))^2}{2\underline{a}}$$

Note that $\underline{X(v(T))}$ is convex and monotonically increases and $\overline{X(v(T))}$ is concave and monotonically increases. Hence, we have,

$$\underline{X(v(T))}_{max} = \underline{X(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}$$
$$\underline{X(v(T))}_{min} = \underline{X(T)}$$
$$\overline{X(v(T))}_{max} = \overline{X(T)}$$
$$\overline{X(v(T))}_{min} = \overline{X(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$$

Furthermore,

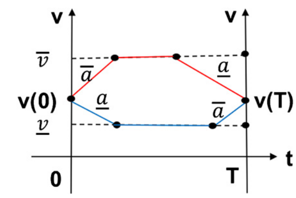

Fig. 11. Furthest and closest position.

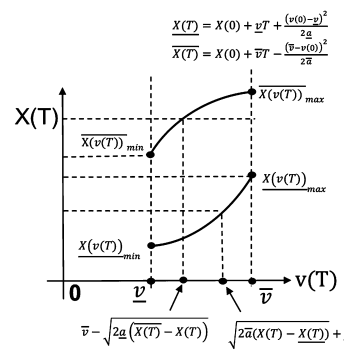

Fig. 12. v(T)-X(T) relationship.

$$\overline{X(v(T))}_{min} - \underline{X(v(T))}_{max} = \overline{X(T)} - \underline{X(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}$$
$$= \overline{v}T - \underline{v}T - \frac{(v(0)-\underline{v})^2}{2\underline{a}} - \frac{(\overline{v}-v(0))^2}{2\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}$$
$$> \overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}$$
$$= \overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{\underline{a}} > 0 \quad \text{(see Property 1)}$$

Hence,

$$\overline{X(v(T))}_{max} > \overline{X(v(T))}_{min} > \underline{X(v(T))}_{max} > \underline{X(v(T))}_{min}$$

In addition, since $\underline{X(v(T))}$ is convex and $\overline{X(v(T))}$ is concave, $\overline{X(v(T))} > \underline{X(v(T))}$ can be easily proved and the relationship between $v(T)$ and $X(T)$ can be plotted as Fig. 12.

Fig. 12 shows the implication that $X(T)$ has on $v(T)$. An extreme example is when $X(T) = \overline{X(T)}$, $v(T)$ can only be equal to $\overline{v}$ instead of the full range of $[\underline{v}, \overline{v}]$.

In summary, given $X(0)$, $v(0)$ and $X(T)$, all the possible $v(T)$ is shown as follows.

1) When $X(T) \in [\underline{X(v(T))}_{min}, \underline{X(v(T))}_{max}]$

$$v(T) \in [\underline{v}, \underline{v} + \sqrt{2\overline{a}(X(T) - \underline{X(T)})}]$$

2) When $X(T) \in [\underline{X(v(T))}_{max}, \overline{X(v(T))}_{min}]$

$$v(T) \in [\underline{v}, \overline{v}]$$

3) When $X(T) \in [\overline{X(v(T))}_{min}, \overline{X(v(T))}_{max}]$

$$v(T) \in [\overline{v} - \sqrt{2\underline{a}(\overline{X(T)} - X(T))}, \overline{v}]$$

where,

$$\underline{X(T)} = X(0) + \underline{v}T + \frac{(v(0)-\underline{v})^2}{2\underline{a}}$$
$$\overline{X(T)} = X(0) + \overline{v}T - \frac{(\overline{v}-v(0))^2}{2\overline{a}} \quad \square$$

Appendix B. Applicable conditions

The applicable conditions refer to the sets of $X1(0)$, $X2(0)$, $v1(0)$ and $v2(0)$ that make the strategies applicable. In this section, we provide the detailed proof of those conditions.

Strategy (a):

$$X1(T) \in [max(X2(0) + \bar{v}T, X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}),$$
$$X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}}]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X2(0) + \bar{v}T - \frac{(\bar{v}-v2(0))^2}{2\bar{a}}]$$

For this strategy to be applicable, the following has to hold:

$$X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}}$$
$$> max(X2(0) + \bar{v}T, X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}})$$

which is equivalent to:

$$\begin{cases} X2(0) + \bar{v}T > X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}} \\ X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}} > X2(0) + \bar{v}T \end{cases}$$

or,

$$\begin{cases} X2(0) + \bar{v}T < X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}} \\ X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}} > X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}} \\ \quad (\text{true by Property 1}) \end{cases}$$

After further simplified, it can be written as:

$$\frac{(\bar{v}-v1(0))^2}{2\bar{a}} < X1(0) - X2(0) < \bar{v}T - \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

or

$$X1(0) - X2(0) > \bar{v}T - \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

Therefore, as long as $X1(0) - X2(0) > \frac{(\bar{v}-v1(0))^2}{2\bar{a}}$, Strategy (a) is applicable. The corresponding conflict-free conditions are summarized in Section 5.3.

Strategy (b):

$$X1(T) \in [X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}},$$
$$min(X2(0) + \bar{v}T, X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}})]$$
$$X2(T) \in [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}, X1(0) + \underline{v}T]$$

For this strategy to be applicable, the following has to hold:

$$min(X2(0) + \bar{v}T, X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}})$$
$$> X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

and,

$$X1(0) + \underline{v}T > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}$$

which is equivalent to:

$$\begin{cases} X2(0) + \bar{v}T < X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}} \\ X2(0) + \bar{v}T > X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}} \\ X1(0) + \underline{v}T > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}} \end{cases}$$

or,

$$\begin{cases} X2(0) + \bar{v}T > X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}} \\ X1(0) + \bar{v}T - \frac{(\bar{v}-v1(0))^2}{2\bar{a}} > X1(0) + \underline{v}T + \frac{(v1(0)-\underline{v})^2}{2\underline{a}} \\ \quad (\text{true by Property 1}) \\ X1(0) + \underline{v}T > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}} \end{cases}$$

After further simplified, it can be written as:

$$max(\frac{(\bar{v}-v1(0))^2}{2\bar{a}}, \frac{(v2(0)-\underline{v})^2}{2\underline{a}}) < X1(0) - X2(0)$$
$$< \bar{v}T - \underline{v}T - \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$$

or,

$$\frac{(v2(0)-\underline{v})^2}{2\underline{a}} < X1(0) - X2(0) < \frac{(\bar{v}-v1(0))^2}{2\bar{a}}$$

Therefore, as long as $\frac{(v2(0)-\underline{v})^2}{2\underline{a}} < X1(0) - X2(0) < \bar{v}T - \underline{v}T - \frac{(v1(0)-\underline{v})^2}{2\underline{a}}$ Strategy (b) is applicable. The corresponding conflict-free conditions are summarized in Section 5.3. □

Appendix C. $X1(T)$ and $X2(T)$ for the existence of conflict-free trajectories during $[T, 2T]$

Formula (8) can be expanded into the following set A of $v2(T)$, meaning as long as $v2(T)$ is within set A, (8) is true.

$$A = \left\{v2(T) | \underline{v} < v2(T) < \sqrt{2\underline{a}(X1(T) - X2(T))} + \underline{v}\right\}$$

According to Property 2, each $X2(T)$ implies a set B of all possible $v2(T)$ if it is to be successfully reached at $t = T$. Therefore, for any $X2(T)$, if B is subset of A, then (8) is always true. In this appendix, we will prove the relationship between $X1(T)$ and $X2(T)$ to make B a subset of A for any $X2(T)$.

Recall in Property 2:

1) When $X2(T) \in [\underline{X2(T)}, \underline{X2(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}}]$, we have

$$B = \left\{v2(T) | v2(T) \in [\underline{v}, \underline{v} + \sqrt{2\bar{a}(X2(T) - \underline{X2(T)})}]\right\}$$

To make sure $B \subset A$, we need,

$$\underline{v} + \sqrt{2\bar{a}(X2(T) - \underline{X2(T)})} < \sqrt{2\underline{a}(X1(T) - X2(T))} + \underline{v}$$

Expand it and then we get,

$$X2(T) < \frac{\bar{a}\underline{X2(T)} + \underline{a}X1(T)}{\bar{a} + \underline{a}}$$

To make sure $X2(T)$ at least exists, we need,

$$\underline{X2(T)} < \frac{\bar{a}\underline{X2(T)} + \underline{a}X1(T)}{\bar{a} + \underline{a}}$$

Expand it and then we get,

$$X1(T) > \underline{X2(T)}$$

Therefore,

$$X2(T) \in [\underline{X2(T)}, min\{\frac{\bar{a}\underline{X2(T)} + \underline{a}X1(T)}{\bar{a} + \underline{a}}, \underline{X2(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}}\}]$$

Specifically, to further expand it,

if $\underline{X2(T)} < X1(T) < \underline{X2(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}} + \frac{(\bar{v}-\underline{v})^2}{2\underline{a}}$

$$X2(T) \in [\underline{X2(T)}, \frac{\bar{a}\underline{X2(T)} + \underline{a}X1(T)}{\bar{a} + \underline{a}}]$$

if $X1(T) > \underline{X2(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}} + \frac{(\bar{v}-\underline{v})^2}{2\underline{a}}$

$$X2(T) \in [\underline{X2(T)}, \underline{X2(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}}]$$

2) When $X2(T) \in [\underline{X(T)} + \frac{(\bar{v}-\underline{v})^2}{2\bar{a}}, \overline{X2(T)} - \frac{(\bar{v}-\underline{v})^2}{2\underline{a}}]$, we have

$$B = \left\{v2(T) | v2(T) \in [\underline{v}, \bar{v}]\right\}$$

To make sure $B \subset A$, we need,

$\overline{v} < \sqrt{2\underline{a}(X1(T) - X2(T))} + \underline{v}$

Expand it and then we get,

$X2(T) < X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$

To make sure $X2(T)$ at least exists, we need,

$X1(T) > \underline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}} + \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$

Therefore,

$$X2(T) \in [\underline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}, \min\{\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}, X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}\}]$$

Specifically, to further expand it,

if $\underline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}} + \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} < X1(T) < \overline{X2(T)}$

$$X2(T) \in [\underline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}, X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}]$$

if $X1(T) > \overline{X2(T)}$

$$X2(T) \in [\underline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\overline{a}}, \overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}]$$

3) When $X2(T) \in [\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}, \overline{X2(T)}]$, we have

$$B = \{v2(T) | v2(T) \in [\overline{v} - \sqrt{2\underline{a}(\overline{X2(T)} - X2(T))}, \overline{v}]\}$$

To make sure $B \subset A$, we need,

$$\overline{v} < \sqrt{2\underline{a}(X1(T) - X2(T))} + \underline{v} \;\; \underline{v} < \overline{v} - \sqrt{2\underline{a}(\overline{X2(T)} - X2(T))}$$

Expand it and then we get,

$\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}} < X2(T) < X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$

To make sure $X2(T)$ at least exists, we need,

$X1(T) > \overline{X2(T)}$

Therefore,

$$X2(T) \in [\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}, \min\{\overline{X2(T)}, X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}\}]$$

Specifically, to further expand it,

if $\overline{X2(T)} < X1(T) < \overline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$

$$X2(T) \in [\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}, X1(T) - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}]$$

if $X1(T) > \overline{X2(T)} + \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$

$$X2(T) \in [\overline{X2(T)} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}, \overline{X2(T)}]$$

Appendix D

In this appendix, we prove $X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}} > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}$ and $X2(0) + \overline{v}T > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}$.

First:

$$[X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}}] - [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}]$$

$$= X1(0) - X2(0) + \overline{v}T - \underline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}} - \frac{(v2(0)-\underline{v})^2}{2\underline{a}}$$

$$> X1(0) - X2(0) + \overline{v}T + \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{2\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{2\underline{a}}$$

$$> X1(0) - X2(0) + \overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{\underline{a}}$$

According to Property 1, $\overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{\underline{a}} > 0$, and obviously $X1(0) - X2(0) > 0$, hence:

$$X1(0) - X2(0) + \overline{v}T - \underline{v}T - \frac{(\overline{v}-\underline{v})^2}{\overline{a}} - \frac{(\overline{v}-\underline{v})^2}{\underline{a}} > 0$$

$$\Longrightarrow X1(0) + \overline{v}T - \frac{(\overline{v}-v1(0))^2}{2\overline{a}} > X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}} \quad \square$$

Second:

$$[X2(0) + \overline{v}T] - [X2(0) + \underline{v}T + \frac{(v2(0)-\underline{v})^2}{2\underline{a}}]$$

$$= \overline{v}T - \underline{v}T - \frac{(v2(0)-\underline{v})^2}{2\underline{a}} > 0 \quad \text{(similar with the proof above)} \quad \square$$

References

[1] D.L. Bakowski, B.L. Hooey, D.C. Foyle, Flight deck surface trajectory-based operations (STBO): a four-dimensional trajectory (4DT) simulation, in: 2017 IEEE/AIAA 36th Digital Avionics Systems Conference (DASC), 2017.

[2] Lesley A. Weitz, Roland Sgorcea, Alexander Boyd, Designing instrument approach procedures compatible with the use of ATC automation for trajectory-based operations, in: AIAA Scitech 2019 Forum, AIAA SciTech Forum, (AIAA 2019-0434), https://doi.org/10.2514/6.2019-0434.

[3] A. Gardi, R. Sabatini, T. Kistan, Y. Lim, S. Ramasamy, 4 Dimensional trajectory functionalities for air traffic management systems, in: Integrated Communication, Navigation and Surveillance Conference (ICNS), 2015.

[4] P. Ostwald, T. Topiwala, J. Dearmon, The miles-in-trail impact assessment capability, in: 6th AIAA Aviation Technology, Integration and Operations Conference (ATIO), 2006.

[5] NextGen Advisory Committee, Priorities for Improving Operational Performance in the Northeast Corridor through CY2021, 2018.

[6] Federal Aviation Administration, Concept of operations for Time-Based Flow Management (TBFM), available at https://faaco.faa.gov/index.cfm/attachment/download/94539. (Accessed March 2019).

[7] M.R.C. Jackson, B.E. O'Laughlin, Airborne required time of arrival control and integration with ATM, in: Seventh AIAA Aviation Technology, Integration and Operations Conference (ATIO), 2007, p. 1820.

[8] M. Sun, X. Ge, C. Fleming, Resolving a dilemma: guaranteeing conflict-free 4D-trajectories while leaving autonomy to airlines, in: 2018 IEEE/AIAA 37th Digital Avionics Systems Conference (DASC), 2018.

[9] D. Hicok, B. Barmore, Concept of operations for interval management arrivals and approach, in: AIAA Guidance, Navigation, and Control Conference, AIAA SciTech Forum, 2016.

[10] X. Olive, J. Morio, Trajectory clustering of air traffic flows around airports 84 (2019) 776–781.

[11] Federal Aviation Administration, Regulation, ORDER JO 7210.3AA.

[12] Radio Technical Commission for Aeronautics, PBN time, speed, spacing, www.rtca.org/sites/default/files/2016_pbn_tss_report_fnl_oct5_nac_approved.pdf. (Accessed March 2019).

[13] US Department of Transportation, FAA Has Not Effectively Deployed Controller Automation Tools that Optimize Benefits of Performance-Based Navigation, Report No. AV-2015-081, 2015.

[14] I.M. Levitt, A.W. Lesley, M.W. Castle, Modeling delivery accuracy for metering operations to support RNAV arrivals, in: Proceedings of the Tenth USA/Europe Air Traffic Management Research and Development Seminar, 2013.

[15] Federal Aviation Administration (FAA), Time Based Flow Management (TBFM) Study Report, V24: 4-17-2014, 2014.

[16] Air Traffic Management (ATM) Technology Demonstration – 1 (ATD-1), www.aviationsystemsdivision.arc.nasa.gov/research/tactical/atd1.shtml. (Accessed March 2019).

[17] A. Globus, R. Jacoby, T. Prevot, J.K. Wong, Rapid generation and utilization of four-dimensional trajectories for air traffic control and management applications in MACS, in: Infotech@Aerospace, 2011, p. 117.

[18] W. Liu, I. Hwang, Probabilistic trajectory prediction and conflict detection for air traffic control, J. Guid. Control Dyn. 34 (2011) 1779–1789, https://doi.org/10.2514/1.53645.

[19] Y. Matsuno, Stochastic conflict-free 4D trajectory optimization in the presence of uncertainty, in: 29th Congress of the International Council of the Aeronautical Sciences, ICAS 2014, 2014, p. 121.

[20] E. Calvo-Fernndez, L. Perez-Sanz, J.M. Cordero-Garca, R.M. ArnaldoValds, Conflict-free trajectory planning based on a data-driven conflict-resolution model, J. Guid. Control Dyn. 40 (2017) 615–627, https://doi.org/10.2514/1.G000691.

[21] S. Vilardaga, X. Prats, Conflict-free trajectory optimisation for complex departure procedures, in: International Conference on Research in Air Transportation, 2014.

[22] M.R. Jardin, Real-time conflict-free trajectory optimization, in: 5th USA/Europe ATM 2003 R&D Seminar, 2003, p. 2327.

[23] J. Gu, X. Tang, W. Hong, P. Chen, T. Li, Real-time optimization of short-term flight profiles to control time of arrival, Aerosp. Sci. Technol. 84 (2019) 1164–1174, https://doi.org/10.1016/j.ast.2018.09.029.
[24] J. Zhang, J. Liu, R. Hu, H. Zhu, Online four dimensional trajectory prediction method based on aircraft intent updating, Aerosp. Sci. Technol. 77 (2018) 774–787.
[25] S. Hao, S. Cheng, Y. Zhang, A multi-aircraft conflict detection and resolution method for 4-dimensional trajectory-based operation, Chin. J. Aeronaut. 31 (2018) 1579–1593, https://doi.org/10.1016/j.cja.2018.04.017.
[26] D. Rey, C. Rapine, R. Fondacci, N.E. El Faouzi, Subliminal speed control in air traffic management: optimization and simulation, Transp. Sci. 50 (2016) 240–262, https://doi.org/10.1287/trsc.2015.0602.
[27] B.L. Hooey, V.H.L. Cheng, D.C. Foyle, A Concept of Operations for Far-Term Surface Trajectory-Based Operations (STBO), 2014.
[28] S. Johnson, Characterization of metering, merging and spacing requirements for future trajectory-based operations, https://doi.org/10.2514/6.2017-3932, 2017.
[29] C.R. Spitzert, The Avionics Handbook, CRC press, Boca Raton, FL, 2001.
[30] B. Lascara, L.A. Weitz, T. Monson, R. Mount, Measuring performance of initial ground-based interval management – spacing (GIM-S) operations, in: 12th USA/Europe Air Traffic Management Research and Development Seminar, 2017.
[31] I.M. Levitt, L.A. Weitz, B.E. Barmore, M.W. Castle, Feasibility Criteria for Interval Management Operations as Part of Arrival Management Operations, AIAA, 2014.
[32] C. Moehlenbrink, et al., Evaluation of integrated demand management looking into strategic & tactical flow management, in: Proceedings of the Tenth USA/Europe Air Traffic Management Research and Development Seminar, 2017.
[33] N.M. Smith, et al., Integrated demand management: coordinating strategic and tactical flow scheduling operations, in: 16th AIAA Aviation Technology, Integration, and Operations Conference, 2016, p. 4221.