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General Section

p-adic quotient sets II: Quadratic forms ☆

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ABSTRACT

For $A \subseteq \{1, 2, \ldots\}$, we consider $R(A) = \{a/a' : a, a' \in A\}$. If A is the set of nonzero values assumed by a quadratic form, when is $R(A)$ dense in the p-adic numbers? We show that for a binary quadratic form Q, $R(A)$ is dense in $\mathbb{Q}_p$ if and only if the discriminant of Q is a nonzero square in $\mathbb{Q}_p$, and for a quadratic form in at least three variables, $R(A)$ is always dense in $\mathbb{Q}_p$. This answers a question posed by several authors in 2017.

1. Introduction

For a subset $A \subseteq \mathbb{N} = \{1, 2, 3, \ldots\}$, let $R(A) = \{a/a' : a, a' \in A\}$ denote the corresponding *ratio set* (or *quotient set*). The question of when $R(A)$ is dense in the positive

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real numbers has been examined by many authors over the years [6,10,12,13,21,20,5,16, 17,27,2–4,22,23,28,29]. Analogues in the Gaussian integers [8] and, more generally, in algebraic number fields [26], have recently been considered.

The study of quotient sets in the p-adic setting was initiated by Florian Luca and the second author [9]. Shortly thereafter several other papers on the topic appeared [7,19,24, 18]. In [7] it was shown that if $A = \{x^2 + y^2 : x, y \in \mathbb{Z}\}\backslash\{0\}$, then $R(A)$ is dense in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 4$. It is natural to wonder about possible extensions to other quadratic forms.

Fix a prime number p and observe that each nonzero rational number has a unique representation of the form $r = \pm p^k a/b$, in which $k \in \mathbb{Z}$, $a, b \in \mathbb{N}$, and $\gcd(a,p) = \gcd(b,p) = \gcd(a,b) = 1$. The *p-adic valuation* of such an r is $\nu_p(r) = k$ and its *p-adic absolute value* is $\|r\|_p = p^{-k}$. By convention, $\nu_p(0) = \infty$ and $\|0\|_p = 0$. The *p-adic metric* on $\mathbb{Q}$ is $d(x,y) = \|x - y\|_p$. We write $\|\cdot\|$ in place of $\|\cdot\|_p$ when no confusion can arise. The field $\mathbb{Q}_p$ of *p-adic numbers* is the completion of $\mathbb{Q}$ with respect to the p-adic metric [11,14]. We let $\mathbb{Q}_p^\times = \mathbb{Q}_p\backslash\{0\}$.

A *quadratic form* is a homogeneous polynomial

$$Q(x_1, x_2, \ldots, x_r) = \sum_{i=1}^{r} \sum_{j=i}^{r} a_{ij} x_i x_j, \tag{1.1}$$

of degree 2. We say that Q is *integral* if $a_{ij} \in \mathbb{Z}$ for all i, j, and we say that Q is *primitive* if there is no positive integer $k > 1$ so that $k | a_{ij}$ for all i and j. We can write $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A \vec{x}$ for an $r \times r$ symmetric matrix A (which will have even diagonal entries, and integral off-diagonal entries). Two forms Q and Q' are *equivalent* if there is an $r \times r$ matrix M with integer entries and $\det(M) = \pm 1$ so that $Q'(\vec{x}) = Q(M\vec{x})$.

In the case of binary forms, we will distinguish *proper* equivalence (the case that $\det(M) = 1$) from *improper* equivalence (the case that $\det(M) = -1$). Given a binary form

$$Q(x, y) = ax^2 + bxy + cy^2, \tag{1.2}$$

the *discriminant* of Q is $b^2 - 4ac$. Equivalent binary forms assume the same values and have the same discriminants.

Let $\mathbb{F}$ be a field. We say that Q is *nonsingular* over $\mathbb{F}$ if $\det(A) \neq 0$ (and *singular* otherwise). We say that Q is *isotropic* over $\mathbb{F}$ if there is a nonzero vector $\vec{x} \in \mathbb{F}^r$ so that $Q(\vec{x}) = 0$. Otherwise, Q is *anisotropic* over $\mathbb{F}$. If Q represents every value in $\mathbb{F}$, then Q is *universal* over $\mathbb{F}$. It is known that if Q is isotropic and nonsingular over $\mathbb{F}$, then Q is universal over $\mathbb{F}$ [15, Thm. I.3.4].

For brevity, the term "quadratic form" hereafter refers to a quadratic form that is nonsingular over $\mathbb{Q}$, integral, and primitive. The quotient set generated by a quadratic form Q is

$$R(Q) = \{Q(\vec{x})/Q(\vec{y}) : \vec{x}, \vec{y} \in \mathbb{Z}^r, Q(\vec{y}) \neq 0\}.$$

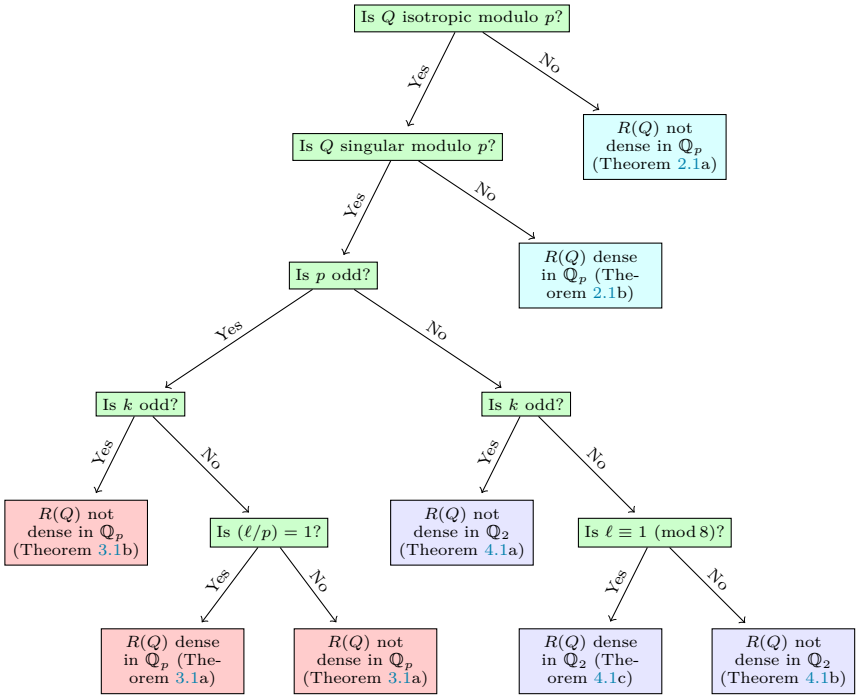

Fig. 1. How to decide if $R(Q)$ is dense in $\mathbb{Q}_p$. Here Q is an integral, binary, and primitive quadratic form of discriminant $p^k\ell$, in which $\gcd(p, \ell) = 1$. Here (ℓ/p) denotes a Legendre symbol.

If Q and Q' are equivalent, then $R(Q) = R(Q')$. It has been asked when $R(Q)$ is dense in $\mathbb{Q}_p$ [7, Problem 4.4]. The main result of this paper is a complete answer to this question.

Theorem 1.3. *Let Q be an integral quadratic form in r variables. Assume that Q is primitive and is nonsingular over $\mathbb{Q}$ and let p be a prime number.*

(a) *If Q is binary, then $R(Q)$ is dense in $\mathbb{Q}_p$ if and only if the discriminant of Q is a square in $\mathbb{Q}_p$.*
(b) *If $r \geq 3$, then $R(Q)$ is dense in $\mathbb{Q}_p$.*

We give two proofs of Theorem 1.3a. Our first approach is longer (Fig. 1), but completely elementary. The second approach is shorter, but requires the classification of values represented by quadratic forms over $\mathbb{Q}_p$ (as can be found in Serre's book [25]). This same tool is used to prove Theorem 1.3b.

The organization of this paper is as follows. The elementary proof of Theorem 1.3a constitutes sections 2, 3, and 4. In Section 2 we handle binary quadratic forms that are nonsingular over $\mathbb{F}_p$; the results therein apply to all primes. Section 3 concerns binary quadratic forms that are singular modulo an odd prime and Section 4 treats forms that are singular modulo 2. In Section 5, we give a more sophisticated proof of Theorem 1.3a as well as the proof of Theorem 1.3b.

2. Nonsingular (all primes)

Our aim in this section is to prove the following theorem, which addresses the two uppermost terminal nodes (blue) in Fig. 1.

Theorem 2.1. *Let $Q(x,y) = ax^2 + bxy + cy^2$ be primitive and integral.*

(a) *If Q is anisotropic modulo p, then $R(Q)$ is not dense in $\mathbb{Q}_p$.*
(b) *If Q is isotropic and nonsingular modulo p, then $R(Q)$ is dense in $\mathbb{Q}_p$.*

2.1. Proof of Theorem 2.1a

Suppose that Q is anisotropic over $\mathbb{Z}/p\mathbb{Z}$. We claim that $\nu_p(Q(x,y))$ is even for all $x, y \in \mathbb{Z}$. If $Q(x,y) \not\equiv 0 \pmod p$, then $\nu_p(Q(x,y)) = 0$, which is even. Suppose that $Q(x,y) \equiv 0 \pmod p$. Then $(x,y) \equiv (0,0) \pmod p$ since Q is anisotropic; that is, $x = mp^j$ and $y = np^k$, in which $j, k \geq 1$, $p \nmid m$, and $p \nmid n$. Without loss of generality, assume that $j \geq k$. Then

$$\begin{aligned}\nu_p(Q(x,y)) &= \nu_p(am^2p^{2j} + bmnp^{j+k} + cn^2p^{2k}) \\ &= \nu_p\big(p^{2k}(am^2p^{2(j-k)} + bmnp^{j-k} + cn^2)\big) \\ &= 2k + \nu_p(Q(mp^{j-k}, n)) = 2k\end{aligned}$$

since $p \nmid n$ and Q is anisotropic. Thus, $\nu_p(Q(x,y)) - \nu_p(Q(z,w)) \neq 1 = \nu_p(p)$ for all $x, y, z, w \in \mathbb{Z}$ and hence $R(Q)$ is bounded away from p in $\mathbb{Q}_p$. Consequently, $R(Q)$ is not dense in $\mathbb{Q}_p$. □

2.2. Proof of Theorem 2.1b for p odd

Before proceeding, we need two lemmas.

Lemma 2.2 *(Lemma 2.3 of [7]). Let $A \subset \mathbb{N}$ and let p be a prime.*

(a) *If A is p-adically dense in $\mathbb{N}$, then $R(A)$ is dense in $\mathbb{Q}_p$.*
(b) *$R(A)$ is p-adically dense in $\mathbb{N}$ if and only if $R(A)$ is dense in $\mathbb{Q}_p$.*

Proof. (a) If A is p-adically dense in $\mathbb{N}$, it is p-adically dense in $\mathbb{Z}$. Inversion is continuous on $\mathbb{Q}_p^\times$, so $R(A)$ is p-adically dense in $\mathbb{Q}$, which is dense in $\mathbb{Q}_p$.
(b) Suppose that $R(A)$ is p-adically dense in $\mathbb{N}$. Since inversion is continuous on $\mathbb{Q}_p^\times$, the result follows from the fact that $\mathbb{N}$ is p-adically dense in $\{x \in \mathbb{Q} : \nu_p(x) \geq 0\}$. □

Lemma 2.3. *Let Q be nonsingular modulo an odd prime p. If $(x,y) \not\equiv (0,0) \pmod p$ and $Q(x,y) \equiv 0 \pmod p$, then $2ax + by \not\equiv 0 \pmod p$ or $bx + 2cy \not\equiv 0 \pmod p$.*

Proof. We prove the contrapositive. Suppose that

$$2ax + by \equiv bx + 2cy \equiv 0 \pmod{p}. \tag{2.4}$$

Since Q is nonsingular, $b^2 \not\equiv 4ac \pmod{p}$. If $p|b$, then $p\nmid a$ and $p\nmid c$. Thus, there are two cases: $p\nmid a$ and $p\nmid c$, or $p\nmid b$.

Case 1: If $p\nmid a$ and $p\nmid c$, then (2.4) implies that

$$x \equiv -\frac{by}{2a} \pmod{p}. \tag{2.5}$$

Thus,

$$0 \equiv Q\left(-\frac{by}{2a}, y\right) \equiv \left(\frac{-b^2 + 4ac}{4a}\right) y^2 \pmod{p}$$

and hence $y \equiv 0 \pmod{p}$. Then (2.5) implies that $(x, y) \equiv (0, 0) \pmod{p}$.

Case 2: If $p\nmid b$, then

$$x \equiv -\frac{2cy}{b} \pmod{p} \tag{2.6}$$

and hence

$$0 \equiv Q\left(-\frac{2cy}{b}, y\right) \equiv -cy^2 \left(\frac{b^2 - 4ac}{b^2}\right) \pmod{p}.$$

Consequently, $p|y$ or $p|c$.

- If $p|y$, then (2.6) implies that $(x, y) \equiv (0, 0) \pmod{p}$.
- If $p|c$, then (2.6) implies that $p|x$. Since $p\nmid b$, (2.4) ensures that $p|y$. Thus, $(x, y) \equiv (0, 0) \pmod{p}$. □

Suppose that Q is isotropic and nonsingular modulo an odd prime p. By Lemma 2.2, it suffices to show that for each $n \in \mathbb{Z}$ and $r \geq 1$, there exists an $(x, y) \in \mathbb{Z}^2$ such that $Q(x, y) \equiv n \pmod{p^r}$. To this, we add the requirement

$$p\nmid (2ax + by) \qquad \text{or} \qquad p\nmid (bx + 2cy)\,. \tag{2.7}$$

We induct on r. The base case is $r = 1$.

- If $n \equiv 0 \pmod{p}$, then since Q is isotropic we may find $(x, y) \not\equiv (0, 0) \pmod{p}$ so that $Q(x, y) \equiv 0 \pmod{p}$. Lemma 2.3 ensures that at least one of the two conditions in (2.7) hold.

- If $n \not\equiv 0 \pmod{p}$, then there is an (x, y) so that $Q(x, y) \equiv n \pmod{p}$ since Q is isotropic and nonsingular [15, Prop. 3.4]. Since p is odd,

$$0 \not\equiv n \equiv Q(x, y) \equiv \frac{x}{2}(2ax + by) + \frac{y}{2}(bx + 2cy) \pmod{p},$$

  which implies that (2.7) holds.

Now suppose that $Q(x, y) \equiv n \pmod{p^r}$ and, without loss of generality, that $p \nmid (2ax + by)$. Then $Q(x, y) = n + mp^r$ for some $m \in \mathbb{Z}$. If

$$i \equiv -(2ax + by)^{-1} m \pmod{p},$$

then the identity

$$Q(x + z, y) = Q(x, y) + az^2 + bzy + 2axz \tag{2.8}$$

yields

$$\begin{aligned} Q(x + ip^r, y) &= Q(x, y) + ai^2p^{2r} + bip^r y + 2axip^r \\ &= n + mp^r + ai^2p^{2r} + bip^r y + 2axip^r \\ &\equiv n + mp^r + bip^r y + 2axip^r \pmod{p^{r+1}} \\ &\equiv n + p^r(m + (2ax + by)i) \pmod{p^{r+1}} \\ &\equiv n \pmod{p^{r+1}}, \end{aligned}$$

in which $2a(x + ip^r) + by = (2ax + by) + 2aip^r$ is not divisible by p. This completes the induction. □

2.3. *Proof of Theorem 2.1b for $p = 2$*

Suppose that Q is isotropic and nonsingular modulo 2. Since $2 \nmid (b^2 - 4ac)$, it follows that b is odd and hence

$$Q(x, y) \equiv ax^2 + xy + cy^2 \pmod{2}.$$

Because Q is isotropic, a or c is even; see Table 1. Without loss of generality, suppose that a is even. By Lemma 2.2, it suffices to show that for each $n \in \mathbb{Z}$ and $r \geq 1$, there is an $(x, y) \in \mathbb{Z}^2$ such that

$$Q(x, y) \equiv n \pmod{2^r} \qquad \text{and} \qquad y \not\equiv 0 \pmod{2}. \tag{2.9}$$

We proceed by induction on r. For the base case $r = 1$, we may let $(x, y) = (n - c, 1)$.

Table 1
Values of $Q(x,y) \equiv ax^2 + xy + cy^2 \pmod 2$.

x	y	$Q(x,y) \pmod 2$	a, c odd	a, c even	a even, c odd	a odd, c even
0	0	0	0	0	0	0
0	1	c	1	0	1	0
1	0	a	1	0	0	1
1	1	$1 + a + c$	1	1	0	0

Now suppose that (2.9) holds for some r. Then $Q(x,y) = n + m2^r$ for some $m \in \mathbb{Z}$. If $i \equiv mb^{-1}y^{-1} \pmod 2$, then (2.8) yields

$$\begin{aligned} Q(x + 2^r i, y) &= Q(x,y) + a(2^r i)^2 + b(2^r i)y + 2ax(2^r i) \\ &= (n + m2^r) + 2^{2r}ai^2 + 2^r biy + 2^{r+1}aix \\ &\equiv n + m2^r + 2^r biy \pmod{2^{r+1}} \\ &\equiv n + 2^r(m + biy) \pmod{2^{r+1}} \\ &\equiv n \pmod{2^{r+1}}. \end{aligned}$$

This completes the induction. □

3. Singular modulo an odd prime

Our aim in this section is to prove the following theorem, which addresses the three lower-left terminal nodes (red) in Fig. 1. Below (ℓ/p) is a Legendre symbol.

Theorem 3.1. *Let $Q(x,y) = ax^2 + bxy + cy^2$ be primitive and integral with discriminant $p^k\ell$, in which $k \geq 1$ and p is an odd prime that does not divide ℓ.*

(a) *If k is even, then $R(Q)$ is dense in $\mathbb{Q}_p$ if and only if $(\ell/p) = 1$.*
(b) *If k is odd, then $R(Q)$ is not dense in $\mathbb{Q}_p$.*

3.1. Proof of Theorem 3.1a

We have $b^2 - 4ac = p^k\ell$ with $k \geq 2$ even. Because Q is primitive, p cannot divide both a and c since otherwise it would divide a, b, and c. Without loss of generality, suppose that $p \nmid a$. Let $u \equiv 2^{-1}a^{-1}b \pmod{p^k}$, so that $2ua - b \equiv 0 \pmod{p^k}$. The forms $Q(x,y)$ and

$$Q'(x,y) = Q(-x - uy, y) = ax^2 + (2ua - b)xy + (u^2a - ub + c)y^2$$

are (improperly) equivalent. Thus, Q and Q' have the same discriminant and assume the same values, hence $R(Q) = R(Q')$. Since $p \nmid 4a$ and

$$4a(u^2a - ub + c) \equiv (2au - b)^2 - (b^2 - 4ac) \equiv 0 \pmod{p^k},$$

it follows that

$$B = \frac{2ua - b}{p^{k/2}} \qquad \text{and} \qquad C = \frac{u^2 a - ub + c}{p^k}$$

are integers. We may write

$$Q'(x, y) = ax^2 + p^{k/2} Bxy + p^k Cy^2, \tag{3.2}$$

which has discriminant

$$p^k(B^2 - 4aC) = p^k \ell.$$

Consequently, the integral quadratic form

$$Q''(x, y) = ax^2 + Bxy + Cy^2 \tag{3.3}$$

has discriminant $B^2 - 4aC = \ell$. Moreover,

$$4aQ''(x, y) = (2ax + By)^2 - \ell y^2. \tag{3.4}$$

Case 1: Suppose that $(\ell/p) = -1$. If $Q''(x_0, y_0) \equiv 0 \pmod p$, then (3.4) implies

$$(2ax_0 + By_0)^2 \equiv \ell y_0^2 \pmod p$$

since $p \nmid 4a$. The Legendre symbol of the left-hand side is 0 or 1; the Legendre symbol of the right-hand side is 0 or -1. Thus, both sides are congruent to 0 modulo p and hence $y_0 \equiv 0 \pmod p$. Since $p \nmid 2a$, it follows that $x_0 \equiv 0 \pmod p$ and hence Q'' is anisotropic modulo p. Theorem 2.1 ensures that $R(Q'')$ is not dense in $\mathbb{Q}_p$. Since $Q'(x, y) = Q''(x, p^{k/2} y)$, we conclude that $R(Q')$, which equals $R(Q)$, is not dense in $\mathbb{Q}_p$.

Case 2: Suppose that $(\ell/p) = 1$. Let $\sqrt{\ell}$ denote a square root of ℓ modulo p and let $(x_0, y_0) \equiv (\sqrt{\ell} - B, 2a) \pmod p$, which is not congruent to $(0, 0)$ modulo p since $p \nmid 2a$. Then (3.3) yields

$$4aQ''(x_0, y_0) \equiv \big(2a(\sqrt{\ell} - B) + B(2a)\big)^2 - \ell(2a)^2 \equiv a\ell - a\ell \equiv 0 \pmod p.$$

Since $p \nmid 4a$, it follows that Q'' is isotropic modulo p. Since the discriminant ℓ of Q'' is not divisible by p, Theorem 2.1b implies that $R(Q'')$ is dense in $\mathbb{Q}_p$. If $Q''(z, w) \neq 0$, then (3.2) provides

$$\frac{Q''(x, y)}{Q''(z, w)} = \frac{p^k Q''(x, y)}{p^k Q''(z, w)} = \frac{Q''(p^{k/2} x, p^{k/2} y)}{Q''(p^{k/2} z, p^{k/2} w)} = \frac{Q'(p^{k/2} x, y)}{Q'(p^{k/2} z, w)},$$

and hence $R(Q')$ is dense in $\mathbb{Q}_p$. Since Q and Q' are equivalent, $R(Q) = R(Q')$ is also dense in $\mathbb{Q}_p$. □

3.2. Proof of Theorem 3.1b

As in the proof of Theorem 3.1a, we may assume that $p\nmid a$. Since $R(Q) = R(4aQ)$ and

$$4aQ(x,y) = (2ax+by)^2 - (b^2-4ac)y^2,$$

we may assume without loss of generality that

$$Q(x,y) = x^2 - p^k \ell y^2.$$

Suppose toward a contradiction that $R(Q)$ is dense in $\mathbb{Q}_p$. Let n be a quadratic nonresidue modulo p. Then there are $x,y,z,w\in\mathbb{Z}$, not all multiples of p, so that $Q(z,w)\neq 0$ and

$$\left\| \frac{Q(x,y)}{Q(z,w)} - n \right\| < \frac{1}{p^k}. \tag{3.5}$$

In particular, $\|Q(x,y)\| = \|Q(z,w)\|$. Multiplying (3.5) by $Q(z,w)$ gives

$$\left\|(x^2-nz^2) - p^k\ell(y^2-nw^2)\right\| = \|Q(x,y)-nQ(z,w)\| < \frac{\|Q(z,w)\|}{p^k} \le \frac{1}{p^k}.$$

If $p\nmid x$ or $p\nmid z$, then $x^2-nz^2 \not\equiv 0 \pmod p$ and hence $\|Q(x,y)-nQ(z,w)\| = 1$, which is a contradiction.

Since $p|x$ and $p|z$, we get $p\nmid y$ or $p\nmid w$. Thus, $y^2-nw^2\not\equiv 0 \pmod p$. Now observe that x^2-nz^2 has even p-adic valuation (the form u^2-nv^2 is anisotropic and nonsingular modulo p and the proof of Theorem 2.1a ensures that it has even p-adic valuation for all u,v). Consequently, $Q(x,y)-nQ(z,w)$ is the sum of a p-adic integer with even valuation, and one with odd valuation k. Thus, $\|Q(x,y)-nQ(z,w)\| \ge p^{-k}$, which is a contradiction. Since n cannot be arbitrarily well approximated by elements of $R(Q)$, it follows that $R(Q)$ is not dense in $\mathbb{Q}_p$. □

4. Singular modulo 2

Our aim in this section is to prove the following theorem, which addresses the three lower-right terminal nodes (purple) in Fig. 1.

Theorem 4.1. *Let $Q(x,y) = ax^2+bxy+cy^2$ be primitive and integral with discriminant $2^k\ell$, in which ℓ is odd.*

(a) *If k is odd, then $R(Q)$ is not dense in $\mathbb{Q}_2$.*
(b) *If k is even and $\ell\not\equiv 1 \pmod 8$, then $R(Q)$ is not dense in $\mathbb{Q}_2$.*
(c) *If k is even and $\ell\equiv 1 \pmod 8$, then $R(Q)$ is dense in $\mathbb{Q}_2$.*

4.1. Proof of Theorem 4.1a

The proof is similar in flavor to that of Theorem 3.1b, although there are a couple modifications. Since $R(Q) = R(4aQ)$ and $4aQ(x,y) = (2ax+by)^2 - (b^2-4ac)y^2$, we may assume without loss of generality that $Q(x,y) = x^2 - 2^k\ell y^2$. Suppose that $R(Q)$ is dense in $\mathbb{Q}_2$. Then there are $x,y,z,w \in \mathbb{Z}$, not all even, so that $Q(z,w) \neq 0$ and

$$\left\| \frac{Q(x,y)}{Q(z,w)} - 5 \right\| < \frac{1}{2^{k+2}}.$$

We also see that $\|Q(x,y)\| = \|Q(z,w)\|$ and from this we get

$$\|(x^2-5z^2) - 2^k\ell(y^2-5w^2)\| = \|Q(x,y) - 5Q(z,w)\| < \frac{1}{2^{k+2}}. \tag{4.2}$$

If x or z is odd, then $x^2-5z^2 \equiv 1, 3, \text{or } 4 \pmod 8$. It follows that the power of 2 dividing x^2-5z^2 is even. If x and z are odd, then $\|Q(x,y)-5Q(z,w)\| \geq 1/4$, which contradicts (4.2). Thus, x and z are both even. However, in this case, the power of 2 dividing x^2-5z^2 is even, and the power of 2 dividing $2^k\ell(y^2-5w^2)$ is odd and at most 2^{k+2}. It follows that

$$\|Q(x,y) - 5Q(z,w)\| \geq \frac{1}{2^{k+2}},$$

which is a contradiction. Thus, $R(Q)$ is not dense is $\mathbb{Q}_2$. □

4.2. Proof of Theorem 4.1b

In this section, we show that if $b^2-4ac = 2^k\ell$ with k even and $\ell \equiv 3, 5 \text{ or } 7 \pmod 8$, then $R(Q)$ is not dense in $\mathbb{Q}_2$. As before, if $Q = ax^2+bxy+cy^2$, then $R(Q) = R(4aQ) = (2ax+by)^2 - (b^2-4ac)y^2$ and so if $Q'(x,y) = x^2 - 2^k\ell y^2$, then $R(Q) \subseteq R(Q')$. Letting $Q''(x,y) = x^2 - \ell y^2$, we have

$$\frac{Q'(x,y)}{Q'(z,w)} = \frac{Q''(x, 2^{k/2}y)}{Q''(z, 2^{k/2}w)}$$

for $x,y,z,w \in \mathbb{Z}$ and hence $R(Q') \subseteq R(Q'')$. Consequently, it suffices to show that $R(Q'')$ is not dense in $\mathbb{Q}_2$. We require a couple computational lemmas.

Lemma 4.3. *If $\ell \equiv 5 \pmod 8$, then $R(Q'')$ is not dense in $\mathbb{Q}_2$.*

Proof. Write $x = 2^j\tilde{x}$ and $y = 2^k\tilde{y}$, in which $j,k \geq 0$ and $\tilde{x},\tilde{y}$ are odd.

- If $j < k$, then

$$\begin{aligned}\nu_2(Q''(x,y)) &= \nu_2\big((2^j\tilde{x})^2 - \ell(2^k\tilde{y})^2\big)\\ &= \nu_2\big(2^{2j}\tilde{x}^2 - 2^{2k}\ell\tilde{y}^2\big)\\ &= 2j + \nu_2(\tilde{x}^2 - 2^{2(k-j)}\ell\tilde{y}^2)\\ &= 2j.\end{aligned}$$

- If $j > k$, then

$$\begin{aligned}\nu_2(Q''(x,y)) &= \nu_2\big((2^j\tilde{x})^2 - \ell(2^k\tilde{y})^2\big)\\ &= \nu_2\big(2^{2j}\tilde{x}^2 - 2^{2k}\ell\tilde{y}^2\big)\\ &= 2k + \nu_2(2^{2(j-k)}\tilde{x}^2 - \ell\tilde{y}^2)\\ &= 2k.\end{aligned}$$

- If $j = k$, then

$$\begin{aligned}\nu_2(Q''(x,y)) &= \nu_2\big((2^j\tilde{x})^2 - \ell(2^k\tilde{y})^2\big)\\ &= 2j + \nu_2\big(\tilde{x}^2 - \ell\tilde{y}^2\big).\end{aligned}$$

  If $\ell \equiv 5 \pmod 8$, then

$$\tilde{x}^2 - \ell\tilde{y}^2 \equiv 4 \pmod 8$$

  since $\tilde{x}^2 \equiv \tilde{y}^2 \equiv 1 \pmod 8$. Thus, $\nu_2(Q''(x,y))$ is even.

It follows that $\nu_2(Q''(x,y)/Q''(z,w))$ is even, and so there are no solutions to

$$\left\| \frac{Q''(x,y)}{Q''(z,w)} - 2 \right\| < \frac{1}{2}.$$

Thus, $R(Q'')$ is not dense in $\mathbb{Q}_2$. □

Lemma 4.4. *If $\ell \equiv 3$ or $7 \pmod 8$, then $R(Q'')$ is not dense in $\mathbb{Q}_2$.*

Proof. Suppose that $R(Q'')$ is dense in $\mathbb{Q}_2$. Then there are $x, y, z, w \in \mathbb{Z}$ so that

$$\left\| \frac{Q''(x,y)}{Q''(z,w)} - 3 \right\| \leq \frac{1}{2^3}.$$

We may assume at least one of x, y, z, w is odd. Multiplying by $\|Q''(z,w)\|$ gives

$$\big\|(x^2 - \ell y^2) - 3(z^2 - \ell w^2)\big\| \leq \frac{1}{2^3}.$$

For $\ell = 3$, a computation confirms that there are no solutions to $x^2 - 3y^2 - 3z^2 + 9w^2 \equiv 0 \pmod 8$ with at least one of x, y, z, w is odd. For $\ell = 7$, there are no solutions to $x^2 - 7y^2 - 3z^2 + 21w^2 \equiv 0 \pmod 8$ with at least one of x, y, z, w is odd. This contradiction tells us that $R(Q'')$ is not dense in $\mathbb{Q}_2$. □

4.3. Proof of Theorem 4.1c

Suppose that $Q(x, y) = ax^2 + bxy + cy^2$ is primitive and $b^2 - 4ac = 2^k\ell$ where $k \geq 2$ is even and $\ell \equiv 1 \pmod 8$. Since $b^2 - 4ac \equiv 0 \pmod 4$, b must be even. By switching a and c if necessary, we may assume that a is odd. The form $Q(x, y)$ is equivalent to

$$Q'(x, y) = Q(x + qy, y) = ax^2 + (2aq + b)xy + (aq^2 + bq + c)y^2$$

and hence $R(Q) = R(Q')$. We claim that we can choose a q such that

$$2aq + b \equiv 0 \pmod{2^{k/2}} \quad \text{and} \quad aq^2 + bq + c \equiv 0 \pmod{2^k}. \tag{4.5}$$

Let

$$q \equiv -\frac{b}{2a} + 2^{k/2-1} \pmod{2^k}.$$

Then

$$2aq + b \equiv (2a)\left(-\frac{b}{2a} + 2^{k/2-1} \right) + b \equiv -b + a2^{k/2} + b \equiv 0 \pmod{2^{k/2}},$$

which is the first condition in (4.5). The second condition follows from

$$\begin{aligned} aq^2 + bq + c &\equiv a\left(-\frac{b}{2a} + 2^{k/2-1} \right)^2 + b\left(-\frac{b}{2a} + 2^{k/2-1} \right) + c \\ &\equiv 2^{k-2}a - \frac{b^2}{4a} + c \\ &\equiv 2^{k-2}a - \frac{2^k\ell + 4ac}{4a} + c \\ &\equiv \frac{1}{a}\left(2^{k-2}a^2 - 2^{k-2}\ell\right) \pmod{2^k} \\ &\equiv 0 \pmod{2^k} \end{aligned}$$

since a is odd and $\ell \equiv 1 \pmod 8$. Thus, we may define the integers

$$B = \frac{2aq + b}{2^{k/2}} \qquad \text{and} \qquad C = \frac{c + bq + aq^2}{2^k}$$

so that the form

$$Q''(x,y) = ax^2 + Bxy + Cy^2$$

has discriminant

$$B^2 - 4aC = \frac{(2aq+b)^2 - 4a(c+bq+aq^2)}{2^k} = \frac{b^2-4ac}{2^k} = \ell \equiv 1 \pmod 8. \tag{4.6}$$

Since $Q'(x,y) = Q''(x, 2^{k/2}y)$, we have $R(Q') \subseteq R(Q'')$. Since

$$\frac{Q'(x,y)}{Q'(z,w)} = \frac{Q'(2^{k/2}x, 2^{k/2}y)}{Q'(2^{k/2}z, 2^{k/2}w)} = \frac{Q''(2^{k/2}x, y)}{Q''(2^{k/2}z, w)},$$

we get $R(Q'') \subseteq R(Q')$. Thus, $R(Q') = R(Q'')$.

From (4.6), it follows that B is odd and hence $B^2 \equiv 1 \pmod 8$. Thus, either a or C is even and it follows that Q'' is isotropic modulo 2. Theorem 2.1b ensures that $R(Q'')$ is dense in $\mathbb{Q}_2$. □

5. An alternative approach

In this section, we present an alternative approach to the proof of Theorems 2.1, 3.1, and 4.1. We also prove that if Q is a non-degenerate quadratic form in $r \geq 3$ variables, then $R(Q)$ is dense in $\mathbb{Q}_p$ for all p. While the arguments given here are shorter, they rely heavily on the classification of quadratic forms over $\mathbb{Q}_p$ and the values they represent. One convenient source for this material is [25].

Over a field, any quadratic form Q is equivalent to a diagonal one (by [25, Thm. IV.1]), namely

$$Q' = a_1x_1^2 + a_2x_2^2 + \cdots + a_rx_r^2.$$

For the remainder of this section, we will use the classification of squares in $\mathbb{Q}_p$ (see [25, Thms. 2.3 & 2.4]). If $p > 2$, then an element $x = p^n u \in \mathbb{Q}_p$ with $u \in \mathbb{Z}_p$ and $\nu_p(u) = 0$ is a square if and only if n is even and $u \bmod p \in \mathbb{F}_p$ is a square. If $p = 2$, then an element $x = 2^n u \in \mathbb{Q}_2$ is a square if and only if n is even and $u \equiv 1 \pmod 8$. It follows from this that $\mathbb{Q}_p^\times$ has four square classes if $p > 2$ and eight square classes if $p = 2$.

The corollary on page 37 of [25] gives a classification of the values represented by a quadratic form over $\mathbb{Q}_p$. We wish to record some consequences of this corollary. In particular, a binary quadratic form over $\mathbb{Q}_p$ whose discriminant is not a square represents half of the square classes, while a binary quadratic form over $\mathbb{Q}_p$ whose discriminant is a square represents everything in $\mathbb{Q}_p$. A quadratic form in three variables either represents everything in $\mathbb{Q}_p$, or represents all but one square class. Finally, a quadratic form in four or more variables over $\mathbb{Q}_p$ is universal.

We begin by reproving Theorems 2.1, 3.1, and 4.1. We start with a result of Arnold (which he attributes to F. Aicardi) [1, Thm. 1].

Lemma 5.1 *(Arnold). Let $Q(x,y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. If Q represents A, B and C, then it represents ABC.*

One way of interpreting this statement is that the inverse of $Q(x,y) = ax^2+bxy+cy^2$ in the class group is $ax^2 - bxy + cy^2$, which is improperly equivalent to Q. Because $Q \circ Q^{-1} \circ Q = Q$ in the class group, if Q represents A, Q^{-1} represents B, and Q represents C, then $Q = Q \circ Q^{-1} \circ Q$ represents ABC.

Proof. If $Q(x_1,y_1) = A$, $Q(x_2,y_2) = B$ and $Q(x_3,y_3) = C$, then $Q(x,y) = Q(x_1,y_1)Q(x_2,y_2)Q(x_3,y_3)$, in which

$$\begin{aligned} x &= (ax_1x_2 - cy_1y_2)x_3 + (c(y_1x_2 + x_1y_2) + bx_1x_2)y_3 \\ y &= (a(x_1y_2 + x_2y_1) + by_1y_2)x_3 + (-ax_1x_2 + cy_1y_2)y_3. \quad \square \end{aligned}$$

The following result provides an alternate representation of $R(Q)$ based upon Arnold's lemma.

Lemma 5.2. *Let Q be a binary quadratic form and let a be a nonzero integer represented by Q. Then*

$$R(Q) = \left\{ \frac{Q(x,y)}{a} : x, y \in \mathbb{Q} \right\}.$$

Proof. Suppose that $b = Q(x,y)/a$, in which $x, y \in \mathbb{Q}$ and $a = Q(z,w)$ for some $z, w \in \mathbb{Z}$. Write $x = c/f$ and $y = d/f$, in which $c, d, f \in \mathbb{Z}$ and $f \neq 0$. Then

$$b = \frac{Q(c/f, d/f)}{a} = \frac{Q(c,d)/f^2}{a} = \frac{Q(c,d)}{af^2} = \frac{Q(c,d)}{Q(fz,fw)} \in R(Q).$$

Now suppose that $b \in R(Q)$. Then there are $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ so that

$$b = \frac{Q(x_1,y_1)}{Q(x_2,y_2)} = \frac{aQ(x_1,y_1)Q(x_2,y_2)}{aQ(x_2,y_2)^2}.$$

By Lemma 5.1, there are $X, Y \in \mathbb{Z}$ so that $Q(X,Y) = aQ(x_1,y_1)Q(x_2,y_2)$. Thus,

$$b = \frac{Q(X/Q(x_2,y_2), Y/Q(x_2,y_2))}{a} \in \left\{ \frac{Q(x,y)}{a} : x, y \in \mathbb{Q} \right\}. \quad \square$$

Next we require an analogue of Lemma 5.2 that describes the p-adic closure $R(Q)^-$ of $R(Q)$.

Lemma 5.3. *If a is a nonzero integer represented by Q, then*

$$R(Q)^- = \left\{ \frac{Q(x,y)}{a} : x, y \in \mathbb{Q}_p \right\}.$$

Proof. Suppose that $b = Q(x,y)/a$ with $x, y \in \mathbb{Q}_p$. Write $a = Q(z,w)$ with $z, w \in \mathbb{Q}_p$ and choose sequences x_n, y_n of rational numbers such that

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y$$

in $\mathbb{Q}_p$. The continuity of Q ensures that

$$\frac{Q(x,y)}{a} = \frac{\lim_{n\to\infty} Q(x_n,y_n)}{a} = \lim_{n\to\infty} \frac{Q(x_n,y_n)}{a},$$

so $Q(x,y)/a$ is a limit point of $R(Q)$ by Lemma 5.2. Thus, $Q(x,y)/a \in R(Q)^-$.

Now suppose that $b \in R(Q)^-$. If $b = 0$, then $b = Q(0,0)/a$ and we are done. If $b \neq 0$, Lemma 5.2 provides $x, y \in \mathbb{Q}$ such that

$$\left\| b - \frac{Q(x,y)}{a} \right\| \leq \frac{\|b\|}{p^3}, \qquad \text{and hence} \qquad \left\| 1 - \frac{Q(x,y)}{ab} \right\| \leq \frac{1}{p^3},$$

which implies that

$$1 - \frac{Q(x,y)}{ab} \in \mathbb{Z}_p \qquad \text{and} \qquad \frac{Q(x,y)}{ab} \equiv 1 \pmod{p^3}.$$

Since every element of $\mathbb{Z}_p$ that is congruent to 1 modulo p^3 is a square (by [25, Thms. II3 & II.4] mentioned above), there is a $w \in \mathbb{Z}_p$ such that $Q(x,y)/(ab) = w^2$. Then

$$b = \frac{Q(x,y)}{aw^2} = \frac{Q(x/w, y/w)}{a} \in \left\{ \frac{Q(x,y)}{a} : x, y \in \mathbb{Q}_p \right\}. \quad \square$$

We can now reprove Theorems 2.1, 3.1, and 4.1. Lemma 5.3 implies that the p-adic closure of $R(Q)$ depends only on the $\mathbb{Q}_p$-equivalence class of Q. A quadratic form over a field can be diagonalized, and so up to scaling, any binary quadratic form is equivalent to $Q(x,y) = x^2 - dy^2$, where d is a representative of the $\mathbb{Q}_p$-square class of the discriminant of Q. As mentioned earlier, the corollary on page 37 of [25] shows that Q represents every element of $\mathbb{Q}_p$ if and only if d is a square in $\mathbb{Q}_p$. For this reason, $R(Q)$ is dense in $\mathbb{Q}_p$ if and only if the discriminant of Q is a square in $\mathbb{Q}_p$. In particular, if $p > 2$ and $b^2 - 4ac = p^k\ell$, then $R(Q)$ is dense in $\mathbb{Q}_p$ if and only if k is even and $(\ell/p) = 1$. If $p = 2$, and $b^2 - 4ac = 2^k\ell$, then $R(Q)$ is dense in $\mathbb{Q}_2$ if and only if k is even and $\ell \equiv 1 \pmod 8$.

Now, we turn to the situation of quadratic forms in $r \geq 3$ variables. Suppose that $Q(\vec{x}) = x^T A\vec{x}$ is an integral quadratic form in $r \geq 3$ variables and $\det(A) \neq 0$. The special case $A = I$ and $r = 3$ was settled by Miska, Murru, and Sanna [19, Thm. 1.8c].

Theorem 5.4. *If $r \geq 3$, then $R(Q)$ is dense in $\mathbb{Q}_p$ for all primes p.*

Proof. Fix an $n \in \mathbb{Q}_p$. If $n = 0$, then it is clear that n is in the p-adic closure of $R(Q)$, since we can take a vector $\vec{y} \in \mathbb{Z}^r$ so that $Q(\vec{y}) \neq 0$, and note that $0 = \frac{Q(\vec{x})}{Q(\vec{y})} \in R(Q)$.

Assume therefore that $n \neq 0$. By the same corollary from page 37 of [25] quoted above, the forms Q and nQ each represent either everything in $\mathbb{Q}_p$ or all but one square class in $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ has four square classes if $p > 2$ (and eight if $p = 2$), there must be some nonzero element $d \in \mathbb{Q}_p$ represented by both Q and nQ. By scaling these representations by a power of p, we can assume that there are vectors $\vec{x} \in \mathbb{Z}_p^r$ and $\vec{y} \in \mathbb{Z}_p^r$ so that $Q(\vec{x}) = nQ(\vec{y}) = k$ with $k \in \mathbb{Z}_p$ and $k \neq 0$.

Fix $\epsilon > 0$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, there are vectors $\vec{z} \in \mathbb{Z}^r$ and $\vec{w} \in \mathbb{Z}^r$ (with components $z_1, \ldots, z_r$ and $w_1, \ldots, w_r$) so that

$$\|z_i - w_i\| < \delta := \min\{\epsilon\|k/n\|, \|k/n\|, \epsilon\|k/n^2\|\}$$

for all i (and similarly $\|y_i - w_i\| < \delta$ for all i). Since Q is a polynomial with integer coefficients, Q is p-adically continuous. In fact, the ultrametric inequality implies that if $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are elements of $\mathbb{Q}_p$ with $\|a_i - b_i\| < \epsilon$ for all i, then

$$\|Q(a_1, a_2, \ldots, a_r) - Q(b_1, b_2, \ldots, b_r)\| < \epsilon.$$

Using this, we have that

$$\begin{aligned}\|Q(\vec{z}) - nQ(\vec{w})\| &= \|Q(\vec{z}) - Q(\vec{x}) + Q(\vec{x}) - nQ(\vec{y}) + nQ(\vec{y}) - nQ(\vec{w})\| \\ &\leq \max(\|Q(\vec{z}) - Q(\vec{x})\|, \|Q(\vec{x}) - nQ(\vec{y})\|, \|nQ(\vec{y}) - nQ(\vec{w})\|) \\ &< \max(\epsilon\|k/n\|, 0, \|n\|\epsilon\|k/n^2\|) \leq \epsilon\|k/n\|.\end{aligned}$$

Since $\|Q(\vec{w}) - Q(\vec{y})\| < \|k/n\|$ and $Q(\vec{y}) = k/n$, it follows that $\|Q(\vec{w})\| = \|Q(\vec{y})\| = \|k/n\|$. Thus,

$$\begin{aligned}\left\|\frac{Q(\vec{z})}{Q(\vec{w})} - n\right\| &= \frac{1}{\|Q(\vec{w})\|} \cdot \|Q(\vec{z}) - nQ(\vec{w})\| = \frac{1}{\|Q(\vec{y})\|} \cdot \|Q(\vec{z}) - nQ(\vec{w})\| \\ &< \frac{1}{\|k/n\|}(\epsilon\|k/n\|) < \epsilon.\end{aligned}$$

This proves that n is in the p-adic closure of $R(Q)$, as desired. □

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