Journal of Number Theory 208 (2020) 400–417

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Computational Section

Primitive root bias for twin primes II: Schinzel-type theorems for totient quotients and the sum-of-divisors function ☆

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ARTICLE INFO

Article history:
Received 14 June 2019
Received in revised form 28 August 2019
Accepted 29 August 2019
Available online 24 September 2019
Communicated by F. Pellarin

MSC:
11A07
11A41
11N05
11N37

ABSTRACT

Garcia, Kahoro, and Luca showed that the Bateman–Horn conjecture implies $\varphi(p-1) \geq \varphi(p+1)$ for a majority of twin-primes pairs $p, p+2$ and that the reverse inequality holds for a small positive proportion of the twin primes. That is, p tends to have more primitive roots than does $p+2$. We prove that Dickson's conjecture, which is much weaker than Bateman–Horn, implies that the quotients $\frac{\varphi(p+1)}{\varphi(p-1)}$, as $p, p+2$ range over the twin primes, are dense in the positive reals. We also establish several Schinzel-type theorems, some of them

☆ SRG supported by NSF grant DMS-1800123.
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[1] FL supported in part by NRF grant CPRR160325161141, the Focus Area Number Theory grant RTNUM19 from CoEMaSS Wits, and by CGA grant no. 17-02804S.

https://doi.org/10.1016/j.jnt.2019.08.003

11N56

Keywords:
Prime
Twin prime
Primitive root
Bateman–Horn conjecture
Twin prime conjecture
Prime bias
Dickson's conjecture
Totient function
Chen's theorem

unconditional, about the behavior of $\frac{\varphi(p+1)}{\varphi(p)}$ and $\frac{\sigma(p+1)}{\sigma(p)}$, in which σ denotes the sum-of-divisors function.

1. Introduction

The number of primitive roots modulo a prime p is $\varphi(p-1)$, in which

$$\varphi(n) = \Big|\big\{i \in \{1,2,\ldots,n\} : (i,n) = 1\big\}\Big| \;=\; n\prod_{q\mid n}\left(1-\frac{1}{q}\right) \tag{1.1}$$

is the Euler totient function. In other words, $\varphi(p-1)$ is the number of generators of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. We reserve p, q for prime numbers and use (a,b) to denote the greatest common divisor of a and b.

For twin primes $p, p+2$, it is natural to ask about the relationship between $\varphi(p-1)$ and $\varphi(p+1)$. Assuming the Bateman–Horn conjecture, Garcia, Kahoro, and Luca proved that

$$\varphi(p-1) \geq \varphi(p+1) \tag{1.2}$$

for a majority of the twin primes [10]. Such proportions are computed relative to the conjectured twin-prime counting function

$$\pi_2(x) \;\sim\; 2C_2\int_2^x \frac{dt}{(\log t)^2},$$

in which

$$C_2 = \prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2} \approx 0.660161815$$

is the *twin primes constant* [1,13]. Here $\sim$ stands for asymptotic equivalence: $f \sim g$ means $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$. The proportion of twin primes that satisfy (1.2) is at least 65% (assuming the Bateman–Horn conjecture), although computations suggest something around 98%. Moreover, at least 0.46% of the twin primes satisfy the reverse inequality $\varphi(p-1) \leq \varphi(p+1)$ [10]. Analogous results for prime pairs p, $p+k$ were obtained by

Table 1
The ratio $\frac{\varphi(p+1)}{\varphi(p-1)}$ for the first 100 twin-prime pairs with $p \geq 5$ for which (1.2) fails. The initial numerical evidence suggests that $\frac{\varphi(p+1)}{\varphi(p-1)}$ is bounded above as $p, p+2$ runs over the twin primes. Dickson's conjecture (and more extensive computation) suggest otherwise (Theorem 1).

p	$\frac{\varphi(p+1)}{\varphi(p-1)}$	p	$\frac{\varphi(p+1)}{\varphi(p-1)}$	p	$\frac{\varphi(p+1)}{\varphi(p-1)}$	p	$\frac{\varphi(p+1)}{\varphi(p-1)}$
2381	1.03125	119771	1.0234	230861	1.03125	348461	1.02981
3851	1.06	126491	1.03986	232961	1.02648	354971	1.07174
14561	1.05208	129221	1.06786	237161	1.01061	356441	1.04177
17291	1.00309	134681	1.08247	241781	1.05652	357281	1.05826
20021	1.11806	136991	1.03558	246611	1.0571	361901	1.09268
20231	1.02941	142871	1.05983	251231	1.00926	362951	1.03542
26951	1.06857	145601	1.05313	259211	1.01637	371141	1.02375
34511	1.06845	150221	1.03489	270131	1.03752	399491	1.04795
41231	1.05926	156941	1.04382	274121	1.06364	402221	1.09235
47741	1.08	165551	1.0946	275591	1.01252	404321	1.01206
50051	1.12	166601	1.03296	278741	1.07537	406631	1.00558
52361	1.13594	167861	1.06481	282101	1.08833	410411	1.13514
55931	1.02446	173741	1.00101	282311	1.00772	413141	1.02876
57191	1.05026	175631	1.05845	298691	1.037	416501	1.03179
65171	1.02608	188861	1.04087	300581	1.03534	418601	1.1011
67211	1.01413	197891	1.02266	301841	1.04082	424271	1.16905
67271	1.0043	202931	1.05743	312551	1.04783	427421	1.00958
70841	1.11799	203771	1.01071	315701	1.09613	438131	1.03357
82811	1.02747	205031	1.0169	316031	1.05385	440441	1.15852
87011	1.07857	205661	1.05097	322631	1.07177	448631	1.10491
98561	1.0694	206081	1.00692	325781	1.05864	454721	1.00694
101501	1.00679	219311	1.05694	328511	1.05523	464171	1.00607
101531	1.00714	222041	1.02361	330821	1.04042	464381	1.01407
108461	1.00871	225611	1.0726	341321	1.02666	465011	1.06779
117041	1.12882	225941	1.00577	345731	1.04732	470471	1.1837

Garcia, Luca, and Schaaff [12]. Garcia and Luca showed unconditionally that the split is 50/50 if only p is assumed to be prime [11].

A glance at the numerical evidence suggests that $\frac{\varphi(p+1)}{\varphi(p-1)}$ is bounded as p, $p+2$ range over the twin primes; see Table 1. Our first theorem, whose proof is in Section 2, demonstrates that this is far from the truth.

Theorem 1. *Dickson's conjecture implies that*

$$\left\{ \frac{\varphi(p+1)}{\varphi(p-1)} \,:\, p,\, p+2 \text{ are prime} \right\} \text{ is dense in } [0,\infty).$$

Before proceeding, we require a few words about Dickson's conjecture. The assertion that there are infinitely many twin primes is the twin prime conjecture, which remains unresolved despite significant recent work [16,18,19,27]. Thus, some unproved conjecture must be assumed to say anything nontrivial about the large-scale behavior of the twin primes. Dickson's conjecture is among the weakest general assertions that implies the twin prime conjecture [1,6,20].

Table 2
Running leaders among twin primes $p, p+2$ with $p \geq 5$ for which (1.2) fails. Theorem 1 suggests that $\frac{\varphi(p+1)}{\varphi(p-1)}$ can be arbitrarily large.

p	$\frac{\varphi(p+1)}{\varphi(p-1)}$	p	$\frac{\varphi(p+1)}{\varphi(p-1)}$
2381	1.03125	17497481	1.27427
3851	1.06	69989921	1.27484
20021	1.11806	78278201	1.28693
50051	1.12	183953771	1.30984
52361	1.13594	242662421	1.32797
424271	1.16905	468818351	1.34577
470471	1.1837	2156564411	1.37262
602141	1.18793	24912037151	1.37901
2302301	1.2058	43874931101	1.37949
6806801	1.23097	73769375681	1.39837
16926911	1.23678	131104243271	1.42545

Dickson's conjecture. *If $f_1, f_2, \ldots, f_k \in \mathbb{Z}[t]$ are linear polynomials with positive leading coefficients and $f = f_1 f_2 \cdots f_k$ does not vanish identically modulo any prime, then $f_1(t), f_2(t), \ldots, f_k(t)$ are simultaneously prime infinitely often.*

The twin prime conjecture is the special case $f_1(t) = t$ and $f_2(t) = t + 2$. Dickson's conjecture is weaker than the Bateman–Horn conjecture, which concerns polynomials of arbitrary degree and makes asymptotic predictions [1–3]. More extensive computations suggest the truth of Theorem 1; see Table 2.

Totient quotients have a long and storied history [21, Ch. 1]. Schinzel established a curious result in 1954 [22], when he showed that

$$\left\{ \frac{\varphi(n+1)}{\varphi(n)} : n = 1, 2, \ldots \right\} \text{ is dense in } [0, \infty). \tag{1.3}$$

This inspired later research by Schinzel, Sierpiński, Erdős, and others [7,8,23–26].

The prime analogue of (1.3) is false since $\limsup_{p\to\infty} \frac{\varphi(p+1)}{\varphi(p)} \leq \frac{1}{2}$ because $p+1$ is even when p is odd. Taking this into account, we prove in Section 3 that the following modified analogue of Schinzel's theorem holds unconditionally. The main ingredient is a generalization of Chen's theorem [9, Thm. 25.11].

Theorem 2. *(Unconditional)*

$$\left\{ \frac{\varphi(p+1)}{\varphi(p)} : p \text{ prime} \right\} \text{ is dense in } \left[0, \tfrac{1}{2}\right].$$

The corresponding twin-prime analogue of Schinzel's theorem (1.3) is the following result, whose proof is in Section 4.

Theorem 3. *Dickson's conjecture implies that*

$$\left\{ \frac{\varphi(p+1)}{\varphi(p)} : p,\ p+2 \text{ prime} \right\} \text{ is dense in } \left[0, \tfrac{1}{3}\right].$$

Our proofs are transparent enough to permit the construction of striking numerical examples that cannot be obtained easily through brute force alone. For example, the twin primes

$$p = 764285639860212468862974993419856587131254042904630389577091619295190673487065915056193496684464602708406281503155825518749784559242226359109916595661252353313029368701555134300787209725331177359111091752818831641459967249879112699863157166992751455469619725727863427512872427912818920974627178112772897119083576682100470569543193160046268059953665344021121664462737410334017428033077318532039713892151321174957272918842474077633173437349691652015152100258928315754383111794468807078557250176673988290542562419309668559632085007869827690141104045394494128202306648516704435434657074923111882378956454135472132525452823675738966289401997978475200583675379095724691820411030387379563258017200594242538450669651202374919121495415188829356968237769795398150669455404888422416864506510115044610100083342495472810975227134398867224393738033365491602127364628893351092398323186473558398511298470433390035546551418171276020082303320049094093469744509978525764118599399852277163603374344979783281151840116220500409194140872519378759045299325118348565045824669410375351276245344689367388445556640560175550802 1

and $p+2$ yield a ratio $\frac{\varphi(p+1)}{\varphi(p-1)} = 3.11615\ldots$, which is far larger than those displayed in Table 2. As another example, consider $\frac{\pi}{10} = 0.31415\ldots \in \left[0, \frac{1}{3}\right]$. The method of proof of Theorem 3 (with slight modifications) and a computer search yields the twin prime pair $p = 772627482100447485208656616013827857576370161313315 7$ and $p+2$, which satisfies (the underlined digits agree with those of $\frac{\pi}{10}$)

$$\frac{\varphi(p+1)}{\varphi(p)} = \underline{0.31415926535897}93213 41\ldots.$$

Theorem 1, Theorem 2, and Theorem 3 each have analogues for the sum-of-divisors function $\sigma(n) = \sum_{d|n} d$. We collect these results in the following theorem, whose proof is in Section 5.

Theorem 4.

(a) *Dickson's conjecture implies that*

$$\left\{\frac{\sigma(p+1)}{\sigma(p-1)} : p,\ p+2 \text{ prime}\right\} \text{ is dense in } [0,\infty).$$

(b) *(Unconditional)*

$$\left\{\frac{\sigma(p+1)}{\sigma(p)} : p \text{ prime}\right\} \text{ is dense in } \left[\tfrac{3}{2},\infty\right).$$

(c) *Dickson's conjecture implies that*

$$\left\{\frac{\sigma(p+1)}{\sigma(p)} : p,\ p+2 \text{ prime}\right\} \text{ is dense in } [2,\infty).$$

2. Proof of Theorem 1

2.1. A folk lemma

Mertens' third theorem asserts that

$$\prod_{q\le x}\left(1-\frac{1}{q}\right) \sim \frac{e^{-\gamma}}{\log x}, \tag{2.1}$$

in which γ is the Euler–Mascheroni constant [14,17]. A more elementary proof of the following lemma can be based on [5, Prop. 8.8] instead.

Lemma 5. *Let $\mathcal{P}$ denote a finite set of primes. Then*

$$\left\{\frac{\varphi(n)}{n} : n \text{ squarefree, } p\nmid n \text{ for all } p\in\mathcal{P}\right\} \text{ is dense in } [0,1].$$

Proof. Let $\xi\in(0,1]$ and $n_t=\prod_{e^{\xi t}\le q<e^t} q$, in which $t>\frac{1}{\xi}\log\max\mathcal{P}$. Then n_t is squarefree, $p\nmid n_t$ for all $p\in\mathcal{P}$, and

$$\frac{\varphi(n_t)}{n_t}=\prod_{e^{\xi t}\le q<e^t}\left(1-\frac{1}{q}\right)\sim\frac{e^{-\gamma}/\log(e^t)}{e^{-\gamma}/\log(e^{\xi t})}\sim\frac{\log(e^{\xi t})}{\log(e^t)}=\xi$$

as $t\to\infty$. □

2.2. Initial setup

It suffices to show that Dickson's conjecture implies that for each fixed $\xi\in(0,\infty)$ and $0<\delta<1$, there is a twin-prime pair $p,p+2$ such that $\frac{\varphi(p+1)}{\varphi(p-1)}\in\big(\xi(1-\delta),\xi(1+\delta)\big)$.

Let $0<x\le\min\left\{\frac{2}{3},\xi\right\}$. Lemma 5 provides a squarefree b such that

$$(b,6)=1 \quad\text{and}\quad \frac{\varphi(b)}{b}\in\left(\frac{x}{\xi}\left(\frac{1}{1+\delta}\right),\frac{x}{\xi}\right).$$

A second appeal to Lemma 5 yields a squarefree a' such that

$$(a',6b)=1 \quad\text{and}\quad \frac{\varphi(a')}{a'}\in\left(\frac{3x}{2}(1-\delta),\frac{3x}{2}\right).$$

Our choice of x ensures that the intervals specified are contained in $(0,1)$. Let $a = 3a'$ and observe that

$$\frac{\varphi(a)}{a} = \frac{2}{3}\frac{\varphi(a')}{a'} \in \big(x(1-\delta), x\big).$$

Consequently,

$$\frac{\varphi(a)}{a}\frac{b}{\varphi(b)} \in \big(\xi(1-\delta), \xi(1+\delta)\big). \tag{2.2}$$

2.3. The polynomials

Our strategy is to produce linear polynomials f_1, f_2, f_3, f_4 to which Dickson's conjecture can be applied, using f_1, f_2 to produce twin primes p, $p+2$, and using f_3, f_4 to ensure that $\frac{\varphi(p+1)}{\varphi(p-1)}$ falls in the desired interval.

Since $8, a^2, b^2$ are pairwise relatively prime, the Chinese remainder theorem provides c such that

$$c \equiv 5 \pmod 8, \tag{2.3}$$

$$c \equiv a - 1 \pmod{a^2}, \quad \text{and} \tag{2.4}$$

$$c \equiv b + 1 \pmod{b^2}. \tag{2.5}$$

Since $3 \mid a$, it follows from (2.4) that

$$c \equiv 2 \pmod 3. \tag{2.6}$$

Define

$$h(t) = 24a^2b^2t + c$$

and let

$$\begin{aligned} f_1(t) &= h(t), \\ f_2(t) &= h(t) + 2, \\ f_3(t) &= \frac{h(t)-1}{4b} = \frac{24a^2b^2t + (c-1)}{4b} = 6a^2bt + \frac{c-1}{4b}, \quad \text{and} \\ f_4(t) &= \frac{h(t)+1}{2a} = \frac{24a^2b^2t + (c+1)}{2a} = 12ab^2t + \frac{c+1}{2a}. \end{aligned}$$

Clearly $f_1, f_2 \in \mathbb{Z}[t]$. Observe that (2.3) and (2.5) ensure that f_3 has integral coefficients. Similarly, (2.3) and (2.4) ensure that f_4 has integral coefficients. Thus, all four polynomials are in $\mathbb{Z}[t]$ and have positive leading coefficients.

2.4. Nonvanishing product

We claim that $f = f_1 f_2 f_3 f_4$ does not vanish identically modulo any prime. Since

$$\begin{aligned} f_1(t) &\equiv f_2(t) \equiv c \equiv 1 \pmod 2, && \text{by (2.3)},\\ f_3(t) &\equiv \frac{c-1}{4b} \equiv 1 \pmod 2, && \text{by } (b,2)=1 \text{ and (2.3)},\\ f_4(t) &\equiv \frac{c+1}{2a} \equiv 1 \pmod 2, && \text{by } (a,2)=1 \text{ and (2.3)}, \end{aligned}$$

it follows that f does not vanish modulo 2. Similarly,

$$\begin{aligned} f_1(t) &\equiv c \equiv 2 \pmod 3, && \text{by (2.6)},\\ f_2(t) &\equiv c+2 \equiv 1 \pmod 3, && \text{by (2.6)},\\ f_3(t) &\equiv \frac{c-1}{4b} \equiv b \not\equiv 0 \pmod 3, && \text{by } (b,3)=1 \text{ and (2.6)},\\ f_4(t) &\equiv \frac{c+1}{2a} \equiv 2 \pmod 3, && \text{by (2.4)}, \end{aligned}$$

so f does not vanish modulo 3. The final statement perhaps deserves a bit of explanation. From (2.4) we have $c+1 \equiv a \pmod{a^2}$ and hence $\frac{c+1}{a} \equiv 1 \pmod a$. Since $3 \mid a$, it follows that $\frac{c+1}{a} \equiv 1 \pmod 3$ from which the desired statement follows.

For any prime $q \geq 5$ such that $q \nmid ab$, the polynomial f has degree four and hence cannot vanish identically modulo q. Now suppose that $q \geq 5$ is prime and $q \mid ab$. Then $h(t) \equiv c \pmod q$. Since (2.4) and (2.5) ensure that

$$c \equiv \begin{cases} -1 \pmod q & \text{if } q \mid a,\\ 1 \pmod q & \text{if } q \mid b, \end{cases} \tag{2.7}$$

it follows that f_1 and f_2 do not vanish modulo q. Similarly,

$$f_3(t) \equiv \frac{c-1}{4b} \equiv \begin{cases} -2^{-1}b^{-1} \pmod q & \text{if } q \mid a \text{ (by (2.7))},\\ 4^{-1} \pmod q & \text{if } q \mid b \text{ (by (2.5))}, \end{cases}$$

and

$$f_4(t) \equiv \frac{c+1}{2a} \equiv \begin{cases} 2^{-1} \pmod q & \text{if } q \mid a \text{ (by (2.4))},\\ a^{-1} \pmod q & \text{if } q \mid b \text{ (by (2.7))}. \end{cases}$$

Thus, f does not vanish identically modulo any prime.

2.5. Conclusion

Dickson's conjecture provides infinitely many T such that $f_1(T)$, $f_2(T)$, $f_3(T)$, and $f_4(T)$ are prime. For such T, the primes

$$p = f_1(T) \qquad \text{and} \qquad p+2 = f_2(T)$$

satisfy

$$p+1 = 2a\, f_4(T) \qquad \text{and} \qquad p-1 = 4b\, f_3(T).$$

Consequently, (2.2) ensures that

$$\begin{aligned}
\frac{\varphi(p+1)}{\varphi(p-1)} &= \frac{\varphi(p+1)}{p+1}\,\frac{p-1}{\varphi(p-1)}\,\frac{p+1}{p-1} \\
&= \frac{\varphi(p+1)}{p+1}\,\frac{p-1}{\varphi(p-1)}\big(1+o(1)\big) \\
&= \frac{\varphi(2a\, f_4(T))}{2a\, f_4(T)}\,\frac{4b\, f_3(T)}{\varphi(4b\, f_3(T))}\big(1+o(1)\big) \\
&= \frac{\varphi(2)\,\varphi(a)\,\varphi(f_4(T))}{2a\, f_4(T)}\,\frac{4b\, f_3(T)}{\varphi(4)\,\varphi(b)\,\varphi(f_3(T))}\big(1+o(1)\big) \\
&= \frac{1\varphi(a)}{2a}\,\frac{f_4(T)-1}{f_4(T)}\,\frac{4b}{2\varphi(b)}\,\frac{f_3(T)}{f_3(T)-1}\big(1+o(1)\big) \\
&= \frac{\varphi(a)}{a}\,\frac{b}{\varphi(b)}\big(1+o(1)\big)
\end{aligned}$$

belongs to $\big(\xi(1-\delta), \xi(1+\delta)\big)$ for large T. Here we have used the facts $(2a, f_4(T)) = 1$ and $(4b, f_3(T)) = 1$, which follow from (2.3) and (2.4), and from (2.3) and (2.5), respectively. □

3. Proof of Theorem 2

Chen's theorem asserts that every sufficiently large even number is the sum of two primes, or a sum of a prime and a semiprime (a number with precisely two prime factors) [4,15]. We require a generalization of Chen's theorem to linear forms. The version below is due to Friedlander and Iwaniec [9, Thm. 25.11].

Theorem 6 *(Chen, Friedlander–Iwaniec). Let $a, c \geq 1$ and $b \neq 0$ be pairwise coprime integers with $2 \mid abc$. For t sufficiently large (in terms of abc),*

$$\big|\{p \leq t : ap + b = cs\}\big| \;\geq\; \frac{W(abc)}{31c}\,\frac{Bt}{(\log t)^2},$$

in which s has at most two prime factors, each one larger than $t^{3/11}$,

$$W(d) = \prod_{\substack{p \mid d \\ p>2}} \left(1 - \frac{1}{p-1}\right)^{-1} \qquad \text{and} \qquad B = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right).$$

Let $\xi \in \left[0, \frac{1}{2}\right]$ and $\delta > 0$. For $x \geq \frac{\log 2}{2\xi}$, the integer $Q = Q(x)$ defined by

$$Q = 2 \prod_{e^{2\xi x} < q \leq e^x} q$$

is divisible by 2, but not 4. As $x \to \infty$, (1.1) and (2.1) imply

$$\frac{\varphi(Q)}{Q} = \frac{1}{2} \prod_{e^{2\xi x} < q \leq e^x} \left(1 - \frac{1}{q}\right) \sim \frac{1}{2} \frac{e^{-\gamma}}{\log(e^x)} \frac{\log(e^{2\xi x})}{e^{-\gamma}} = \xi.$$

For each x, apply Theorem 6 with $a = b = 1$ and $c = Q$ to obtain an $S = S(x)$ with at most two prime factors, both of which are greater than $\max\{Q, x\}$, such that $p = p(x) = QS - 1$ is prime. Then

$$\lim_{x \to \infty} \frac{\varphi(S)}{S} = \lim_{x \to \infty} \prod_{q \mid S} \left(1 - \frac{1}{q}\right) = 1$$

and hence

$$\frac{\varphi(p+1)}{\varphi(p)} = \frac{\varphi(QS)}{\varphi(QS-1)} = \frac{\varphi(Q)\,\varphi(S)}{QS-2} = \frac{\varphi(Q)}{Q} \frac{\varphi(S)}{S} \frac{S}{S - 2/Q} \to \xi$$

as $x \to \infty$. This concludes the proof. □

4. Proof of Theorem 3

Fix $\xi \in \left(0, \frac{1}{3}\right)$ and let $0 < \delta < \frac{1-3\xi}{3\xi}$. Lemma 5 yields a squarefree Q' such that

$$(Q', 6) = 1 \qquad \text{and} \qquad \frac{\varphi(Q')}{Q'} \in \big(3\xi(1-\delta), 3\xi(1+\delta)\big).$$

Let $Q = 6Q'$, and observe that

$$\frac{\varphi(Q)}{Q} = \frac{1}{3} \frac{\varphi(Q')}{Q'} \in \big(\xi(1-\delta), \xi(1+\delta)\big).$$

Define the polynomials

$$f_1(t) = t, \qquad f_2(t) = Qt - 1, \quad \text{and} \quad f_3(t) = Qt + 1.$$

If $q \geq 5$ and $q \nmid Q$, then $f = f_1 f_2 f_3$ has degree three and cannot vanish identically modulo q. If $q \mid Q$, then $f(1) = Q^2 - 1 \equiv -1 \pmod{q}$ and hence f does not vanish identically modulo q. In particular, f does not vanish identically modulo 2 or 3. Thus, f does not vanish identically modulo any prime.

Dickson's conjecture provides infinitely many T such that $f_1(T)$, $f_2(T)$, and $f_3(T)$ are prime. In particular, we may assume that the prime $f_1(T) = T$ is greater than Q so that $(Q, T) = 1$. Then $p = QT - 1$ and $p + 2 = QT + 1$ are twin primes and $p + 1 = QT$. Then

$$\begin{aligned}\frac{\varphi(p+1)}{\varphi(p)} &= \frac{\varphi(QT)}{\varphi(QT-1)} \\ &= \frac{\varphi(Q)\,\varphi(T)}{\varphi(QT-1)} \\ &= \frac{\varphi(Q)(T-1)}{QT-2} \\ &= \frac{\varphi(Q)}{Q}\frac{T-1}{T-2/Q} \\ &= \frac{\varphi(Q)}{Q}\big(1+o(1)\big)\end{aligned}$$

is in $\big(\xi(1-\delta), \xi(1+\delta)\big)$ for sufficiently large T. □

5. Proof of Theorem 4

5.1. Proof of Theorem 4a

The proof of Theorem 4a is similar to the proof of Theorem 1. We first require the following version of Lemma 5 for the sum-of-divisors function.

Lemma 7. *Let $\mathcal{P}$ denote a finite set of primes. Then*

$$\left\{\frac{\sigma(n)}{n} : n \text{ squarefree, } p \nmid n \text{ for all } p \in \mathcal{P}\right\} \text{ is dense in } [1, \infty).$$

Proof. Let $Q = Q(t) = \prod_{q \leq t} q$. Then Mertens' third theorem (2.1), the Euler product formula, and the evaluation $\zeta(2) = \frac{\pi^2}{6}$ yield

$$\frac{\sigma(Q)}{Q} = \prod_{q \leq t} \frac{1+q}{q} = \prod_{q \leq t}\left(1 + \frac{1}{q}\right) = \frac{\prod_{q \leq t}(1 - 1/q^2)}{\prod_{q \leq t}(1 - 1/q)} \sim \frac{6/\pi^2}{e^{-\gamma}/\log t} = \frac{6e^{\gamma}}{\pi^2}\log t$$

as $t \to \infty$. Let $\xi \geq 1$ and define $n_t = \prod_{e^t \leq q < e^{\xi t}} q$, in which $\log t > \max \mathcal{P}$. Then n_t is squarefree, $p \nmid n_t$ for all $p \in \mathcal{P}$, and

$$\frac{\sigma(n_t)}{n_t} \sim \frac{\log(e^{\xi t})}{\log(e^t)} = \xi$$

as $t \to \infty$. $\square$

Fix $\xi \in (0,\infty)$ and $0 < \delta < 1$. Let $x \geq \max\left\{\frac{4}{3}, \frac{7\xi}{6}\right\}$. Then Lemma 7 provides a squarefree b such that

$$(b,6) = 1 \quad \text{and} \quad \frac{\sigma(b)}{b} \in \left(\frac{6x}{7\xi}, \frac{6x}{7\xi}\left(\frac{1}{1-\delta}\right)\right).$$

A second appeal to Lemma 7 yields a squarefree a' such that

$$(a', 6b) = 1 \quad \text{and} \quad \frac{\sigma(a')}{a'} \in \left(\frac{3x}{4}, \frac{3x}{4}(1+\delta)\right).$$

Our choice of x ensures that the intervals specified are contained in $(1,\infty)$. Let $a = 3a'$ and observe that

$$\frac{\sigma(a)}{a} = \frac{4}{3}\frac{\sigma(a')}{a'} \in \big(x, x(1+\delta)\big).$$

Consequently,

$$\frac{\sigma(a)}{a}\frac{b}{\sigma(b)} \in \left(\frac{7}{6}\xi(1-\delta), \frac{7}{6}\xi(1+\delta)\right). \tag{5.1}$$

Define the polynomials f_1, f_2, f_3, f_4 as in the proof of Theorem 1, in which we showed that the application of Dickson's conjecture to this family is permissible. Dickson's conjecture provides infinitely many T such that $f_1(T)$, $f_2(T)$, $f_3(T)$, and $f_4(T)$ are prime. For such T, the primes

$$p = f_1(T) \qquad \text{and} \qquad p+2 = f_2(T)$$

satisfy $p+1 = 2a\, f_4(T)$ and $p-1 = 4b\, f_3(T)$. Consequently, (5.1) ensures that

$$\begin{aligned}
\frac{\sigma(p+1)}{\sigma(p-1)} &= \frac{\sigma(p+1)}{p+1}\frac{p-1}{\sigma(p-1)}\frac{p+1}{p-1} \\
&= \frac{\sigma(p+1)}{p+1}\frac{p-1}{\sigma(p-1)}\big(1+o(1)\big) \\
&= \frac{\sigma(2a\, f_4(T))}{2a\, f_4(T)}\frac{4b\, f_3(T)}{\sigma(4b\, f_3(T))}\big(1+o(1)\big) \\
&= \frac{\sigma(2)\,\sigma(a)\,\sigma(f_4(T))}{2a f_4(T)}\frac{4b\, f_3(T)}{\sigma(4)\,\sigma(b)\,\sigma(f_3(T))}\big(1+o(1)\big)
\end{aligned}$$

$$= \frac{3\sigma(a)}{2a}\frac{f_4(T)+1}{f_4(T)}\frac{4b}{7\sigma(b)}\frac{f_3(T)}{f_3(T)+1}\big(1+o(1)\big)$$
$$= \frac{6}{7}\frac{\sigma(a)}{a}\frac{b}{\sigma(b)}\big(1+o(1)\big)$$

belongs to $\big(\xi(1-\delta),\xi(1+\delta)\big)$ for large T. □

5.2. Proof of Theorem 4b

Since the proof of Theorem 4b is similar to the proof of Theorem 2, we only sketch the details. First, a simple modification of Lemma 7 shows that for any finite set $\mathcal{P}$ of primes that does not contain 2, the set

$$\left\{\frac{\sigma(n)}{n} : n \text{ squarefree and even, } p \nmid n \text{ for all } p \in \mathcal{P}\right\}$$

is dense in $\left[\frac{3}{2},\infty\right)$. Let $\xi \in \left[\frac{3}{2},\infty\right)$ and mimic the proof of Theorem 2 to find an even squarefree $Q = Q(x)$ such that $\frac{\sigma(Q)}{Q} \to \xi$ as $x \to \infty$. Apply Theorem 6 and obtain an $S = S(x)$ with at most two prime factors, both of which are greater than $\max\{Q, x\}$, such that $p = p(x) = QS - 1$ is prime. Then $\frac{\sigma(S)}{S} \to 1$ as $x \to \infty$ and hence

$$\frac{\sigma(p+1)}{\sigma(p)} = \frac{\sigma(QS)}{\sigma(QS-1)} = \frac{\sigma(Q)\,\sigma(S)}{QS} = \frac{\sigma(Q)}{Q}\frac{\sigma(S)}{S} \to \xi. \quad \square$$

5.3. Proof of Theorem 4c

Since the proof of Theorem 4c is similar to the proof of Theorem 3, we only sketch the details. Let $\xi \in [2,\infty)$ and mimic the proof of Theorem 3 to find a squarefree $Q = Q(x)$ which is divisible by 6 such that $\frac{\sigma(Q)}{Q} \to \xi$ as $x \to \infty$.

Define the polynomials f_1, f_2, f_3 as in the proof of Section 4 in which we showed that the application of Dickson's conjecture to this family is permissible. Thus, we can find arbitrarily large T such that $f_1(T) = T$, $p = f_2(T) = QT-1$, and $p+2 = f_3(T) = QT+1$ are simultaneously prime and hence

$$\frac{\sigma(p+1)}{\sigma(p)} = \frac{\sigma(QT)}{\sigma(QT-1)} = \frac{\sigma(Q)\,\sigma(T)}{QT} = \frac{\sigma(Q)}{Q}\frac{T+1}{T} \to \xi. \quad \square$$

6. Numerical examples

Our methods of proof are transparent enough that they permit us to construct numerical examples whose totient and divisor-sum quotients approximate various mathematical constants surprisingly well (much better than can be obtained by brute force alone). Tables 3, 4, 5, 6, 7, and 8 showcase various examples for each of the theorems proven above.

Table 3
Numerical examples for Theorem 1.

ξ	p	$\frac{\varphi(p+1)}{\varphi(p-1)}$
$\gamma = 0.577215664901532960\ldots$	42372380883463421517732313631034725791798152943 7	0.57721566490153289 5 . . .
$\frac{\pi}{10} = 0.31415926535897932\ldots$	821951741475735357573957878449898348628555998383837563816565056272351869	0.31415926535897929 . . .
$\frac{\pi}{2} = 1.570796326794\ldots$	166699917024666766393292717283478616682999425902809932261	1.57079632678 2 . . .
$\frac{e}{10} = 0.271828182845904523\ldots$	340751879321709831822472985002608295768606685410340636965145727573941026087045198646133 89	0.27182818284590450 1 . . .
$\frac{1}{e} = 0.36787944117144232\ldots$	113683657087500964781899014960071623256577731144273048059639590773073268 29	0.36787944117144229 . . .
$\sqrt{2} = 1.414213562373095048\ldots$	999363429694171504049296941944028692211326710 4	1.41421356237309503 4 . . .
$\sqrt{3} = 1.732050807568877\ldots$	684404914906660046091876747014620547593630589653842274957065135628352504896296766161	1.73205080756886 2 . . .
$\frac{\sqrt{5}+1}{2} = 1.6180339887498948\ldots$	130696490486524307951434975404588722262181891430565948118397386374299 81	1.61803398874989 32 . . .
$\frac{\sqrt{5}-1}{2} = 0.61803398874989484\ldots$	113819885725207836594181379451145300826370298475149690990486037	0.61803398874989493 . . .
$\log 2 = 0.693147180\ldots$	763278080829361407712397	0.69314717 2 . . .
$\log 3 = 1.09861228866810969\ldots\ldots$	864353706215229152175362749211971296467930344 61	1.09861228866810905 . . .

Table 4
Numerical examples for Theorem 2.

ξ	p	$\frac{\varphi(p+1)}{\varphi(p)}$
$\frac{\pi}{10} = 0.314159265\ldots$	1902037158772097	0.31415923 3 . . .
$\frac{\pi}{20} = 0.1570796326794\ldots$	652305109481531435513874189 89	0.15707961697 22 . . .
$\frac{e}{10} = 0.2718281828\ldots$	9240530296299581	0.27182815 56 . . .
$\frac{1}{e} = 0.367879441\ldots$	5309646891817189	0.36787940 4 . . .
$\frac{\sqrt{2}}{10} = 0.141421356\ldots$	400277093654122670523115304726 9	0.14142134 2 . . .
$\sqrt{2}-1 = 0.414213562\ldots$	233570456771714761	0.41421352 0 . . .
$\frac{\sqrt{3}}{10} = 0.173205080\ldots$	721221963089661856995482309	0.17320506 3 . . .
$\frac{\sqrt{5}}{10} = 0.223606797\ldots$	1061017350953476949129	0.22360677 5 . . .
$\frac{\sqrt{7}}{10} = 0.264575131\ldots$	184295506315169	0.26457510 4 . . .
$\frac{\sqrt{5}+1}{20} = 0.161803398\ldots$	9472109613048955830568610 9	0.16180338 2 . . .

Table 5
Numerical examples for Theorem 3.

ξ	p	$\frac{\varphi(p+1)}{\varphi(p)}$
$\frac{\pi}{10} = 0.314159265\ldots$	1902037158772097	$\underline{0.314159233}\ldots$
$\pi - 3 = 0.141592653\ldots$	37850921999916257860282849163969	$\underline{0.141592639}\ldots$
$\frac{\pi}{20} = 0.15707963\ldots$	2762774807373943331969	$\underline{0.15707947}\ldots$
$\frac{e}{10} = 0.2718281828\ldots$	9240621837106421	$\underline{0.2718281556}\ldots$
$\frac{\sqrt{2}}{10} = 0.14142135623\ldots$	400359963884787589865194871858 9	$\underline{0.1414213420}9\ldots$
$\frac{\sqrt{3}}{10} = 0.173205080\ldots$	72123162724845651734344028 9	$\underline{0.17320506}3\ldots$
$\frac{\sqrt{5}}{10} = 0.223606797\ldots$	1061064248215841845709	$\underline{0.22360677}5\ldots$
$\frac{\sqrt{7}}{10} = 0.264575131\ldots$	184327276293689	$\underline{0.26457510}4\ldots$
$\frac{\sqrt{5}+1}{20} = 0.161803398\ldots$	94721096130489558305686109	$\underline{0.16180338}2\ldots$

Table 6
Numerical examples for Theorem 4a.

ξ	p	$\frac{\sigma(p+1)}{\sigma(p-1)}$
$\gamma = 0.577215664901532\ldots$	285659715165349572896202438481583805634555808139263750257162232532026578053941	$\underline{0.57721566490152}7\ldots$
$\pi = 3.1415\ldots$	2646921222256841983860921264087468600029815823870109671748234126884557984956967613160956654655736936734927913892299227076354442 9	$\underline{3.14}07$
$\frac{\pi}{2} = 1.57079632679489661\ldots$	2714452215896122969174543982802497861083412480831708685636987789 89	$\underline{1.5707963267948967}5\ldots$
$e = 2.718281\ldots$	2014265861114963900599828590523787295285725793612168528391507448185004321202 29	$\underline{2.718277}\ldots$
$\sqrt{2} = 1.414213562373095\ldots$	1300421337603424629894146725516266587797672109490592798687340114429	$\underline{1.41421356237308}9\ldots$
$\sqrt{3} = 1.73205080756887729\ldots$	5001304679832232346811636913292219681806190173052913912408611416987702190648767132709	$\underline{1.7320508075688774}4\ldots$
$\sqrt{5} = 2.2360679774997\ldots$	2013817027153422288458176851766980884996440299332870782957602465382085798 9	$\underline{2.236067977498}0\ldots$
$\frac{\sqrt{5}+1}{2} = 1.618033988749894\ldots$	2768409745128994233528433338776786616521251020681096425364422069	$\underline{1.61803398874988}8\ldots$
$\frac{\sqrt{5}-1}{2} = 0.6180339887498948\ldots$	1085428540928425422657480890616152917531265123430976066042852147231541940154141	$\underline{0.618033988749895}4\ldots$
$\log 2 = 0.6931471805599\ldots$	1583151329945483227597515367762019399440086162355445042258 21	$\underline{0.693147180560}1\ldots$
$\log 3 = 1.09861228866810969\ldots$	2122045296053250978265208866926658824798283021311953124598289453 57	$\underline{1.0986122886681098}9\ldots$

Table 7
Numerical examples for Theorem 4b.

ξ	p	$\frac{\sigma(p+1)}{\sigma(p)}$
$10\gamma = 5.77215\ldots$	828783459080807955117965628923827918979743569551497487292089745700491612278956238	$\underline{5.7722}1\ldots$
$\pi = 3.1415926\ldots$	203351964077675489	$\underline{3.1415926}\ldots$
$2\pi = 6.283185\ldots$	655193122132808448338702066088021133128673476003092904430789475344820749138959120591709442615982404569457060616806991988883790296018540849009	$\underline{6.283185}1\ldots$
$\frac{\pi}{2} = 1.57079632\ldots$	22955076440560177	$\underline{1.5707963}48\ldots$
$e = 2.7182818\ldots$	1716574977543884369	$\underline{2.7182818}21\ldots$
$\sqrt{3} = 1.73205080\ldots$	156513047792653	$\underline{1.7320508}98\ldots$
$\sqrt{5} = 2.236067\ldots$	4852141797161	$\underline{2.23607}0\ldots$
$\frac{\sqrt{5}+1}{2} = 1.6180339\ldots$	18106083326748793	$\underline{1.6180339}1\ldots$

Table 8
Numerical examples for Theorem 4c.

ξ	p	$\frac{\sigma(p+1)}{\sigma(p)}$
$10\gamma = 5.7721566\ldots$	2595684868506848043313701558218369992865598970206944461532321445567887878042009056424590269	$\underline{5.7721572}\ldots$
$\pi = 3.14159265\ldots$	2007224256303311429	$\underline{3.1415929}6\ldots$
$2\pi = 6.28318530\ldots$	6561189247647575857894512417012071976627284592861948711670209688265586319709280421920002799868898085933789604550135691104643715461617425356689	$\underline{6.2831859}3\ldots$
$e = 2.7182818\ldots$	1717018302510268229	$\underline{2.7182818}21\ldots$
$\sqrt{5} = 2.2360679\ldots$	285009842420045757101	$\underline{2.2360679}2\ldots$

6.1. Computational differences

For the sake of optimization, our computation of numerical examples involves slightly different methods than those provided in the proofs. In particular, our provided proofs of Lemmas 5 and 7 construct a product of consecutive primes between $e^{\xi t}$ and e^t. Our computation takes a more naïve but more efficient process: begin with 1, and repeatedly multiply by the next smallest $q \notin \mathcal{P}$ so that $\frac{\varphi(n)}{n} \geq \xi$ (resp., for Lemma 7, $\frac{\sigma(n)}{n} \leq \xi$); convergence of this process is guaranteed by the fact that $\prod_q (1-1/q)$ diverges to 0 (resp., $\prod_q (1+1/q)$ diverges to ∞), so the sequence we construct is monotonically decreasing (resp., increasing) and is bounded tightly below (resp., above) by ξ.

For Theorem 1, Theorem 3, Theorem 4a, and Theorem 4c, the method of construction is otherwise the same, relying on the same polynomial-based approach together with Dickson's conjecture. For Theorem 2 and Theorem 4b, instead of the unconditional method of proof based on Theorem 6 provided in the paper, we instead took a polynomial/Dickson approach similar to that of Theorem 3 and Theorem 4c based on Lemma 5 and Lemma 7, since we found no straightforward numerical implementation of Theorem 6.

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