RESEARCH ARTICLE | JULY 27 2020

Spectroscopy from the lattice: The scalar glueball FREE

Ruairí Brett ✉; John Bulava; Daniel Darvish; Jacob Fallica; Andrew Hanlon; Ben Hörz; Colin Morningstar

AIP Conf. Proc. 2249, 030032 (2020)
https://doi.org/10.1063/5.0008566

Spectroscopy From The Lattice: The Scalar Glueball

Ruairí Brett,[1, a)] John Bulava,[2] Daniel Darvish,[3] Jacob Fallica,[4]
Andrew Hanlon,[5] Ben Hörz,[6] and Colin Morningstar[3]

[1)]*Department of Physics, The George Washington University, Washington, DC 20052, USA*
[2)]*CP3-Origins, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark*
[3)]*Department of Physics, Carnegie Mellon University, Pittsburgh, PA 15213, USA*
[4)]*Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA*
[5)]*Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany*
[6)]*Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA*

[a)]*Corresponding author: rbrett@gwu.edu*

Abstract. Lattice calculations allow us to probe the low-lying, non-perturbative spectrum of QCD using first principles numerical methods. Here we present the low-lying spectrum in the scalar sector with vacuum quantum numbers including, in fully dynamical QCD for the first time, the mixing between glueball, q-qbar, and meson-meson operators.

INTRODUCTION

The three-gluon and four-gluon coupling terms in the QCD Lagrangian suggest the existence of composite states consisting solely of gluons, called glueballs. Such states are of great interest especially as they are distinct from the prototypical $\overline{q}q$ and qqq hadronic states predicted by constituent quark models. However, incontrovertible experimental evidence for their existence remains elusive. There are several leading candidates for the lightest scalar glueball, including the $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ states, yet none have been unambiguously identified as a glueball state [1]. To identify which of the three is most likely a glueball or gluon-dominated state, model independent, first principles lattice calculations are required.

The glueball spectrum in pure Yang-Mills gauge theory has been extensively mapped out [2, 3, 4]. The lowest-lying scalar and tensor glueballs have previously been studied in quenched QCD, but the quenched approximation makes such studies unreliable. For the scalar glueball, quenched calculations yield a glueball mass in the range $1.5-1.7$ GeV. More recent studies which have included the effects of sea quarks on glueballs [5, 6, 7, 8, 9, 10], largely agree with one another and with the quenched calculations in the scalar, and tensor sectors, but such studies have not included meson-meson operators.

As the candidate glueball states lie well above many hadron thresholds, the inclusion of glueball, $\overline{q}q$, and meson-meson operators is crucial for making any definitive conclusions about the nature, or even existence of such glueball states. Furthermore, as these states in infinite-volume manifest as unstable resonances, forming any infinite-volume conclusions will require the determination of coupled channel infinite-volume scattering amplitudes from finite-volume energies. Here we present the low-lying finite-volume spectrum from Ref. [11] in the scalar sector with vacuum quantum numbers, where glueball, $\overline{q}q$, and meson-meson operators have been included for the first time in lattice QCD.

ANALYSIS DETAILS

Temporal correlation functions involving glueball operators are notoriously difficult to measure in lattice QCD, requiring prohibitively large computational resources to achieve even modest statistical precision. In the scalar sector the signal-to-noise ratio for such observables falls extremely rapidly with increasing separation between source and sink, as the relevant interpolating operators have large vacuum expectation values. This prohibits the lattice from being too large, as the magnitude of these vacuum fluctuations will scale with the lattice volume. On the other hand, due to the large masses of these states, lattice studies of glueballs require very fine temporal lattice spacings so that a reliable signal can be measured. As both of these considerations have a significant effect on the required computational resources, we employ an anisotropic lattice that is spatially coarse and temporally fine [12].

We use a single anisotropic ensemble of $N_f = 2+1$ clover-improved Wilson fermions with $m_\pi \approx 390$ MeV, generated by the Hadron Spectrum collaboration [13, 14]. Various ensemble parameters are listed in table I. Throughout

Proceedings of the 15th International Conference on Meson-Nucleon Physics and the Structure of the Nucleon
AIP Conf. Proc. 2249, 030032-1–030032-7; https://doi.org/10.1063/5.0008566
Published by AIP Publishing. 978-0-7354-2008-3/$30.00

TABLE I. Details of the anisotropic ensemble used in the scalar glueball study. The anisotropy $\xi = a_s/a_t$ has been determined by enforcing the relativistic dispersion relation for the pion, though the value is insensitive to the hadron used.

$(L/a_s)^3 \times (T/a_t)$	N_{cfgs}	a_s	ξ_π	$a_t m_\pi$	$a_t m_K$	$m_\pi L$
$24^3 \times 128$	551	0.12 fm	3.4464(71)	0.06901(17)	0.09689(15)	5.7

we will quote energies as dimensionless ratios using a reference mass: $m_{\text{ref}} = 2m_K$ where m_K is the kaon mass.

Correlation Matrix Analysis

Finite-volume stationary state energies are extracted from the matrix of temporal correlation functions, $\mathscr{C}_{ij}(t) = \langle 0|\mathscr{O}_i(t)\overline{\mathscr{O}}_j(0)|0\rangle$, for which all-to-all quark propagation is evaluated using the stochastic LapH method [15]. To extract the finite-volume energies E_n, and operator overlap factors $Z_j^n \equiv \langle 0|\mathscr{O}_j|n\rangle$, we solve the generalized eigenvalue problem [16]

$$C(t)v_n(t,\tau_0) = \lambda_n(t,\tau_0)C(\tau_0)v_n(t,\tau_0), \tag{1}$$

where $C_{AB}(t) \equiv \mathscr{C}_{AA}(\tau_N)^{-1/2}\mathscr{C}_{AB}(t)\mathscr{C}_{BB}(\tau_N)^{-1/2}$ is the normalized correlation matrix, with normalization time τ_N, and τ_0 is referred to as the metric time. To do this, we define the "rotated" correlation matrix by

$$\widetilde{D}(t) \equiv U^\dagger C(\tau_0)^{-1/2}C(t)C(\tau_0)^{-1/2}U, \tag{2}$$

where the matrix U is formed using the eigenvectors of $C(\tau_0)^{-1/2}C(\tau_D)C(\tau_0)^{-1/2}$, for a single choice of the metric and diagonalization times (τ_0, τ_D). By diagonalizing only for a single time separation τ_D, we avoid diagonalizing the correlation matrix for late times where significantly increased statistical noise can lead to a significant bias in the final results. The diagonal elements of $\widetilde{D}(t)$ can then be shown to tend to, in the limit of large time separations, $\lambda_n(t) \propto e^{-E_n t}$ [17]. Then we can use single- and multi-exponential (to account for excited state contamination) fits to the diagonal elements of $\widetilde{D}(t)$ to determine the energies E_n and overlaps $|Z_j^n|^2$.

The basis of single- and two-hadron interpolating operators used is constructed to overlap maximally with the states of interest as described in Ref. [18], including the so-called TrLapH scalar glueball operator constructed using the eigenvalues of the covariant Laplacian:

$$\mathscr{O}_G \equiv -\operatorname{Tr}[\Theta(\sigma_s^2 + \widetilde{\Delta})\widetilde{\Delta}]. \tag{3}$$

The operator basis is chosen so as to saturate the spectrum of single- and two-particle stationary states below $\approx 2m_{\text{ref}}$. Along with conventional isoscalar $\overline{q}q$ single-hadron operators and a scalar glueball operator, we include $\pi\pi$, $\eta\eta$, and $K\bar{K}$ two-hadron operators with various definite back-to-back momenta for each of the allowed two-body decays in the sector.

As we are concerned with states that share quantum numbers with the vacuum, interpolating operators designed to transform irreducibly in the at-rest A_{1g}^+ irrep (where g and $+$ denote positive spatial parity and G-parity, respectively) are expected to have non-zero vacuum expectation values (VEVs). These VEVs must be subtracted in order to extract the signal of interest:

$$C_{ij}(t) \to \langle 0|\mathscr{O}_i(t)\overline{\mathscr{O}}_j(0)|0\rangle - \langle 0|\mathscr{O}_i|0\rangle\langle 0|\overline{\mathscr{O}}_j|0\rangle. \tag{4}$$

For correlation functions featuring the scalar glueball operator, this subtraction presents some additional difficulty. Even in the moderately sized volume employed here, the magnitude of $\langle 0|\mathscr{O}_G|0\rangle$ is very large, and statistically very noisy. The large uncertainties that manifest on the inclusion of the scalar glueball operator in the operator basis requires us to use quite aggressive noise reduction techniques in order to reliably extract a signal.

Symmetry arguments based on the behaviour of our operators under time reversal, etc. tell us that the correlation matrices here must be both real and symmetric. In practice however, stochastic estimates of $\text{Im}C_{ij}(t)$ will only be statistically consistent with zero rather than exactly zero. This poses a problem when we include the scalar glueball operator in our correlation matrix. See for example the matrix element $\text{Im}\langle 0|\mathscr{O}_G\mathscr{O}_{\pi(2)\pi(2)}|0\rangle$ shown in fig. 1. The mean value is systematically shifted away from zero, hinting at the difficulty in accurately estimating the large VEVs

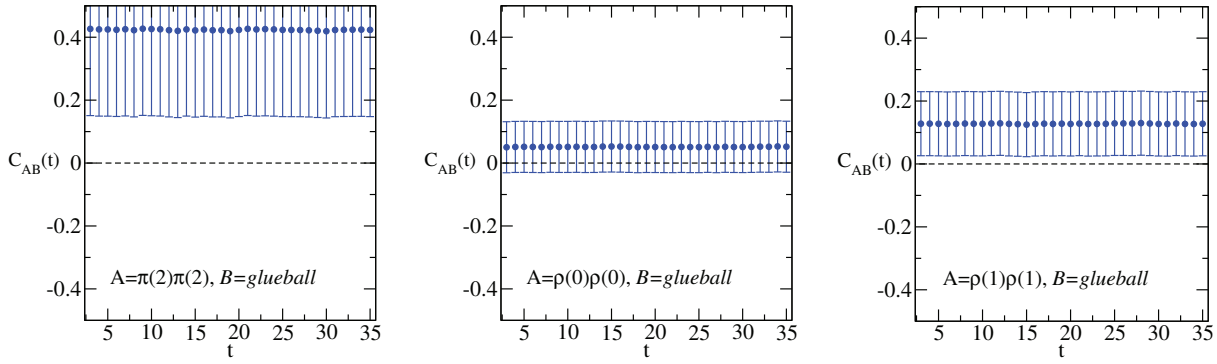

FIGURE 1. $\mathrm{Im}C_{AB}(t)$ for three normalized matrix elements including the scalar glueball operator. Each element has been normalized using $C_{AB}(t) \equiv \mathscr{C}_{AA}(\tau_N)^{-1/2}\mathscr{C}_{AB}(t)\mathscr{C}_{BB}(\tau_N)^{-1/2}$ where $\mathscr{C}_{ij}(t) = \langle 0|\mathscr{O}_i(t)\overline{\mathscr{O}}_j(0)|0\rangle$, and $\tau_N = 3$.

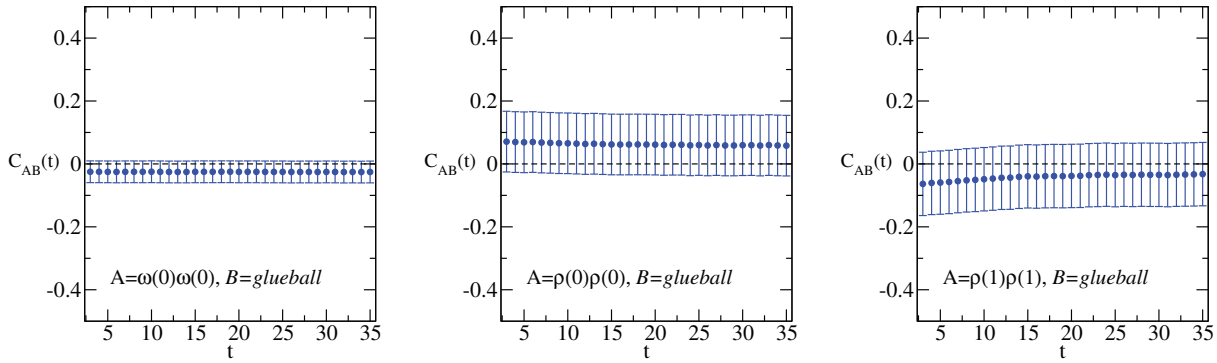

FIGURE 2. $\mathrm{Re}C_{AB}(t)$ for three normalized matrix elements including the scalar glueball operator and a vector-vector two-hadron operator. Each element has been normalized using $C_{AB}(t) \equiv \mathscr{C}_{AA}(\tau_N)^{-1/2}\mathscr{C}_{AB}(t)\mathscr{C}_{BB}(\tau_N)^{-1/2}$ where $\mathscr{C}_{ij}(t) = \langle 0|\mathscr{O}_i(t)\overline{\mathscr{O}}_j(0)|0\rangle$, and $\tau_N = 3$.

for the scalar glueball operator. We find that in order to maintain a strictly positive definite correlation matrix when the basis includes the scalar glueball operator, we must explicitly set the imaginary components of the correlation matrix to be zero. We find that when the glueball operator is omitted, the finite-volume spectrum extracted is unaffected by setting $\mathrm{Im}C_{ij} = 0$.

The significant statistical noise present in the VEV-subtracted correlators presents additional difficulty when we consider the "single-pivot" method in eq. (2). As we perform the diagonalization on only the full sample estimate of the correlation matrix (i.e. the matrix U), significant statistical noise in the matrix elements can introduce a bias. Usually this is easily avoided by choosing early diagonalization times for which statistical noise is minimized, however the significant increase in noise, even at very early time separations, in the presence of the glueball operator can have a drastic effect on the pivot. This is shown in the $\langle 0|\mathscr{O}_G\mathscr{O}_{VV}|0\rangle$ elements in fig. 2, where $\mathscr{O}_{VV}$ is a vector-vector two-hadron operator. We found that a significant bias in the pivot could be mitigated by setting these elements that are statistically consistent with zero to be exactly zero in our analysis.

SPECTRUM RESULTS

Our main goal is to discover if any finite-volume states below $2m_{\mathrm{ref}}$ in the vacuum sector are *missed* when no glueball operators are included. We assume that our operators couple minimally to states involving three or more hadrons. First we will consider the finite-volume spectrum determined using a basis of interpolating operators *excluding* the scalar

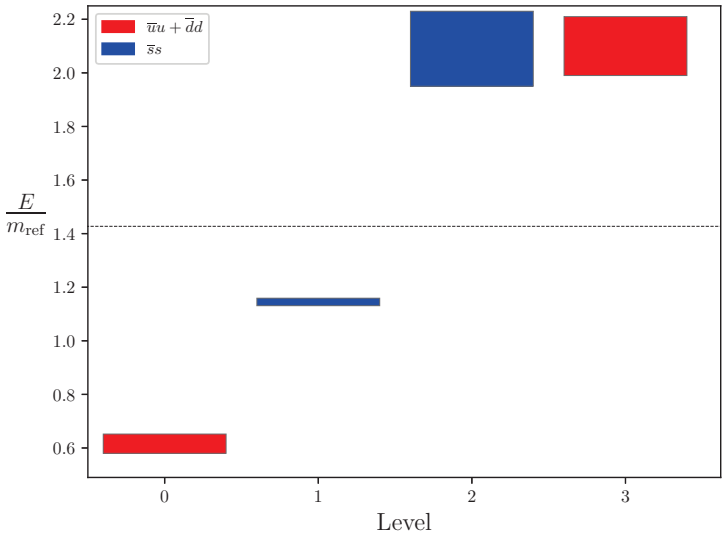

FIGURE 3. Finite-volume energies in the $I = 0, S = 0, A_{1g}^{+}$ channel for levels with significant overlap onto states produced only by quark-antiquark operators. A 4×4 correlation matrix including only $\overline{q}q$ operators is used to extract these levels. 1σ uncertainties are denoted by the box heights. Levels are coloured indicating the operator flavour type with maximal overlap onto that state. The horizontal dashed black line indicates the 4π threshold, and $m_{\text{ref}} = 2m_K$. These energies do not change appreciably when other $\overline{q}q$ operators are included in a larger correlation matrix with meson-meson operators and the glueball operator.

glueball operator. We begin by including a two-hadron (meson-meson) operator for each expected non-interacting level, adding additional operators with various flavor, spin, etc. structure until no new finite-volume levels are found below $\sim 2m_{\text{ref}}$. Single-hadron $\overline{q}q$ operators are chosen in a similar way, including one of each isoscalar flavor structure: $(\overline{u}u + \overline{d}d, \overline{s}s)$ with various spatial displacements until no new states are seen in the energy region of interest. We find only two such finite-volume states below $2m_{\text{ref}}$ using $\overline{q}q$ operators, shown in fig. 3. Hence, we need only include two of these operators in the final operator set, one of each flavor structure. This has also been confirmed by including additional $\overline{q}q$ interpolating operators in the final operator set and observing no deviation of the finite-volume spectrum below $2m_{\text{ref}}$.

The finite-volume spectrum extracted using an operator basis *excluding* the glueball operator is shown on the left in fig. 4. The significant statistical noise present in many of the operators used here necessitates rather early GEVP metric and diagonalization times of $(\tau_0, \tau_D) = (3,6)$, though we have used various combinations of $\tau_0 = 3,4$, $\tau_D = 4,5,6,7,8$ in order to ensure the spectrum does not change. With these choices the correlation matrices remain well conditioned, having condition numbers < 10 at τ_0 and τ_D. We also ensure that the off-diagonal elements of $\widetilde{D}(t)$ remain statistically consistent with zero for $t > \tau_D$. We then *include* the scalar glueball operator in the basis and extract the finite-volume spectrum as above using $(\tau_0, \tau_D) = (3,6)$, shown on the right in fig. 4.

Looking first at the states below 4π in fig. 4, indicated by the horizontal dashed line, with the exception of some increased statistical noise, the spectrum below 4π is insensitive to the addition of the glueball operator. The overlap factors in fig. 5 show minimal mixing in this region and so level identification is relatively straightforward and is indicated by the colouring of the energy levels. As level 0 is predominantly created by the $(\overline{u}u + \overline{d}d)$ quark-antiquark operator, along with the glueball operator, we can interpret this state as the finite-volume counterpart of the σ resonance. This is consistent with the $\pi\pi$ scattering study of Ref. [19] where a bound state σ meson is found below the $\pi\pi$ threshold. Similarly, from figs. 4(b) and 4(f), levels 1 and 2 are created by the $\pi(0)\pi(0)$ and $s\bar{s}$ quark-antiquark operators, respectively, where the integers indicate the square of the hadron momentum, in units of $2\pi/L$. As level 2 is predominantly created by a $\overline{q}q$ interpolating operator, we identify level 2 as the finite-volume counterpart of the $f_0(980)$ resonance, just above the $K\bar{K}$ threshold.

Above the 4π threshold, we can assess the effect that including the glueball operator has on the finite-volume spectrum. Note that these levels in fig. 4 have been reordered slightly. From fig. 5(k-m) we can identify the rightmost three levels as being predominantly created by the vector-vector $\omega(0)\omega(0)$, $\rho(0)\rho(0)$, and $\rho(1)\rho(1)$ operators. With the exception of an increase in statistical noise, these levels are largely unaffected by the inclusion of the glueball

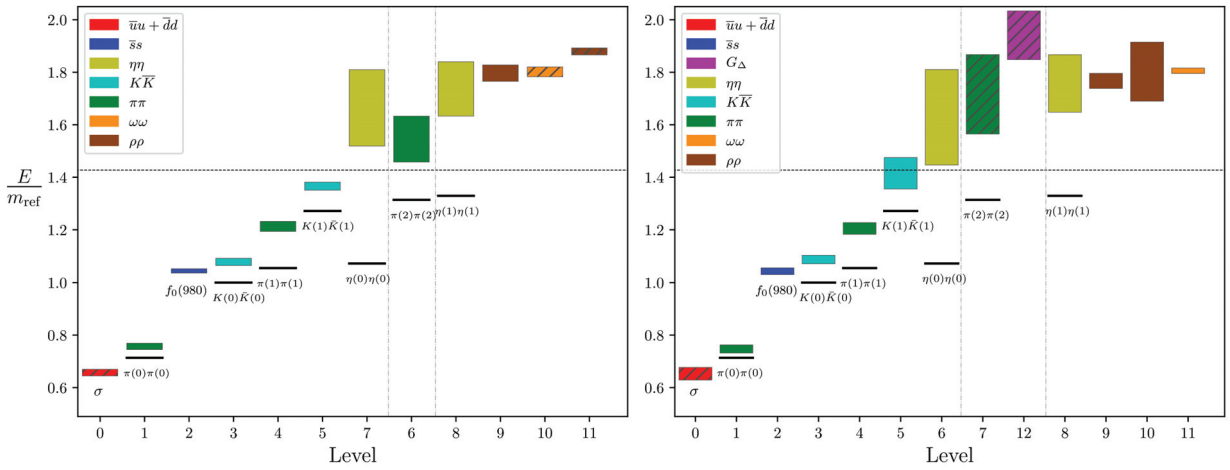

FIGURE 4. Finite-volume stationary state energies in the $I = 0$, $S = 0$, A_{1g}^{+} channel extracted using a 12×12 correlation matrix, *excluding* the scalar glueball operator on the left, and using a 13×13 correlation matrix *including* the scalar glueball operator on the right. 1σ uncertainties are denoted by the box heights. If a level is created predominantly by a single operator, the level is colored to indicate the flavor content of that operator. If a level is created predominantly by more than one operator, a hatched box is used to denote the presence of operator overlaps within 75% of the maximum, indicating significant mixing. Level numbers indicate order in terms of increasing mean energy, but the levels have been rearranged horizontally to highlight the area of interest involving the glueball operator. Note that these finite-volume energies should *not* be directly compared to the spectrum of experimental resonance states, in particular the two-hadron dominated levels. See text for further discussion. Short black lines indicate the non-interacting two-hadron levels, and the dashed horizontal black line indicates the 4π threshold, and $m_{\text{ref}} = 2m_K$.

operator. The same can be said for levels 6 and 8 (as numbered on the right-hand side of fig. 4), identifiable as being created dominantly by the $\eta(0)\eta(0)$ and $\eta(1)\eta(1)$ type operators, respectively. Note however the significant shift of these levels from the corresponding non-interacting values, especially when compared to the shifts seen in the lower lying levels.

The remaining states, where the effect of the glueball operator is seen the most, are highlighted by the vertical dashed lines. Figure 5(a) shows that the glueball operator mainly creates levels 0, 7, and 12. Remarkably, a new state is *not* created near 1.5-1.7m_{ref}. When the glueball operator is included, there are two effects: the uncertainty in level 7 is greatly increased and *an additional state appears at a very high energy*. Based on both fig. 5(d) and the overlap factors when the glueball operator is excluded, we can identify level 7 as being dominantly created by $\pi(2)\pi(2)$. When the glueball operator is included, it has significant overlap with this state. More notable is that the additional state we extract with the enlarged operator basis lies above all other extracted states in fig. 4. This indicates that we have saturated the spectrum of single and two-particle states in this region without a glueball operator. Hence, we identify no finite-volume energy eigenstate predominantly created by a scalar glueball operator below $\sim 1.9m_{\text{ref}}$. As this new energy occurs above the region where our operator set is designed to create states, this level appears most likely just as a consequence of the enlarged operator basis. We cannot conclude that a pure glueball state has been created.

While these finite-volume results are insufficient to make any definitive statements regarding the infinite-volume resonances in this channel, we can make some qualitative comparisons to experiment. In finding only two $\bar{q}q$ dominated states below $2m_{\text{ref}}$, we have observed no clearly identifiable counterpart finite-volume $\bar{q}q$ states to the $f_0(1370)$, $f_0(1500)$, or $f_0(1710)$ resonances in this region. This *suggests* that these resonances are molecular in nature rather than conventional $\bar{q}q$ or pure glueball states.

CONCLUSIONS

We have presented here the first study of the low-lying spectrum in the scalar sector of QCD with vacuum quantum numbers to include the mixing between $\bar{q}q$, two-hadron, and glueball operators in fully dynamical lattice QCD. When a scalar glueball operator is included in the operator basis, we observe an additional state lying near $2m_{\text{ref}}$, the upper

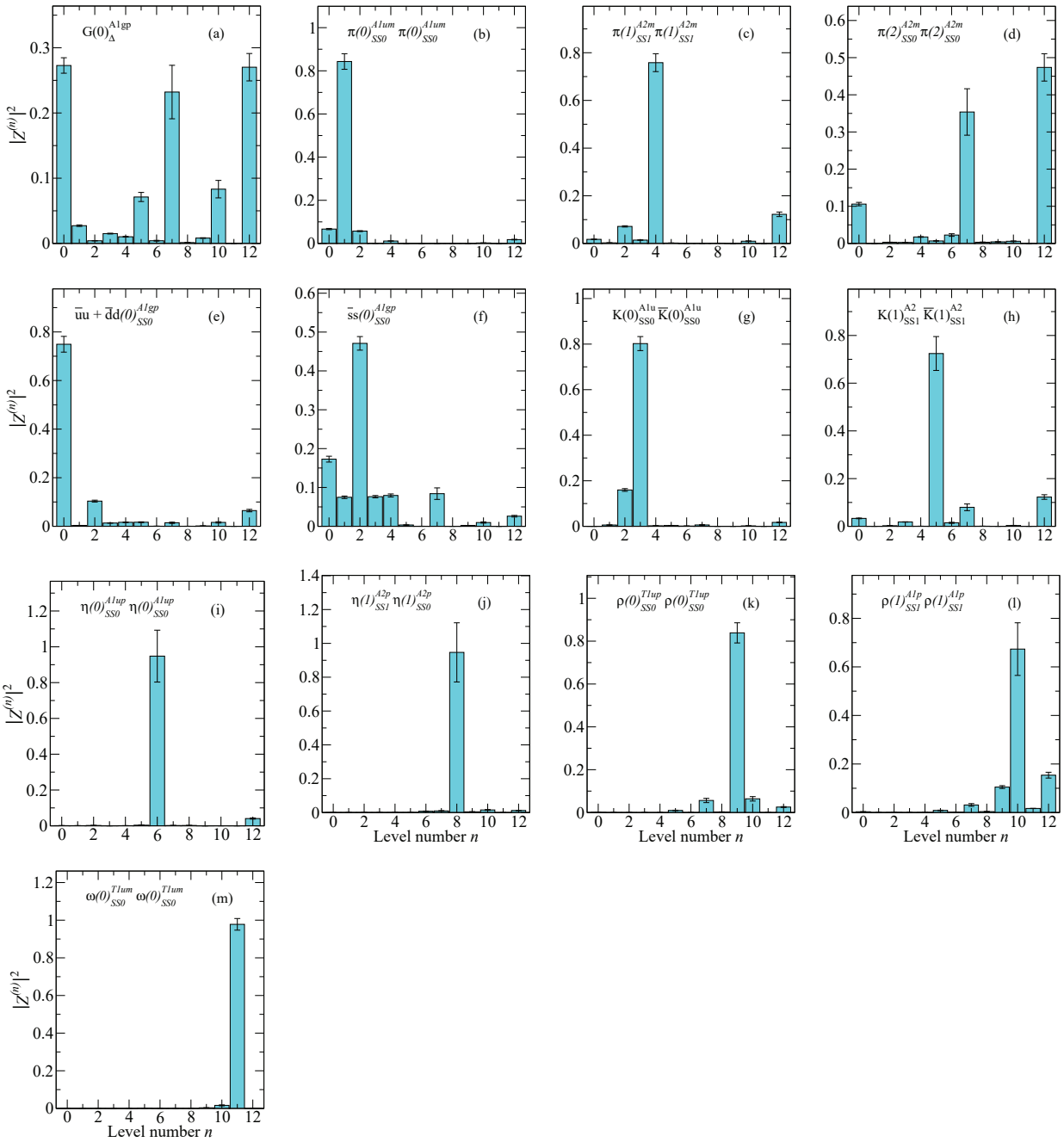

FIGURE 5. Overlaps $|Z^{(n)}|^2$ for the operators used in the A_{1g}^+ correlation matrix *including* the scalar glueball operator.

limit we can study here. As the extracted spectrum, up to an increase in statistical noise, is insensitive to the inclusion of the glueball operator, we have concluded that the energies presented saturate the spectrum of single and two-particle states in this region. Considering the leading experimental glueball candidates in this region, the $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$, we found no quark-antiquark dominated levels identifiable as finite-volume counterpart to these states, suggesting that these states are likely to be molecular in nature.

Our results reinforce, via the significant coupling of the glueball operator to the $\pi(2)\pi(2)$ and σ finite-volume states, the need for extensive operator bases in a proper determination of the excited state spectrum in this sector of QCD. To date, previous studies of glueballs in lattice QCD have included only glueball interpolating operators. Form-

ing definite infinite-volume conclusions about these states will also require the determination of coupled-channel scattering amplitudes above the 4π threshold. Hence, 3- and 4-hadron interpolating operators, along with a formalism for extracting infinite-volume scattering information from finite-volume energies will be required. Work in this direction is underway, with the spectrum of three-pion states with maximal isospin reported recently in Ref. [20], and a review of the current state of amplitude extraction above the three particle threshold in Ref. [21].

ACKNOWLEDGMENTS

This work was supported by the U.S. National Science Foundation under award PHY-1613449, and the John Peoples Jr. Research Fellowship at CMU. Computing resources were provided by the Extreme Science and Engineering Discovery Environment (XSEDE) under grant number TG-MCA07S017.

REFERENCES

1. V. Crede and C. A. Meyer, "The Experimental Status of Glueballs," Prog. Part. Nucl. Phys. **63**, 74–116 (2009), arXiv:0812.0600 [hep-ex].
2. G. S. Bali, K. Schilling, A. Hulsebos, A. C. Irving, C. Michael, and P. W. Stephenson (UKQCD), "A Comprehensive lattice study of SU(3) glueballs," Phys. Lett. **B309**, 378–384 (1993), arXiv:hep-lat/9304012 [hep-lat].
3. C. J. Morningstar and M. J. Peardon, "The Glueball spectrum from an anisotropic lattice study," Phys. Rev. **D60**, 034509 (1999), arXiv:hep-lat/9901004 [hep-lat].
4. Y. Chen *et al.*, "Glueball spectrum and matrix elements on anisotropic lattices," Phys. Rev. **D73**, 014516 (2006), arXiv:hep-lat/0510074 [hep-lat].
5. G. S. Bali, B. Bolder, N. Eicker, T. Lippert, B. Orth, P. Ueberholz, K. Schilling, and T. Struckmann (TXL, T(X)L), "Static potentials and glueball masses from QCD simulations with Wilson sea quarks," Phys. Rev. **D62**, 054503 (2000), arXiv:hep-lat/0003012 [hep-lat].
6. A. Hart and M. Teper, "On the glueball spectrum in O(a) improved lattice QCD," Phys. Rev. **D65**, 034502 (2002), arXiv:hep-lat/0108022 [hep-lat].
7. A. Hart, C. McNeile, C. Michael, and J. Pickavance, "A Lattice study of the masses of singlet 0^{++} mesons," Phys. Rev. **D74**, 114504 (2006), arXiv:hep-lat/0608026 [hep-lat].
8. C. M. Richards, A. C. Irving, E. B. Gregory, and C. McNeile, "Glueball mass measurements from improved staggered fermion simulations," Phys. Rev. **D82**, 034501 (2010), arXiv:1005.2473 [hep-lat].
9. E. Gregory, A. Irving, B. Lucini, C. McNeile, A. Rago, C. Richards, and E. Rinaldi, "Towards the glueball spectrum from unquenched lattice QCD," JHEP **10**, 170 (2012), arXiv:1208.1858 [hep-lat].
10. W. Sun, L.-C. Gui, Y. Chen, M. Gong, C. Liu, Y.-B. Liu, Z. Liu, J.-P. Ma, and J.-B. Zhang, "Glueball spectrum from $N_f = 2$ lattice QCD study on anisotropic lattices," Chin. Phys. **C42**, 093103 (2018), arXiv:1702.08174 [hep-lat].
11. R. Brett, *The Scalar Glueball and $K\pi$ Scattering from Lattice QCD*, Ph.D. thesis.
12. C. J. Morningstar and M. J. Peardon, "Efficient glueball simulations on anisotropic lattices," Phys. Rev. **D56**, 4043–4061 (1997), arXiv:hep-lat/9704011 [hep-lat].
13. R. G. Edwards, B. Joo, and H.-W. Lin, "Tuning for Three-flavors of Anisotropic Clover Fermions with Stout-link Smearing," Phys. Rev. **D78**, 054501 (2008), arXiv:0803.3960 [hep-lat].
14. H.-W. Lin *et al.* (Hadron Spectrum), "First results from 2+1 dynamical quark flavors on an anisotropic lattice: Light-hadron spectroscopy and setting the strange-quark mass," Phys. Rev. **D79**, 034502 (2009), arXiv:0810.3588 [hep-lat].
15. C. Morningstar, J. Bulava, J. Foley, K. J. Juge, D. Lenkner, M. Peardon, and C. H. Wong, "Improved stochastic estimation of quark propagation with Laplacian Heaviside smearing in lattice QCD," Phys. Rev. **D83**, 114505 (2011), arXiv:1104.3870 [hep-lat].
16. M. Luscher and U. Wolff, "How to Calculate the Elastic Scattering Matrix in Two-dimensional Quantum Field Theories by Numerical Simulation," Nucl. Phys. **B339**, 222–252 (1990).
17. B. Blossier, M. Della Morte, G. von Hippel, T. Mendes, and R. Sommer, "On the generalized eigenvalue method for energies and matrix elements in lattice field theory," JHEP **04**, 094 (2009), arXiv:0902.1265 [hep-lat].
18. C. Morningstar, J. Bulava, B. Fahy, J. Foley, Y. C. Jhang, K. J. Juge, D. Lenkner, and C. H. Wong, "Extended hadron and two-hadron operators of definite momentum for spectrum calculations in lattice QCD," Phys. Rev. **D88**, 014511 (2013), arXiv:1303.6816 [hep-lat].
19. R. A. Briceno, J. J. Dudek, R. G. Edwards, and D. J. Wilson, "Isoscalar $\pi\pi$ scattering and the σ meson resonance from QCD," Phys. Rev. Lett. **118**, 022002 (2017), arXiv:1607.05900 [hep-ph].
20. B. Hörz and A. Hanlon, "Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD," (2019), arXiv:1905.04277 [hep-lat].
21. M. T. Hansen and S. R. Sharpe, "Lattice QCD and Three-particle Decays of Resonances," (2019), 10.1146/annurev-nucl-101918-023723, arXiv:1901.00483 [hep-lat].