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Nonlinear Differential Equations and Applications NoDEA

Exponential decay of quasilinear Maxwell equations with interior conductivity

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Abstract. We consider a quasilinear nonhomogeneous, anisotropic Maxwell system in a bounded smooth domain of $\mathbb{R}^3$ with a strictly positive conductivity subject to the boundary conditions of a perfect conductor. Under appropriate regularity conditions, adopting a classical L^2-Sobolev solution framework, a nonlinear energy barrier estimate is established for local-in-time H^3-solutions to the Maxwell system by a proper combination of higher-order energy and observability-type estimates under a smallness assumption on the initial data. Technical complications due to quasilinearity, anisotropy, the lack of solenoidality, and the fact that only partial dissipation is imposed on the system. Finally, provided the initial data are small, the barrier method is applied to prove that local solutions exist globally and exhibit an exponential decay rate.

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1. Introduction

The Maxwell equations are one of the fundamental equations of mathematical physics describing electro-magnetic theory. In this work we establish global existence and exponential decay for the quasilinear Maxwell system with strictly positive conductivity and small initial fields. The evolutionary part of the Maxwell system

$$\partial_t D = \operatorname{curl} H - J \quad \text{and} \quad \partial_t B = -\operatorname{curl} E$$

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connects the electric fields E and D, the magnetic fields B and H, and the current J via Ampère's circuital law and Faraday's law of induction.

In our analysis, we take (E, H) as the state variables and postulate the *instantaneous nonlinear material laws*

$$D = \varepsilon(x, E)E \quad \text{and} \quad B = \mu(x, H)H$$

with nonlinear, nonhomogeneous, anisotropic tensor-valued permittivity ε and permeability μ. We further employ linear Ohm's law

$$J = \sigma(x)E$$

with a nonhomogeneous, anisotropic, strictly positive conductivity tensor σ. Imposing the boundary conditions of a perfect conductor, we arrive at the *quasilinear* Maxwell system

$$\begin{aligned} \partial_t\big(\varepsilon(x, E(t,x))E(t,x)\big) &= \operatorname{curl} H(t,x) - \sigma(x)E(t,x), && t \geq 0,\ x \in \Omega, \\ \partial_t\big(\mu(x, H(t,x))H(t,x)\big) &= -\operatorname{curl} E(t,x), && t \geq 0,\ x \in \Omega, \\ E(t,x) \times \nu(x) = 0, \quad \nu(x) \cdot \mu\big(x, H(t,x)\big)&H(t,x) = 0, && t \geq 0,\ x \in \Gamma, \\ E(0,x) = E_0(x), \qquad H(0,x) &= H_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

defined on a simply connected, bounded domain $\Omega \subseteq \mathbb{R}^3$, with a smooth boundary Γ in C^5 and outer unit normal ν. The initial fields (E_0, H_0) satisfy the compatibility conditions given below in (2.4). In (2.1) and (2.2) we state our assumptions on ε, μ and σ under which (1.1) becomes a symmetric quasilinear hyperbolic system. We note that the Gaussian laws (3.4) for the charges and the magnetic boundary condition in (1.1) follow from (2.4) and the other equations in (1.1).

In this work we are interested in global-in-time solvability of Maxwell's system along with a quantitified long time behavior. More specifically, accounting for a dissipative effect of conductivity we aim at proving that the solutions to Maxwell's system exist globally and exhibit exponentially fast decay rates when time t goes to infinity, provided that the initial data are small enough. This kind of result, of independent interest on its own, is also of critical importance in control/stabilization theory of electromagnetic materials and plasma physics [3].

Challenges and known results As well known, global solvability of quasilinear systems is intimately connected with the following two properties: (i) *construction* of the solutions at a sufficiently high regularity level, (ii) *exponential decay* of the energy established first for the linearization and then propagated to higher derivatives.

Regarding point (i): This process consists of two steps—a priori estimates for the solutions *assumed* to exist and the actual construction of the solutions exhibiting the needed regularity. The latter construction is particularly challenging in hyperbolic dynamics when there is a loss of derivative in the apriori estimates. This requires the construction of a good regularity theory for the fully non-autonomous problem with variable coefficients in the principal part.

In the case of the Maxwell systems under consideration, in the full space case $\Omega = \mathbb{R}^3$, one has a well-developed local well-posedness theory of $\mathcal{H}^3$–valued solutions due to Kato [13]. However, more specific approaches are needed in the case of bounded domains and a characteristic boundary, which could lead to a loss of regularity in the normal direction. The available general existence results work in Sobolev spaces of very high order and with weights vanishing at the boundary, see [12,27] as well as [25] for tangential regularity. For absorbing boundary conditions, generating additional regularity on the boundary, local existence results in $\mathcal{H}^3$ were given in [23] (without uniqueness). However, in our case, the space and time variable permitivity and permeability along with the presence of a conductivity term spoils a nice div-curl framework which could used to recover the "lost derivatives". In fact, this is evident from the constructions based on subtle approximations leading to the local well-posedness theory in $\mathcal{H}^3$ which was established only recently in the papers [30,31]. We will strongly rely on these results in our construction of solutions with sufficiently high regularity.

Regarding point (ii): While decay rates for the Maxwell system have been studied in a number of works (viz. [1,8–10,14,15,21,22] and references therein), the cited studies only allow for linear permittivity and permeability and partly deal with constant isotropic coefficients. Stabilization results for general hyperbolic systems typically concern damping mechanism acting on all components of the solutions, see [26], whereas in our Maxwell system the dissipation via conductivity *only* affects the electric field. For $\Omega = \mathbb{R}^m$ the paper [2] allows for partial damping even in the quasilinear case, but its assumptions exclude the Maxwell equations. For the quasilinear Maxwell system we are only aware of a few results on the full space $\Omega = \mathbb{R}^3$ that establish global existence and decay for small and smooth solutions, see [17,24,28]. These works rely on dispersive estimates which are not available on bounded domains. On the other hand, it is known that blow-up in $W^{1,\infty}$ or $\mathcal{H}(\text{curl})$ can occur in various cases, see [5] and the references therein. To deal with the issue, we shall adapt a technique presented in recent work [8] which establishes decay rates for a *linear* anisotropic system with a strictly positive conductivity *only* assuming that the 3D domain is simply connected. The latter is also a necessary condition, as shown in [8].

Summing up: Space and particularly time dependence of permitivity and permeability along with a presence of conductivity spoil nice divergence relations on the electric field which could be used to restore the "lost derivatives". This point and the only *partial* dissipation imposed on the system require a particular subtle treatment for both the construction of regular solutions and the observability estimates leading to the globality and decay rates obtained for the solution. This explains why up to date, no global results on *quasilinear* Maxwell systems *with boundary conditions* and *only partial* dissipation are known in the literature.

2. Problem setting and main result

2.1. The main result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $\Gamma := \partial\Omega$ of class C^5 and the outer unit normal vector ν. Our basic assumptions are

$$\begin{aligned} &\varepsilon, \mu \in C^3(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}^{3\times 3}_{\rm sym}), \quad \sigma \in C^3(\overline{\Omega}, \mathbb{R}^{3\times 3}_{\rm sym}) \quad \text{ and} \\ &\varepsilon(x,0) \geq 2\eta I, \quad \mu(x,0) \geq 2\eta I, \quad \sigma(x) \geq \eta I \qquad \text{for all } x \in \overline{\Omega} \end{aligned} \tag{2.1}$$

and for some constant $\eta > 0$, where $C^3(\overline{\Omega})$ is the space of C^3–functions v such that v and its derivatives up to the third order possess a continuous extension to Γ. We introduce the matrix-valued functions $\varepsilon^{\rm d}$ and $\mu^{\rm d}$ given by

$$\begin{aligned} \varepsilon^{\rm d}_{jk}(x,\xi) &= \varepsilon_{jk}(x,\xi) + \sum_{l=1}^{3} \partial_{\xi_k} \varepsilon_{jl}(x,\xi)\,\xi_l, \\ \mu^{\rm d}_{jk}(x,\xi) &= \mu_{jk}(x,\xi) + \sum_{l=1}^{3} \partial_{\xi_k} \mu_{jl}(x,\xi)\,\xi_l \end{aligned}$$

for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^3$ and $j,k \in \{1,2,3\}$, which arise when differentiating the left-hand side of (1.1). We further assume that

$$\partial_\xi \varepsilon, \partial_\xi \mu \in C^3(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}^{3\times 3}), \quad \varepsilon^{\rm d} = (\varepsilon^{\rm d})^\top, \quad \text{ and } \quad \mu^{\rm d} = (\mu^{\rm d})^\top. \tag{2.2}$$

Example 2.1. Let $\varepsilon_{\rm lin} \in C^3(\overline{\Omega}, \mathbb{R}^{3\times 3}_{\rm sym})$ satisfy $\varepsilon_{\rm lin} \geq 2\eta I$. We specify two nonlinear terms in the sum $\varepsilon(x,E) = \varepsilon_{\rm lin}(x) + \varepsilon_{\rm nl}(x,E)$ so that the conditions (2.1) and (2.2) are valid. One can take Kerr-type isotropic nonlinearities $\varepsilon_{\rm nl}(x,E) = a(x)\varphi(|E|^2)I$ for scalar functions $a \in C^3(\overline{\Omega})$ and $\varphi \in C^4([0,\infty))$ with $\varphi(0) = 0$. A typical anistropic example is furnished by

$$\varepsilon_{\rm nl}(x,E) = \Big(\sum\nolimits_{j,k=1}^{3} \chi_i^{jkl}(x) E_j E_k\Big)_{il}$$

for scalar coefficients $\chi_i^{jkl} \in C^3(\overline{\Omega})$, cf. [19]. Because of the triple sum in $\varepsilon_{\rm nl}(x,E)E$, the tensor $(\chi_i^{jkl})_{i,j,k,l}$ has to be symmetric in $\{j,k,l\}$. Our assumptions also require symmetry in $\{i,l\}$, i.e., we can only prescribe χ_i^{jkl} for, say, $1 \leq i \leq j \leq l \leq 3$.

We write $\operatorname{tr}_n u$ for the trace of the normal component $u\cdot\nu$ on Γ, while $\operatorname{tr}_t u$ stands for the tangential trace $u \times \nu$ on Γ. We also use its rotated counterpart $\operatorname{tr}_\tau u = \nu \times (\operatorname{tr}_t u)$, which is the tangential component $\operatorname{tr} u - (\operatorname{tr}_n u)\nu$ of the full trace $\operatorname{tr} u$. Let $E_0, H_0 \in \mathcal{H}^3(\Omega)^3$. To express the compatibility conditions, we set

$$\begin{aligned} E_0^1 &= \varepsilon^{\rm d}(E_0)^{-1}[\operatorname{curl} H_0 - \sigma E_0], \qquad H_0^1 = -\mu^{\rm d}(H_0)^{-1} \operatorname{curl} E_0, \\ E_0^2 &= \varepsilon^{\rm d}(E_0)^{-1}\big[\operatorname{curl} H_0^1 - \sigma E_0^1 - (\nabla_E \varepsilon^{\rm d}(E_0) E_0^1) \cdot E_0^1\big], \\ H_0^2 &= -\mu^{\rm d}(H_0)^{-1}\big[\operatorname{curl} E_0^1 + (\nabla_H \mu^{\rm d}(H_0) H_0^1) \cdot H_0^1\big], \end{aligned} \tag{2.3}$$

where we put $((\nabla A)\xi \cdot \eta)_j = \big(\sum_{i,k} \partial_i A_{jk}\xi_k\eta_i\big)_j$. The initial fields $E_0, H_0 \in \mathcal{H}^3(\Omega)^3$ shall satisfy the divergence and boundary conditions

$$\operatorname{div}(\mu(H_0)H_0) = 0, \quad \operatorname{tr}_n(\mu(H_0)H_0) = 0, \qquad \operatorname{tr}_t E_0 = \operatorname{tr}_t E_0^1 = \operatorname{tr}_t E_0^2 = 0 \tag{2.4}$$

as well as the smallness assumption

$$\|E_0\|^2_{\mathcal{H}^3(\Omega)} + \|H_0\|^2_{\mathcal{H}^3(\Omega)} \le r^2. \tag{2.5}$$

The main goal of this paper is to establish the global existence of solutions in (1.1), assuming that the initial data are small enough. This smallness assumption is needed to ensure the positivity of the coefficients tensors and in several perturbation arguments. It is well known that global existence for quasilinear systems is closely related to the exponential decay of the resulting dynamics. The big bulk of the paper is thus devoted to the proof of this latter property. Our main result reads as follows.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with $\partial\Omega \in C^5$, the coefficients satisfy (2.1) and (2.2), and the initial data $E_0, H_0 \in \mathcal{H}^3(\Omega)^3$ fulfill (2.4) and (2.5). Then there exists a radius $r > 0$ in assumption (2.5) and constants $M, \omega > 0$ such that the solution (E, H) of the Maxwell system (1.1) exists for all $t \ge 0$ and is bounded by*

$$\max_{j\in\mathbb{N}_0, 0\le j\le 3} \big(\|\partial_t^j E(t)\|^2_{\mathcal{H}^{3-j}(\Omega)} + \|\partial_t^j H(t)\|^2_{\mathcal{H}^{3-j}(\Omega)}\big) \le M e^{-\omega t}\|(E_0, H_0)\|^2_{\mathcal{H}^3(\Omega)} \quad \textit{for all } t \ge 0.$$

The proof of the theorem is given at the end of Sect. 5.

2.2. Comments about the proof

We first have to make sure that our solutions of the quasilinear problem stay in a ball of prescribed size. In the next section, based on [31] we indeed find a radius $r(\delta) > 0$ for the initial data in (2.5) so that solution is bounded by δ in $\mathcal{H}^3$ up to a certain time T_*.

The next fundamental step are the dissipation and observability-type estimates in Proposition 4.1 and 4.4 for the fields (E, H) and their time derivatives. These estimates contain correction terms arising from the quasilinear nature of the problem. The dissipation bound relies on the energy identity Lemma 4.2 for a related linear problem with limited space-time regularity of the coefficients, which follows from the theory developed in [8] and [30]. The limited regularity forces us to perform a delicate approximation procedure. In the *autonomous linear* case, only recently an obervability estimate was shown in [8] for tensor-valued coefficients (see also [22] for constant scalar coefficients). We use ideas from [8], but we have to deal with extra terms because of the quasilinear setting. The proof of Proposition 4.4 is based on Helmholtz-type decompositions of $(E(t), H(t))$ and their time derivatives. We construct the decompositions in Lemma 4.3 where we have to cope with the anisotropy of the coefficients and, in particular, with the presence of nontrivial electrical charges. The latter problem can be tackled since there are no magnetic charges in our system, see (3.4).

The correction terms in the above inequalities are small up to T_* for our small data, but they are small in much stronger norms than the quantities controlled by dissipation. Therefore, to deal with the quasilinear situation one has to establish an additional, rather deep regularity result with constants independent of the time interval. It is provided by our Proposition 7.1, where we bound the spatial derivatives up to order $3-k$ of the fields $\partial_t^k(E,H)$ by the L^2 norms of the time derivatives alone, where $k \in \{0,1,2\}$. With this result at hand, one can then easily show Theorem 2.2 by adopting the widely known "barrier method" from the linear case and using smallness, see Sect. 5. In a different situation, a similar procedure was used in [18, 20] and recently in [16] for quasilinear thermoelastic plate equations.

The lengthy proof of Proposition 7.1 is relegated to the last section. Its very first step still is rather simple. Using the Maxwell system (1.1) one can estimate the L^2 norm of $\operatorname{curl} H(t)$ by that of $\partial_t E(t)$. (We recall that we work on a time interval where we can control, e.g., the $\mathcal{H}^3$ norm of the fields uniformly.) We further have the magnetic boundary condition in (1.1) and the divergence relation $\operatorname{div}(\mu(H)H) = 0$ from (3.4). A variant of well-known "elliptic" curl-div estimates then yields the bound $\|H(t)\|_{\mathcal{H}^1} \leq c\,\|\partial_t E(t)\|_{L^2}$. This procedure also works for time and tangential space derivatives of H, but not for normal ones since they destroy the boundary conditions. For the electric field this approach entirely fails because the divergence relation (3.4) for E is spoiled by the conductivity term. For E and its tangential derivatives we have to resort to the (weaker) energy estimate taking advantage of the better properties of H. The normal derivatives of both fields are then treated by a "curl-div-strategy:" Using formulas derived in Sect. 6 one can solve in the Maxwell system for the normal derivative of the tangential components of the fields, and in the divergence relations for the normal derivative of the normal components. In the latter case the anisotropy of the material laws becomes a major problem. In the proof of Proposition 7.1 one has to apply these ideas on differentiated modifications of the Maxwell system and perform an intricate iteration over the regularity levels.

We briefly outline the rest of the paper. In the next section, our functional-analytic setting along with basic notations is introduced. In Sect. 4, we establish energy and observability-type inequalities for local solutions to system (1.1). Subsequently, in Sect. 5, these estimates are improved to incorporate higher-order spatial derivatives which then allows us to show the main result. Sect. 6 presents elliptic-type curl-div-estimates and introduces our curl-div-strategy, which are subsequently adopted in Sect. 7 to prove our core regularity result, i.e., Proposition 7.1.

3. Preparation for the Proof

We recall that $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary $\Gamma = \partial\Omega$ of class C^5 and outer unit normal vector ν. For $T > 0$, we set $J = J_T = [0, T]$, $\Omega_T = (0, T) \times \Omega$ and $\Gamma_T = (0, T) \times \Gamma$. For the sake of brevity, the same notation will often be used for spaces of scalar and vector-valued functions. Also, we sometimes write $\mathcal{H}^k$ instead of the Sobolev space $\mathcal{H}^k(\Omega)$, etc., if the domain of integration is clear from the context. Spaces on Γ are always equipped with the surface measure denoted by $\mathrm{d}x$.

As stated above, the trace of the normal component $u \cdot \nu$ on Γ is denoted by $\operatorname{tr}_n u$, while $\operatorname{tr}_t u$ stands for the tangential trace $u \times \nu$ on Γ. It is well known that the mappings

$$\operatorname{tr}_n : \mathcal{H}(\operatorname{div}) \to \mathcal{H}^{-1/2}(\Gamma) \quad \text{and} \quad \operatorname{tr}_t : \mathcal{H}(\operatorname{curl}) \to \mathcal{H}^{-1/2}(\Gamma)^3$$

are continuous, where the Hilbert spaces

$$\mathcal{H}(\operatorname{curl}) = \{u \in L^2(\Omega)^3 \mid \operatorname{curl} u \in L^2(\Omega)^3\} \quad \text{and}$$
$$\mathcal{H}(\operatorname{div}) = \{u \in L^2(\Omega)^3 \mid \operatorname{div} u \in L^2(\Omega)\}$$

are endowed with their natural norms. See Theorems IX.1.1 and IX.1.2 in [6].

Let the assumptions (2.1) and (2.2) be satisfied. By (2.1) and continuity, there exists a radius $\tilde{\delta} \in (0, 1]$ such that

$$\varepsilon(x, \xi), \mu(x, \xi), \varepsilon^{\mathrm{d}}(x, \xi), \mu^{\mathrm{d}}(x, \xi), \sigma(x) \geq \eta I \tag{3.1}$$

for all $\xi \in \mathbb{R}^3$ with $|\xi| \leq \tilde{\delta}$ and $x \in \overline{\Omega}$. Writing $C_S > 0$ for the norm of the Sobolev embedding $\mathcal{H}^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we put $\delta_0 = \min\{1, \tilde{\delta}/C_S\}$.

Take $T > 0$ and $\delta \in (0, \delta_0]$. The (small) parameter $\delta > 0$ will be fixed in subsequent proofs. The local well-posedness result Theorem 5.3 in [31] provides a radius $r(T, \delta) \in (0, r(T, \delta_0)]$ such that, for all $r \in (0, r(T, \delta)]$ and initial data $E_0, H_0 \in \mathcal{H}^3(\Omega)^3$ fulfilling the compatibility conditions (2.4) and the smallness assumption (2.5), there exists a maximal existence time $T_{\max} \in (T, \infty]$ and a unique solution

$$(E, H) \in \bigcap_{k=0}^{3} C^k\big([0, T_{\max}), \mathcal{H}^{3-k}(\Omega)\big)^6 =: G^3 \tag{3.2}$$

to the quasilinear Maxwell system (1.1). The fields (E, H) further satisfy the estimate

$$\max_{k \in \{0,1,2,3\}} \max_{t \in [0,T]} \big(\|\partial_t^k E(t)\|^2_{\mathcal{H}^{3-k}(\Omega)} + \|\partial_t^k H(t)\|^2_{\mathcal{H}^{3-k}(\Omega)}\big) \leq \delta^2 \leq 1 \tag{3.3}$$

and the divergence equations

$$\begin{aligned} &\operatorname{div}\big(\mu(H(t))H(t)\big) = 0, \\ &\operatorname{div}\big(\varepsilon(E(t))E(t)\big) = \operatorname{div}\big(\varepsilon(E_0)E_0\big) - \int_0^t \operatorname{div}\big(\sigma E(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

on Ω for all $t \in [0, T_{\max}) =: J_{\max}$. (We write $E(t)$ instead of $E(t, \cdot)$, etc.)

We note that Theorem 5.3 in [31] is not concerned with (3.4) and the second boundary condition in the system (1.1). These formulas follow from the other equations in (1.1) and the assumption (2.4) in a standard way, see Lemma 7.25 of [29].

Inequality (3.3) will frequently be invoked in this article, sometimes without being explicitly mentioned. In addition to rendering the solution small, it also provides a crucial uniform bound. Observe that along solutions to (1.1) fulfilling (3.3), the lower bound (3.1) is valid for $t \in [0, T]$.

We now fix $T = 1$ yielding the radius $r(\delta) := r(\delta, 1)$. Given initial fields (E_0, H_0) satisfying (2.4) and (2.5), we introduce the time

$$T_* = \sup\big\{T \in [1, T_{\max}) \,|\, (3.3) \text{ is valid for } t \in [0,T]\big\}. \tag{3.5}$$

The bound (3.3) is thus true on $[0, T_*) =: J_*$. If $T_* < \infty$, then the blow-up condition in Theorem 5.3 of [31] implies that $T_{\max} > T_*$ and hence

$$z(T_*) := \max_{k\in\{0,1,2,3\}} \big(\|\partial_t^k E(T_*)\|^2_{\mathcal{H}^{3-k}(\Omega)} + \|\partial_t^k H(T_*)\|^2_{\mathcal{H}^{3-k}(\Omega)}\big) = \delta^2 \quad (\text{if } T_* < \infty) \tag{3.6}$$

by continuity.

We work with time-differentiated versions of (1.1). For the sake of brevity, we set

$$\widehat{\varepsilon}_k = \begin{cases} \varepsilon(E), & k = 0, \\ \varepsilon^{\mathrm{d}}(E), & k \in \{1,2,3\}, \end{cases} \qquad \widehat{\mu}_k = \begin{cases} \mu(H), & k = 0, \\ \mu^{\mathrm{d}}(H), & k \in \{1,2,3\}. \end{cases} \tag{3.7}$$

For $k \in \{0, 1, 2, 3\}$, we then obtain the system

$$\begin{aligned} \partial_t(\widehat{\varepsilon}_k \partial_t^k E) &= \operatorname{curl} \partial_t^k H - \sigma \partial_t^k E - \partial_t f_k, && t \in J_{\max},\ x \in \Omega, \\ \partial_t(\widehat{\mu}_k \partial_t^k H) &= -\operatorname{curl} \partial_t^k E - \partial_t g_k, && t \in J_{\max},\ x \in \Omega, \\ \operatorname{tr}_t \partial_t^k E = 0, &\quad \operatorname{tr}_n(\widehat{\mu}_k \partial_t^k H) = -\operatorname{tr}_n g_k, && t \in J_{\max},\ x \in \Gamma, \end{aligned} \tag{3.8}$$

with the commutator terms

$$\begin{aligned} &f_0 = f_1 = 0, \quad f_2 = \partial_t \varepsilon^{\mathrm{d}}(E)\, \partial_t E, \quad f_3 = \partial_t^2 \varepsilon^{\mathrm{d}}(E)\, \partial_t E + 2\partial_t \varepsilon^{\mathrm{d}}(E)\, \partial_t^2 E, \\ &g_0 = g_1 = 0, \quad g_2 = \partial_t \mu^{\mathrm{d}}(H)\, \partial_t H, \quad g_3 = \partial_t^2 \mu^{\mathrm{d}}(H)\, \partial_t H + 2\partial_t \mu^{\mathrm{d}}(H)\, \partial_t^2 H. \end{aligned} \tag{3.9}$$

Equation (3.4) further yields the divergence relations

$$\operatorname{div}(\mu^{\mathrm{d}}(H) \partial_t^k H) = -\operatorname{div} g_k, \qquad \operatorname{div}(\varepsilon^{\mathrm{d}}(E) \partial_t^k E) = -\operatorname{div}(\sigma \partial_t^{k-1} E + f_k) \tag{3.10}$$

for $k \in \{1, 2, 3\}$. Estimate (3.14) below shows that all functions $\partial_t f_k$, $\partial_t g_k$, $\operatorname{div} f_k$, and $\operatorname{div} g_k$ belong to $L^\infty(J_*, L^2(\Omega))$. For $k = 3$, the evolution equations in (3.8) are interpreted in $\mathcal{H}^{-1}(\Omega_T)$ while the divergence operator in (3.10) is understood in $\mathcal{H}^{-1}(\Omega)$. Since the inhomogenities belong to L^2, the traces in (3.8) exist in $\mathcal{H}^{-1/2}(\Gamma)$, cf. Sect. 2.1 of [29].

For the energy estimate, it is useful to consider an equivalent version of (3.8), viz.

$$\begin{aligned} \varepsilon^{\mathrm{d}}(E)\, \partial_t \partial_t^k E &= \operatorname{curl} \partial_t^k H - \sigma \partial_t^k E - \tilde{f}_k, && t \in J_{\max},\ x \in \Omega, \\ \mu^{\mathrm{d}}(H)\, \partial_t \partial_t^k H &= -\operatorname{curl} \partial_t^k E - \tilde{g}_k, && t \in J_{\max},\ x \in \Omega, \end{aligned}$$

$$\operatorname{tr}_t \partial_t^k E = 0, \qquad t \in J_{\max},\ x \in \Gamma, \tag{3.11}$$

for $k \in \{0,1,2,3\}$ with the new commutator terms

$$\tilde{f}_k = \sum_{j=1}^{k} \binom{k}{j} \partial_t^j \varepsilon^{\mathrm{d}}(E)\, \partial_t^{k+1-j} E, \qquad \tilde{g}_k = \sum_{j=1}^{k} \binom{k}{j} \partial_t^j \mu^{\mathrm{d}}(H)\, \partial_t^{k+1-j} H, \tag{3.12}$$

where we put $\tilde{f}_0 = \tilde{g}_0 = 0$. We further introduce the quantities

$$\begin{aligned} e_k(t) &= \frac{1}{2} \max_{j\in\mathbb{N}_0, j\le k} \big(\|\widehat{\varepsilon}_k^{1/2} \partial_t^j E(t)\|_{L^2(\Omega)}^2 + \|\widehat{\mu}_k^{1/2} \partial_t^j H(t)\|_{L^2(\Omega)}^2\big), \qquad e = e_3, \\ d_k(t) &= \max_{j\in\mathbb{N}_0, j\le k} \|\sigma^{1/2} \partial_t^j E(t)\|_{L^2(\Omega)}^2, \qquad d = d_3, \\ z_k(t) &= \max_{j\in\mathbb{N}_0, j\le k} \big(\|\partial_t^j E(t)\|_{\mathcal{H}^{k-j}(\Omega)}^2 + \|\partial_t^j H(t)\|_{\mathcal{H}^{k-j}(\Omega)}^2\big), \qquad z = z_3, \end{aligned} \tag{3.13}$$

for $k \in \{0,1,2,3\}$ and $t \in J_{\max}$. The coefficients of the energies e_k are chosen in view of Lemma 4.3. Throughout the paper, c_k or c are positive constants that do not depend on $t \in [0, T_*)$, T_*, $\delta \in (0, \delta_0]$, $r \in (0, r(\delta_0)]$, and (E_0, H_0) satisfying the conditions (2.4) and (2.5).

Using standard methods (as in Sect. 2 of [31]) and the estimate (3.3), one can show that

$$\begin{aligned} \|\widehat{\varepsilon}_k(t)\|_\infty, \|\widehat{\mu}_k(t)\|_\infty, \|\widehat{\varepsilon}_k^{-1}(t)\|_\infty, \|\widehat{\mu}_k^{-1}(t)\|_\infty &\le c, \\ \|\partial^\alpha \widehat{\varepsilon}_j(t)\|_{L^2(\Omega)}, \|\partial^\alpha \widehat{\mu}_j(t)\|_{L^2(\Omega)} &\le c(z_k^{1/2}(t) + \delta_{\alpha_0=0}), \\ \max_{k\in\{2,3\}, j\in\{0,1\}} \big(\|\partial_t^j f_k(t)\|_{\mathcal{H}^{4-j-k}(\Omega)} + \|\partial_t^j g_k(t)\|_{\mathcal{H}^{4-j-k}(\Omega)}\big) &\le cz(t), \\ \|f_2(t)\|_{L^2(\Omega)}, \|g_2(t)\|_{L^2(\Omega)}, \|f_3(t)\|_{L^2(\Omega)}, \|g_3(t)\|_{L^2(\Omega)} &\le ce_2^{1/2}(t), \\ \|\tilde{f}_k(t)\|_{\mathcal{H}^{3-k}(\Omega)}, \|\tilde{g}_k(t)\|_{\mathcal{H}^{3-k}(\Omega)} &\le cz(t) \end{aligned} \tag{3.14}$$

for $j, k \in \{0,1,2,3\}$, $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k > 0$, and $t \in J_*$. The constants c do not depend on t, and we set $\partial_0 = \partial_t$, $\delta_{\alpha_0=0} = 1$ if $\alpha_0 = 0$, and $\delta_{\alpha_0=0} = 0$ if $\alpha_0 > 0$. The term $+c$ on the right-hand side of the second line in (3.14) arises if all derivatives in ∂^α are applied to the x–variable of ε or μ.

4. Energy and observability-type inequalities

We start with a basic higher-order energy estimate establishing an explicit dissipation in the system due to the electric conductivity.

Proposition 4.1. *We assume the conditions of Theorem 2.2 except for the simple connectedness of* Ω. *For* $0 \le s \le t < T_*$ *and* $k \in \{0,1,2,3\}$, *we obtain the inequality*

$$e_k(t) + \int_s^t d_k(\tau)\,\mathrm{d}\tau \le e_k(s) + c_1 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau, \tag{4.1}$$

where the constant c_1 *does not depend on* s *and* t.

We first give the short proof for the case $k=0$. Since our solutions (E,H) of (1.1) are regular, see (3.2), integration by parts and (1.1) easily yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\int_\Omega \big(\varepsilon(E(t))E(t)\cdot E(t)+\mu(H(t))H(t)\cdot H(t)\big)\,\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega \big(\partial_t(\varepsilon(E)E)\cdot E+\varepsilon(E)E\cdot(\varepsilon(E)^{-1}\partial_t(\varepsilon(E)E))\\
&\qquad+\varepsilon(E)E\cdot(\partial_t\varepsilon(E)^{-1}\,\varepsilon(E)E))+\partial_t(\mu(H)H)\cdot H\\
&\qquad+\mu(H)H\cdot(\mu(H)^{-1}\partial_t(\mu(H)H))+\mu(H)H\cdot(\partial_t\mu(H)^{-1}\,\mu(H)H))\big)\,\mathrm{d}x\\
&=\int_\Omega\big(\operatorname{curl}H\cdot E-\sigma E\cdot E-\operatorname{curl}E\cdot H-\tfrac12\partial_t\varepsilon(E)\,E\cdot E\\
&\qquad-\tfrac12\partial_t\mu(H)\,H\cdot H\big)\,\mathrm{d}x\\
&=-\int_\Omega\big(\sigma E\cdot E+\tfrac12\partial_t\varepsilon(E)\,E\cdot E+\tfrac12\partial_t\mu(H)\,H\cdot H\big)\,\mathrm{d}x.
\end{aligned}$$

We thus obtain the energy inequality

$$\begin{aligned}
e_0(t)+&\int_s^t d_0(\tau)\,\mathrm{d}\tau\\
&=e_0(t)-\frac12\int_{(s,t)\times\Omega}\big(\partial_t\varepsilon(E)\,E\cdot E+\partial_t\mu(H)\,H\cdot H\big)\,\mathrm{d}(\tau,x). \qquad (4.2)
\end{aligned}$$

Combined with estimate (3.14), we derive (4.1) for the case $k=0$.

For $k\in\{1,2,3\}$ in Proposition 4.1, we have different coefficients in the energy e_k defined in (3.13). In this case, (4.1) follows from Lemma 4.2 below, the system (3.11) and the estimates (3.14). This lemma provides an energy identity in a more general situation to be encountered later.

For some $T>0$, let the coefficients $a,b\in W^{1,\infty}(\Omega_T,\mathbb{R}^{3+3}_{\mathrm{sym}})$ satisfy $a,b\geq \eta I$. Take data $\varphi,\psi\in L^2(\Omega_T)^3$, $\chi\in L^2(J,\mathcal{H}^{1/2}(\Gamma))^3$ with $\nu\cdot\chi=0$, and $u_0,v_0\in L^2(\Omega)^3$. Theorem 1.4 of [7] yields a solution $(u,v)\in C(J,L^2(\Omega))^6$ with $\operatorname{tr}_t v\in L^2(J,\mathcal{H}^{-1/2}(\Gamma))^3$ to the linear system

$$\begin{aligned}
a\partial_t u&=\operatorname{curl}v-\sigma u-\varphi, && t\in J,\ x\in\Omega,\\
b\partial_t v&=-\operatorname{curl}u-\psi, && t\in J,\ x\in\Omega,\\
\operatorname{tr}_t u&=\chi, && t\in J,\ x\in\Gamma.\\
u(0)&=u_0,\quad v(0)=v_0. && \qquad (4.3)
\end{aligned}$$

(As noted before (3.11), the tangential trace of u exists in $L^2(J,\mathcal{H}^{-1/2}(\Gamma))$.)

Lemma 4.2. *Under the assumptions above, for $0\leq s\leq t\leq T$ we have*

$$\begin{aligned}
\frac12\int_\Omega\big(a(t)u(t)\cdot u(t)+b(t)v(t)\cdot v(t)\big)\mathrm{d}x+\int_s^t\int_\Omega\sigma u\cdot u\,\mathrm{d}x\,\mathrm{d}\tau&\\
=\frac12\int_\Omega\big(a(0)u_0\cdot u_0+b(0)v_0\cdot v_0\big)\mathrm{d}x+\int_s^t\int_\Gamma\chi\cdot\operatorname{tr}_\tau v\,\mathrm{d}x\,\mathrm{d}\tau&\\
+\int_s^t\int_\Omega\big(\tfrac12\partial_t a\,u\cdot u+\tfrac12\partial_t b\,v\cdot v+\varphi\cdot u+\psi\cdot v\big)\mathrm{d}x\,\mathrm{d}\tau.&\qquad (4.4)
\end{aligned}$$

For $\mathcal{H}^1$–solutions (u, v), the claim easily follows from the system (4.3) and integration by parts, see step 3) below. We thus have to regularize the given data and coefficients to obtain $\mathcal{H}^1$–solutions for these regularized problems. Afterwards one passes to the limit in the resulting variant of (4.4). In view of the available a priori estimates and regularity results from [7,29] or [30], one has to approximate the data and the coefficients separately. The assertion is closely related to [7], but not stated there. Since the reasoning is somewhat involved, we give a (partly sketchy) proof.

(1) We approximate the initial data u_0 and v_0 and the forcing terms φ and ψ in L^2 by test functions $u_{0,n}$, $v_{0,n}$, φ_n and ψ_n, respectively. The boundary inhomogeneity χ is approximated in $L^2(J, \mathcal{H}^{1/2}(\Gamma))$ by mappings $\chi_n \in \mathcal{H}^1(J, \mathcal{H}^{3/2}(\Gamma))$ which vanish at $t = 0$. Moreover, we take coefficients $a_m, b_m \in C^3(J \times \overline{\Omega}, \mathbb{R}^{3\times3}_{\text{sym}})$ which are uniformly positive definite and uniformly bounded in $W^{1,\infty}(J \times \overline{\Omega}, \mathbb{R}^{3\times3}_{\text{sym}})$, that converge to a and b uniformly, and whose derivatives tend pointwise a.e. to $\nabla_{t,x} a$ and $\nabla_{t,x} b$, respectively, as $m \to \infty$.

(2) Theorem 1.1 of [30] yields functions $(u_{n,m}, v_{n,m})$ in $G^1 := C^1(J, L^2(\Omega)) \cap C(J, \mathcal{H}^1(\Omega))$ which solve the problem (4.3) with the coefficients and the data from step 1). We note that the required compatibility condition $\operatorname{tr}_t u_{0,n} = \chi_n(0)$ is trivially satisfied. The a priori estimates in this theorem are *not* uniform in m or n. However, Corollary 3.12 of [29] allows us to dominate $(u_{n,m}, v_{n,m})$ in G^1 by constants depending on the (uniformly bounded) $W^{1,\infty}$–norms of a_m and b_m as well as on the norms of $u_{0,n}$, $v_{0,n}$, φ_n and ψ_n in $\mathcal{H}^1$ and of χ_n in $L^2(J, \mathcal{H}^{3/2}(\Gamma)) \cap \mathcal{H}^1(J, \mathcal{H}^{1/2}(\Gamma))$. (Note that these norms of the data may blow up as $n \to \infty$.) This corollary actually deals with the localized problem on the half-space $\mathbb{R}^3_+$, but it can be transfered to our system (4.3) on Ω in a standard way, cf. Chapter 5 of [29] or Sect. 2 of [30]. Moreover, Theorem 1.4 of [7] shows a uniform estimate of the norms of the solutions in $C(J, L^2(\Omega))$ and of their tangential traces in $L^2(J, \mathcal{H}^{-1/2}(\Gamma))$ by the norms of the data in L^2 or the boundary forcing in $L^2(J, \mathcal{H}^{1/2}(\Gamma))$.

We first keep $n \in \mathbb{N}$ fixed. The aforementioned results from [29] and [7] imply that a subsequence of $(u_{n,m}, v_{n,m})_m$ has a weak-$*$ accumulation point (u_n, v_n) in $W^{1,\infty}(J, L^2(\Omega)) \cap L^\infty(J, \mathcal{H}^1(\Omega))$, that $(u_{n,m}, v_{n,m})_m$ converges to (u_n, v_n) in $C(J, L^2(\Omega))$ and that $(\operatorname{tr}_\tau u_{n,m}, \operatorname{tr}_\tau v_{n,m})_m$ tends to $(\operatorname{tr}_\tau u_n, \operatorname{tr}_\tau v_n)$ in $L^2(J, \mathcal{H}^{-1/2}(\Gamma))$. It is then routine to check that the functions (u_n, v_n) solve (4.3) with the coefficients a and b and for the data $u_{0,n}$, $v_{0,n}$, φ_n, ψ_n, and χ_n.

(3) Using the system (4.3) and integrating by parts, we calculate

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2} \int_\Omega \big(a(t)u_n(t) \cdot u_n(t) + b(t)v_n(t) \cdot v_n(t)\big)\, \mathrm{d}x \\
&\quad = \int_\Omega \big(\tfrac{1}{2}\partial_t a\, u_n \cdot u_n + \tfrac{1}{2}\partial_t b\, v_n \cdot v_n + a\partial_t u_n \cdot u_n + b\partial_t v_n \cdot v_n\big)\, \mathrm{d}x \\
&\quad = \int_\Omega \big(\tfrac{1}{2}\partial_t a\, u_n \cdot u_n + \tfrac{1}{2}\partial_t b\, v_n \cdot v_n + (\operatorname{curl} v_n - \sigma u_n + \varphi_n) \cdot u_n \\
&\qquad\quad + (-\operatorname{curl} u_n + \psi_n) \cdot v_n\big)\, \mathrm{d}x
\end{aligned}
$$

$$= \int_\Omega \big(\tfrac{1}{2}\partial_t a\, u_n \cdot u_n + \tfrac{1}{2}\partial_t b\, v_n \cdot v_n + \varphi_n \cdot u_n + \psi_n \cdot v_n - \sigma u_n \cdot u_n\big)\,\mathrm{d}x \\ + \int_\Gamma \chi_n \cdot \operatorname{tr}_\tau v_n \,\mathrm{d}x. \tag{4.5}$$

(4) The estimate in Theorem 1.4 of [7] indicated above now implies the convergence of $((u_n, v_n))_n$ to (u, v) in $C(J, L^2(\Omega))$ and of $((\operatorname{tr}_\tau u_n, \operatorname{tr}_\tau v_n))_n$ to $(\operatorname{tr}_\tau u, \operatorname{tr}_\tau v)$ in $L^2(J, \mathcal{H}^{-1/2}(\Gamma))$. Here (u, v) is the solution to (4.3) provided by Theorem 1.4 of [7]. After integrating the identity (4.5) in time, we can finally pass to the limit $n \to \infty$ obtaining (4.4). □

We now assume that Ω is simply connected in order to derive our observability-type estimate. Following [8] or [22] in the linear autonomous case, we use Helmholtz decompositions of the fields $(E(t), H(t))$ and the spaces

$$\mathcal{H}(\operatorname{curl} 0) = \{u \in L^2(\Omega)^3 \mid \operatorname{curl} u = 0\}, \quad \mathcal{H}_0(\operatorname{curl} 0) = \{u \in \mathcal{H}(\operatorname{curl} 0) \mid \operatorname{tr}_t u = 0\},$$
$$\mathcal{H}(\operatorname{div} 0) = \{u \in L^2(\Omega)^3 \mid \operatorname{div} u = 0\}, \qquad \mathcal{H}_0(\operatorname{div} 0) = \{u \in \mathcal{H}(\operatorname{div} 0) \mid \operatorname{tr}_n u = 0\},$$
$$\mathcal{H}^\Gamma(\operatorname{div} 0) = \big\{u \in \mathcal{H}(\operatorname{div} 0) \,\big|\, \textstyle\int_{\Gamma_j} \operatorname{tr}_n u \,\mathrm{d}x = 0 \quad \text{for all components } \Gamma_j \text{ of } \Gamma\big\},$$
$$\mathcal{H}^1_{t0}(\Omega) = \{u \in \mathcal{H}^1(\Omega)^3 \mid \operatorname{tr}_t u = 0\} = \{u \in \mathcal{H}(\operatorname{div}) \cap \mathcal{H}(\operatorname{curl}) \mid \operatorname{tr}_t u = 0\},$$

The last identity is shown in Theorem XI.1.3 of [6]. The first five spaces are endowed with the L^2—norm, while $\mathcal{H}^1_{t0}(\Omega)$ and its subspace $\mathcal{H}(\operatorname{div} 0)\cap\mathcal{H}_0(\operatorname{curl} 0)$ are equipped with that of $\mathcal{H}^1$. We next establish the Helmholtz decomposition needed in the sequel. Our result is a variant of Proposition 2 in [8], where the case of time-independent ε and μ and less regular solutions was treated.

Lemma 4.3. *Let the assumptions of Theorem 2.2 be satisfied. We take the fields (E, H) from (3.2) solving the Maxwell system (1.1). Then there exist functions w in $C^3\big(J_{\max}, \mathcal{H}^1_{t0}(\Omega)^3 \cap \mathcal{H}^\Gamma(\operatorname{div} 0)\big) \cap C^4\big(J_{\max}, L^2(\Omega)\big)^3$, p in $C^3(J_{\max}, \mathcal{H}^1_0(\Omega))$ and h in $C^3\big(J_{\max}, \mathcal{H}(\operatorname{div} 0) \cap \mathcal{H}_0(\operatorname{curl} 0)\big)$ such that*

$$\partial_t^k E = -\partial_t^{k+1} w + \nabla \partial_t^k p + \partial_t^k h, \tag{4.6}$$
$$\widehat{\mu}_k \partial_t^k H = \operatorname{curl} \partial_t^k w - g_k \tag{4.7}$$

for $k \in \{0, 1, 2, 3\}$, cf. (3.7) and (3.9), where the sum in (4.6) is orthogonal in $L^2(\Omega)^3$.

Let $t \in J_{\max}$. Equations (1.1) and (3.4) imply that the function $\mu\big(H(t)\big) H(t)$ is contained in $\mathcal{H}_0(\operatorname{div} 0)$. Because Ω is simply connected, Theorem 2.8 of [4] then yields a vector field $w(t)$ in $\mathcal{H}^1_{t0}(\Omega)^3 \cap \mathcal{H}^\Gamma(\operatorname{div} 0)$ satisfying

$$\operatorname{curl} w(t) = \mu(H(t))H(t). \tag{4.8}$$

Moreover, the mapping $\operatorname{curl}\colon \mathcal{H}^1_{t0}(\Omega)^3 \cap \mathcal{H}^\Gamma(\operatorname{div} 0) \to \mathcal{H}(\operatorname{div} 0)$ is invertible on the strength of Theorem 2.9 in [4]. In view of (3.2), the map w thus belongs to $C^3(J_{\max}, \mathcal{H}^1_{t0}(\Omega)^3 \cap \mathcal{H}^\Gamma(\operatorname{div} 0))$. Differentiating equation (4.8) in t, we deduce

$$\operatorname{curl} \partial_t^k w = \partial_t^k(\mu(H)H) = \mu^{\mathrm{d}}(H)\partial_t^k H + g_k$$

for $k \in \{1, 2, 3\}$ which proves (4.7). Comparing this relation for $k = 1$ with (1.1), we infer $\operatorname{curl}(E + \partial_t w) = 0$. Morever, the sum $E + \partial_t w$ belongs to the kernel of tr_t. Theorem 2.8 of [4] now provides functions $p(t) \in \mathcal{H}_0^1(\Omega)$ and $h(t) \in \mathcal{H}(\operatorname{div} 0) \cap \mathcal{H}_0(\operatorname{curl} 0)$ such that

$$E(t) = -\partial_t w(t) + \nabla p(t) + h(t) \tag{4.9}$$

for $t \in J_{\max}$. The spaces $\mathcal{H}^\Gamma(\operatorname{div} 0)$, $\nabla \mathcal{H}_0^1(\Omega)$ and $\mathcal{H}(\operatorname{div} 0) \cap \mathcal{H}_0(\operatorname{curl} 0)$ are orthogonal in $L^2(\Omega)^3$ and span this space, see Theorem 2.10' of [4]. This fact furnishes the remaining regularity assertions. We can now differentiate the identity (4.9) in time, proving (4.6). □

The energy inequality in Proposition 4.1 allows us to control the time integral of energy of the electric field E by the initial data and a higher order term. However, it is necessary to bound the time integrals of the energy of both E and H to obtain the desired global existence of solutions along with corresponding decay rates. This will be achieved by means of the Helmholtz decomposition established in Lemma 4.3. We now show a lower bound for the dissipation (up to correction terms) using the quantities introduced in (3.13).

Proposition 4.4. *Let the conditions of Theorem 2.2 be satisfied. For $0 \le s \le t < T_*$ and $k \in \{0,1,2,3\}$, we can estimate*

$$\int_s^t e_k(\tau)\,\mathrm{d}\tau \le c_2 \int_s^t d_k(\tau)\,\mathrm{d}\tau + c_3(e_k(t) + e_k(s)) + c_4 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau,$$

where the constants c_j do not depend on the times s and t.

Let $k \in \{0,1,2,3\}$. To simplify, we take $s = 0$. Equality (4.7) yields the identity

$$\int_{\Omega_t} \widehat{\mu}_k \partial_t^k H \cdot \partial_t^k H \,\mathrm{d}(x,\tau) = \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) - \int_{\Omega_t} g_k \cdot \partial_t^k H \,\mathrm{d}(x,\tau), \tag{4.10}$$

where $\Omega_t = \Omega \times (0,t)$. Using the regularity $\partial_t^k w \in C(J_{\max}, \mathcal{H}_{t0}^1(\Omega))^3$ established in Lemma 4.3, we integrate by parts and then invoke the first line of the system (3.8). It follows

$$\begin{aligned}
\int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) &= \langle \partial_t^k w, \operatorname{curl} \partial_t^k H \rangle_{L^2((0,t),\mathcal{H}^{-1}(\Omega))} \\
&= \langle \partial_t^k w, \partial_t(\widehat{\varepsilon}_k \partial_t^k E) \rangle_{L^2((0,t),\mathcal{H}^{-1}(\Omega))} + \int_{\Omega_t} \partial_t^k w \cdot (\sigma \partial_t^k E + \partial_t f_k)\,\mathrm{d}(x,\tau) \\
&= \int_\Omega \partial_t^k w(t,\cdot) \cdot \widehat{\varepsilon}_k(t,\cdot) \partial_t^k E(t,\cdot)\,\mathrm{d}x - \int_\Omega \partial_t^k w(0,\cdot) \cdot \widehat{\varepsilon}_k(0,\cdot) \partial_t^k E(0,\cdot)\,\mathrm{d}x \\
&\quad - \int_{\Omega_t} \partial_t^{k+1} w \cdot \widehat{\varepsilon}_k \partial_t^k E \,\mathrm{d}(x,\tau) + \int_{\Omega_t} \partial_t^k w \cdot (\sigma \partial_t^k E + \partial_t f_k)\,\mathrm{d}(x,\tau).
\end{aligned} \tag{4.11}$$

Since $\partial_t^k w(t,\cdot)$ belongs to $\mathcal{H}_{t0}^1(\Omega)^3 \cap \mathcal{H}^\Gamma(\operatorname{div} 0)$, Theorem 2.9 in [4] yields the Poincaré-type estimate $\|\partial_t^k w(\tau)\|_{L^2} \le c\,\|\operatorname{curl} \partial_t^k w(\tau)\|_{L^2}$. From formulas (4.7) and (3.14), we then infer the bound

$$\begin{aligned}
\|\partial_t^k w(\tau)\|_{L^2(\Omega)} \le c\,\|\operatorname{curl} \partial_t^k w(\tau)\|_{L^2(\Omega)} &\le c\,\|\hat{\mu}_k \partial_t^k H(\tau) + g_k(\tau)\|_{L_2(\Omega)} \\
&\le c e_k^{1/2}(\tau).
\end{aligned} \tag{4.12}$$

The orthogonality in equation (4.6) implies

$$\|\partial_t^{k+1} w(\tau)\|_{L^2(\Omega)} \le \|\partial_t^k E(\tau)\|_{L^2(\Omega)}.$$

For any $\theta > 0$, these inequalities along with (4.11) and (3.14) lead to the estimate

$$\begin{aligned} \Big| \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| & \\ \le c(e_k(t) + e_k(0)) + c \int_{\Omega_t} |\partial_t^k E|^2 \,\mathrm{d}(x,\tau) + \theta \int_{\Omega_t} |\partial_t^k w|^2 \mathrm{d}(x,\tau) & \\ + c_\theta \int_{\Omega_t} |\partial_t^k E|^2 \,\mathrm{d}(x,\tau) + c \int_0^t z_k^{3/2}(\tau)\,\mathrm{d}\tau. & \qquad (4.13) \end{aligned}$$

As in (4.12), we further compute

$$\begin{aligned} \int_{\Omega_t} |\partial_t^k w|^2 \,\mathrm{d}(x,\tau) &\le c \int_{\Omega_t} |\operatorname{curl} \partial_t^k w|^2 \,\mathrm{d}(x,\tau) \\ &\le c \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \widehat{\mu}_k^{-1} \operatorname{curl} \partial_t^k w \,\mathrm{d}(x,\tau) \\ &= c \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot (\partial_t^k H + \widehat{\mu}_k^{-1} g_k) \,\mathrm{d}(x,\tau) \\ &\le c \Big| \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| + c \int_0^t z^{3/2}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Fixing a small number $\theta > 0$, the term with $|\partial_t^k w|^2$ in equation (4.13) can now be absorbed by the left-hand side and by the integral of $z^{3/2}$. Employing also the condition $\sigma \ge \eta I$, we arrive at

$$\begin{aligned} \Big| \int_{\Omega_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| &\le c(e_k(t) + e_k(0)) + c \int_0^t d_k(\tau)\,\mathrm{d}\tau \\ &\quad + c \int_0^t z_k^{3/2}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Equation (4.10), the last inequality, and the estimates (3.14) yield the claim. Note that the constants c depend neither on t nor on s. $\square$

Combining the results of Propositions 4.1 and 4.4, we arrive at the following energy bound.

Corollary 4.5. *Under the conditions of Theorem 2.2, we have the inequality*

$$e_k(t) + \int_s^t e_k(s) ds \le C_1 e_k(s) + C_2 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau$$

for $0 \le s \le t < T_*$ *and* $k \in \{0,1,2,3\}$, *where the constants* C_k *do not depend on* t *and* s.

We multiply the inequality in Proposition 4.4 by $\theta := \min\{c_2^{-1}, (2c_3)^{-1}\}$ and add it to (4.1) from Proposition 4.1, obtaining

$$e_k(t) + 2\theta \int_s^t e_k(\tau)\,\mathrm{d}\tau \le 3 e_k(s) + 2(c_1 + \theta c_4) \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau. \qquad \square$$

Corollary 4.5 bounds the full energy (over the time interval (s,t)) by the initial energy and superlinear higher-order energies. The quasilinear character of the equation requires to involve higher topological levels (up to the third order). To control these higher order terms, we need to closely investigate higher regularity of solutions. While such an analysis has been developed in [11] at the local level for the linear stationary problem, our task is to globally extend the estimates by exploiting higher-order decay rates of the energy. To this end, both observability and regularity theories need to be developed—a formidable task on its own and of independent interest.

5. Higher-order energy observability and proof of Theorem 2.2

The central aim of this section is to strengthen the inequality in Corollary 4.5 by including higher-order space derivatives represented by the terms $z(t)$ and $\int_s^t z(\tau)\,\mathrm{d}\tau$ on the left-hand side of the estimate. Such inequalities are often referred to as "higher energy observability estimates" and are used to derive decay rates of energies. To be more specific, the following estimate is the key step in the proof of the main result.

Proposition 5.1. *Suppose the conditions of Theorem 2.2 are satisfied. Then there exists a radius $\delta \in (0,\delta_0]$ such that for all radii $r \in \big(0, r(\delta)\big]$ from equation (2.5), the solutions (E,H) satisfy*

$$z(t) + \int_s^t z(\tau)\,\mathrm{d}\tau \le C z(s)$$

for all $0 \le s \le t < T_*$, where z is defined in (3.13) and the constant C does not depend on time or $r \in (0, r(\delta))$, but it depends on δ.

The result easily follows from Proposition 7.1 below and Corollary 4.5. The proof of Proposition 7.1 is relegated to subsequent sections. We take it for granted here. We note that the proof of Proposition 5.1 actually yields a radius $\delta_1 \in (0,\delta_0]$ such that the above statement is true for all $\delta \in (0,\delta_1]$ with a constant C depending on δ_1, but not on δ.

Proof of Proposition 5.1. Let $0 \le s \le t < T_*$. Proposition 7.1 provides the estimate

$$z(t) + \int_s^t z(\tau)\,\mathrm{d}\tau \le c_5(z(s) + e(t) + z^2(t)) + c_6 \int_s^t \big(e(\tau) + z^{3/2}(\tau)\big)\,\mathrm{d}\tau$$

for some constants c_j independent of s and t. Corollary 4.5 thus yields the inequality

$$\begin{aligned} z(t) + \int_s^t z(\tau)\,\mathrm{d}\tau &\le (c_5 + C_1(c_5 + c_6)) z(s) \\ &\quad + c_5 z^2(t) + (c_6 + C_2(c_5 + c_6)) \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Recall that $z(\tau)$ is bounded by δ^2 on $[0, T_*)$ by (3.3). Fixing a sufficiently small radius $\delta \in (0,\delta_0]$, we can now absorb the superlinear terms involving z^2 and $z^{3/2}$ by the left-hand side. ☐

We first discuss the linear case, which was recently treated in [8] in the autonomous case. After that we prove our main result based on Proposition 5.1.

Remark 5.2. For linear material laws $\varepsilon(x,E)=\varepsilon(x)$ and $\mu(x,H)=\mu(x)$, one can show the variant

$$e_0(t)+\int_s^t e_0(\tau)\,\mathrm{d}\tau \le Ce_0(s) \tag{5.1}$$

of Proposition 5.1 for *all* $t\ge s\ge 0$ and all initial data, see [8]. Here we have replaced z by the usual 0-th order energy e_0. This estimate easily yields the exponential decay for all data by a standard argument. Indeed, since (5.1) implies $e_0(\tau)\ge C^{-1}e_0(t)$, we infer the inequality

$$(1+(t-s)C^{-1})e_0(t)\le Ce_0(s) \tag{5.2}$$

for all $t\ge s\ge 0$. Fix the time $T>0$ with $C^2/(C+T)=1/2$. Estimate (5.2) then provides the bound $e_0(nT)\le \frac{1}{2}e_0((n-1)T)$ for all $n\in\mathbb{N}$. Inductively, it follows $e_0(nT)\le 2^{-n}e_0(0)$ and hence the exponential decay

$$e_0(t)\le M\mathrm{e}^{-\omega t}e_0(0) \tag{5.3}$$

for suitable constants $\omega, M>0$, where we use (5.1) once more.

Let now the coefficients $\varepsilon(t,x)$ and $\mu(t,x)$ depend on time $t\in\mathbb{R}_+$. If the supremum norms of $\partial_t\varepsilon$ and $\partial_t\mu$ are small enough, formula (4.2) and the proof of Proposition 4.4 for $k=0$ imply the estimate (5.1) also in this case. Then the exponential decay (5.3) follows as above.

Proof of Theorem 2.2. We first show that $T_*=\infty$ if the radius $r>0$ in (2.5) is small enough. We suppose that $T_*<\infty$. Equation (3.6) then yields $z(T_*)=\delta^2$, where δ is given by Proposition 5.1. On the other hand, as in Remark 5.2 we deduce the inequality

$$(1+(t-s)C^{-1})z(t)\le Cz(s) \tag{5.4}$$

from Proposition 5.1, but now only for $0\le s\le t<T_*$ and initial data with $\|(E_0,H_0)\|^2_{\mathcal{H}^3}\le r^2$ for all radii $r\in(0,r(\delta)]$, where $r(\delta)>0$ was introduced before (2.5). The differentiated Maxwell system (3.11) and the bounds from (3.14) next yield

$$z(0)\le c_0\,\|(E_0,H_0)\|^2_{\mathcal{H}^3}\le c_0r^2$$

for a constant $c_0>0$. We now fix the radius

$$r:=\min\Big\{r(\delta),\frac{\delta}{\sqrt{2c_0C}}\Big\}. \tag{5.5}$$

Because of (5.4) with $s=0$, the number $z(t)$ is bounded by $\delta^2/2$ for $t<T_*$ and by continuity also for $t=T_*$. This fact contradicts $z(T_*)=\delta^2$, and hence it follows $T_*=\infty$. We can now conclude the proof exactly as in Remark 5.2. □

In order to establish the main result of the paper, it thus remains to prove Proposition 7.1. Necessary preparations are done in the following section.

6. Auxiliary results

6.1. Curl–div estimates

One can bound the $\mathcal{H}^1$-norm of a field u by its norms in $\mathcal{H}(\mathrm{curl})\cap\mathcal{H}(\mathrm{div})$ and the $\mathcal{H}^{1/2}$-norm of $\mathrm{tr}_t\, u$ or $\mathrm{tr}_n\, u$, see Corollary XI.1.1 of [6]. In the next section, we will need a version of this result with regular, matrix-valued coefficients a. This fact does not directly follow from the case $a = I$—unless a is scalar. It is stated in Remark 4 of [8] with a brief indication of a proof. For the convenience of the reader we present a (different) proof below.

Proposition 6.1. *Let $a \in W^{1,\infty}(\Omega, \mathbb{R}^{3+3}_{\mathrm{sym}})$ satisfy $a \geq \eta I$. Suppose $u \in \mathcal{H}(\mathrm{curl})$ fulfills $\mathrm{div}(au) \in L^2(\Omega)$ and $\mathrm{tr}_n(au) \in \mathcal{H}^{1/2}(\Gamma)$. Then the vector field u belongs to $\mathcal{H}^1(\Omega)^3$ and fulfills*

$$\|u\|_{\mathcal{H}^1(\Omega)} \leq c\big(\|u\|_{\mathcal{H}(\mathrm{curl})} + \|\,\mathrm{div}(au)\|_{L^2(\Omega)} + \|\,\mathrm{tr}_n(au)\|_{\mathcal{H}^{1/2}(\Gamma)}\big) =: c\kappa(u).$$

There exists a finite partition of unity $\{\chi_i\}_i$ on $\overline{\Omega}$ such that the support of each χ_i is contained in a simply connected subset of $\overline{\Omega}$ with a connected C^2-boundary. Since each χ_i is scalar, we obtain the estimate

$$\|\chi_i u\|_{L^2(\Omega)} + \|\,\mathrm{curl}\,(\chi_i u)\|_{L^2(\Omega)} + \|\,\mathrm{div}\,(a\chi_i u)\|_{L^2(\Omega)} + \|\,\mathrm{tr}\,_n(a\chi_i u)\|_{\mathcal{H}^{1/2}(\Gamma)} \leq c\kappa(u).$$

We can thus assume that Γ is connected. In this case, $\mathrm{curl}\, u$ belongs to $\mathcal{H}^\Gamma(\mathrm{div}\, 0)$ and Theorem 2.9 of [4] yields a vector field $w \in \mathcal{H}^1(\Omega) \cap H_0(\mathrm{div}\, 0)$ with $\mathrm{curl}\, u = \mathrm{curl}\, w$ and $\|w\|_{\mathcal{H}^1} \leq c\,\|\,\mathrm{curl}\, u\|_{L^2}$. As the difference $u-w$ is an element of $\mathcal{H}(\mathrm{curl}\, 0)$, it is represented by $u - w = \nabla\varphi$ for a function $\varphi \in \mathcal{H}^1(\Omega)$ by Theorem 2.8 in [4]. We obtain

$$\mathrm{div}(a\nabla\varphi) = \mathrm{div}(au) - \mathrm{div}(aw) \ \in L^2(\Omega),$$
$$\mathrm{tr}_n(a\nabla\varphi) = \mathrm{tr}_n(au) - \mathrm{tr}_n(aw) \ \in \mathcal{H}^{1/2}(\Gamma),$$

because of the assumptions and the fact $w \in \mathcal{H}^1(\Omega)$. Due to the uniform ellipticity, φ thus is an element of $\mathcal{H}^2(\Omega)$ satisfying

$$\|\varphi\|_{\mathcal{H}^2(\Omega)} \leq c\big(\|\,\mathrm{div}(au)\|_{L^2(\Omega)} + \|\,\mathrm{tr}_n(au)\|_{\mathcal{H}^{1/2}(\Gamma)} + \|w\|_{\mathcal{H}^1(\Omega)}\big) \leq c\kappa(u).$$

The assertion now follows from the equation $u = w + \nabla\varphi$. □

6.2. Geometry: coordinate transformation and differential calculus

For a fixed distance $\rho > 0$, on the collar $\Gamma_\rho = \{x \in \overline{\Omega} \mid \mathrm{dist}(x,\Gamma) < \rho\}$, we can find functions $\tau^1, \tau^2, \nu \in C^4(\Gamma_\rho, \mathbb{R}^3)$ such that the vectors $\{\tau^1(x), \tau^2(x), \nu(x)\}$ form an orthonormal basis of $\mathbb{R}^3$ for each point $x \in \Gamma_\rho$ and ν extends the outer unit normal at Γ. Hence, τ^1 and τ^2 span the tangential planes at Γ. For $\xi, \zeta \in \{\tau^1, \tau^2, \nu\}$, $u \in \mathbb{R}^3$ and $a \in \mathbb{R}^{3\times 3}$, we set

$$\partial_\xi = \sum\nolimits_j \xi_j \partial_j, \quad u_\xi = u \cdot \xi, \quad u^\xi = u_\xi \xi, \quad u^\tau = u_{\tau^1}\tau^1 + u_{\tau^2}\tau^2, \quad a_{\xi\zeta} = \xi^\top a\zeta.$$

We state several calculus formulas, which are extensively exploited in the next section. In the following, it is always assumed that the functions involved are sufficiently regular. We can switch between the derivatives of the coefficient u_ξ and the component u^ξ up to a lower-order term since

$$\partial_\zeta u^\xi = \partial_\zeta u_\xi \xi + u_\xi \partial_\zeta \xi.$$

The commutator of tangential derivatives and traces

$$\partial_\tau \operatorname{tr}_t u = \partial_\tau (u \times \nu) = \operatorname{tr}_t \partial_\tau u + u \times \partial_\tau \nu \qquad \text{on } \Gamma$$

is also of lower order. The gradient of a scalar function φ is expanded as

$$\nabla\varphi = \sum\nolimits_\xi \xi \cdot (\nabla\varphi)\, \xi = \sum\nolimits_\xi \xi \partial_\xi \varphi,$$

so that $\partial_j = \sum_\xi \xi_j \partial_\xi$ for $j \in \{1,2,3\}$. To express the curl operator, we use the matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$J(\xi) = \sum\nolimits_j \xi_j J_j \,.$$

Because of

$$\operatorname{curl} u = \partial_1 [0, -u_3, u_2]^\top + \partial_2 [u_3, 0, -u_1]^\top + \partial_3 [-u_2, u_1, 0]^\top,$$

we have

$$\operatorname{curl} = \sum\nolimits_j J_j \partial_j = \sum\nolimits_{j,\xi} J_j \xi_j \partial_\xi = \sum\nolimits_\xi J(\xi) \partial_\xi.$$

Observe that the kernel of $J(\nu)$ is spanned by ν. Hence, after factoring out the null space, we can write $J(\nu)u = J(\nu)u^\tau$, and the restriction of $J(\nu)$ to $\operatorname{span}\{\tau^1, \tau^2\}$ has an inverse $R(\nu)$.

In order to produce estimates with additional, say $\mathcal{H}^1(\Omega)$–, spatial regularity, one typically exploits that the boundary value problem is non-characteristic. However, the Maxwell system is characteristic since $J(\nu)$ has the kernel $\operatorname{span}\{\nu\}$. In order to obtain regularity in the normal direction, we employ the "curl-div-strategy." The curl operator contains the normal derivative of the tangential components, while the divergence condition will provide estimates for the normal derivatives of the normal component via an ordinary differential equation. This procedure is carried out in the next subsection.

6.3. Representation of normal derivatives

The following construction is based on an adaptation of the well-known ADN (Agmon–Douglis–Nirenberg) method from the elliptic theory. We begin by solving the equation $\operatorname{curl} u = f$ for normal derivatives of the tangential components of u. By expanding

$$\operatorname{curl} u = J(\nu)(\partial_\nu u)^\tau + J(\tau^1)\partial_{\tau^1} u + J(\tau^2)\partial_{\tau^2} u,$$

we obtain

$$\partial_\nu u^\tau = \sum\nolimits_i (\partial_\nu \tau^i \, u_{\tau^i} + \tau^i \partial_\nu \tau^i \cdot u) + R(\nu)\Big(f - \sum\nolimits_i J(\tau^i)\partial_{\tau^i} u\Big) \tag{6.1}$$

and hence

$$\partial_\nu u^\tau = R(\nu)\Big(f - \sum\nolimits_i J(\tau^i)\partial_{\tau^i} u\Big) + \text{l.o.t.}(u), \tag{6.2}$$

where l.o.t.(u) denote lower-order terms depending on u, but not on its derivatives.

In order to recover the normal derivative of the normal component of u, we resort to the divergence operator. The divergence of a vector field u can be expressed as

$$\operatorname{div} u = \sum\nolimits_j \partial_j \sum\nolimits_\xi u_\xi \xi_j = \sum\nolimits_\xi \big(\partial_\xi u_\xi + \operatorname{div}(\xi) u_\xi\big).$$

Letting $\varphi = \operatorname{div}(au)$ for a matrix-valued function a, we derive

$$\begin{aligned}\operatorname{div}(au) &= \sum\nolimits_{\xi,\zeta} \partial_\xi(\xi^\top a \zeta u_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\,\xi^\top a u \\ &= \sum\nolimits_{\xi,\zeta} (a_{\xi\zeta}\partial_\xi u_\zeta + \partial_\xi a_{\xi\zeta} u_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\,\xi^\top a u, \\ a_{\nu\nu}\partial_\nu u_\nu &= \varphi - \sum_{(\xi,\zeta)\neq(\nu,\nu)} a_{\xi\zeta}\partial_\xi u_\zeta - \sum_{\xi,\zeta} \partial_\xi a_{\xi\zeta} u_\zeta - \sum_\xi \operatorname{div}(\xi)\,\xi^\top a u \\ &=: \varphi - D(a)u, \end{aligned} \tag{6.3}$$

where $D(a)u$ contains all tangential derivatives and normal derivatives of tangential components of u plus lower order terms. Next, let $a \in W^{1,\infty}(\Omega_T, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ be uniformly positive definite, $u \in C^1\big(J, \mathcal{H}^1(\Omega)\big)^3$, and $\psi \in L^2(\Omega_J)$. In view of formula (3.4), we look at the equation

$$\operatorname{div}\big(a(t)u(t)\big) = \operatorname{div}\big(a(0)u(0)\big) - \int_0^t \big(\operatorname{div}(\sigma u(s)) + \psi(s)\big)\,\mathrm{d}s \tag{6.4}$$

for $0 \le t \le T$. (In (3.4) we have $\psi = 0$.) We set $\gamma = \sigma_{\nu\nu}/a_{\nu\nu}$ and $\Gamma(t,s) = \exp(-\int_s^t \gamma(\tau)\,\mathrm{d}\tau)$. Equations (6.3) and (6.4) yield

$$\begin{aligned} a_{\nu\nu}(t)\partial_\nu u_\nu(t) &= \operatorname{div}\big(a(0)u(0)\big) - D\big(a(t)\big)u(t) \\ &\quad - \int_0^t \big(\gamma(s)a_{\nu\nu}(s)\partial_\nu u_\nu(s) + D\big(\sigma\big)u(s) + \psi(s)\big)\,\mathrm{d}s. \end{aligned}$$

Differentiating with respect to t and solving the resulting ODE, we obtain

$$\begin{aligned} a_{\nu\nu}(t)\partial_\nu u_\nu(t) &= \Gamma(t,0)a_{\nu\nu}(0)\partial_\nu u_\nu(0) - \int_0^t \Gamma(t,s)\big(D(\sigma)u(s) + \psi(s) \\ &\qquad + \partial_s\big(D(a(s))u(s)\big)\big)\,\mathrm{d}s \\ &= \Gamma(t,0)\operatorname{div}(a(0)u(0)) - D(a(t))u(t) \\ &\qquad + \int_0^t \Gamma(t,s)\big(\gamma(s)D\big(a(s)\big)u(s) - D(\sigma)u(s) - \psi(s)\big)\,\mathrm{d}s, \end{aligned} \tag{6.5}$$

where ψ is the same as in (6.4) and $D(a)$ is defined in (6.3).

7. A regularity result: higher order energy bounds

In this section we show that $\partial_t^k E$ and $\partial_t^k H$ can be bounded in $\mathcal{H}^{3-k}$ for $k \in \{0,1,2\}$ by the L^2-norms of $\partial_t^k E$ and $\partial_t^k H$. This astonishing fact is a crucial ingredient of our reasoning, and its proof is quite demanding. In contrast to our situation, when studying linear autonomous problems such regularity estimates readily follow from semigroup theory combined with a "good" characterization of the domains of generators and their powers—the latter often is a consequence of the theory of strongly elliptic operators. In our case, semigroup tools are not available. Instead we proceed in line with the ADN approach and the div-curl-strategy using the techniques discussed in the previous section. We sketch the main ideas.

The $\mathcal{H}^1$—norm of $\partial_t^k H$ with $k \in \{0, 1, 2\}$ can easily be estimated by means of the "elliptic" curl–div estimates from Proposition 6.1 because we control the curl and the divergence of $\partial_t^k H$ via the time differentiated Maxwell system (3.8) and (3.10). Aiming at higher space regularity, we can apply the above strategy to tangential derivatives of $\partial_t^k H$ only, whereas non-tangential derivatives destroy the boundary condition in (3.8). The normal derivatives of the fields are treated similarly as in the local well-posedness theory from [29–31]: Their tangential components are read off the differentiated Maxwell system using the expansion (6.1) of the curl-operator, while the normal components are bounded employing the divergence condition (3.10) and the formula (6.3). In these arguments we have to restrict ourselves to fields localized near the boundary. The localized fields in the interior can be controlled more easily since the boundary conditions become trivial for them.

The electric fields have less favorable divergence properties because of the conductivity term in (3.8). Instead of the curl-div estimates from Proposition 6.1, we thus employ the energy bound of the system (7.2) that arises by differentiating the Maxwell equations in time and tangential directions. The normal components are again treated by the curl-div-strategy indicated in the previous paragraph. However, to handle the extra divergence term in (3.10) caused by the conductivity, we need the more sophisticated divergence formula (6.5) which relies on an ODE derived from (3.10). This program is carried out by iteration on the space regularity. In each step one has to start with the magnetic fields in order to use their better properties when estimating the electric ones.

The following result is the main technical ingredient of the paper. As explained in Sect. 5, Propositions 4.1, 4.4 and 7.1 imply Proposition 5.1 which in turn yields our main Theorem 2.2.

Proposition 7.1. *We impose the conditions of Theorem 2.2 with the exception of the simple connectedness of Ω. Then the solutions (E, H) to the Maxwell system (1.1) satisfy the inequality*

$$z(t) + \int_s^t z(\tau)\,\mathrm{d}\tau \le c_5(z(s) + e(t) + z^2(t)) + c_6 \int_s^t \big(e(\tau) + z^{3/2}(\tau)\big)\,\mathrm{d}\tau$$

for all $0 \le s \le t < T_*$, where z and e are defined in (3.13) and the constants c_j do not depend on t and s.

Let (E, H) be a solution of (1.1) on $J_* = [0, T_*)$ satisfying the bound (3.3) and the divergence equations (3.4). Take $k \in \{0, 1, 2\}$ and $0 \le t < T_*$, where we let $s = 0$ for simplicity. To localize the fields, we take scalar functions χ and $1-\chi$ in $C^5(\overline{\Omega})$ having compact support in $\Omega \setminus \Gamma_{a/2}$ and Γ_a, respectively.

We already outlined our methods above. The proof is divided into several steps which we list before presenting the details.

(1) We estimate the $\mathcal{H}^1$-norm of $\partial_t^k H$ using the curl-div-estimates from Proposition 6.1.
(2) We bound all relevant derivatives of E and H in the interior using the time-space differentiated Maxwell system (7.2). Here and in the other steps, highest order terms of E appear on the right-hand side which are absorbed later.
(3) To complete the $\mathcal{H}^1$-estimate, we treat E near the boundary employing energy estimates for the tangential derivatives and the curl-div-strategy for the normal derivatives. These arguments rely on the differentiated Maxwell system (7.7) and variants thereof.
(4) The $\mathcal{H}^2$-estimates of E, $\partial_t E$, H, and $\partial_t H$ near the boundary are carried out in a similar way, based on steps (1)–(3).
(5) We handle the $\mathcal{H}^3$–norm of E and H by iterating our techniques.

(1) Estimate of $\partial_t^k H$ in $\mathcal{H}^1(\Omega)$. These bounds are a direct consequence of the elliptic curl-div-estimate in Proposition 6.1 since we can control the relevant quantities through the (time differentiated) Maxwell system (3.8) and the divergence equation (3.10). Indeed, using also the estimates (3.14), we obtain

$$\begin{aligned}
\left\| \operatorname{curl} \partial_t^k H(t) \right\|_{L^2(\Omega)} &\le c e_{k+1}^{1/2}(t) + c z(t)\delta_{k2},\\
\left\| \operatorname{div}(\widehat{\mu}_k \partial_t^k H(t)) \right\|_{L^2(\Omega)} &\le c z(t)\delta_{k2},\\
\left\| \operatorname{tr}_n(\widehat{\mu}_k \partial_t^k H(t)) \right\|_{\mathcal{H}^{1/2}(\Omega)} &\le c z(t)\delta_{k2},
\end{aligned}$$

where $\delta_{k2} = 1$ for $k = 2$ and $\delta_{k2} = 0$ for $k \in \{0, 1\}$. Proposition 6.1 thus implies

$$\begin{aligned}
\left\| \partial_t^k H(t) \right\|_{\mathcal{H}^1(\Omega)}^2 &\le c e_{k+1}(t) + c z^2(t)\delta_{k2},\\
\int_0^t \left\| \partial_t^k H(\tau) \right\|_{\mathcal{H}^1(\Omega)}^2 \mathrm{d}\tau &\le c \int_0^t (e_{k+1}(\tau) + z^2(\tau)\delta_{k2})\, \mathrm{d}\tau. \qquad (7.1)
\end{aligned}$$

We stress that the inhomogeneities in (3.8) and (3.10) involving f_2 and g_2 are quadratic in (E, H) and can thus be bounded by z via (3.14). This fact is essential in future estimates.

(2) Estimates in the interior for E and H. We look at the localized fields $\partial_t^k(\chi E)$ and $\partial_t^k(\chi H)$ whose support $\operatorname{supp}\chi$ is strictly separated from the boundary. Hence, their spatial derivatives satisfy the boundary conditions of the Maxwell system so that we can treat the electric fields via energy bounds and the magnetic ones via the curl-div estimates.

(a) Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 3 - k$. We apply $\partial_x^\alpha \chi$ to the Maxwell system (3.11), deriving the equations

$$\begin{aligned}
\varepsilon^{\mathrm{d}}(E)\, \partial_t \partial_x^\alpha \partial_t^k(\chi E) &= \operatorname{curl} \partial_x^\alpha \partial_t^k(\chi H) - \sigma \partial_x^\alpha \partial_t^k(\chi E) + \partial_x^\alpha([\chi, \operatorname{curl}]\partial_t^k H)\\
&\quad - \sum_{0 \le \beta < \alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta}(\sigma + \varepsilon^{\mathrm{d}}(E))\, \partial_x^\beta \partial_t^k(\chi E) - \partial_x^\alpha(\chi \tilde{f}_k),
\end{aligned}$$

$$\begin{aligned}\mu^{\mathrm{d}}(H)\,\partial_t\partial_x^\alpha\partial_t^k(\chi H) &= -\operatorname{curl}\partial_x^\alpha\partial_t^k(\chi E) - \partial_x^\alpha([\chi,\operatorname{curl}]\partial_t^k E) - \partial_x^\alpha(\chi\tilde g_k)\\ &\quad - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_x^{\alpha-\beta}\mu^{\mathrm{d}}(H)\,\partial_x^\beta\partial_t^k(\chi H),\\ \operatorname{tr}_t\partial_x^\alpha\partial_t^k(\chi E) &= 0, \qquad \operatorname{tr}_n\partial_x^\alpha\partial_t^k(\chi H) = 0. \end{aligned} \tag{7.2}$$

Note that the commutator $m := [\chi, \operatorname{curl}]$ is merely a multiplication operator. Lemma 4.2 and the estimates (3.14) thus yield

$$\begin{aligned}&\|\partial_x^\alpha\partial_t^k(\chi E)(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_x^\alpha\partial_t^k(\chi E)(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau\\ &\quad\le cz(0) + c\int_0^t \big(z^{3/2}(\tau) + \|\partial_t^k(\chi E(\tau))\|_{\mathcal{H}^{|\alpha|-1}(\Omega)}^2 + \|\partial_t^k(\chi H(\tau))\|_{\mathcal{H}^{|\alpha|-1}(\Omega)}^2\big)\,\mathrm{d}\tau\\ &\qquad + c\int_{\Omega_t}\big(\partial_x^\alpha(m\partial_t^k H)\cdot\partial_x^\alpha\partial_t^k(\chi E)) - \partial_x^\alpha(m\partial_t^k E)\cdot\partial_x^\alpha\partial_t^k(\chi H)\big)\,\mathrm{d}(x,\tau).\end{aligned}$$

The former part of the last integral can be estimated by

$$\frac14\int_0^t \big\|\partial_x^\alpha\partial_t^k(\chi E)(\tau)\big\|_{L^2(\Omega)}^2\,\mathrm{d}\tau + c\int_0^t \big\|\tilde\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\,\mathrm{d}\tau,$$

where $\tilde\chi \in C_c^\infty(\Omega\setminus\Gamma_{a/2})$ is another cut-off function being equal to 1 on $\operatorname{supp}\chi$. The first summand is absorbed by the left-hand side, while the second one only involves H and can be treated separately. The latter part of the integral on Ω_t is similarly bounded by

$$\theta\int_0^t \big\|\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\,\mathrm{d}\tau + c(\theta)\int_0^t \big\|\tilde\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\,\mathrm{d}\tau$$

for an arbitrary (small) $\theta > 0$. It follows

$$\begin{aligned}&\big\|\partial_x^\alpha\partial_t^k(\chi E)(t)\big\|_{L^2(\Omega)}^2 + \int_0^t \big\|\partial_x^\alpha\partial_t^k(\chi E)(\tau)\big\|_{L^2(\Omega)}^2\,\mathrm{d}\tau\\ &\quad\le cz(0) + c\int_0^t (z^{3/2}(\tau) + \big\|\partial_t^k(\chi E(\tau)\big\|_{\mathcal{H}^{|\alpha|-1}(\Omega)}^2)\,\mathrm{d}\tau\\ &\qquad + \theta\int_0^t \big\|\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\,\mathrm{d}\tau + c(\theta)\int_0^t \big\|\tilde\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\,\mathrm{d}\tau.\end{aligned} \tag{7.3}$$

(b) To treat H, we only need to look at the case $|\alpha| \le 2 - k$. Equations (3.4) and (3.10) yield

$$\begin{aligned}\operatorname{div}\big(\widehat\mu_k\partial_x^\alpha\partial_t^k(\chi H)\big) &= \partial_x^\alpha([\operatorname{div},\chi]\widehat\mu_k\partial_t^k H) - \partial_x^\alpha(\chi g_k)\\ &\quad - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\operatorname{div}\big(\partial_x^{\alpha-\beta}\widehat\mu_k\,\partial_x^\beta(\partial_t^k(\chi H)\big).\end{aligned} \tag{7.4}$$

Recalling formulas (7.2) and (3.14), we deduce

$$\begin{aligned}&\big\|\operatorname{curl}\partial_x^\alpha\partial_t^k(\chi H(t))\big\|_{L^2(\Omega)} + \big\|\operatorname{div}\partial_x^\alpha\partial_t^k(\chi H(t))\big\|_{L^2(\Omega)}\\ &\quad\le c\Big(z(t) + \big\|\partial_t^k\tilde\chi H(t)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)} + \big\|\partial_t^{k+1}(\chi E(t))\big\|_{\mathcal{H}^{|\alpha|}(\Omega)} + \big\|\partial_t^k(\chi E(t))\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}\Big)\end{aligned}$$

Proposition 6.1 now implies the inequalities

$$\begin{aligned}
&\left\|\partial_t^k \chi H(t)\right\|_{\mathcal{H}^{|\alpha|+1}(\Omega)}^2 \\
&\quad \leq c\big(z^2(t)+\left\|\partial_t^k \tilde{\chi} H(t)\right\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2+\max_{j\leq k+1}\|\partial_t^j(\chi E(t))\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\big),\\
&\int_0^t \left\|\partial_t^k \chi H(\tau)\right\|_{\mathcal{H}^{|\alpha|+1}(\Omega)}^2 \mathrm{d}\tau \\
&\quad \leq c\int_0^t \big(z^2(\tau)+\left\|\partial_t^k \tilde{\chi} H(\tau)\right\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2+\max_{j\leq k+1}\left\|\partial_t^j(\chi E(\tau))\right\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\big)\,\mathrm{d}\tau. \qquad (7.5)
\end{aligned}$$

Here, we can replace χ by $\tilde{\chi}$ from inequality (7.3) and $\tilde{\chi}$ by a function $\breve{\chi} \in C_c^\infty(\Omega\setminus\Gamma_{a/2})$ which is equal to 1 on $\operatorname{supp}\tilde{\chi}$.

We set $y_j(t)=\max_{0\leq k\leq 3-j}\left\|\partial_t^k \chi(E(t),H(t))\right\|_{\mathcal{H}^j}^2$. The estimates (7.1), (7.3) and (7.5) iteratively imply

$$\begin{aligned}
y_j(t)+\int_0^t y_j(\tau)\,\mathrm{d}\tau &\leq cz(0)+c\big(e(t)+z^2(t)\big)+c(\theta)\int_0^t (e(\tau)+z^{3/2}(\tau))\,\mathrm{d}\tau\\
&\quad +\theta \max_{0\leq k\leq 3-j}\int_0^t \|\partial_t^k E(\tau)\|_{\mathcal{H}^j(\Omega)}^2\,\mathrm{d}\tau \qquad (7.6)
\end{aligned}$$

for any $\theta>0$ and $j\in\{1,2,3\}$.

(3) Boundary-collar estimate of $\partial_t^k E$ in $\mathcal{H}^1$. (a) We write $\hat{\chi}=1-\chi$ and $\partial_\tau=(\partial_{\tau^1},\partial_{\tau^2})$. Let $\alpha\in\mathbb{N}_0^2$ with $0<|\alpha|\leq 3-k$. (For the later use, also higher-order space derivatives are treated.) We localize the system near the boundary by including the cut-off $\hat{\chi}$ into the equations (3.11), and then apply ∂_τ^α to the resulting system. The localized tangential-time derivatives of (E,H) thus satisfy

$$\begin{aligned}
\varepsilon^{\mathrm{d}}(E)\,\partial_t\partial_\tau^\alpha\partial_t^k(\hat{\chi}E) &= \operatorname{curl}\partial_\tau^\alpha\partial_t^k(\hat{\chi}H)-\sigma\partial_\tau^\alpha\partial_t^k(\hat{\chi}E)+[\partial_\tau^\alpha,\operatorname{curl}]\partial_t^k(\hat{\chi}H)\\
&\quad +\partial_\tau^\alpha([\hat{\chi},\operatorname{curl}]\partial_t^k H)\\
&\quad -\sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}(\sigma+\varepsilon^{\mathrm{d}}(E))\,\partial_\tau^\beta\partial_t^k(\hat{\chi}E)-\partial_\tau^\alpha(\hat{\chi}\tilde{f}_k),\\
\mu^{\mathrm{d}}(H)\,\partial_t\partial_\tau^\alpha\partial_t^k(\hat{\chi}H) &= -\operatorname{curl}\partial_\tau^\alpha\partial_t^k(\hat{\chi}E)-\partial_\tau^\alpha([\hat{\chi},\operatorname{curl}]\partial_t^k E)-[\partial_\tau^\alpha,\operatorname{curl}]\partial_t^k(\hat{\chi}E)\\
&\quad -\sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}\mu^{\mathrm{d}}(H)\,\partial_\tau^\beta\partial_t^k(\hat{\chi}H)-\partial_\tau^\alpha(\hat{\chi}\tilde{g}_k),\\
\operatorname{tr}_t\partial_\tau^\alpha\partial_t^k(\hat{\chi}E) &= [\partial_\tau^\alpha,\operatorname{tr}_\tau]\partial_t^k(\hat{\chi}E)=:\chi. \qquad (7.7)
\end{aligned}$$

The commutators $[\partial_\tau^\alpha,\operatorname{curl}]$ are differential operators of order $|\alpha|$ with bounded coefficients, whereas $[\partial_\tau^\alpha,\operatorname{tr}_\tau]$ is of order $|\alpha|-1$ on the boundary and hence a bounded operator from $\mathcal{H}^{|\alpha|-1/2}(\Gamma)$ to $\mathcal{H}^{1/2}(\Gamma)$. We now use the energy identity in Lemma 4.2 with $a=\varepsilon^{\mathrm{d}}(E)$, $b=\mu^{\mathrm{d}}(H)$, $u=\partial_\tau^\alpha\partial_t^k(\hat{\chi}E)$, and $v=\partial_\tau^\alpha\partial_t^k(\hat{\chi}H)$. The commutator terms, the sums and the summands with f_k and g_k yield the inhomogeneities φ and ψ, respectively. From Lemma 4.2 we deduce the inequality

$$\begin{aligned}
&\big\|\partial_\tau^\alpha \partial_t^k (\hat\chi E)(t)\big\|_{L^2(\Omega)}^2 + \int_0^t \big\|\partial_\tau^\alpha \partial_t^k (\hat\chi E)(\tau)\big\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau \\
&\quad \le cz(0) + c\int_{\Omega_t} \big(|\partial_t a u\cdot u| + |\partial_t b v\cdot v| + |\varphi\cdot u| + |\psi\cdot v|\big)\,\mathrm{d}(\tau,x) \\
&\qquad + c\int_{\Gamma_t} |\chi\cdot \mathrm{tr}_\tau\, v|\,\mathrm{d}(\tau,x).
\end{aligned} \tag{7.8}$$

Several terms on the right-hand side are super-quadratic in (E,H) and can be bounded by $cz^{3/2}$ due to (3.14). The quadratic ones need more care. The summands in $\varphi\cdot u$ and $\psi\cdot v$ containing the commutators are less or equal to

$$\theta\int_0^t \big\|\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2 \,\mathrm{d}\tau + c(\theta)\int_0^t \big\|\tilde\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2 \,\mathrm{d}\tau$$

with any (small) constant $\theta > 0$ and a cut-off $\tilde\chi \in C_c^\infty(\Gamma_a)$ being equal to 1 on $\operatorname{supp}\hat\chi$. The boundary integral is estimated by the same expression, where we use the dual paring $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ and that ∂_{τ^i} belongs to $\mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. The sums over β give rise to the terms

$$\begin{aligned}
&\frac14\int_0^t \big\|\partial_\tau^\alpha(\partial_t^k\hat\chi E(\tau))\big\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau + c\int_0^t \big\|\hat\chi\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|-1}(\Omega)}^2 \,\mathrm{d}\tau \\
&\quad + c\int_0^t \big\|\hat\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2 \,\mathrm{d}\tau
\end{aligned}$$

plus super-quadratic terms. We thus arrive at

$$\begin{aligned}
&\big\|\partial_\tau^\alpha \partial_t^k (\hat\chi E)(t)\big\|_{L^2(\Omega)}^2 + \int_0^t \big\|\partial_\tau^\alpha \partial_t^k (\hat\chi E)(\tau)\big\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau \\
&\quad \le cz(0) + c(\theta)\int_0^t \Big(\big\|\hat\chi\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|-1}(\Omega)}^2 + \big\|\tilde\chi\partial_t^k H(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2\Big)\,\mathrm{d}\tau \\
&\qquad + \theta\int_0^t \big\|\partial_t^k E(\tau)\big\|_{\mathcal{H}^{|\alpha|}(\Omega)}^2 \,\mathrm{d}\tau + c\int_0^t z^{3/2}(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{7.9}$$

(b) In order to finalize the $\mathcal{H}^1$-estimate for the electric field, we must control the normal derivatives. Their tangential component is determined by the curl-term in the Maxwell system. More precisely, the second equation in (3.11), formula (6.1) and the estimate (3.14) imply

$$\big\|\partial_\nu(\partial_t^k(\hat\chi E(t))^\tau\big\|_{L^2(\Omega)}^2 \le c\big(e_{k+1}(t) + z^2(t) + \big\|\partial_\tau\partial_t^k(\hat\chi E(t))\big\|_{L^2(\Omega)}^2\big). \tag{7.10}$$

For the normal component we use the div-relations, where we also consider higher tangential derivatives for later use. We first look at the case $k\in\{1,2\}$ and apply $\partial_\tau^\alpha\hat\chi$ to equation (3.10) with $|\alpha|\le 2-k$. It follows

$$\begin{aligned}
\operatorname{div}\big(\varepsilon^{\mathrm{d}}(E)\partial_\tau^\alpha\partial_t^k(\hat\chi E)\big) &= -D(\varepsilon^{\mathrm{d}}(E),\alpha)\partial_t^k E - \operatorname{div}(\sigma\partial_\tau^\alpha(\hat\chi\partial_t^{k-1}E)) \\
&\quad - D(\sigma,\alpha)\partial_t^{k-1}E - \partial_\tau^\alpha\big(\hat\chi\operatorname{div} f_k\big).
\end{aligned} \tag{7.11}$$

Here we abbreviate the commutator terms

$$D(a,\alpha)u := \partial_\tau^\alpha\big([\hat\chi,\, \mathrm{div}\,](au)\big) + [\partial_\tau^\alpha,\, \mathrm{div}\,](\hat\chi a u) + \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\,\mathrm{div}\,\big(\partial_\tau^{\alpha-\beta}a\,\partial_\tau^\beta(\hat\chi u)\big)$$

for a matrix-valued function a and a vector function u. Observe that $D(a,\alpha)$ is a differential operator of order $|\alpha|$ and that $|D(a,0)u| \le c\,|u|$. Below we treat the equality (7.11) by means of formula (6.3). For $k=0$ the divergence equation contains a time integral and initial data which are handled by means of identity (6.5). To avoid terms which grow linearly in time, we have to derive another equation from (1.1), namely,

$$\partial_t(\varepsilon(E)\partial_\tau^\alpha(\hat\chi E)) = \mathrm{curl}\,\partial_\tau^\alpha(\hat\chi H) - \sigma\partial_\tau^\alpha(\hat\chi E) - [\mathrm{curl}, \partial_\tau^\alpha](\hat\chi H) - \partial_\tau^\alpha([\mathrm{curl}, \hat\chi]H) - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}(\sigma+\varepsilon(E))\,\partial_\tau^\beta(\hat\chi E). \tag{7.12}$$

Writing h for the sum of the three commutator terms, we derive the divergence relation

$$\mathrm{div}\,\big(\varepsilon(E(t))\partial_\tau^\alpha(\hat\chi E(t))\big) = \mathrm{div}\,\big(\varepsilon(E_0)\partial_\tau^\alpha(\hat\chi E_0)\big) - \int_0^t \big(\,\mathrm{div}(\sigma\partial_\tau^\alpha(\hat\chi E(\tau))) + \mathrm{div}\,h(\tau)\big)\,\mathrm{d}\tau. \tag{7.13}$$

(c) To control $\partial_\nu E_\nu$, we use equation (7.13) with $\alpha=0$ and identity (6.5), where we put $a=\varepsilon(E)$, $u=\hat\chi E$, and $\psi = \mathrm{div}\,h$. The function $\gamma = \sigma_{\nu\nu}/a_{\nu\nu}$ is bounded from below by $\gamma_0 = c\eta > 0$. We then get the estimate

$$\begin{aligned}&\big\|\partial_\nu(\hat\chi E(t))_\nu\big\|_{L^2(\Omega)}^2\\ &\quad\le c\mathrm{e}^{-\gamma_0 t}z(0) + c\,\big(\|E(t)\|_{L^2(\Omega)}^2 + \|\partial_\tau(\hat\chi E(t))\|_{L^2(\Omega)}^2 + \|\partial_\nu(\hat\chi E(t))^\tau\|_{L^2(\Omega)}^2\big)\\ &\qquad + c\int_0^t \mathrm{e}^{-\gamma_0(t-\tau)}\big(\|E(\tau)\|_{L^2}^2 + \|\partial_\tau(\hat\chi E(\tau))\|_{L^2}^2\\ &\qquad\qquad + \|\partial_\nu(\hat\chi E(\tau))^\tau\|_{L^2}^2 + \|H(\tau)\|_{\mathcal{H}^1}^2 + z^2(\tau)\big)\,\mathrm{d}s.\end{aligned}$$

This bound together with equations (7.9), (7.10) and (7.1) now implies

$$\begin{aligned}&\big\|\partial_\nu(\hat\chi E(t))_\nu\big\|_{L^2(\Omega)}^2 + \int_0^t \big\|\partial_\nu(\hat\chi E(s))_\nu\big\|_{L^2(\Omega)}^2\,\mathrm{d}s\\ &\quad\le c\big(z(0)+e(t)+z^2(t)\big) + \theta\int_0^t \|E(s)\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\ &\qquad + c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s,\end{aligned} \tag{7.14}$$

where the small number θ comes from (7.9). Combining (7.9), (7.10), (7.14) and (7.1), we conclude

$$\begin{aligned}&\big\|\hat\chi E(t)\big\|_{\mathcal{H}^1(\Omega)}^2+\int_0^t\big\|\hat\chi E(s)\big\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\&\quad\le c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t\|E(s)\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

For $k\in\{1,2\}$ we proceed similarly using equation (7.11) with $\alpha=0$ and formula (6.3) for the normal component. Here the term $\|\partial_t^{k-1}\hat\chi E(t)\|_{\mathcal{H}^1(\Omega)}^2$ appears on the right-hand side, which can be treated iteratively. We thus show the inequality

$$\begin{aligned}&\big\|\partial_t^k\hat\chi E(t)\big\|_{\mathcal{H}^1(\Omega)}^2+\int_0^t\big\|\partial_t^k\hat\chi E(s)\big\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\&\quad\le c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t\big\|\partial_t^k E(s)\big\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\&\qquad+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s \qquad (7.15)\end{aligned}$$

for all $k\in\{0,1,2\}$. Both in this relation and in inequality (7.3) for $|\alpha|=1$, we now choose a sufficiently small $\theta>0$. Together with (7.1), we derive our first order bound. □

Lemma 7.2. ($\mathcal{H}^1$-estimate) *Let $k\in\{0,1,2\}$ and the assumptions of Theorem 2.2 with the exception of the simple connectedness of Ω be satisfied. Then we can estimate*

$$\begin{aligned}&\big\|\partial_t^k(E(t),H(t))\big\|_{\mathcal{H}^1(\Omega)}^2+\int_0^t\big\|\partial_t^k(E(s),H(s))\big\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\&\quad\le c\big(z(0)+e(t)+z^2(t)\big)+c\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s, \qquad (7.16)\end{aligned}$$

where the constant c does not depend on time t.

(4) Estimate in $\mathcal{H}^2$. While the bound of H in $\mathcal{H}^1$ was entirely based on the curl-div-estimates of Proposition 6.1, this is only partly possible in $\mathcal{H}^2$ or $\mathcal{H}^3$ since normal derivatives violate the boundary conditions. We thus have to employ the curl-div strategy of Sect. 6.3 also for H, proceeding in multiple steps. We let $k\in\{0,1\}$.

(a) We first control tangential space-time derivatives of H in $\mathcal{H}^1$ by means of curl-div estimates. Proposition 6.1 yields

$$\begin{aligned}&\big\|\partial_\tau\partial_t^k\hat\chi H\big\|_{\mathcal{H}^1(\Omega)}\\&\quad\le c\big(\|\operatorname{curl}\partial_\tau\partial_t^k\hat\chi H\|_{L^2(\Omega)}+\|\operatorname{div}\partial_\tau\partial_t^k\hat\chi H\|_{L^2(\Omega)}+\|\operatorname{tr}_n\partial_\tau\partial_t^k\hat\chi H\|_{H^{1/2}(\Gamma)}\big).\end{aligned} \qquad (7.17)$$

From equations (3.8), (3.4) and (3.10) we deduce

$$\begin{aligned}\operatorname{tr}_n\big(\widehat\mu_k\partial_\tau\partial_t^k(\hat\chi H)\big)&=[\operatorname{tr}_n,\partial_\tau](\partial_t^k\hat\chi H)-\operatorname{tr}_n\big(\partial_\tau\widehat\mu_k\,\partial_t^k(\hat\chi H)\big),\\ \operatorname{div}\big(\widehat\mu_k\partial_\tau\partial_t^k(\hat\chi H)\big)&=\partial_\tau([\operatorname{div},\hat\chi]\widehat\mu_k\partial_t^k H)\\&\quad-[\partial_\tau,\operatorname{div}](\widehat\mu_k\partial_t^k(\hat\chi H))-\operatorname{div}(\partial_\tau\widehat\mu_k\,\partial_t^k(\hat\chi H)).\end{aligned}$$

The commutator $[\partial_\tau, \operatorname{div}]$ is of order one and the others are of order zero. For the curl-relation we can use the first equation in (7.7) with $|\alpha| = 1$. By means of (3.14), we estimate

$$\|\operatorname{div}\big(\widehat{\mu}_k \partial_\tau \partial_t^k (\hat{\chi} H(t))\big)\|_{L^2(\Omega)} \le c\,\|\partial_t^k H(t)\|_{\mathcal{H}^1(\Omega)},$$
$$\|\operatorname{curl}\big(\partial_\tau \partial_t^k \hat{\chi} H(t)\big)\|_{L^2(\Omega)} \le c\,\big(\|\partial_t^{k+1} E(t)\|_{\mathcal{H}^1(\Omega)} + \|\partial_t^k (E(t), H(t))\|_{\mathcal{H}^1(\Omega)}\big),$$
$$\|\operatorname{tr}_n\big(\widehat{\mu}_k \partial_\tau \partial_t^k (\hat{\chi} H(t))\big)\|_{\mathcal{H}^{1/2}(\Gamma)} \le c\,\|\partial_t^k H(t)\|_{\mathcal{H}^1(\Omega)}. \tag{7.18}$$

Since $k+1 \le 2$, inequalities (7.16), (7.17) and (7.18) now imply

$$\big\|\partial_\tau \partial_t^k (\hat{\chi} H(t))\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \big\|\partial_\tau \partial_t^k (\hat{\chi} H(s))\big\|_{\mathcal{H}^1(\Omega)}^2 \,\mathrm{d}s$$
$$\le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \tag{7.19}$$

(b) The estimate of the normal derivative of H in $\mathcal{H}^1$ will be based on the curl-div strategy. We first solve in the first equation of (7.7) with $\alpha = 0$ for $\partial_\nu(\partial_t^k \hat{\chi} H(t))^\tau$ using formula (6.1), and derive the inequality

$$\|\partial_\nu \partial_t^k (\hat{\chi} H(t))^\tau\|_{\mathcal{H}^1(\Omega)}$$
$$\le c\,\big(\|\partial_\tau \partial_t^k (\hat{\chi} H(t))\|_{\mathcal{H}^1(\Omega)} + \|\partial_t^k (E(t), H(t))\|_{\mathcal{H}^1(\Omega)} + \|\partial_t^{k+1} E(t)\|_{\mathcal{H}^1(\Omega)}\big).$$

Equations (7.16) and (7.19) thus allow us to bound the tangential components by

$$\big\|\partial_\nu \partial_t^k (\hat{\chi} H(t))^\tau\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \big\|\partial_\nu \partial_t^k (\hat{\chi} H(s))^\tau\big\|_{\mathcal{H}^1(\Omega)}^2 \,\mathrm{d}s$$
$$\le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \tag{7.20}$$

As to the normal component, we apply formula (6.3) to the divergence equation (7.4) with $\alpha = 0$ and $\hat{\chi}$ instead of χ. The $\mathcal{H}^1$-norm of $\partial_\nu(\hat{\chi} H(t))_\nu$ is thus controlled by that of $\hat{\chi} H(t)$, $\partial_\tau(\hat{\chi} H(t))$, and $\partial_\nu(\hat{\chi} H(t))^\tau$. From (7.16), (7.19) and (7.20), we now conclude

$$\big\|\partial_\nu \partial_t^k (\hat{\chi} H(t))_\nu\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \big\|\partial_\nu \partial_t^k (\hat{\chi} H(s))_\nu\big\|_{\mathcal{H}^1(\Omega)}^2 \,\mathrm{d}s$$
$$\le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \tag{7.21}$$

Combining the inequalities (7.19), (7.20), (7.21), (7.5) and (7.16), we arrive at the $\mathcal{H}^2$-estimate for the fields H and $\partial_t H$

$$\big\|\partial_t^k H(t)\big\|_{\mathcal{H}^2(\Omega)}^2 + \int_0^t \big\|\partial_t^k H(s)\big\|_{\mathcal{H}^2(\Omega)}^2 \,\mathrm{d}s$$
$$\le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \tag{7.22}$$

(c) We now turn our attention to E. Let $|\alpha| = 2$. The L^2-norms of the tangential derivatives $\partial_\tau^\alpha(\hat{\chi} \partial_t^k E)$ is already controlled via inequalities (7.9), (7.16), and (7.22) up to the term

$$\theta \int_0^t \left\|\partial_t^k E(\tau)\right\|_{\mathcal{H}^2(\Omega)}^2 \mathrm{d}\tau.$$

The second equation in (7.7) with $|\alpha| = 1$ and formula (6.1) lead to the estimate

$$\left\|\partial_\nu \left[\partial_\tau \partial_t^k (\hat{\chi} E(t))\right]^\tau\right\|_{L^2(\Omega)} \le c\big[\|\partial_\tau^2 \partial_t^k (\hat{\chi} E(t))\|_{L^2(\Omega)} + \|\partial_t^k (E(t),H(t))\|_{\mathcal{H}^1(\Omega)} \\ + \|\partial_t^{k+1} E(t)\|_{\mathcal{H}^1(\Omega)} + z(t)\big].$$

Combined with the above mentioned tangential bound and the $\mathcal{H}^1$–result (7.16), we obtain

$$\begin{aligned} &\left\|\partial_\nu \left(\partial_\tau \partial_t^k (\hat{\chi} E(t)\right)^\tau\right\|_{L^2}^2 + \left\|\partial_\tau^\alpha \partial_t^k (\hat{\chi} E(t))\right\|_{L^2}^2 + \int_0^t \Big[\left\|\partial_\nu \left(\partial_\tau \partial_t^k (\hat{\chi} E(s))\right)^\tau\right\|_{L^2}^2 \\ &\quad + \left\|\partial_\tau^\alpha \partial_t^k (\hat{\chi} E(s))\right\|_{L^2}^2\Big] \mathrm{d}s \\ &\quad \le c(z(0) + e(t) + z^2(t)) + \theta \int_0^t \|\partial_t^k E(s)\|_{\mathcal{H}^2(\Omega)}^2 \mathrm{d}s \\ &\quad + c(\theta) \int_0^t \left(e(s) + z^{3/2}(s)\right) \mathrm{d}s \end{aligned} \tag{7.23}$$

(d) For the normal component and $k = 0$, we look at the divergence relation (7.13) with $|\alpha| = 1$. As in (7.14), we deduce from (6.5) the estimate

$$\begin{aligned} &\left\|\partial_\nu \left(\partial_\tau (\hat{\chi} E(t))\right)_\nu\right\|_{L^2(\Omega)}^2 + \int_0^t \left\|\partial_\nu \left(\partial_\tau (\hat{\chi} E(s))\right)_\nu\right\|_{L^2(\Omega)}^2 \mathrm{d}s \\ &\quad \le c(z(0) + e(t) + z^2(t)) + \theta \int_0^t \|E(s)\|_{\mathcal{H}^2(\Omega)}^2 \mathrm{d}s \\ &\quad + c(\theta) \int_0^t \left(e(s) + z^{3/2}(s)\right) \mathrm{d}s. \end{aligned} \tag{7.24}$$

The two above inqualities imply

$$\begin{aligned} &\left\|\partial_\tau (\hat{\chi} E(t))\right\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \left\|\partial_\tau (\hat{\chi} E(s))\right\|_{\mathcal{H}^1(\Omega)}^2 \mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + \theta \int_0^t \|E(s)\|_{\mathcal{H}^2(\Omega)}^2 \mathrm{d}s + c(\theta) \int_0^t \left(e(s) + z^{3/2}(s)\right) \mathrm{d}s. \end{aligned} \tag{7.25}$$

To treat the case $k = 1$, we start from the divergence equation (7.11) with $|\alpha| = 1$ and use formula (6.3). Employing also estimates (7.23), (7.25) and (3.14), we get

$$\begin{aligned} &\left\|\partial_\nu \left(\partial_\tau (\hat{\chi} \partial_t E(t))\right)_\nu\right\|_{L^2(\Omega)}^2 + \int_0^t \left\|\partial_\nu \left(\partial_\tau (\hat{\chi} \partial_t E(s))\right)_\nu\right\|_{L^2(\Omega)}^2 \mathrm{d}s \\ &\quad \le c(z(0) + e(t) + z^2(t)) + \theta \int_0^t \left(\|E(s)\|_{\mathcal{H}^2(\Omega)}^2 + \|\partial_t E(s)\|_{\mathcal{H}^2(\Omega)}^2\right) \mathrm{d}s \\ &\quad + c(\theta) \int_0^t \left(e(s) + z^{3/2}(s)\right) \mathrm{d}s. \end{aligned} \tag{7.26}$$

Combined with inequality (7.23), this relation leads to

$$\begin{aligned} &\|\partial_\tau \partial_t(\hat\chi E(t))\|^2_{\mathcal{H}^1(\Omega)} + \int_0^t \|\partial_\tau \partial_t(\hat\chi E(s))\|^2_{\mathcal{H}^1(\Omega)}\,\mathrm{d}s \\ &\quad \le c(z(0) + e(t) + z^2(t)) + \theta \int_0^t \big(\|E(s)\|^2_{\mathcal{H}^2(\Omega)} + \|\partial_t E(s)\|^2_{\mathcal{H}^2(\Omega)}\big)\,\mathrm{d}s \\ &\qquad + c(\theta) \int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.27}$$

(e) It remains to control the term $\partial_\nu^2(\partial_t^k \hat\chi E)$. We first replace in system (7.7) the derivative ∂_τ^α by ∂_ν. The resulting second equation, the curl-formula (6.1) and estimates (3.14) allow us to bound

$$\begin{aligned} \big\|\partial_\nu\big(\partial_\nu \partial_t^k(\hat\chi E(t))\big)^\tau\|_{L^2(\Omega)} &\le c\,\big(\big\|\partial_\tau \partial_\nu \partial_t^k(\hat\chi E(t))\big\|_{L^2(\Omega)} \\ &\quad + \max_{j\le 2}\big\|\partial_t^j(E(t), H(t))\big\|_{\mathcal{H}^1(\Omega)} + z(t)\big). \end{aligned}$$

The right-hand side can be controlled via inequalities (7.16), (7.25), and (7.27).

For the normal component we use the modifications of the divergence relations (7.13) and (7.11) with ∂_ν instead of ∂_τ^α. We then estimate ∂_ν $\big(\partial_\nu \partial_t^k(\hat\chi E(t))\big)_\nu$ for $k \in \{0,1\}$ as in inequalities (7.24) and (7.26). Here and in (7.6), (7.25) and (7.27) we take a small $\theta > 0$ to absorb the $\mathcal{H}^2$-norms of $\partial_t^k E$ on the right-hand side. Using also (7.22) for the H field, we derive the desired bound in $\mathcal{H}^2$.

Lemma 7.3. ($\mathcal{H}^2$-estimate) *Let $k \in \{0,1\}$ and the assumptions of Theorem 2.2 with the exception of the simple connectedness of Ω be satisfied. Then we can estimate*

$$\begin{aligned} &\big\|\partial_t^k(E(t), H(t))\big\|^2_{\mathcal{H}^2(\Omega)} + \int_0^t \|\partial_t^k(E(s), H(s))\|^2_{\mathcal{H}^2(\Omega)}\,\mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s, \end{aligned} \tag{7.28}$$

where the constant c does not depend on time t.

(5) Estimate in $\mathcal{H}^3$. Since the reasoning is similar to the one presented above, we will omit unnecessesary details here. Let $k = 0$.

(a) We again begin with the magnetic field H. We first look at the tangential derivative $\partial_\tau^\alpha(\hat\chi E)$ with $|\alpha| = 2$, where we proceed as in (7.19) using curl-div estimates. For $\xi, \zeta \in \{\nu, \tau^1, \tau^2\}$, differentiating the divergence relation (3.10) we obtain

$$\begin{aligned} \operatorname{div}\big(\mu(H)\partial_\xi \partial_\zeta(\hat\chi H)\big) &= \partial_\xi \partial_\zeta([\operatorname{div}, \hat\chi]\mu(H)H) - [\partial_\xi\partial_\zeta, \operatorname{div}](\mu(H)\hat\chi H) \\ &\quad - \operatorname{div}(\partial_\zeta \mu(H)\,\partial_\xi(\hat\chi H)) - \operatorname{div}(\partial_\xi \mu(H)\,\partial_\zeta(\hat\chi H)) \\ &\quad - \operatorname{div}(\partial_\xi\partial_\zeta \mu(H)\,\hat\chi H). \end{aligned} \tag{7.29}$$

Similary, the magnetic boundary condition in (1.1) yields

$$\operatorname{tr}_n\big(\mu(H)\partial_\tau^\alpha(\hat\chi H)\big) = [\operatorname{tr}_n, \partial_\tau^\alpha](\mu(H)\hat\chi H) + \operatorname{tr}_n \sum_{0\le\beta<\alpha} \binom{\alpha}{\beta} \partial_\tau^{\alpha-\beta}\mu(H)\,\partial_\tau^\beta(\hat\chi H).$$

Using (3.14), from (7.7) and the above formulas we deduce the estimates

$$\begin{aligned} \|\operatorname{curl}\big(\partial_\tau^\alpha \hat\chi H(t)\big)\|_{L^2(\Omega)} &\le c\big(\|\partial_t H(t)\|_{\mathcal{H}^2(\Omega)} + \|(E(t),H(t))\|_{\mathcal{H}^2(\Omega)} + z(t)\big),\\ \|\operatorname{div}\big(\mu(H(t))\partial_\tau^\alpha(\hat\chi H(t))\big)\|_{L^2(\Omega)} &\le c\,(\|H(t)\|_{\mathcal{H}^2(\Omega)} + z(t)),\\ \|\operatorname{tr}_n\big(\mu(H(t))\partial_\tau^\alpha(\hat\chi H(t))\big)\|_{\mathcal{H}^{1/2}(\Gamma)} &\le c\,(\|H(t)\|_{\mathcal{H}^2(\Omega)} + z(t)). \end{aligned} \tag{7.30}$$

The second-order bound (7.28) and Proposition 6.1 thus imply

$$\begin{aligned} &\big\|\partial_\tau^\alpha(\hat\chi H(t))\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \|\partial_\tau^\alpha(\hat\chi H(s))\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\ &\quad\le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.31}$$

To handle the mixed derivative $\partial_\nu\partial_\tau$, we use the first equation in (7.7) with $|\alpha|=1$ and the curl-formula (6.1). We can then bound the $\mathcal{H}^1$-norm of $\partial_\nu(\partial_\tau(\hat\chi H(t)))^\tau$ by

$$\|\partial_\tau^2(\hat\chi H(t))\|_{\mathcal{H}^1(\Omega)} + \max_{j\le 1}\big\|\partial_t^j(E(t),H(t))\big\|_{\mathcal{H}^2(\Omega)} + z(t).$$

The normal component is treated as in (7.21), based on the divergence relation (7.4) with $|\alpha|=1$, χ replaced by $\hat\chi$, and ∂_x^α by ∂_τ. By means of (6.3) and (3.14), the $\mathcal{H}^1$-norm of the function $\partial_\nu(\partial_\tau(\hat\chi H(t)))_\nu$ is thus controlled by that of $\partial_\nu(\partial_\tau(\hat\chi H(t)))^\tau$ and $\partial_\tau^2(\hat\chi H(t))$ plus lower order terms. Combing these inequalities with (7.28) and (7.31), we infer

$$\begin{aligned} &\big\|\partial_\nu\partial_\tau(\hat\chi H(t))\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \|\partial_\nu\partial_\tau(\hat\chi H(s))\|_{\mathcal{H}^1(\Omega)}^2\,\mathrm{d}s\\ &\quad\le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.32}$$

In this reasoning we can replace ∂_τ by ∂_ν, arriving at

$$\begin{aligned} &\big\|\partial_\nu^2(\hat\chi H(t))\big\|_{\mathcal{H}^1}^2 + \int_0^t \|\partial_\nu^2(\hat\chi H(s))\|_{\mathcal{H}^1}^2\,\mathrm{d}s\\ &\quad\le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.33}$$

Combined with (7.6), the estimates (7.31), (7.32) and (7.33) lead to

$$\begin{aligned} &\big\|H(t)\big\|_{\mathcal{H}^3(\Omega)}^2 + \int_0^t \|H(s)\|_{\mathcal{H}^3(\Omega)}^2\,\mathrm{d}s\\ &\quad\le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.34}$$

(b) We finally tackle E in $\mathcal{H}^3$. The third-order tangential derivatives $\partial_\tau^\alpha(\hat\chi E)$ were already treated in estimate (7.9) with $k=0$, where the lower order-terms on the right-hand side are now dominated by (7.28) and (7.34). Let $|\beta|=2$. The second equation in (7.7) with $|\alpha|=2$ and the curl-formula (6.1) allow us to bound $\partial_\nu(\partial_\tau^\beta(\hat\chi E))^\tau$ in the same fashion. The normal component

$\partial_\nu(\partial_\tau^\beta(\hat{\chi}E))_\nu$ can also be controlled via equations (7.13) and (6.5). We thus arrive at

$$\begin{aligned} &\big\|\partial_\tau^\beta(\hat{\chi}E(t))\big\|_{\mathcal{H}^1(\Omega)}^2 + \int_0^t \big\|\partial_\tau^\beta(\hat{\chi}E(s))\big\|_{\mathcal{H}^1(\Omega)}^2 \,\mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + \theta \int_0^t \big\|E(s)\big\|_{\mathcal{H}^3(\Omega)}^2 \,\mathrm{d}s \\ &\qquad + c(\theta) \int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.35}$$

We replace in system (7.7) the tangential derivative ∂_τ^α with $\partial_\nu\partial_\tau$. The second equation therein and formula (6.1) provide control of the tangential component $\partial_\nu(\partial_\nu\partial_\tau(\hat{\chi}E))^\tau$ in L^2 via inequalities (7.35) and (7.28). The related normal component can then be handled through the formula (6.5) and the divergence identity (7.13) with $\partial_\nu\partial_\tau$ instead of ∂_τ^α. In this way we show the estimate

$$\begin{aligned} &\big\|\partial_\tau(\hat{\chi}E(t))\big\|_{\mathcal{H}^2(\Omega)}^2 + \int_0^t \big\|\partial_\tau(\hat{\chi}E(s))\big\|_{\mathcal{H}^2(\Omega)}^2 \,\mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + \theta \int_0^t \big\|E(s,t)\big\|_{\mathcal{H}^3(\Omega)}^2 \,\mathrm{d} \\ &\qquad + c(\theta) \int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned}$$

The remaining $\partial_\nu^3(\hat{\chi}E)$-term is managed analogously, resulting in the inequality

$$\begin{aligned} &\big\|\hat{\chi}E(t)\big\|_{\mathcal{H}^3(\Omega)}^2 + \int_0^t \big\|\hat{\chi}E(s)\big\|_{\mathcal{H}^3(\Omega)}^2 \,\mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + \theta \int_0^t \big\|E(s)\big\|_{\mathcal{H}^3(\Omega)}^2 \,\mathrm{d}s \\ &\qquad + c(\theta) \int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned}$$

Fixing a sufficiently small number $\theta > 0$, the above inequalities and the interior estimate (7.6) lead to the final bound

$$\begin{aligned} &\big\|E(t)\big\|_{\mathcal{H}^3(\Omega)}^2 + \int_0^t \big\|E(s)\big\|_{\mathcal{H}^3(\Omega)}^2 \,\mathrm{d}s \\ &\quad \le c\big(z(0) + e(t) + z^2(t)\big) + c \int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{7.36}$$

Equation (7.34) and (7.36) now furnish our last result.

Lemma 7.4. ($\mathcal{H}^3$ estimate) *Let the assumptions of Theorem 2.2 with the exception of the simple connectedness of* Ω *be satisfied. Then we can estimate*

$$\begin{aligned}
&\big\|(E(t), H(t))\big\|^2_{\mathcal{H}^3(\Omega)} + \int_0^t \big\|(E(s), H(s))\big\|^3_{\mathcal{H}^3(\Omega)}\,\mathrm{d}s \\
&\quad \le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\mathrm{d}s,
\end{aligned}$$

where the constant c does not depend on time t.

Lemmas 7.2, 7.3 and 7.4 complete the proof of Proposition 7.1. □

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