

MINIMIZERS OF CONVEX FUNCTIONALS WITH SMALL DEGENERACY SET

CONNOR MOONEY

ABSTRACT. We study the question whether Lipschitz minimizers of $\int F(\nabla u) dx$ in $\mathbb{R}^n$ are C^1 when F is strictly convex. Building on work of De Silva-Savin, we confirm the C^1 regularity when D^2F is positive and bounded away from finitely many points that lie in a 2-plane. We then construct a counterexample in $\mathbb{R}^4$, where F is strictly convex but D^2F degenerates on the intersection of a Simons cone with S^3 . Finally we highlight a connection between the case $n = 3$ and a result of Alexandrov in classical differential geometry, and we make a conjecture about this case.

1. INTRODUCTION

In this paper we study the regularity of Lipschitz minimizers of

$$(1) \quad E(u) = \int_{B_1} F(\nabla u) dx$$

in $\mathbb{R}^n$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex. By Lipschitz minimizer we mean a function $u \in W^{1,\infty}(B_1)$ that satisfies $E(u + \varphi) \geq E(u)$ for all $\varphi \in C_0^1(B_1)$. It is straightforward to show that Lipschitz minimizers solve the Euler-Lagrange equation

$$(2) \quad \operatorname{div}(\nabla F(\nabla u)) = 0$$

in the weak sense. Conversely, any Lipschitz weak solution of (2) is a minimizer of E by the convexity of F .

In the extreme case that the graph of F contains a line segment, minimizers are no better than Lipschitz by simple examples. In the other extreme that F is smooth and uniformly convex, De Giorgi and Nash proved that Lipschitz minimizers are smooth and solve the Euler-Lagrange equation $F_{ij}(\nabla u)u_{ij} = 0$ classically ([DG], [Na]). It remains largely open what happens in the intermediate case where F is strictly convex, but the eigenvalues of D^2F go to 0 or ∞ on some set D_F . Such functionals arise naturally in the study of anisotropic surface tensions [DMMN], traffic flow [CF], and statistical mechanics ([CKP], [KOS]).

In [DS] the authors raise the natural question:

$$(3) \quad \textit{Are Lipschitz minimizers in } C^1 \textit{ when } F \textit{ is strictly convex?}$$

They give evidence that the answer may be “yes,” at least in two dimensions. In particular, they show that if $n = 2$ and D_F consists of finitely many points, then Lipschitz minimizers of E are C^1 . In this paper we study this question in higher dimensions. We first confirm the C^1 regularity of Lipschitz minimizers when D_F is a finite set in some 2-plane. In particular, this covers the case that D_F consists of three points. We then show the answer to Question (3) is “no” in general, by constructing a singular Lipschitz minimizer in $\mathbb{R}^4$. In our example, F is in fact uniformly convex and C^1 , but one eigenvalue of D^2F goes to ∞ on the intersection

of a Simons cone with S^3 . This leaves open the possibility that Lipschitz minimizers are C^1 in dimension $n \geq 3$ in the interesting case that D_F consists of finitely many points. To address this problem we connect it to a result of Alexandrov in the classical differential geometry of convex surfaces, and we propose a possible counterexample in $\mathbb{R}^3$ where D_F consists of four non-coplanar points.

Remark 1.1. Guided by the observation that the Legendre transform F^* of F solves $\operatorname{div}(\nabla F(\nabla F^*)) = \operatorname{div}(x) = n$, one could (more ambitiously) ask whether the minimizers are as regular as F^* . This is known in some special cases, e.g. for the p -Laplace case $F(x) = |x|^p$ when $p > 2$ and $n = 2$ (see [ATO] and also [IM], where it is shown that in this case minimizers are in fact *more* regular than F^*).

Remark 1.2. The case that D_F consists of a single point (e.g. p -Laplace) is well-studied (see [E], [Uh], [Ur]). The case that D_F is “large” is also understood: in [CF] the authors show that if D_F is convex and $F = 0$ on D_F , then for $x \in B_1$ the gradients $\nabla u(B_r(x))$ localize as $r \rightarrow 0$ either to a point outside D_F or to D_F .

Remark 1.3. One can show the existence of Lipschitz minimizers with additional hypotheses on the behavior of F at infinity. For example, if F has quadratic growth, then for $g \in H^1(B_1)$ the direct method gives the existence of a minimizer $u \in H^1(B_1)$ with $u - g \in H_0^1(B_1)$. If g is smooth enough ($C^{1,1}$ suffices) then u is Lipschitz by the comparison principle. Alternatively, if F is uniformly convex with bounded second derivatives at infinity, then u is locally Lipschitz. For a proof of this result, see [Ma2]. The local Lipschitz regularity of minimizers is in fact true under assumptions that allow for growth of D^2F at infinity (the so called (p, q) growth conditions); see [Ma1], [Ma2], and the references therein. However, if the growth is sufficiently anisotropic, then there can be discontinuous and even unbounded minimizers (see [Gi], [Ma3]).

The paper is organized as follows. In Section 2 we give precise statements of our results, and we discuss a connection between the problem in dimension $n = 3$ and a result of Alexandrov. In Section 3 we prove the C^1 regularity result. In Section 4 we construct the counterexample. Finally, in the Appendix we record some technical results that we used to construct the counterexample.

2. STATEMENTS OF RESULTS

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 convex function, and let $D_F \subset \mathbb{R}^n$ be a compact set such that

$$F \in C^2(\mathbb{R}^n \setminus D_F), \quad D_F = \mathbb{R}^n \setminus \left(\bigcup_{k>1} \{k^{-1}I < D^2F < kI\} \right).$$

Here and below, dependence on F means dependence on the sets

$$\mathcal{O}_k := \{k^{-1}I < D^2F < kI\} \subset \mathbb{R}^n \setminus D_F$$

(in particular, the geometry of D_F), and on the moduli of continuity of D^2F in compact sets that exhaust $\mathbb{R}^n \setminus D_F$. Our first theorem is:

Theorem 2.1. *Let u be a Lipschitz solution of (2). If D_F is finite and is contained in a two-dimensional affine subspace of $\mathbb{R}^n$, then $u \in C^1(B_1)$, and the modulus of continuity of ∇u in $B_{1/2}$ depends only on n , F , and $\|\nabla u\|_{L^\infty(B_1)}$.*

Remark 2.2. We conjecture that the assumption in Theorem 2.1 is optimal. That is, that there exists a singular minimizer in $\mathbb{R}^3$ where D_F consists of four non-coplanar points (see the discussion in Section 2.1).

The starting point of Theorem 2.1 is the well-known fact that convex functions of ∇u are sub-solutions to the linearized Euler-Lagrange equation. Using this fact we show that $\nabla u(B_r)$ localizes as $r \rightarrow 0$ either to a point outside D_F (in which case we are done), or to the convex hull of D_F . This was observed in [CF] in the case that D_F is a convex set and $F = 0$ on D_F , motivated by models of traffic congestion. The key observation in [DS] is that in two dimensions, certain slightly non-convex functions of ∇u are also sub-solutions to the linearized equation. If the convex hull of D_F is two-dimensional, we can use higher-dimensional versions of these functions to further localize the gradients to a point.

To state our second result we let $x = (x_1, x_2) \in \mathbb{R}^{2n}$ with $x_i \in \mathbb{R}^n$. We define

$$(4) \quad w(x) := \frac{1}{\sqrt{2}} \frac{|x_2|^2 - |x_1|^2}{|x|}.$$

Then w is a nontrivial one-homogeneous function on $\mathbb{R}^{2n}$ that is analytic outside of the origin. We show:

Theorem 2.3. *When $n \geq 2$, w is a minimizer of a functional of the form (1) with F uniformly convex and C^1 , and $D_F = \{|x_1|^2 = |x_2|^2\} \cap \sqrt{2} S^{2n-1}$.*

Our approach to Theorem 2.3 is based on the observation that when $n \geq 2$, the gradient image $\Sigma_w := \nabla w(\mathbb{R}^{2n} \setminus \{0\}) = \nabla w(S^{2n-1})$ is a saddle-shaped hypersurface that is smooth away from a “cusp” singularity on $\{|x_1|^2 = |x_2|^2\} \cap \sqrt{2} S^{2n-1}$. This reflects that $D^2 w$ has positive and negative eigenvalues, and thus solves some elliptic equation. We then build the integrand F near Σ_w so that the Euler-Lagrange equation (2) is satisfied, and finally we make a global convex extension. In previous work with Savin we took a similar approach to construct singular minimizers of functionals with large degeneracy set in $\mathbb{R}^3$, where D_F consists of two disconnected convex sets with nonempty interior [MS].

2.1. The Case $n = 3$ and Hyperbolic Hedgehogs. To conclude the section we highlight a connection between our approach to Theorem 2.3 and classical differential geometry.

Natural candidates for singular minimizers are one-homogeneous functions with Hessians that have indefinite sign. Indeed, such functions are invariant under the rescalings that preserve (2), and they solve some elliptic PDE. It is useful to identify a one-homogeneous function u with its gradient image, a (possibly singular) hypersurface Σ_u . The function u is the support function of Σ_u , and the eigenvalues of $D^2 u$ on S^{n-1} are the principal radii of Σ_u . The set Σ_u is the parallel set a distance A in the direction of the *inward* unit normal from the convex body $\Sigma_{u+A|x|}$, where A is chosen large enough that $D^2 u + AI > 0$ on S^{n-1} . Such parallel surfaces to a convex body are known in the literature as “hedgehogs” (see e.g. [MM2]).

In dimension $n = 3$, a natural candidate for a singular minimizer thus corresponds to a hedgehog that is saddle-shaped away from its singularities, i.e. a parallel set a distance A in the inward direction from a convex surface with principal radii $r_1, r_2 > 0$ that satisfy $(r_1 - A)(r_2 - A) \leq 0$. Alexandrov originally conjectured that the only such convex surfaces in $\mathbb{R}^3$ are spheres. He proved his conjecture for analytic convex surfaces ([A1], [A2]). Thus, we cannot construct with our method a singular minimizer in $\mathbb{R}^3$ that is analytic outside of the origin (compare to Theorem 2.3). For C^2 surfaces, Alexandrov’s conjecture remained open for a long time

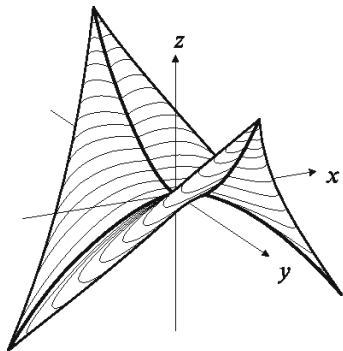


FIGURE 1. The hyperbolic hedgehog Σ_h from [MM1].

(with at least one incorrect proof). It was resolved in 2001 by a beautiful counterexample of Martinez-Maure ([MM1]). Martinez-Maure's hedgehog is built by gluing together four self-intersecting "cross caps" with figure-eight cross sections that shrink to cusps (see Figure 1). Motivated by this discussion and Theorem 2.1 we conjecture:

Conjecture 2.4. *The support function h of the hedgehog from [MM1] is a one-homogeneous singular minimizer of a functional of the type (1), where D_F consists exactly of the four cusps of Σ_h .*

This would show that the geometric conditions on D_F in Theorem 2.1 are optimal.

Remark 2.5. The surface Σ_h can be written as the union of two graphs, which makes writing the Euler-Lagrange equation on Σ_h relatively simple. Using this observation we can show that it is possible to construct F locally (in particular, in a small neighborhood of a cusp), with some tedious calculation. It seems challenging to construct F globally, but so far we do not see a fundamental obstruction. Our approach may benefit from calculations by a computer, as in [MM1].

Remark 2.6. The regularity of h in the example from ([MM1]) is C^2 . Smooth counterexamples to Alexandrov's conjecture, with (a version of) Σ_h as a special case, were later constructed by Panina [P].

3. PROOF OF THEOREM 2.1

Choose M_0 large so that $D_F \subset B_{M_0}$, and let $M = \max\{M_0, \|u\|_{L^\infty(B_1)}\}$. By a standard approximation argument, to prove Theorem 2.1 it suffices to assume $u, F \in C^\infty$ and show that the modulus of continuity of ∇u in $B_{1/2}$ depends only

on M , the sets $\{\mathcal{O}_k\}$, and the moduli of continuity of D^2F in the sets $\{B_M \cap \mathcal{O}_k\}$ (see e.g. [CF]).

3.1. Preliminaries. We record some important preliminary results. Our argument is based on applying the following estimate of De Giorgi (the “weak Harnack inequality”) to various functions of ∇u :

Proposition 3.1. *Assume that $v \geq 0$ is in $H^1(B_1)$ and solves $\partial_i(a_{ij}(x)v_j) \geq 0$, with a_{ij} bounded measurable and $\lambda I \leq (a_{ij}) \leq \lambda^{-1}I$ for some $\lambda > 0$. Then for all $\mu > 0$, there exists $\nu(\mu, n, \lambda) > 0$ such that if*

$$\frac{|\{v > 0\} \cap B_1|}{|B_1|} \leq 1 - \mu$$

then

$$\sup_{B_{1/2}} v \leq (1 - \nu) \sup_{B_1} v.$$

To prove Proposition 3.1 apply the weak Harnack inequality for supersolutions (Theorem 8.18 in [GT]) to $\sup_{B_1} v - v$.

We now discuss the types of functions of ∇u that Proposition 3.1 applies to. We denote the linearized Euler-Lagrange operator by L_F . That is,

$$L_F(v) := \operatorname{div}(D^2F(\nabla u)\nabla v) = \partial_i(F_{ij}(\nabla u)v_j).$$

The key observation is that if η is slightly concave in only one direction, then $\eta(\nabla u)$ is a subsolution of L_F where ∇u avoids D_F . For $\Omega \subset \mathbb{R}^n$ let $\mathcal{N}_\delta(\Omega)$ denote the δ -neighborhood of Ω . We have:

Lemma 3.2. *Assume η is a smooth function in a neighborhood of $\nabla u(B_1)$. For any $\rho \in (0, 1)$, there exists $\lambda(\rho, F, M, n) > 0$ such that if $\nabla u(B_1) \cap \{\eta > 0\} \subset B_M \setminus \mathcal{N}_\rho(D_F)$, and in $\nabla u(B_1) \cap \{\eta > 0\}$ the eigenvalues $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n$ of $D^2\eta$ satisfy $\gamma_2 > 0$ and $\gamma_1 \geq -\lambda\gamma_2$, then*

$$L_F(\eta_+(\nabla u)) \geq 0.$$

Here $\eta_+ := \max\{\eta, 0\}$.

Proof. Using that $L_F(u_k) = 0$ we compute

$$L_F(\eta(\nabla u)) = \operatorname{div}(D^2F(\nabla u)\nabla(\eta(\nabla u))) = F_{ij}u_{jk}\eta_{kl}u_{li}.$$

At a fixed point $x_0 \in \{\eta(\nabla u) > 0\}$ choose coordinates so that $\eta_{kl}(\nabla u(x_0)) = \gamma_k\delta_{kl}$. Summing over l we obtain

$$L_F(\eta(\nabla u))(x_0) = \sum_{k=1}^n \gamma_k \sum_{i,j=1}^n F_{ij}u_{ki}u_{kj}.$$

For some m large depending on ρ, F, M we have $\nabla u(B_1) \cap \{\eta > 0\} \subset \mathcal{O}_m$. Since $\gamma_1 \geq -\lambda\gamma_2$ in $\{\eta(\nabla u) > 0\}$ we conclude that

$$\gamma_2^{-1}L_F(\eta(\nabla u))(x_0) \geq m^{-1} \sum_{k=2}^n |\nabla u_k(x_0)|^2 - m\lambda |\nabla u_1(x_0)|^2.$$

If $L_F(\eta(\nabla u))(x_0) < 0$ then the above inequality gives

$$(\lambda^{-1}m^{-2} - 1) \sum_{(i,j) \neq (1,1)} u_{ij}^2(x_0) < u_{11}^2(x_0).$$

On the other hand, the equation $F_{11}u_{11} = -F_{ij}u_{ij}|_{(i,j) \neq (1,1)}$ gives

$$u_{11}^2 \leq C(n, m) \sum_{(i,j) \neq (1,1)} u_{ij}^2$$

in $\{\eta(\nabla u) > 0\}$. The previous two inequalities contradict each other for $\lambda(n, m)$ small. $\square$

In order to apply Proposition 3.1 and Lemma 3.2, we need ∇u to be close to D_F in sets of positive measure. The alternative is that u nearly solves a non-degenerate equation. To handle this situation we will also use a “flatness implies regularity” result for u :

Proposition 3.3. *Assume that a_{ij} are smooth elliptic coefficients on $\mathbb{R}^n$ that satisfy $\lambda I \leq (a_{ij}) \leq \lambda^{-1}I$ in $B_\rho(p)$ for some fixed $\lambda, \rho > 0, p \in \mathbb{R}^n$. There exists $\epsilon > 0$ depending on ρ, n, λ and the modulus of continuity of a_{ij} in $B_\rho(p)$ such that if $v \in C^\infty(B_1)$ solves $a_{ij}(\nabla v)v_{ij} = 0$ and*

$$\|v - l_p\|_{L^\infty(B_1)} \leq \epsilon$$

for some linear function l_p with $\nabla l_p = p$, then

$$\nabla v(B_{1/2}) \subset B_\rho(p).$$

Heuristically, $w := \epsilon^{-1}(v - l_p)$ solves $a_{ij}(p + \epsilon \nabla w)w_{ij} = 0$ which is nearly a constant-coefficient equation for ϵ small. The idea of Proposition 3.3 is due to Savin [S], who treated equations with degeneracy in the Hessian of v . For a proof of the proposition as stated (with degeneracy in the gradient of v) see [CF].

An easy consequence of Proposition 3.3 is:

Lemma 3.4. *For any $\rho \in (0, 1)$ there exist $\epsilon_1, \mu_1(\rho, F, M, n) > 0$ such that if*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p)\} \cap B_r|}{|B_r|} \geq 1 - \mu_1$$

for some $p \in B_{2M} \setminus \mathcal{N}_{2\rho}(D_F)$ then

$$\nabla u(B_{r/2}) \subset B_\rho(p).$$

Proof. After taking $u \rightarrow r^{-1}u(rx)$ we may assume that $r = 1$. Since u solves $F_{ij}(\nabla u)u_{ij} = 0$, by Proposition 3.3 there exists $\epsilon_0 > 0$ depending on ρ, F, M, n such that if

$$\|u - l_p\|_{L^\infty(B_1)} \leq \epsilon_0,$$

for some linear function l_p with $\nabla l_p = p$, then $\nabla u(B_{1/2}) \subset B_\rho(p)$. The above inequality holds by standard embeddings if we take e.g. $\epsilon_1 < c(n)\epsilon_0$ and take μ_1 small depending on M, n, ϵ_0 . $\square$

Our approach to Theorem 2.1 is to first show that as $r \rightarrow 0$, the sets $\nabla u(B_r)$ localize to the convex hull of D_F , and then to show that if this set is two-dimensional, they localize to a point. We treat these two results separately in the following subsections, and then combine them.

3.2. Localization to the Convex Hull. Let K_F denote the convex hull of D_F . In this subsection we show:

Proposition 3.5. *For any $\rho > 0$, there exists $s(\rho, F, M, n) > 0$ such that either $\nabla u(B_s) \subset B_\rho(p)$ for some $p \in B_{2M} \setminus \mathcal{N}_{2\rho}(D_F)$, or $\nabla u(B_s) \subset \mathcal{N}_{4\rho}(K_F)$.*

In this subsection we call a constant universal if it depends only on ρ, M, F, n . Let β be a smooth uniformly convex function on $\mathbb{R}^n$ such that

$$\mathcal{N}_{3\rho}(K_F) \subset \{\beta \leq 0\} \subset \mathcal{N}_{4\rho}(K_F),$$

with $B_{2M} \subset \{\beta < \tilde{M}\}$ for some universal $\tilde{M}$. Let $\epsilon_1, \mu_1 > 0$ be the universal constants from Lemma 3.4, corresponding to ρ .

Lemma 3.6. *There exists $\delta > 0$ universal such that if $\sup_{B_r} \beta(\nabla u) > 0$ and*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p)\} \cap B_r|}{|B_r|} < 1 - \mu_1$$

for all $p \in B_{2M} \setminus \mathcal{N}_{2\rho}(D_F)$, then

$$\sup_{B_{r/2}} \beta(\nabla u) \leq \sup_{B_r} \beta(\nabla u) - \delta.$$

Proof. After taking $u \rightarrow r^{-1}u(rx)$ we may assume that $r = 1$. Let

$$0 < t := \sup_{B_1} \beta(\nabla u) \leq \tilde{M}.$$

For any unit vector e let $s_e := \sup_{\{\beta < t\}} (p \cdot e)$. There is some universal $\delta_0 > 0$ (independent of e) such that for some $c_e \leq s_e - \delta_0$ we have

$$\text{diam}(\{p \cdot e \geq c_e\} \cap \{\beta < t\}) < \epsilon_1$$

and that D^2F has universal ellipticity constant in $\{p \cdot e > c_e\}$. We denote the function $\nabla u \cdot e$ by u_e , and we note that $u_e \leq s_e$ in B_1 by the definition of s_e . By the hypotheses we may apply Proposition 3.1 to $v := (u_e - c_e)_+$ with $\mu = \mu_1$ to conclude that

$$u_e \leq (1 - \nu_1)s_e + \nu_1 c_e \leq s_e - \nu_1 \delta_0$$

in $B_{1/2}$, with $\nu_1 > 0$ universal. (Here we use that the coefficients of L_F have universal ellipticity constant in $\{v > 0\}$; we can replace the coefficients by e.g. δ_{ij} in $\{v = 0\}$ without changing the equation for v). Since

$$\cap_{e \in S^{n-1}} \{p \cdot e \leq s_e - \nu_1 \delta_0\} \subset \{\beta < t - \delta\}$$

for some universal $\delta > 0$ the proof is complete. $\square$

Proof of Proposition 3.5. Apply the following algorithm for $k \geq 0$: if one of the hypotheses of Lemma 3.6 is not satisfied in $B_{2^{-k}}$, then stop. We either have $\nabla u(B_{2^{-k}}) \subset \{\beta \leq 0\} \subset \mathcal{N}_{4\rho}(K_F)$, or we can apply Lemma 3.4 to conclude $\nabla u(B_{2^{-k-1}}) \subset B_\rho(p)$ for some $p \in B_{2M} \setminus \mathcal{N}_{2\rho}(D_F)$. Otherwise, we apply Lemma 3.6. The algorithm terminates after at most k_0 steps with $\tilde{M} - k_0 \delta \leq 0$. $\square$

3.3. Localization Beyond the Convex Hull. In this subsection we show that if $\nabla u(B_1)$ is sufficiently close to a two-dimensional affine subspace, then as $r \rightarrow 0$ the gradients $\nabla u(B_r)$ localize to a connected component of D_F .

Let $(p, q) \in \mathbb{R}^n$ with $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^{n-2}$, and assume that $D_F \subset \{q = 0\}$. Let $8\rho_0$ be the smallest distance between a pair of points in D_F . In this subsection we call constants depending on ρ_0, F, M, n universal.

Proposition 3.7. *There exist $\sigma_0, r_0 > 0$ universal such that if $\nabla u(B_1) \subset \{|q| < \sigma_0\}$ then either $\nabla u(B_{r_0}) \subset B_{\rho_0}(p)$ for some $p \in B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F)$ or $\nabla u(B_{r_0}) \subset \mathcal{N}_{3\rho_0}(D_F)$.*

In particular, since ∇u is (qualitatively) continuous the set $\nabla u(B_{r_0})$ is connected, so it is contained in a ball with at most one point of D_F , and is a distance at least ρ_0 from the remaining points in D_F .

The idea is to localize the gradients using the level sets of non-convex functions of ∇u . Let $\epsilon_1, \mu_1 > 0$ be the (universal) constants from Lemma 3.4 with $\rho = \rho_0$. We assume by taking ϵ_1 smaller if necessary that $\epsilon_1 \leq \rho_0$. Let λ_1 be the constant from Lemma 3.2 with $\rho = \rho_0$. Finally, let ν_1 be the constant from Proposition 3.1 corresponding to $\mu = \mu_1$ and the ellipticity constants of D^2F in $B_{2M} \setminus \mathcal{N}_{\rho_0}(D_F)$. The following lemma says that when the gradient image is sufficiently close to $\{q = 0\}$, we can “chop” at its projection to $\{q = 0\}$ with circles (see Figure 2):

Lemma 3.8. *Let $(p_0, 0) \in B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F)$. There exist $\sigma_0, \delta_0 > 0$ universal such that if $\nabla u(B_r) \subset \{|q| < \sigma_0\} \cap \{|p - p_0| \geq \epsilon_1/4\}$ and*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p_0, 0)\} \cap B_r|}{|B_r|} < 1 - \mu_1,$$

then

$$\nabla u(B_{r/2}) \subset \{|p - p_0| \geq \epsilon_1/4 + \delta_0\}.$$

Proof. We may assume that $r = 1$ after a Lipschitz rescaling. Define

$$\eta_A(p, q) := e^{A^2|q|^2/2 - A|p|}, \quad \eta_{A, p_0} := \eta_A(p - p_0, q) - e^{-A\epsilon_1/2}.$$

In an appropriate system of coordinates we have in $\{|q| < A^{-3}\}$ that

$$(A^2\eta_A)^{-1}D^2\eta_A = \text{diag}(-(A|p|)^{-1}, 1, \dots, 1) + O(A^{-2}),$$

and that $\{\eta_A > e^{-A\epsilon_1/2}\} \subset \{|p| < \epsilon_1/2 + A^{-5}\}$. Then by our first hypothesis, for A large universal and $\sigma_0 < A^{-3}$ we have $\nabla u(B_1) \cap \{\eta_{A, p_0} > 0\} \subset B_M \cap B_{\epsilon_1}(p_0, 0) \subset B_M \setminus \mathcal{N}_{\rho_0}(D_F)$, and that the eigenvalues $\gamma_1 \leq \dots \leq \gamma_n$ of $D^2\eta_{A, p_0}$ satisfy $\gamma_2 > 0$ and $\gamma_1 > -\lambda_1\gamma_2$ in $\nabla u(B_1) \cap \{\eta_{A, p_0} > 0\}$. We conclude using Lemma 3.2 that the function $v_{p_0} := (\eta_{A, p_0})_+(\nabla u)$ satisfies $L_F(v_{p_0}) \geq 0$. By our second hypothesis we can apply Proposition 3.1 to v_{p_0} . In the extreme case that $\sigma_0 = 0$, Proposition 3.1 gives

$$\nabla u(B_{1/2}) \subset \{\eta_{A, p_0} < (1 - \nu_1)(e^{-A\epsilon_1/4} - e^{-A\epsilon_1/2})\} \subset \{|p - p_0| \geq \epsilon_1/4 + 2\delta_0\}$$

for some $\delta_0 > 0$ universal. By continuity we have the same inclusion with $2\delta_0$ replaced by δ_0 for sufficiently small $\sigma_0 < A^{-3}$, completing the proof. $\square$

We can now prove Proposition 3.7.

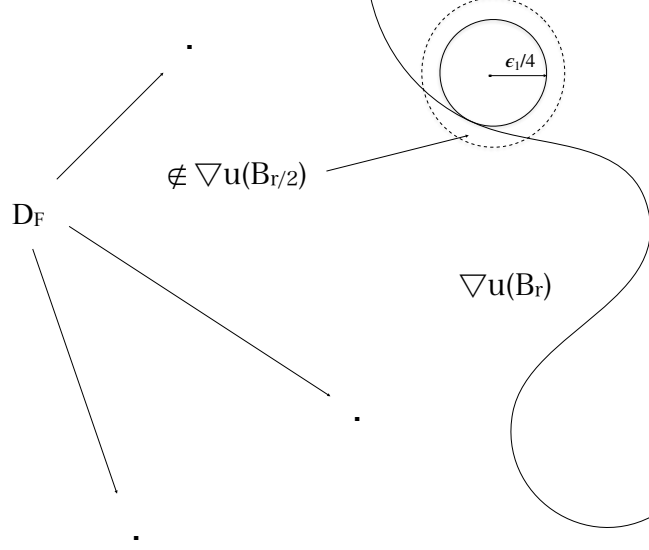


FIGURE 2. If $\nabla u(B_r)$ is “nearly $2D$ ” we can chop its projection to $\{q = 0\}$ with circles. The picture illustrates the case $n = 2$.

Proof of Proposition 3.7: Take σ_0 as in Lemma 3.8, and apply the following algorithm for $k \geq 0$: If

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p_0, 0)\} \cap B_{2^{-k}}|}{|B_{2^{-k}}|} \geq 1 - \mu_1$$

for some $(p_0, 0) \in B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F)$, then stop. We have by Lemma 3.4 that $\nabla u(B_{2^{-k-1}}) \subset B_{\rho_0}(p_0, 0)$. If not, apply Lemma 3.8 to conclude that

$$\nabla u(B_{2^{-k-1}}) \subset \{|p - p_0| \geq \epsilon_1/4 + \delta_0\}$$

for all $(p_0, 0) \in B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F)$ such that $\nabla u(B_{2^{-k}}) \subset \{|p - p_0| \geq \epsilon_1/4\}$. If at this point we can conclude that $\nabla u(B_{2^{-k-1}}) \subset \mathcal{N}_{3\rho_0}(D_F)$, then stop.

To show that this algorithm terminates after a universal number of steps, we use a simple covering argument. Let $\mathcal{S}_k$ be the projections of the sets $\nabla u(B_{2^{-k}})$ to $\{q = 0\}$. Take a finite number of lines $\{l_i\}$ in $\{q = 0\}$ that avoid $\mathcal{N}_{2\rho_0}(D_F)$, whose $\epsilon_1/4$ neighborhoods cover $B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F) \cap \{q = 0\}$. For each l_i take a universal number $J + 1$ of two-dimensional balls $\{B_{ij} := B_{\epsilon_1/4}(p_{ij})\}_{j=1}^{J+1}$ in $\{q = 0\}$ that cover $B_{2M} \cap l_i \cap \{q = 0\}$, with centers $p_{ij} \in B_{2M} \cap l_i \cap \{q = 0\}$ on l_i such that $|p_{i,j+1} - p_{ij}| \leq \delta_0$ for $j = 1, \dots, J$. Since $\mathcal{S}_0 \subset B_M$ we can arrange that $B_{i1} \cap \mathcal{S}_0 = \emptyset$ for all i . By induction, if the algorithm doesn't terminate after k steps, then $\mathcal{S}_k$ has empty intersection with the convex hull of B_{i1} and $B_{i,k+1}$, for each i . In particular, after J steps we have that $\mathcal{S}_J \subset \mathcal{N}_{2\rho_0}(D_F)$, and the proof is complete up to replacing σ_0 with $\min\{\sigma_0, \rho_0\}$. $\square$

3.4. Proof of Theorem 2.1. We are now in position to prove Theorem 2.1. We call constants depending on F, M, n universal.

Proof. For any $\epsilon > 0$ we will show that there is some $\delta(\epsilon, F, M, n) > 0$ such that $\nabla u(B_\delta)$ is contained in a ball of radius ϵ .

Take σ_0 to be the constant from Proposition 3.7. Applying Proposition 3.5 with $\rho = \sigma_0/4$ we obtain $s_0 > 0$ universal such that either $\nabla u(B_{s_0}) \subset B_{\sigma_0/4}(p)$ for some $p \in B_{2M} \setminus \mathcal{N}_{\sigma_0/2}(D_F)$, or $\nabla u(B_{s_0}) \subset \mathcal{N}_{\sigma_0}(K_F)$. In the latter case, apply Proposition 3.7 to $s_0^{-1}u(s_0x)$ to conclude for some $r_0 > 0$ universal that either $\nabla u(B_{r_0s_0}) \subset B_{\rho_0}(p)$ for some $p \in B_{2M} \setminus \mathcal{N}_{2\rho_0}(D_F)$ or $\nabla u(B_{r_0s_0}) \subset \mathcal{N}_{3\rho_0}(D_F)$.

In all cases, $\nabla u(B_{r_0s_0})$ is contained in a ball $\mathcal{B}$ that has at most one point of D_F and is a positive universal distance from the remaining points of D_F . Thus, after restricting our attention to $\tilde{u} = (r_0s_0)^{-1}u(r_0s_0x)$ we may assume that D_F contains at most one point (indeed, we can modify F outside of $\mathcal{B}$ without changing that $\tilde{u}$ is a minimizer). Applying Proposition 3.5 to $\tilde{u}$ with $\rho = \epsilon/4$ completes the proof. $\square$

4. PROOF OF THEOREM 2.3

In this section we construct the examples from Theorem 2.3. Here and below we let $k \geq 1$, and $q = (q_1, q_2)$, $y = (y_1, y_2) \in \mathbb{R}^{2k+2}$ with $q_i, y_i \in \mathbb{R}^{k+1}$, $i = 1, 2$. We will reduce the problem to making a certain one-dimensional construction using the symmetries of w .

4.1. Reduction to Two Dimensions. We first reduce Theorem 2.3 to a problem in two dimensions. Let v be the one-homogeneous function on $\mathbb{R}^2$ given by

$$v(x_1, x_2) := \frac{1}{\sqrt{2}} \frac{x_2^2 - x_1^2}{|x|}.$$

We claim it suffices to construct a C^1 , uniformly convex function $G(p_1, p_2)$ on $\mathbb{R}^2$ that is smooth away from $D_G = \sqrt{2}S^1 \cap \{p_1^2 = p_2^2\}$, such that G is invariant under reflection over the axes and over the lines $\{p_1 = \pm p_2\}$ (that is, $G(p_1, p_2) = G(-p_1, p_2) = G(p_2, p_1)$) and furthermore

$$(5) \quad \text{tr}(D^2G(\nabla v) D^2v) + k \nabla G(\nabla v) \cdot \left(\frac{1}{x_1}, \frac{1}{x_2} \right) = 0$$

for x in the positive quadrant. Indeed, if we manage to do this, note that by the symmetries of G and v , each term on the left is smooth away from $\{x_1^2 = x_2^2\}$, where ∇v maps to D_G . If we then take $F(q) = G(|q_1|, |q_2|)$ we obtain a C^1 , uniformly convex function on $\mathbb{R}^{2k+2}$ that is smooth away from $D_F = \sqrt{2}S^{2k+1} \cap \{|q_1|^2 = |q_2|^2\}$. Using that $w(y) = v(|y_1|, |y_2|)$ we compute

$$\begin{aligned} \text{tr}(D^2F(\nabla w) D^2w)(y) &= \text{tr}(D^2G(\nabla v) D^2v)(|y_1|, |y_2|) \\ &\quad + k \nabla G(\nabla v(|y_1|, |y_2|)) \cdot \left(\frac{1}{|y_1|}, \frac{1}{|y_2|} \right) \\ &= 0 \end{aligned}$$

classically away from the cone $\{|y_1|^2 = |y_2|^2\}$. Here we used that $v_1 < 0$ and $v_2 > 0$ in the positive quadrant. It is not hard to show that the equation $\text{div}(\nabla F(\nabla w)) = 0$ holds in the weak sense in B_1 by integrating away from a thin cone containing $\{|y_1|^2 = |y_2|^2\}$ and a small ball around the origin, using the C^1 regularity of F and the one-homogeneity of w , and taking a limit.

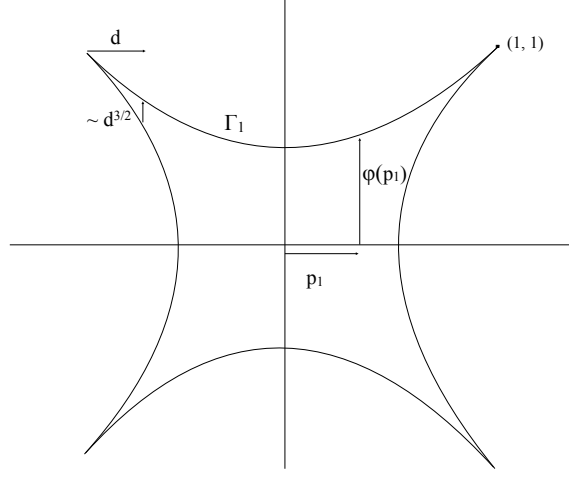


FIGURE 3. The set Σ_v consists of four congruent curves separating from the lines $p_2 = \pm p_1$ like $\text{dist}^{3/2}$.

4.2. Reduction to One Dimension. We now use that $\Sigma_v := \nabla v(S^1)$ is one-dimensional and an extension lemma to reduce our problem to one dimension. The set $\nabla v(S^1 \cap \{x_2 \geq |x_1|\})$ can be written as a graph $\Gamma_1 := \{(p_1, \varphi(p_1))\}$ with $p_1 \in [-1, 1]$, where $\varphi \in C^\infty(-1, 1) \cap C^1([-1, 1])$ is even, uniformly convex, and separates from the lines $p_2 = \pm p_1$ like $\text{dist}^{3/2}$ at the endpoints. See the Appendix for a justification of these properties, as well as an expansion of φ near the endpoints. The set Σ_v consists of four rotations of Γ_1 by $\pi/2$ (see Figure 3). Let $S := \sqrt{2} S^1 \cap \{p_1^2 = p_2^2\}$, and let $\Sigma_0 := \Sigma_v \setminus S$, $\Gamma_0 := \Gamma_1 \setminus S$. We will use the following important extension lemma:

Lemma 4.1. *Assume that $g : \Sigma_v \rightarrow \mathbb{R}$ and $\mathbf{v} : \Sigma_v \rightarrow \mathbb{R}^2$ are smooth on Σ_0 and continuous on Σ_v , and satisfy the condition*

$$(6) \quad g(\tilde{p}) - g(p) - \mathbf{v}(p) \cdot (\tilde{p} - p) \geq \gamma |\tilde{p} - p|^2$$

for some $\gamma > 0$ and all $\tilde{p}, p \in \Sigma_v$. Then there exists a C^1 , uniformly convex function G on $\mathbb{R}^2$ with $G \in C^\infty(\mathbb{R}^2 \setminus S)$, such that $G = g$ and $\nabla G = \mathbf{v}$ on Σ_v .

We delay the proof of Lemma 4.1 to the Appendix, and proceed with the reduction. We claim that it suffices to construct even functions $f, h \in C^\infty(-1, 1) \cap C^1([-1, 1])$ such that

$$(7) \quad h = \frac{1}{1 + 2k} \left(\frac{f''}{\varphi''} + k \frac{f'}{\varphi'} \right)$$

on $(-1, 1)$, and furthermore the pair $(g, \mathbf{v})$ defined on Σ_v by

$$(8) \quad g(s, \varphi(s)) := f(s), \quad \mathbf{v}(s, \varphi(s)) := (f'(s) - h(s)\varphi'(s), h(s))$$

on Γ_1 and extended by reflection over the lines $p_2 = \pm p_1$, satisfies the conditions of Lemma 4.1 for some $\gamma > 0$. Indeed, if this is accomplished, then the extension

G from Lemma 4.1 satisfies the Euler-Lagrange equation (5). To see this, let ν be the upward unit normal to Γ_0 . By the one-homogeneity of v we have

$$x = \nu(\nabla v(x)) = \frac{(-\varphi', 1)}{\sqrt{1 + \varphi'^2}}$$

for $x \in S^1 \cap \{x_2 > |x_1|\}$. Differentiating in x gives

$$D^2v(x) = \frac{1}{\kappa} \tau \otimes \tau = -\frac{(1 + \varphi'^2)^{3/2}}{\varphi''} \tau \otimes \tau$$

where τ is the unit tangent vector to Γ_0 at $\nabla v(x)$ (also the unit tangent to S^1 at x) and κ is the (signed) curvature of Γ_0 . Finally, by (8) we have

$$(9) \quad G(s, \varphi(s)) = f(s), \quad (\nabla G \cdot e_2)(s, \varphi(s)) = h(s)$$

on Γ_0 . Differentiating the first relation in (9) twice we obtain

$$\nabla G = (f' - h\varphi', h), \quad \tau^T \cdot D^2G \cdot \tau = \frac{f'' - h\varphi''}{1 + \varphi'^2}$$

on Γ_0 . Putting these together gives the equivalence of (5) and (7). Finally, since $(G, \nabla G)$ has the desired symmetries on Σ_v , we can arrange that G has the desired symmetries globally by taking the average of its reflections.

Remark 4.2. One can compute the Euler-Lagrange equation (7) directly in $\mathbb{R}^{2k+2}$ using the geometry of $\Sigma_w := \nabla w(S^{2k+1})$, without much trouble. Using the one-homogeneity of w we see as above that the equation reduces to $\text{tr}((II)^{-1}(q) D_T^2 F(q)) = 0$ on Σ_w , where II is the second fundamental form of Σ_w and T is the tangent hyperplane to Σ_w . Since Σ_w is obtained by taking rotations of Γ_1 , it is tangent on one side to second order to a sphere of radius $\varphi\sqrt{1 + \varphi'^2}$ and on the other side to second order to a sphere of radius $\frac{s}{\varphi'}\sqrt{1 + \varphi'^2}$. It thus has one principal curvature $-\varphi''(1 + \varphi'^2)^{-3/2}$, k principal curvatures $\frac{1}{\varphi\sqrt{1 + \varphi'^2}}$ from rotating around the p_1 axis, and k principal curvatures $-\frac{\varphi'}{s\sqrt{1 + \varphi'^2}}$ from rotating around the p_2 axis. The eigenvalues of D^2F corresponding to these directions are $\frac{f'' - h\varphi''}{1 + \varphi'^2}$, $\frac{h}{\varphi}$ and $\frac{f' - h\varphi'}{s}$, where the latter two come from rotations. Putting these together in the original Euler-Lagrange equation $\text{tr}((II)^{-1}(q) D_T^2 F(q)) = 0$ we recover (7).

4.3. The One-Dimensional Construction. We now construct f and h . The Euler-Lagrange equation (7) determines h through our choice of f , so there is only one function f to construct. It is convenient to do this by choosing a positive even function $\eta \in C^\infty(-1, 1) \cap C([-1, 1])$, and then defining f by the relation

$$(10) \quad f''(s) := \eta(s)\varphi''(s), \quad f'(0) = 0.$$

Fix $0 < \delta \ll 1$. We choose η satisfying the following conditions:

- (i) η is even, concave, and $\eta(1) = 1/2$,
- (ii) $\eta \geq \min\{1, \frac{1}{2}(1 + \delta^{-1/2}(1 - s)^{1/2})\}$,
- (iii) $\eta \equiv 1 + \mu$ on $[0, 1 - \delta]$ for some $\mu > 0$,
- (iv) $\int_0^1 \eta(s)\varphi''(s) ds = 1$.

We observe that by the first condition, the function η is decreasing on $(0, 1)$. Furthermore, since $\eta > 0$ we have by (10) that f is convex, and the last condition on

η implies that $f'(1) = 1$. Since $\int_0^1 \varphi''(s) ds = 1$, it is clear that $\mu \rightarrow 0$ as we take $\delta \rightarrow 0$. In particular, $f' \rightarrow \varphi'$ and $h \rightarrow \frac{1+k}{1+2k}$ as $\delta \rightarrow 0$. We will show below that for any such choice of η with δ sufficiently small, the pair $(g, \mathbf{v})$ defined by (8) satisfies the hypotheses of Lemma 4.1.

Continuity Condition: The condition that $\mathbf{v}$ is continuous on Σ_v and invariant under reflection over the diagonal is that

$$(11) \quad (f' - h\varphi', h)(1) \text{ is parallel to } (1, 1).$$

This follows from (7) and (10), using that $\eta(1) = 1/2$ and $f'(1) = \varphi'(1) = 1$.

Convexity Condition Along Top Graph. We check the convexity condition (6) on $\Gamma_1 \cap \{p_1 \geq 0\}$. Take $\tilde{p} = (y, \varphi(y))$ and $p = (x, \varphi(x))$ for some $x, y \in [0, 1]$. The quantity of interest is

$$\begin{aligned} & g(y, \varphi(y)) - g(x, \varphi(x)) - \mathbf{v}(x, \varphi(x)) \cdot (y - x, \varphi(y) - \varphi(x)) \\ &= [f(y) - f(x) - f'(x)(y - x)] - h(x)[\varphi(y) - \varphi(x) - \varphi'(x)(y - x)] \\ &= \int_x^y f''(s)(y - s) ds - h(x) \int_x^y \varphi''(s)(y - s) ds \\ &= \left(\int_x^y \varphi''(s)(y - s) ds \right) H(x, y), \end{aligned}$$

where

$$H(x, y) = \int_x^y (\eta - 1/2)(s) d\mu_y(s) - \frac{1}{1+2k}(\eta - 1/2)(x) - \frac{k}{1+2k} \frac{f' - \varphi'}{\varphi'}(x),$$

and $d\mu_y$ is the probability density

$$d\mu_y := \frac{\varphi''(s)(y - s) ds}{\int_x^y \varphi''(t)(y - t) dt}$$

on the interval from x to y . We claim that

$$(12) \quad H(x, y) \geq c_0 \max\{(1 - x)^{1/2}, (1 - y)^{1/2}\}$$

for some $c_0 > 0$ independent of $x, y \in [0, 1]$. The desired inequality (6) then follows because

$$\frac{1}{(y - x)^2} \int_x^y \varphi''(s)(y - s) ds \geq c \min\{(1 - x)^{-1/2}, (1 - y)^{-1/2}\}$$

for some $c > 0$, using that φ'' is positive, increasing on $[0, 1)$, and has the expansion $\varphi''(s) = \sqrt{\frac{2}{3}}(1 - s)^{-1/2} + O(1)$ near $s = 1$ (see Appendix). To show (12) we check several cases. Below, c_1, c_2 always denote universal positive constants which may change from line to line.

In the case that $x, y \leq 1 - \delta$ it is obvious that $H(x, y) \geq c_0 > 0$ since $\eta - 1/2 > 1/2$ is constant on $[0, 1 - \delta]$ and $f' - \varphi' = \mu\varphi'$ with μ small.

The next simplest case is that $y < x$ and $x > 1 - \delta$. Using that η is decreasing, that $f' \leq 1$, and the expansion of $\varphi'(s)$ near $s = 1$ (see Appendix) we have

$$H(x, y) \geq \frac{2}{3} \int_x^y (\eta - 1/2)(s) d\mu_y(s) - c_1 (1 - x)^{1/2}.$$

It is elementary to check that the mass of $d\mu_y$ in the left half of the interval $[y, x]$ is at least $c_2 > 0$ independent of y using the properties of φ . Since η is decreasing and concave we conclude that

$$H(x, y) \geq \frac{1}{3}c_2(\eta - 1/2)(y) - c_1(1 - x)^{1/2} \geq c_0(1 - y)^{1/2}$$

using definition of η and that δ is small.

The most delicate case is that $x < y$ and $y > 1 - \delta$. By the expansion of $\varphi(s)$ near $s = 1$ we may choose δ so small that $d\mu_y$ is decreasing on $[1 - 2\delta, y]$, independent of $y > 1 - \delta$ (see Appendix). We first claim that

$$\int_x^y (\eta - 1/2)(s) d\mu_y(s) \geq \frac{1}{2}(\eta - 1/2)(x).$$

If $x < 1 - 2\delta$ then since most of the weight of $d\mu_y$ on $[x, y]$ is to the left of $1 - \delta$ the inequality is obvious. When $x \geq 1 - 2\delta$, since both the weight $d\mu_y$ and η are decreasing on $[x, y]$, the average of $\eta - 1/2$ decreases if we redistribute the weight $d\mu_y$ evenly, and the concavity of η gives the result. We conclude that

$$H(x, y) \geq \frac{1}{6}(\eta - 1/2)(x) - \frac{f' - \varphi'}{2\varphi'}(x).$$

If $x \leq 1 - \delta$ we have $f' - \varphi' = \mu\varphi'$ and $\eta - 1/2 > 1/2$ so $H(x, y) \geq c_0 > 0$. If $x > 1 - \delta$ we argue as in the case $y < x$ that

$$H(x, y) \geq \frac{1}{12}\delta^{-1/2}(1 - x)^{1/2} - c_1(1 - x)^{1/2} > (1 - x)^{1/2}$$

for δ small, and inequality (12) follows.

Global Convexity Condition. Here we show that the convexity condition (6) holds on all of Σ_v , using reflection symmetry and the case treated above. Let $p = (p_1, p_2)$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$. We may assume by reflection symmetry that $p \in \Gamma_1$ and $p_1 \geq 0$, and we have already considered the case $\tilde{p}_2 \geq \tilde{p}_1 \geq 0$. To treat the remaining cases we claim that it suffices to prove the inequalities

$$(13) \quad \mathbf{v}(s, \varphi(s)) \cdot (1, 0) \geq c_0 s, \quad \mathbf{v}(s, \varphi(s)) \cdot (-1, 1) \geq c_0(\varphi(s) - s)$$

for some fixed $c_0 > 0$ and all $s \in [0, 1]$. Indeed, consider first the case that $\tilde{p}_1 \geq \tilde{p}_2 \geq 0$. Let $\tilde{q} = (\tilde{p}_2, \tilde{p}_1)$ be the reflection of $\tilde{p}$ over the diagonal. Then by reflection symmetry and the previous case we have

$$g(\tilde{p}) = g(\tilde{q}) \geq g(p) + \mathbf{v}(p) \cdot (\tilde{p} - p) + \mathbf{v}(p) \cdot (\tilde{q} - \tilde{p}) + \gamma|\tilde{q} - p|^2.$$

By the second inequality in (13), the sum of the last two terms is bounded from below by $c_0(\tilde{p}_1 - \tilde{p}_2)(p_2 - p_1) + \gamma|\tilde{q} - p|^2 \geq (\min\{c_0, \gamma\}/2)|\tilde{p} - p|^2$. The case $\tilde{p}_1 \leq 0, \tilde{p}_2 \geq 0$ is then treated similarly, taking instead $\tilde{q} = (-\tilde{p}_1, \tilde{p}_2)$ and using the first inequality in (13). For the remaining case $\tilde{p}_2 \leq 0$ repeat again the argument with $\tilde{q} = (\tilde{p}_1, -\tilde{p}_2)$ and use that $\mathbf{v}(p) \cdot e_2 = h(p_1) > 1/4$.

To finish we confirm the inequalities in (13). For δ small it is straightforward to show that $h < 3/4$, so $f' - h\varphi' \geq c_0\varphi'$, and the first inequality in (13) follows.

For the second we compute

$$\begin{aligned}
(1 + 2k)\mathbf{v} \cdot (-1, 1) &= (1 + 2k)[(1 + \varphi')h - f'] \\
&= (1 + \varphi')\eta + [k(1 + 1/\varphi') - (1 + 2k)]f' \\
&= (1 + \varphi')\eta + [k(1/\varphi' - 1) - 1]f' \\
&\geq (1 + \varphi')\eta - f'.
\end{aligned}$$

When $0 \leq s \leq 1 - \delta$ this quantity is larger than 1 by the definition of η . For $s \geq 1 - \delta$ we use the expansion $\varphi' = 1 - c_1(1 - s)^{1/2} + O(1 - s)$ (see Appendix) and that $f' \leq 1$ to get

$$\begin{aligned}
(1 + \varphi')\eta - f' &\geq (2 - c_1(1 - s)^{1/2})\eta - 1 + O(1 - s) \\
&= (2\eta - 1) - c_1\eta(1 - s)^{1/2} + O(1 - s) \\
&\geq (\delta^{-1/2} - 2c_1)(1 - s)^{1/2} + O(1 - s) \\
&\geq \frac{1}{2}\delta^{-1/2}(1 - s)^{1/2}
\end{aligned}$$

for δ small. We conclude that

$$\mathbf{v} \cdot (-1, 1) \geq c_0 \min\{1, \delta^{-1/2}(1 - s)^{1/2}\}.$$

Since $\varphi(s) - s \leq c_2(1 - s)^{3/2}$ for some $c_2 > 0$, this confirms (13).

4.4. Proof of Theorem 2.3.

Proof. Choose η as in the previous subsection. Then the pair $(g, \mathbf{v})$ determined by η through the relations (10), (7), and (8) satisfies the hypotheses of Lemma 4.1. We showed above that the extension G then satisfies the Euler-Lagrange equation (5) and can be chosen symmetric over the axes and over the lines $p_1 = \pm p_2$, and that the result follows by taking $F(q) = G(|q_1|, |q_2|)$. $\square$

Remark 4.3. If we choose $\eta - 1/2$ to be a multiple of $(1 - s)^{1/2}$ near $s = 1$, then a straightforward computation shows that the $2k + 1$ second derivatives of F in directions tangent to Σ_w are bounded, and the only second derivative of F that tends to ∞ is the one normal to Σ_w . The regularity of F near D_F is in fact $C^{1, \frac{1}{3}}$, by the computation that confirms the second inequality in (13).

5. APPENDIX

In the Appendix we record some properties of φ , and we prove the extension result Lemma 4.1.

5.1. Properties of φ . We recall from [MS] that if we parametrize Γ_1 by the angle $\theta \in [\pi/4, 3\pi/4]$ of its upward unit normal ν , then its curvature is given by $\kappa = \frac{\sqrt{2}}{3} \sec(2\theta)$. It follows easily that φ is smooth, even, and uniformly convex on $(-1, 1)$, and φ'' is increasing on $[0, 1)$. We recall also the expansion

$$\varphi\left(-1 + \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4)\right) = 1 - \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4)$$

from [MS]. By differentiating implicitly and using that φ is even we obtain near $s = 1$ the expansions

$$\varphi'(s) = 1 - 2\sqrt{\frac{2}{3}}(1 - s)^{1/2} + O(1 - s)$$

$$\begin{aligned}\varphi''(s) &= \sqrt{\frac{2}{3}}(1-s)^{-1/2} + O(1) \\ \varphi'''(s) &= \frac{1}{2}\sqrt{\frac{2}{3}}(1-s)^{-3/2} + O((1-s)^{-1}).\end{aligned}$$

In particular, for $0 < s < y$ the derivative of the weight $\varphi''(s)(y-s)$ is bounded above by $\varphi'''(s)(1-s) - \varphi''(s) = -\frac{1}{2}\sqrt{\frac{2}{3}}(1-s)^{-1/2} + O(1) < 0$ for s close to 1.

5.2. Proof of Extension Lemma. We now prove Lemma 4.1. Our strategy is to first construct G in a set containing a neighborhood of every point on Σ_0 . We then apply a global C^1 extension result to this local extension. To complete the construction we use a mollification and gluing procedure.

5.2.1. Local Extension.

Lemma 5.1. *There exists an open set $\mathcal{O}$ containing a neighborhood of each point on Σ_0 and a function $G_0 \in C^\infty(\mathcal{O})$ such that $G_0 = g$ and $\nabla G_0 = \mathbf{v}$ on Σ_0 , and furthermore*

$$(14) \quad G_0(\tilde{p}) - G_0(p) - \nabla G_0(p) \cdot (\tilde{p} - p) \geq \frac{\gamma}{2} |\tilde{p} - p|^2$$

for all $p, \tilde{p} \in \mathcal{O}$.

Proof. The squared distance function $d_{\Sigma_v}^2$ from Σ_v is smooth in a neighborhood of each point on Σ_0 , as is the projection π_{Σ_v} to Σ_v . Let τ be a unit tangent vector field to Σ_v in a neighborhood of a point on Σ_0 , and ν a unit normal vector field. Then $D^2(d_{\Sigma_v}^2/2)$ projects in the normal direction ν on Σ_0 . See e.g. [AS] for proofs of these properties.

Let $A > 0$ be a smooth function on Σ_0 to be chosen, and define

$$G_0(x) := g(\pi_{\Sigma_v}(x)) + \mathbf{v}(\pi_{\Sigma_v}(x)) \cdot (x - \pi_{\Sigma_v}(x)) + A(\pi_{\Sigma_v}(x))d_{\Sigma_v}^2.$$

It is elementary to check using (6) that $\nabla_{\Sigma_0} G_0$ is the tangential component of $\mathbf{v}$ on Σ_0 , and as a consequence that $G_0 = g$, $\nabla G_0 = \mathbf{v}$ on Σ_0 . Furthermore, it follows from (6) that $(G_0)_{\tau\tau} \geq 2\gamma$ on Σ_0 . Since $D^2(A(\pi_{\Sigma_v}(x))d_{\Sigma_v}^2)$ is $2A$ times the matrix that projects in the direction ν on Σ_0 , we have that $(G_0)_{\nu\nu} = 2A$ on Σ_0 , and that $(G_0)_{\tau\nu}$ on Σ_0 depends only on g , $\mathbf{v}$, and the geometry of Σ_0 (in particular, not on A). By choosing $A(p)$ sufficiently large depending on γ and these quantities (and perhaps going to ∞ as $p \rightarrow S$), we have that

$$(15) \quad D^2 G_0 \geq \frac{3}{2} \gamma I$$

on Σ_0 . In particular, (14) holds in a small neighborhood of each point on Σ_0 .

Now, for $\delta, \sigma > 0$ let S_δ be the closed δ -neighborhood of S , and let Σ_σ be the open σ -neighborhood of Σ_v . By (6), (15), and continuity, for each $\delta > 0$ there exists $\sigma(\delta) > 0$ small and an open set O_δ containing $\Sigma_{\sigma(\delta)} \setminus S_\delta$ and a neighborhood of each point in Σ_0 , such that (14) holds for all $p, \tilde{p} \in O_\delta$ such that at least one of $p, \tilde{p}$ is in $\Sigma_{\sigma(\delta)} \setminus S_\delta$. For $\delta = 1/k$ we may choose $\sigma(1/k)$ and $O_{1/k}$ such that $\Sigma_{\sigma(1/k)} \setminus S_{1/k} \subset O_{1/k} \subset O_{1/(k-1)}$ for $k > 1$. Taking

$$\mathcal{O} = \cup_{k>1} (\Sigma_{\sigma(1/k)} \setminus S_{1/k})$$

completes the proof. $\square$

5.2.2. *Global C^1 Extension.*

Lemma 5.2. *There exists a C^1 convex function G_1 on $\mathbb{R}^2$ such that $G_1 = G_0$ on an open set $\mathcal{U} \subset \mathcal{O}$ that contains a neighborhood of each point on Σ_0 , such that $D^2G_1 \geq \frac{\gamma}{2}I$ on $\mathbb{R}^2$.*

Proof. Let K be a compact set containing Σ_v and a neighborhood of each point on Σ_0 , such that $K \setminus S \subset \mathcal{O}$, and $G_0, \nabla G_0$ are continuous up to S in K . Let $H_0 := G_0 - \frac{\gamma}{4}|x|^2$. On S , define $H_0 = g - \frac{\gamma}{2}$, $\nabla H_0 = \mathbf{v} - \frac{\gamma}{2}x$. Then by (14) we have

$$H_0(\tilde{p}) - H_0(p) - \nabla H_0(p) \cdot (\tilde{p} - p) \geq \frac{\gamma}{4}|\tilde{p} - p|^2$$

for all $p, \tilde{p} \in K$. We may thus apply Theorem 1.10 from [AM] to obtain a global C^1 , convex function H_1 on $\mathbb{R}^2$ such that $H_1 = H_0$, $\nabla H_1 = \nabla H_0$ on K . To finish take $G_1 = H_1 + \frac{\gamma}{4}|x|^2$. $\square$

5.2.3. *Smoothing.*

Lemma 5.3. *There exists a convex function $G \in C^1(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus S)$ such that $G = G_1$ in a neighborhood of each point on Σ_0 , and $D^2G \geq \frac{\gamma}{4}I$ on $\mathbb{R}^2$. In particular, $DG = S$, and $(G, \nabla G) = (g, \mathbf{v})$ on Σ_v .*

Proof. We begin with a simple observation. Let F be a C^1 convex function on $\mathbb{R}^n$ with $D^2F \geq \mu I$. Let ρ_ϵ be a standard mollifier, and let $F_\epsilon := \rho_\epsilon * F$. Then if $\varphi \in C_0^\infty(B_R)$ with $0 \leq \varphi \leq 1$ we have

$$\begin{aligned} \|(\varphi F_\epsilon + (1 - \varphi)F) - F\|_{C^1(\mathbb{R}^n)} &= \|\varphi(F_\epsilon - F)\|_{C^1(\mathbb{R}^n)} \\ &\leq C(\varphi)\|F_\epsilon - F\|_{C^1(B_R)} \end{aligned}$$

and

$$\begin{aligned} D^2(\varphi F_\epsilon + (1 - \varphi)F) &= \varphi D^2F_\epsilon + (1 - \varphi)D^2F + D^2\varphi(F_\epsilon - F) \\ &\quad + \nabla\varphi \otimes (\nabla F_\epsilon - \nabla F) + (\nabla F_\epsilon - \nabla F) \otimes \nabla\varphi \\ &\geq (\mu - C(\varphi)\|F_\epsilon - F\|_{C^1(B_R)})I. \end{aligned}$$

Since F is C^1 , by taking $\epsilon(\varphi)$ small we have that $\varphi F_\epsilon + (1 - \varphi)F$ is as close as we like to F in $C^1(\mathbb{R}^n)$, and we have a lower bound for $D^2(\varphi F_\epsilon + (1 - \varphi)F)$ that is as close as we like to μI . We will apply this observation to a sequence of mollifications of G_1 .

Let $\{B_{r_i}(x_i)\}_{i=1}^\infty$ be a Whitney covering of $\mathbb{R}^2 \setminus \Sigma_v$. That is, for

$$r(x) := \min\{1, d_{\Sigma_v}(x)\}/20,$$

we have $r_i = r(x_i)$, the balls $B_{r_i}(x_i)$ are disjoint, $\cup_i B_{5r_i}(x_i) = \mathbb{R}^2 \setminus \Sigma_v$, and for each $x \in \mathbb{R}^2 \setminus \Sigma_v$ the ball $B_{10r(x)}(x)$ intersects at most $(129)^2$ of the $\{B_{10r_i}(x_i)\}$. For the existence of such a covering see for example [EG].

Now take $\varphi_i \in C_0^\infty(B_{10r_i}(x_i))$ such that $\varphi_i = 1$ in $B_{5r_i}(x_i)$, $0 \leq \varphi \leq 1$. For $j \geq 1$ we define G_{j+1} inductively as follows: If $B_{10r_j}(x_j) \subset \mathcal{U}$ then let $G_{j+1} = G_j$. If not, then let

$$G_{j+1} = \varphi_j(\rho_{\epsilon_j} * G_j) + (1 - \varphi_j)G_j,$$

with ϵ_j chosen so small that

$$(16) \quad \|G_{j+1} - G_j\|_{C^1(\mathbb{R}^2)} \leq 2^{-j}$$

and

$$(17) \quad D^2 G_{j+1} \geq \left(\frac{\gamma}{2} - \frac{\gamma}{4} \sum_{i=1}^j 2^{-i} \right) I.$$

By the covering properties, for any $x \in \mathbb{R}^2 \setminus \Sigma_v$, we have that G_j are smooth and remain constant in $B_{10r_j}(x)$ for all j sufficiently large. In addition, for any point on Σ_0 , every $B_{10r_j}(x_j)$ that intersects a small neighborhood of this point is contained in $\mathcal{U}$, so $G_j = G_1$ in a small neighborhood of every point in Σ_0 . We conclude using (16) and (17) that $G := \lim_{j \rightarrow \infty} G_j$ is C^1 , smooth on $\mathbb{R}^2 \setminus S$, agrees with G_1 in a neighborhood of every point on Σ_0 , and satisfies

$$D^2 G \geq \left(\frac{\gamma}{2} - \frac{\gamma}{4} \sum_{i=1}^{\infty} 2^{-i} \right) I = \frac{\gamma}{4} I.$$

□

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DEPARTMENT OF MATHEMATICS, UC IRVINE
 Email address: mooneycr@math.uci.edu