BARTNIK MASS VIA VACUUM EXTENSIONS

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Dedicated to Luen-Fai Tam on the occasion of his 70th birthday.

Abstract. We construct asymptotically flat, scalar flat extensions of Bartnik data (Σ, γ, H), where γ is a metric of positive Gauss curvature on a two-sphere Σ, and H is a function that is either positive or identically zero on Σ, such that the mass of the extension can be made arbitrarily close to the half area radius of (Σ, γ).

In the case of $H \equiv 0$, the result gives an analogue of a theorem of Mantoulidis and Schoen [13], but with extensions that have vanishing scalar curvature. In the context of initial data sets in general relativity, the result produces asymptotically flat, time-symmetric, vacuum initial data with an apparent horizon (Σ, γ), for any metric γ with positive Gauss curvature, such that the mass of the initial data is arbitrarily close to the optimal value in the Riemannian Penrose inequality.

The method we use is the Shi-Tam type metric construction from [19] and a refined Shi-Tam monotonicity, found by the first named author in [16].

1. Introduction

Let Σ be a two-sphere. Given a Riemannian metric γ and a function H on Σ, the Bartnik quasi-local mass [3] of the triple (Σ, γ, H) can be defined as

$$\mathfrak{m}_{_B}(\Sigma, \gamma, H) = \inf \{ \mathfrak{m}(M, g) \mid (M, g) \text{ is an admissible extension of } (\Sigma, \gamma, H) \} . \tag{1.1}$$

Here $\mathfrak{m}(\cdot)$ is the ADM mass [2], and an asymptotically flat Riemannian 3-manifold (M, g) with boundary ∂M is an admissible extension of (Σ, γ, H) if

- i) g has nonnegative scalar curvature;
- ii) ∂M with the induced metric is isometric to (Σ, γ) and, under the isometry, the mean curvature of ∂M in (M, g) equals H; and
- iii) (M, g) satisfies certain non-degeneracy condition that prevents $\mathfrak{m}(M, g)$ from being arbitrarily small; for instance, it is often required that (M, g) contains no closed minimal surfaces (enclosing ∂M), or ∂M is outer-minimizing in (M, g).

We refer readers to [1, 10, 14] for discussion on the numerous variations in the definition of Bartnik mass, and the recent progress on reconciling them.

Let K_γ denote the Gauss curvature of (Σ, γ). If $K_\gamma > 0$, by the work of Nirenberg [17] (and also Pogorelov [18]), (Σ, γ) admits an isometric embedding into $\mathbb{R}^3$ as a

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convex surface, unique up to rigid motions. Let H_0 be the mean curvature of such an isometric embedding. In [19], as a key step in their proof of the positivity of the Brown-York mass [6, 7], Shi and Tam proved the following:

Theorem 1.1 (Shi-Tam [19]). *Suppose $K_\gamma > 0$. Identify Σ with the image of the isometric embedding of (Σ, γ) in $\mathbb{R}^3$ and write the Euclidean metric $g_{_E} = dr^2 + g_r$ on E that is the exterior of Σ. Here g_r is the induced metric on the surface Σ_r that is r-distance away from Σ in $(E, g_{_E})$. Given any function $H > 0$ on Σ, there is a unique function $u > 0$ on E such that the metric $g_u = u^2 d\rho^2 + g_\rho$ is asymptotically flat and*

- *g_u has zero scalar curvature;*
- *the mean curvature of $\Sigma = \partial E$ in (E, g_u) equals H; and*
- *the ADM mass $\mathfrak{m}(E, g_u)$ satisfies*
$$\mathfrak{m}(E, g_u) \leq \frac{1}{8\pi} \int_\Sigma (H_0 - H)\, d\sigma,$$
where $d\sigma$ is the area measure on (Σ, γ).

Consequently, since (E, g_u) is foliated by a 1-parameter family of surfaces $\{\Sigma_r\}_{r \geq 0}$ with positive mean curvature, it follows that

$$\mathfrak{m}_{_B}(\Sigma, \gamma, H) \leq \frac{1}{8\pi} \int_\Sigma (H_0 - H)\, d\sigma \tag{1.2}$$

regardless of the non-degeneracy condition iii) used in the definition of $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$.

In this paper, we apply the method of Shi-Tam in [19] and its variation by the first named author in [16] to exhibit suitable extensions of (Σ, γ, H) such that the ADM mass of these extensions is controlled by $|\Sigma|_\gamma$, the area of γ.

Theorem 1.2. *Let γ be a metric with $K_\gamma > 0$ and H be a positive function on Σ. Given any $\epsilon > 0$, there exists an asymptotically flat manifold (M, g), diffeomorphic to $\Sigma \times [1, \infty)$, such that*

i) *g has zero scalar curvature;*
ii) *∂M with the induced metric is isometric to (Σ, γ) and the mean curvature of ∂M in (M, g) equals H under the isometry;*
iii) *(M, g) is foliated by a 1-parameter family of closed surfaces with positive mean curvature; and*
iv) *the mass of (M, g) satisfies*

$$\mathfrak{m}(M, g) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} + \epsilon. \tag{1.3}$$

Consequently, the Bartnik mass of (Σ, γ, H) satisfies

$$\mathfrak{m}_{_B}(\Sigma, \gamma, H) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}. \tag{1.4}$$

Theorem 1.2 also holds if H is identically zero. We state this case separately since the extensions in this setting represent time-symmetric, vacuum initial data with apparent horizon boundary.

Theorem 1.3. *Let γ be a metric with $K_\gamma > 0$ on Σ. Given any $\epsilon > 0$, there exists an asymptotically flat manifold (M, g), diffeomorphic to $\Sigma \times [1, \infty)$, such that*

i) *g has zero scalar curvature;*
ii) *∂M with the induced metric is isometric to (Σ, γ) and ∂M has zero mean curvature in (M, g);*
iii) *the interior of (M, g) is foliated by a 1-parameter family of closed surfaces with positive mean curvature; and*
iv) *the mass of (M, g) satisfies*

$$\mathfrak{m}(M, g) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} + \epsilon. \tag{1.5}$$

Consequently, the Bartnik mass of $(\Sigma, \gamma, 0)$ satisfies

$$\mathfrak{m}_{_B}(\Sigma, \gamma, 0) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}. \tag{1.6}$$

We give two important remarks regarding Theorems 1.2 and 1.3.

Remark 1.1. In [13], Mantoulidis and Schoen proved the following result: if γ is a metric on Σ such that $\lambda_1(-\Delta_\gamma + K_\gamma) > 0$, where Δ_γ is the Laplacian of (Σ, γ), then given any $\epsilon > 0$, there exists an asymptotically flat manifold $(M_{_{MS}}, g_{_{MS}})$ such that

a) $g_{_{MS}}$ has nonnegative scalar curvature, and has strictly positive scalar curvature somewhere;
b) $\partial M_{_{MS}}$ with the induced metric is isometric to (Σ, γ) and the mean curvature function of $\partial M_{_{MS}}$ in $(M_{_{MS}}, g_{_{MS}})$ equals 0;
c) $(M_{_{MS}}, g_{_{MS}})$ is foliated by a 1-parameter family of closed surfaces with positive mean curvature, and $(M_{_{MS}}, g_{_{MS}})$, outside a compact set, is isometric to a spatial Schwarzschild manifold with mass m; and
d) $\mathfrak{m}(M_{_{MS}}, g_{_{MS}}) = m$ satisfies

$$m < \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} + \epsilon. \tag{1.7}$$

In the context of the Bartnik mass, Mantoulidis-Schoen's result shows

$$\mathfrak{m}_{_B}(\Sigma, \gamma, 0) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \tag{1.8}$$

under the assumption $\lambda_1(-\Delta_\gamma + K_\gamma) > 0$.

Comparing Theorem 1.3 with Mantoulidis-Schoen's theorem, one sees Theorem 1.3 proves (1.8) under a much stronger assumption $K_\gamma > 0$. However, the extension $(M_{_{MS}}, g_{_{MS}})$ in [13] has positive scalar curvature somewhere, while (M, g) in Theorem 1.3 has identically zero scalar curvature.

In the context of initial data sets in general relativity, Theorem 1.3 produces asymptotically flat, time-symmetric, vacuum initial data with an apparent horizon (Σ, γ) whenever $K_\gamma > 0$ such that the mass of the initial data is arbitrarily close to the optimal value in the Riemannian Penrose inequality [5, 9].

Remark 1.2. In light of [13], the bound (1.4) on $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$ for a positive H is well expected and is known among experts (see [10, 14] for instance). This is because the manifold $(M_{_{MS}}, g_{_{MS}})$ from [13] is "almost" an admissible extension of $(\Sigma, \gamma, H > 0)$, except the mean curvature of $\partial M_{_{MS}}$ in $(M_{_{MS}}, g_{_{MS}})$ is zero. Thus, if one enlarges the class of admissible extensions in the definition of $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$ by replacing condition ii) with a condition

$\widetilde{\text{ii}}$) ∂M with the induced metric is isometric to (Σ, γ) and $H \geq H_{\partial M}$

and denotes the resulting Bartnik mass by $\tilde{\mathfrak{m}}_{_B}(\Sigma, \gamma, H)$, then $(M_{_{MS}}, g_{_{MS}})$ would be a legitimate admissible extension of (Σ, γ, H) and hence, by (1.7), one will have

$$\tilde{\mathfrak{m}}_{_B}(\Sigma, \gamma, H) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}. \tag{1.9}$$

The geometric meaning of "$H \geq H_{\partial M}$" used in condition $\widetilde{\text{ii}}$) can be found in [15]. Naturally one would like to know if $\tilde{\mathfrak{m}}_{_B}(\Sigma, \gamma, H)$ agrees with $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$. We refer readers to the recent work of Jauregui [10] and McCormick [14] for results pertinent to this question.

Even if $\tilde{\mathfrak{m}}_{_B}(\Sigma, \gamma, H) = \mathfrak{m}_{_B}(\Sigma, \gamma, H)$, we note Theorem 1.2 reveals more information than (1.4). This is again because the extension (M, g) in Theorem 1.2 has zero scalar curvature. Suppose $K_\gamma > 0$, if one shrinks the class of admissible extensions in the definition of $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$ by replacing condition i) with a condition

$\widetilde{\text{i}}$) g has zero scalar curvature

and denotes the resulting Bartnik mass by $\mathfrak{m}^0_{_B}(\Sigma, \gamma, H)$, then Theorem 1.2 still applies to show

$$\mathfrak{m}^0_{_B}(\Sigma, \gamma, H) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}. \tag{1.10}$$

On the other hand, as the metric $(M_{_{MS}}, g_{_{MS}})$ has strictly positive scalar curvature somewhere, it is not clear if the result in [13] could imply (1.10).

In the definition of $\mathfrak{m}_{_B}(\cdot)$, it is indeed natural to restrict the class of admissible extensions of (Σ, γ, H) to extensions with identically zero scalar curvature. It was conjectured by Bartnik [3] that, under suitable assumptions on (γ, H), a minimizer (M, g) achieving $\mathfrak{m}_{_B}(\Sigma, \gamma, H)$ exists and is a static vacuum initial data set. For this reason, it is reasonable to consider the revised variational problem of minimizing mass over extensions with zero scalar curvature. Besides Theorem 1.1 of Shi-Tam, prior results on estimating the Bartnik mass by constructing scalar flat extensions using PDE methods were also given by Lin and Sormani [11].

We end this section by comparing estimate (1.4) and (1.2). By the classic Minkowski inequality, if Σ is a closed convex surface in $\mathbb{R}^3$ with intrinsic metric γ and mean curvature H_0, then

$$\frac{1}{8\pi}\int_\Sigma H_0\ d\sigma \geq \sqrt{\frac{|\Sigma|_\gamma}{4\pi}}.$$

Thus, if $H > 0$ is relative small, i.e. if $\frac{1}{8\pi}\int_\Sigma H\ d\sigma < \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}$, then

$$\frac{1}{8\pi}\int_\Sigma (H_0 - H)\, d\sigma > \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}. \tag{1.11}$$

In this case, (1.4) is an estimate sharper than (1.2). On the other hand, if H is close to H_0, then (1.2) represents a better estimate.

2. Proof of Theorems 1.2 and 1.3

For simplicity, replacing (γ, H) with $(c^2\gamma, c^{-1}H)$ for a constant $c > 0$, we may assume $|\Sigma|_\gamma = 4\pi$.

We first prove Theorem 1.2 in which H is a positive function on Σ. Suppose $K_\gamma > 0$. Let $\{g(t)\}_{0\leq t\leq 1}$ be a fixed smooth path of metrics on Σ satisfying

(i) $g(0) = \gamma$, $g(1)$ is a round metric;
(ii) $K_{g(t)} > 0$, $\forall\ t \in [0, 1]$; and
(iii) $\mathrm{tr}_{g(t)} g'(t) = 0$, $\forall\ t \in [0, 1]$.

(Existence of such a path can be given by the proof of [13, Lemma 1.2] for instance.) It follows from conditions (i) and (iii) that $g(1) = \sigma$, a standard round metric on Σ with area 4π.

Given any $\delta > 0$, let

$$t_\delta = t_\delta(\rho) : [1, \infty) \to [0, 1]$$

be a smooth function such that

$$t_\delta(1) = 0, \quad \text{and} \quad t_\delta(\rho) = 1,\ \forall\ \rho \geq 1 + \delta. \tag{2.1}$$

Define

$$g_{\delta,\rho} = g(t_\delta(\rho)),\ \ \rho \in [1, \infty). \tag{2.2}$$

In what follows, we suppress the notation δ and denote $g_{\delta,\rho}$ by g_ρ. $\{g_\rho\}_{1\leq\rho<\infty}$ is a smooth family of metrics on Σ satisfying

$$g_1 = \gamma,\ \text{and}\ g_\rho = \sigma,\ \forall\ \rho \geq 1 + \delta. \tag{2.3}$$

On $N = [1, \infty) \times \Sigma$, consider a background metric

$$g^b = d\rho^2 + \rho^2 g_\rho. \tag{2.4}$$

Let $N_\delta = [1 + \delta, \infty) \times \Sigma$, then

$$g^b = d\rho^2 + \rho^2\sigma = g_{_E},\ \text{on}\ N_\delta, \tag{2.5}$$

where $g_{_E}$ is the standard Euclidean metric.

For each ρ, let H^b and A^b denote the mean curvature and the second fundamental form of $\Sigma_\rho = \{\rho\} \times \Sigma$ in (N, g^b) with respect to $\frac{\partial}{\partial \rho}$, respectively. Then

$$H^b = 2\rho^{-1} + \frac{1}{2}\mathrm{tr}_{g_\rho}(\partial_\rho g_\rho) = 2\rho^{-1}, \tag{2.6}$$

where we used (iii). Given any function $u = u(\rho, x) > 0$ on N, following Bartnik [4] and Shi-Tam [19], we consider the metric

$$g_u = u^2 d\rho^2 + \rho^2 g_\rho. \tag{2.7}$$

The induced metric from g_u on Σ_1, which is identified with Σ, is $g_1 = \gamma$. If H_u denotes the mean curvature of Σ_ρ in (N, g_u), then

$$H_u = u^{-1} H^b > 0. \tag{2.8}$$

The following claim follows directly from results in [19] and [8].

Proposition 2.1. *Given the pair (γ, H) on Σ, there exists a function $u > 0$ on N such that*

- *$g_u = u^2 d\rho^2 + \rho^2 g_\rho$ has zero scalar curvature, and*
- *$H_u = H$ at $\Sigma = \partial N$.*

Moreover, $u \to 1$ as $\rho \to \infty$ and (N, g_u) is asymptotically flat and is foliated by $\{\Sigma_\rho\}_{1 \le \rho < \infty}$ with positive mean curvature.

Proof. We recall the PDE that u needs to satisfy so that g_u has zero scalar curvature. Let $\tilde{g}_\rho = \rho^2 g_\rho$. By [19, Equation (1.10)] (also see [8, Equation (5)]),

$$H^b \partial_\rho u = u^2 \Delta_{\tilde{g}_\rho} u - K_{\tilde{g}_\rho} u^3 + \frac{1}{2}\left[(H^b)^2 + |A^b|^2_{\tilde{g}_\rho} + 2\partial_\rho H^b \right] u. \tag{2.9}$$

Since $H^b > 0$ and $K_{\tilde{g}_\rho} = \rho^{-2} K_{g_\rho} > 0$ for every ρ, a positive solution u with an initial condition $u|_\Sigma = H^{-1} H^b$ exists on $[1, T] \times \Sigma$ for all $T > 0$ by [8, Proposition 2]. Since g^b is the Euclidean metric on N_δ, the claim on the asymptotic behavior of u follows from [19, Theorem 2.1]. $\square$

Let u be given in Proposition 2.1, (N, g_u) is an admissible extension of (Σ, γ, H). By definition,

$$\mathfrak{m}_{_B}(\Sigma, \gamma, H) \le \mathfrak{m}(g_u). \tag{2.10}$$

We want to estimate $\mathfrak{m}(g_u)$ in terms of data at $\Sigma_{1+\delta} = \partial N_\delta$. Initially, we could apply [19, Lemma 4.2] to have

$$\int_{\Sigma_\rho} (H^b - H_u)\, d\sigma_\rho \text{ is monotone nonincreasing in } \rho \in [1+\delta, \infty), \tag{2.11}$$

where $d\sigma_\rho$ is the area measure on $(\Sigma_\rho, \rho^2 g_\rho)$. Also, by [19, Theorem 2.1],

$$\lim_{\rho \to \infty} \frac{1}{8\pi} \int_{\Sigma_\rho} (H^b - H_u)\, d\sigma_\rho = \mathfrak{m}(g_u). \tag{2.12}$$

Thus,

$$
\begin{aligned}
\mathfrak{m}(g_u) &\le \frac{1}{8\pi}\int_{\Sigma_{1+\delta}} (H^b - H_u)\, d\sigma_{1+\delta} \\
&= (1+\delta) - \frac{1}{8\pi}\int_{\Sigma_{1+\delta}} H_u \, d\sigma_{1+\delta}.
\end{aligned}
\tag{2.13}
$$

Since $H_u > 0$, omitting the term involving H_u, one has

$$\mathfrak{m}(g_u) < 1 + \delta.$$

However, this estimate is not sufficient to show (1.3).

To verify (1.3), we make use of a refined monotonicity property concerning (N_δ, g_u). The next proposition is essentially a restatement of results from [16, Proposition 1 and Equation (56)].

Proposition 2.2 ([16])**.** *Given any constant $r > 0$, let $E_r = \mathbb{R}^3 \setminus \{|x| < r\}$. Let $g_0^b = d\rho^2 + \rho^2\sigma$ denote a background Euclidean metric on E_r. On E_r, let $u > 0$ be a function such that the metric $g_u = u^2 d\rho^2 + \rho^2 \sigma$ has zero scalar curvature and hence is asymptotically flat. Let $H_u = 2\rho^{-1}u^{-1}$ be the mean curvature of $S_\rho = \{|x| = \rho\}$ in (E_r, g_u). Then, for any constant $m \in (-\infty, \frac{1}{2}r]$,*

$$\mathfrak{m}(g_u) \le m + \frac{1}{8\pi}\int_{S_r} (2r^{-1}N - H_u) N \, d\sigma_r \tag{2.14}$$

where $N = \sqrt{1 - \frac{2m}{r}}$. As a result, by minimizing its right side over m, one has

$$\mathfrak{m}(g_u) \le \sqrt{\frac{|S_r|}{16\pi}} \left[1 - \frac{1}{16\pi |S_r|} \left(\int_{S_r} H_u \, d\sigma_r \right)^2 \right]. \tag{2.15}$$

Proof. Given any $m \in (-\infty, \frac{1}{2}r]$, consider a background Schwarzschild metric

$$g_m^b = \frac{1}{1 - \frac{2m}{\rho}} d\rho^2 + \rho^2 \sigma \tag{2.16}$$

on E_r. Let H_m^b be the mean curvature of S_ρ in (E_r, g_m^b). By [16, Proposition 1],

$$\int_{S_\rho} (H_m^b - H_u) N(\rho)\, d\sigma_\rho \text{ is monotone nonincreasing in } \rho \in [r, \infty), \tag{2.17}$$

where $N(\rho) = \sqrt{1 - \frac{2m}{\rho}}$, and

$$\lim_{\rho\to\infty} \frac{1}{8\pi}\int_{S_\rho} (H_m^b - H_u) N(\rho)\, d\sigma_\rho = \mathfrak{m}(g_u) - m. \tag{2.18}$$

Thus, by (2.17), (2.18) and the fact $H^b_m = 2\rho^{-1}N(\rho)$,

$$\begin{aligned} \mathfrak{m}(g_u) &\le m + \frac{1}{8\pi}\int_{S_r} (H^b_m - H_u) N \, d\sigma_r \\ &= m + \frac{1}{8\pi}\int_{S_r} (2r^{-1}N - H_u) N \, d\sigma_r . \end{aligned} \tag{2.19}$$

Denote the right side of (2.19) by $\Phi(m)$, i.e.

$$\Phi(m) = m + rN^2 - \left(\frac{1}{8\pi} \int_{S_r} H_u \, d\sigma_r \right) N. \tag{2.20}$$

Since $N^2 = 1 - \frac{2m}{r}$, one can rewrite $\Phi(m)$ as

$$\Phi(m) = \frac{1}{2} r \left[N - \frac{1}{8\pi r} \left(\int_{S_r} H_u \, d\sigma_r \right) \right]^2 + \frac{1}{2} r \left[1 - \left(\frac{1}{8\pi r} \int_{S_r} H_u \, d\sigma_r \right)^2 \right]. \tag{2.21}$$

Minimizing $\Phi(m)$ over $m \in (-\infty, \frac{1}{2}r]$, one has

$$\min_m \Phi(m) = \frac{1}{2} r \left[1 - \left(\frac{1}{8\pi r} \int_{S_r} H_u \, d\sigma_r \right)^2 \right]. \tag{2.22}$$

This explains (2.15). □

Remark 2.1. A monotonicity generalizing (2.17) with the background Schwarzschild metric replaced by a general static metric can be found in [12].

We resume to prove (1.3). Applying Proposition 2.2 to (N_δ, g_u), we have

$$\mathfrak{m}(g_u) \le \sqrt{\frac{|\Sigma_{1+\delta}|}{16\pi}} \left[1 - \frac{1}{16\pi |\Sigma_{1+\delta}|} \left(\int_{\Sigma_{1+\delta}} H_u \, d\sigma_{1+\delta} \right)^2 \right]. \tag{2.23}$$

Omitting the term involving H_u, we conclude from (2.23) that

$$\mathfrak{m}(g_u) < \sqrt{\frac{|\Sigma_{1+\delta}|}{16\pi}} = \frac{1}{2}(1+\delta). \tag{2.24}$$

Since δ can be arbitrarily small, (1.3) follows from (2.24) and rescaling. Theorem 1.2 is proved.

Next, we prove Theorem 1.3 in which $H \equiv 0$. In this case, to obtain a solution u satisfying Proposition 2.1, one needs to solve (2.9) with an initial condition $u|_\Sigma = \infty$. While this might be achieved by following the proofs in [4, 19], we revise our choice of $\{g_\rho\}$ to apply a result of Smith [20, 21] which imitates the horizon in a spatial Schwarzschild manifold. More precisely, given any small $\delta > 0$, we modify the choice of $t_\delta(\rho)$ in (2.1) by re-defining it so that

$$\begin{cases} t_\delta(\rho) = 0, & \text{if } 1 \le \rho \le 1 + \frac{1}{2}\delta, \\ t_\delta(\rho) = 1, & \text{if } \rho \ge 1 + \delta. \end{cases} \tag{2.25}$$

With this choice of $t_\delta(\rho)$, define $g_\rho = g(t_\delta(\rho))$ as in (2.2). Then $\{g_\rho\}_{\rho\geq 1}$ satisfies

$$\begin{cases} g_\rho = \gamma, & \text{if } 1 \leq \rho \leq 1 + \frac{1}{2}\delta, \\ g_\rho = \sigma, & \text{if } \rho \geq 1 + \delta. \end{cases} \tag{2.26}$$

On $N = [1, \infty) \times \Sigma$, consider the background metric $g^b = d\rho^2 + \rho^2 g_\rho$. Applying [20, Main Theorem], there exists a function u of the form

$$u = \frac{v}{\sqrt{1 - \frac{1}{\rho}}}, \tag{2.27}$$

where $v > 0$ is a smooth function on N, independent on ρ in $[1, 1 + \frac{1}{2}\delta]$, such that the metric

$$g_u = u^2 d\rho^2 + \rho^2 g_\rho$$

has zero scalar curvature. This (N, g_u) satisfies Proposition 2.1 with $H \equiv 0$. The rest of the proof is then identical to that following Proposition 2.2 above. This proves Theorem 1.3.

Remark 2.2. In Theorems 1.2 and 1.3, the assumption $K_\gamma > 0$ is imposed to have $K_{g(t)} > 0$, which is to guarantee the existence of the solution u to (2.9).

Remark 2.3. By (iii) and (2.2), equation (2.9) on $[1, \infty) \times \Sigma$ takes an explicit form of

$$2\rho\, \partial_\rho u = u^2 \Delta_{g_\rho} u - K_{g_\rho} u^3 + \left[1 + \frac{1}{8}\rho^2 \left| \frac{dg}{dt} \right|_g^2 \left(\frac{dt_\delta}{d\rho} \right)^2 \right] u. \tag{2.28}$$

On $[1 + \delta, \infty) \times \Sigma$, it reduces to

$$2\rho\, \partial_\rho u = u^2 \Delta_\sigma u + (u - u^3), \tag{2.29}$$

which is the equation in [19, Example 1].

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