

Bounds for the number of cohomological automorphic representations of $GL_3/\mathbb{Q}$ in the weight aspect

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We prove a power saving over the trivial bound for the number of cohomological cuspidal automorphic representations of fixed level and growing weight on $GL_3/\mathbb{Q}$, by adapting the methods of our earlier paper on GL_2 .

1 Introduction

The purpose of this article is to prove a power saving over the trivial bound for the number of cohomological cuspidal automorphic representations on $GL_3/\mathbb{Q}$ of fixed level and growing weight. To state our result, let $a \geq b \geq c$ be integers, and let V be the irreducible algebraic representation of $GL_3(\mathbb{R})$ with highest weight (a, b, c) . By a theorem of Borel and Wallach [5, Ch II, Prop 6.12], if an irreducible unitary representation π of $GL_3(\mathbb{R})$ has nonzero $(\mathfrak{g}, K)$ -cohomology with coefficients in V , then $b = 0$ and $c = -a$ (i.e. V is equivalent to its twist by the Cartan involution). We shall therefore restrict our attention to the coefficient systems V_λ with highest weight $(\lambda, 0, -\lambda)$, and say that an irreducible unitary representation π is cohomological of weight λ if it is infinite dimensional and $H^*(\mathfrak{g}, K; \pi \otimes V_\lambda) \neq 0$. (Note that we take $K = SO(3)$.) It is known that there are two such π , which are trivial on the positive scalar matrices and are twists of each other by the sign of the determinant. Moreover, they satisfy

$$H^i(\mathfrak{g}, K; \pi \otimes V_\lambda) = \begin{cases} \mathbb{C} & i = 2, 3, \\ 0 & i \neq 2, 3. \end{cases}$$

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We shall say that an automorphic representation π on $GL_3/\mathbb{Q}$ is cohomological of weight λ if π_∞ has this property. Our main result is the following:

Theorem 1.1. Fix a compact open subgroup $K \subset GL_3(\mathbb{A}_f)$, and let $\mathcal{A}_\lambda$ be the set of cuspidal automorphic representations on $GL_3/\mathbb{Q}$ that are cohomological of weight λ and have level K . We have $|\mathcal{A}_\lambda| \ll_{K,\epsilon} \lambda^{3-4/27+\epsilon}$.

□

We shall deduce Theorem 1.1 from the following theorem on the cohomology of congruence subgroups of $SL(3, \mathbb{Z})$.

Theorem 1.2. Let Γ be a congruence subgroup of $SL(3, \mathbb{Z})$. We have $\dim H^2(\Gamma, V_\lambda) \ll_{\Gamma,\epsilon} \lambda^{3-4/27+\epsilon}$.

□

We note that the trivial bound in Theorems 1.1 and 1.2 is on the order of $\dim V_\lambda \sim \lambda^3$. As a result, these theorems represent a power improvement for the dimension of a space of automorphic forms that are tempered but not essentially square integrable at infinity. This is a difficult problem, which has only been solved in a few cases [7, 8, 14, 17, 18]. (See also the paper [19] of Sardari for an analogous result at finite places.) Moreover, there are currently no results of this type proved using the trace formula alone, despite recent progress in understanding its analytic properties.

To illustrate this point, we shall recall some results on the problem of counting cohomological cuspidal automorphic representations of fixed level and growing weight on GL_2/K where K is imaginary quadratic. This is analogous to Theorem 1.1, as these representations are also tempered but not essentially square integrable at infinity. We let S_d denote the set of cohomological cuspidal automorphic representations of weight d and some fixed unspecified level on GL_2/K , where ‘weight d ’ means having cohomology with respect to the coefficient system $\text{sym}^d \mathbb{C}^2 \otimes \overline{\text{sym}^d \mathbb{C}^2} \otimes |\det|^{-d}$. The trivial bound here is $|S_d| \ll d^2$, which is the dimension of the coefficient system. The best known bound for $|S_d|$ obtained by an analytic study of the trace formula is $|S_d| \ll d^2 / \log d$, due to Finis, Grunewald and Tirao [12]. On the other hand, in [17] we used the theory of completed cohomology developed by Calegari and Emerton [6, 9] to prove that $\dim S_d \ll_\epsilon d^{5/3+\epsilon}$, and this was later improved to $\ll_\epsilon d^{3/2+\epsilon}$ by Hu [14]. It is likely that the best bound for $|\mathcal{A}_\lambda|$ that one could prove using the trace formula is a similar logarithmic improvement over λ^3 .

One has a lower bound for $|\mathcal{A}_\lambda|$ of $|\mathcal{A}_\lambda| \gg \lambda$ from symmetric square lifts [2, Sec 3.4], and the computations of [1] (and those of [12] in the analogous case of $SL_2(\mathbb{C})$) suggest that this is the main contribution so that in fact $|\mathcal{A}_\lambda| \sim \lambda$.

Theorem 1.2 will be proved by combining the methods of our previous paper [17] with a new bound for the growth of invariants in certain $\mathbb{F}_p$ -representations of $SL_3(\mathbb{Z}_p)$ (Proposition 2.3). We in fact prove a version of Proposition 2.3 for a general $SL_d(\mathbb{Z}_p)$ (Corollary 3.3), but at present we are unable to deduce new bounds on cohomology from this. We discuss this further in Section 2.5.

By improving Proposition 2.3, it may be possible to improve the saving in the exponent of our main theorem from $4/27$ to $8/45$, but we have not checked this carefully. More precisely, the improvement would be made in the proof of Proposition 3.6, see Remark 3.8. We do not know if this is the best exponent one could obtain using the methods of this paper.

1.1 Proof of Theorem 1.1 assuming Theorem 1.2

Before proving Theorem 1.2, we show how it implies Theorem 1.1 using the extension of Matsushima's formula to noncompact quotients proved in [3, 4]. Let Z^+ be the positive scalar matrices in $GL_3(\mathbb{R})$, and define $X = GL_3(\mathbb{Q}) \backslash GL_3(\mathbb{A}) / KZ^+$. We have $X = \bigcup \Gamma_i \backslash SL_3(\mathbb{R})$, where Γ_i are congruence subgroups of $SL(3, \mathbb{Z})$.

There is a unique irreducible unitary infinite dimensional representation π_λ of $SL_3(\mathbb{R})$ with $H^2(\mathfrak{g}, K; \pi_\lambda \otimes V_\lambda) \neq 0$. This implies that if $\pi \in \mathcal{A}_\lambda$ then the restriction of π_∞ to $SL_3(\mathbb{R})$ must be isomorphic to π_λ . Moreover, π_∞ is trivial on Z^+ . If we let $m(\pi_\lambda, X)$ denote the multiplicity with which π_λ occurs in $L^2_{\text{cusp}}(X)$, it follows that $|\mathcal{A}_\lambda| \leq m(\pi_\lambda, X)$. If $\Gamma \subset SL(3, \mathbb{Z})$ is a congruence subgroup we also let $m(\pi_\lambda, \Gamma)$ be the multiplicity of π_λ in $L^2_{\text{cusp}}(\Gamma \backslash SL(3, \mathbb{R}))$. We have

$$m(\pi_\lambda, X) \leq \sum_i m(\pi_\lambda, \Gamma_i),$$

so it suffices to prove that $m(\pi_\lambda, \Gamma) \ll_{\Gamma, \epsilon} \lambda^{3-4/27+\epsilon}$ for any Γ . The extension of Matsushima's formula to noncompact quotients proved in [3, 4] implies that $m(\pi_\lambda, \Gamma) \leq \dim H^2(\Gamma, V_\lambda)$, and Theorem 1.2 completes the proof.

2 Proof of Theorem 1.2

In this section we prove Theorem 1.2, assuming the bound for the growth of invariants in $\mathbb{F}_p$ -representations of $SL_3(\mathbb{Z}_p)$ stated in Proposition 2.3; this proposition is proved in Section 3. We introduce notation in Section 2.1, and construct a lattice $\mathcal{V}_\lambda \subset V_\lambda$ in Section 2.2 (after changing coefficients to $\mathbb{Q}_p$) that will later be used in our reduction mod p argument. We recall facts about p -adic Banach space representations in Section 2.3, and complete the proof of Theorem 1.2 in Section 2.4. Section 2.5 discusses the problem of extending our results to other groups.

2.1 Notation

We let $p > 3$ be a prime which will be fixed throughout Section 2. We shall consider V_λ as a representation of $GL_3(\mathbb{Q})$ with $\mathbb{Q}_p$ coefficients. We define $G_n = \{g \in SL(3, \mathbb{Z}_p) : g \equiv 1(p^n)\}$, and let $G = G_1$. We let T and U be the diagonal and strictly upper triangular subgroups of G , and define $P(n) = TUG_n$. We define the non-commutative Iwasawa algebras

$$\Lambda_{\mathbb{Z}_p} = \varprojlim_k \mathbb{Z}_p[G/G_k], \quad \Lambda_{\mathbb{Q}_p} = \Lambda_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \text{and} \quad \Lambda = \varprojlim_k \mathbb{F}_p[G/G_k],$$

where the projections are induced from the maps $G/G_{k'} \rightarrow G/G_k$ for $k' \geq k$. Suppose L is a representation of G over $\mathbb{F}_p$, and let L^* be the dual representation equipped with the weak topology. If L is smooth, then the action of G on L^* extends uniquely to an action of Λ such that for any $\ell \in L^*$ the orbit map $\Lambda \rightarrow L^*$, $x \mapsto x\ell$, is continuous. We use h^i to denote the dimension of H^i , computed with continuous cochains in the case of the group G .

2.2 Choosing a lattice in V_λ

We shall require a lattice $\mathcal{V}_\lambda \subset V_\lambda$ with the properties described in the following lemma.

Lemma 2.1. If $n \geq 1$ satisfies $p^{n-1} > 3\lambda$, there is a G -stable lattice $\mathcal{V}_\lambda \subset V_\lambda$ such that $\mathcal{V}_\lambda/p\mathcal{V}_\lambda$ is isomorphic to a submodule of $\mathbb{F}_p[G/P(n)]$.

□

Proof. Let V_λ^* be the dual of V_λ , and $\langle \cdot, \cdot \rangle$ the pairing between them. Let $w_\lambda^* \in V_\lambda^*$ be a nonzero vector of highest weight. As a representation of $SL_3(\mathbb{Q}_p)$, V_λ is isomorphic to the space of functions on $SL(3, \mathbb{Q}_p)$ of the form

$$f(g) = \langle \pi(g^{-1})v, w_\lambda^* \rangle, \quad v \in V_\lambda,$$

where $SL(3, \mathbb{Q}_p)$ acts by $[\pi(h)f](g) = f(h^{-1}g)$. We define $\mathcal{V}_\lambda$ to be the $\mathbb{Z}_p$ -module of functions whose values on G lie in $\mathbb{Z}_p$, which is clearly a G -stable lattice. This implies that $\mathcal{V}_\lambda/p\mathcal{V}_\lambda$ may be identified with the submodule of $C(G, \mathbb{F}_p)$ obtained by reducing functions in $\mathcal{V}_\lambda$ modulo p , and we must show that these reductions are right-invariant under $P(n)$.

Let $w_{-\lambda} \in V_\lambda$ be the vector of lowest weight with $\langle w_{-\lambda}, w_\lambda^* \rangle = 1$, and define $f_\lambda(g) = \langle \pi(g^{-1})w_{-\lambda}, w_\lambda^* \rangle$. We have $f_\lambda(e) = 1$, and the invariance properties of $w_{-\lambda}$ and w_λ^* imply that for u_- in the lower triangular unipotent and b in the standard Borel we have $f_\lambda(u_-b) = \chi(b)$, where χ is the highest weight character of V_λ^* . By the Bruhat decomposition, these facts specify f_λ uniquely. Moreover, the function

$$h_\lambda : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \rightarrow x_{11}^\lambda \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^\lambda$$

also has these properties, so that $f_\lambda = h_\lambda$.

Let P_λ be the space of polynomials on $M_3(\mathbb{Q}_p)$ spanned by the left translates of h_λ under $SL_3(\mathbb{Q}_p)$, and let $\mathcal{P}_\lambda \subset P_\lambda$ be the lattice of polynomials that are integral on G . It follows that V_λ and $\mathcal{V}_\lambda$ are the restrictions of P_λ and $\mathcal{P}_\lambda$ to $SL_3(\mathbb{Q}_p)$ respectively. Moreover, because all elements of P_λ transform by the same character under the action of scalar matrices, elements of $\mathcal{P}_\lambda$ are integral on $1 + pM_3(\mathbb{Z}_p)$. We also see that all polynomials in P_λ have degree at most 3λ . By combining this with Lemma 2.2 and our assumption that $3\lambda < p^{n-1}$, we see

that, on $1 + pM_3(\mathbb{Z}_p)$, the functions in $\mathcal{P}_\lambda/p\mathcal{P}_\lambda$ are constant on cosets of $p^n M_3(\mathbb{Z}_p)$. It follows that functions in $\mathcal{V}_\lambda/p\mathcal{V}_\lambda$ are invariant under G_n . The invariance under TU follows from our choice of w_λ^* as a highest weight vector, which completes the proof. ■

Lemma 2.2. Let $h \in \mathbb{Q}_p[x]$ have degree d , and suppose that h maps $\mathbb{Z}_p$ to $\mathbb{Z}_p$. Let $\bar{h} : \mathbb{Z}_p \rightarrow \mathbb{F}_p$ be the reduction of h modulo p . If $d < p^t$, then $\bar{h}$ is constant on cosets of $p^t \mathbb{Z}_p$. □

Proof. We first use the integral-valued property of h to show that we have

$$h(x) = \sum_{j=0}^d c_j \binom{x}{j} \quad (1)$$

with $c_j \in \mathbb{Z}_p$ (we trivially have a unique such representation with $c_j \in \mathbb{Q}_p$). We prove this using induction on d , and note that the case $d = 0$ holds trivially. If $d \geq 1$, we consider the polynomial

$$h(x+1) - h(x) = \sum_{j=1}^d c_j \binom{x}{j-1},$$

and note that this is also integral valued on $\mathbb{Z}_p$. Applying the induction hypothesis gives $c_j \in \mathbb{Z}_p$ for $j \geq 1$, and we have $c_0 = h(0) \in \mathbb{Z}_p$.

Having established the representation (1), it suffices to consider the case where h is a binomial coefficient $\binom{x}{d}$ with $d < p^t$. But the lemma in this case follows from a result of Lucas [16] (see also [11]) on congruences of binomial coefficients modulo p . ■

2.3 p -adic Banach space representations

We now recall from [20] some facts about representations of G (or any other compact p -adic Lie group) on a p -adic Banach space.

We recall that a topological $\mathbb{Z}_p$ -module is called linear-topological if the zero element has a fundamental system of open neighbourhoods consisting of $\mathbb{Z}_p$ -submodules. We let $\text{Mod}_{\text{top}}(\mathbb{Z}_p)$ be the category of all Hausdorff linear-topological $\mathbb{Z}_p$ -modules with morphisms being all continuous $\mathbb{Z}_p$ -linear maps. We let $\text{Mod}_c^{\text{tf}}(\mathbb{Z}_p)$ be the full subcategory in $\text{Mod}_{\text{top}}(\mathbb{Z}_p)$ consisting of torsion free and compact linear-topological $\mathbb{Z}_p$ -modules. We recall from [20, Rmk 1.1] that if M is any object in $\text{Mod}_c^{\text{tf}}(\mathbb{Z}_p)$, we have

$$M \simeq \prod_{i \in I} \mathbb{Z}_p \quad (2)$$

for some set I . We let $\text{Ban}(\mathbb{Q}_p)$ be the category of all $\mathbb{Q}_p$ -Banach spaces $(E, \|\cdot\|)$, with morphisms being continuous linear maps. We let $\text{Ban}(\mathbb{Q}_p)^{\leq 1}$ be the subcategory of spaces such that $\|E\| \subset |\mathbb{Q}_p|$, with the morphisms being norm-decreasing linear maps.

In [20, Section 1], Schneider and Teitelbaum define two contravariant functors between the categories $\text{Ban}(\mathbb{Q}_p)^{\leq 1}$ and $\text{Mod}_c^{\text{tf}}(\mathbb{Z}_p)$. For an object M in $\text{Mod}_c^{\text{tf}}(\mathbb{Z}_p)$ they define the $\mathbb{Q}_p$ -Banach space

$$M^d = \text{Hom}_{\mathbb{Z}_p}^{\text{cts}}(M, \mathbb{Q}_p) \quad \text{with norm} \quad \|\ell\| = \max_{m \in M} |\ell(m)|,$$

which defines a contravariant functor $\text{Mod}_c^{\text{tf}}(\mathbb{Z}_p) \rightarrow \text{Ban}(\mathbb{Q}_p)^{\leq 1}$. For a Banach space $(E, \|\cdot\|)$ with unit ball L , they also define

$$E^d = \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p) \quad \text{with the topology of pointwise convergence,}$$

which gives a contravariant functor $\text{Ban}(\mathbb{Q}_p)^{\leq 1} \rightarrow \text{Mod}_c^{\text{tf}}(\mathbb{Z}_p)$. They prove in [20, Thm 1.2] that the functors $M \mapsto M^d$ and $E \mapsto E^d$ define an antiequivalence of categories. In particular, if E is an object in $\text{Ban}(\mathbb{Q}_p)^{\leq 1}$, L is the unit ball in E , and $M = E^d$, then we have $E = M^d$ and $L = \text{Hom}_{\mathbb{Z}_p}^{\text{cts}}(M, \mathbb{Z}_p)$.

We now consider a Banach space E with a continuous representation of G . We say that this representation is unitary if it preserves the norm. The representation induces an action of G on the dual $E' = E^d[1/p]$, which may be completed to an action of $\Lambda_{\mathbb{Q}_p}$. We say that E is admissible as a representation of G if E' is a finitely generated $\Lambda_{\mathbb{Q}_p}$ -module. In [20, Lemma 3.4] and the subsequent discussion, the authors show that admissibility implies that for any G -stable lattice $L \subset E$, the representation of G on L/pL is smooth and admissible (in the usual sense of these terms, recalled at the start of Section 2.4). They also prove that if an object E in $\text{Ban}(\mathbb{Q}_p)^{\leq 1}$ carries an admissible unitary representation of G , then E^d is a finitely generated $\Lambda_{\mathbb{Z}_p}$ -module.

2.4 Completed cohomology

The last ingredient we need before proving Theorem 1.2 is the following bound for the growth of invariants in representations of G . If L is a representation of G over $\mathbb{F}_p$, we recall that L is called smooth if every $v \in L$ has an open stabilizer in G , and admissible if the dimension of L^H is finite for any open subgroup $H < G$.

Proposition 2.3. Let L be a smooth admissible representation of G over $\mathbb{F}_p$ such that L^* is a finitely generated torsion Λ module. Then we have

$$\dim L^{TG_n} \ll_L (100p^{-4/9})^n |G : TG_n|$$

for all n .

□

Note that Proposition 2.3 represents a power saving over the trivial bound of $\dim L^{TG_n} \ll_L |G : TG_n|$. It will be proved in Section 3.

We now complete the proof of Theorem 1.2. Let $\phi : \Gamma \rightarrow SL(3, \mathbb{Z}_p)$ be the natural injection obtained by composing $\Gamma \rightarrow SL(3, \mathbb{Z}) \rightarrow SL(3, \mathbb{Z}_p)$, and note that $\overline{\phi(\Gamma)}$ is open. By choosing p sufficiently large and passing to a subgroup of Γ , we may assume that $\overline{\phi(\Gamma)} = G$. Our assumption that $p > 3$ then implies that Γ is torsion free. For $k \geq 1$, define $\Gamma_k = \Gamma \cap G_k$. Let n be the smallest integer with $p^{n-1} > 3\lambda$ and $3|n$, and let $\mathcal{V}_\lambda \subset V_\lambda$ be obtained by applying Lemma 2.1 to this n . Following Calegari and Emerton, we define

$$\tilde{H}^i(\mathcal{V}_\lambda) = \varprojlim_s \varinjlim_k H^i(\Gamma_k, \mathcal{V}_\lambda/p^s), \quad \tilde{H}^i(\mathcal{V}_\lambda)_{\mathbb{Q}_p} = \tilde{H}^i(\mathcal{V}_\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Because $\mathcal{V}_\lambda$ is continuous as a representation of G , for each fixed s , $\mathcal{V}_\lambda/p^s$ is eventually trivial on Γ_k . If we define

$$\tilde{H}^i = \varprojlim_s \varinjlim_k H^i(\Gamma_k, \mathbb{Z}_p/p^s), \quad \tilde{H}_{\mathbb{Q}_p}^i = \tilde{H}^i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

we therefore have $\tilde{H}^i(\mathcal{V}_\lambda) = \tilde{H}^i \otimes_{\mathbb{Z}_p} \mathcal{V}_\lambda$ and $\tilde{H}^i(\mathcal{V}_\lambda)_{\mathbb{Q}_p} = \tilde{H}_{\mathbb{Q}_p}^i \otimes_{\mathbb{Q}_p} V_\lambda$. We recall the following facts about $\tilde{H}_{\mathbb{Q}_p}^i$.

- $\tilde{H}^i$ is a p -adically separated and complete $\mathbb{Z}_p$ -module with bounded p -torsion exponent, by [6, Thm 1.1] and [9, Lemma 2.1.4]. This implies that the torsion free quotient $\tilde{H}_{\text{tf}}^i$ is also separated and complete, so that $\tilde{H}_{\mathbb{Q}_p}^i = \tilde{H}_{\text{tf}}^i \otimes \mathbb{Q}_p$ is naturally a $\mathbb{Q}_p$ -Banach space in which $\tilde{H}_{\text{tf}}^i$ is the unit ball.
- The natural action of G on $\tilde{H}_{\text{tf}}^i$ induces an admissible unitary representation of G on $\tilde{H}_{\mathbb{Q}_p}^i$ which extends to the group $SL(3, \mathbb{Q}_p)$ [9, Thm 2.1.5 and 2.2.11].
- Because $SL(3, \mathbb{R})$ does not admit discrete series, the dual space of $\tilde{H}_{\mathbb{Q}_p}^i$ is a torsion $\Lambda_{\mathbb{Q}_p}$ -module for all i [7].
- There is a spectral sequence $E_2^{i,j} = H_{\text{cts}}^i(G, \tilde{H}_{\mathbb{Q}_p}^j \otimes_{\mathbb{Q}_p} V_\lambda) \implies H^{i+j}(\Gamma, V_\lambda)$ [9, Prop 2.1.11].

The spectral sequence above implies that

$$h^2(\Gamma, V_\lambda) \leq \sum_{i+j=2} h^i(G, \tilde{H}_{\mathbb{Q}_p}^j \otimes_{\mathbb{Q}_p} V_\lambda), \quad (3)$$

where h^i denotes the dimension of H_{cts}^i . Moreover, the terms in this sum with $j = 0, 1$ vanish. The vanishing of the $j = 1$ term follows from $\tilde{H}^1 = 0$, which is a consequence of the congruence subgroup property for $SL(3, \mathbb{Z})$. For the $j = 0$ term, one has $\tilde{H}^0 = \mathbb{Z}_p$, and so $H^2(G, \tilde{H}_{\mathbb{Q}_p}^0 \otimes_{\mathbb{Q}_p} V_\lambda) = H^2(G, V_\lambda)$. A theorem of Lazard [15, Ch V, Thm 2.4.10] shows that $H^2(G, V_\lambda)$ is a subspace of the Lie algebra cohomology group $H^2(\mathfrak{sl}_3, V_\lambda)$, which vanishes by Whitehead's lemma. We therefore have

$$h^2(\Gamma, V_\lambda) \leq h^0(G, \tilde{H}_{\mathbb{Q}_p}^2 \otimes_{\mathbb{Q}_p} V_\lambda).$$

We let $E = \tilde{H}_{\mathbb{Q}_p}^2$, and let $L = \tilde{H}_{\text{tf}}^2$ be the unit ball in E . We define $\bar{L} = L/pL$. We then have

$$h^0(G, E \otimes_{\mathbb{Q}_p} V_\lambda) \leq h^0(G, L \otimes_{\mathbb{Z}_p} V_\lambda) \leq h^0(G, \bar{L} \otimes_{\mathbb{F}_p} (V_\lambda/p)),$$

where the middle term denotes the rank of the free finitely generated $\mathbb{Z}_p$ -module $(L \otimes_{\mathbb{Z}_p} \mathcal{V}_\lambda)^G$. The inclusion $\mathcal{V}_\lambda/p\mathcal{V}_\lambda \subset \mathbb{F}_p[G/P(n)]$ from Lemma 2.1 gives

$$h^0(G, \bar{L} \otimes_{\mathbb{F}_p} (\mathcal{V}_\lambda/p)) \leq h^0(G, \bar{L} \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/P(n)]), \quad (4)$$

and by Shapiro's Lemma this is equal to $\dim \bar{L}^{P(n)}$. We now use the $SL(3, \mathbb{Q}_p)$ action to conjugate $P(n)$ so that it is closer to TG_n , which lets us apply Proposition 2.3. If we define $x = \text{diag}(p^{n/3}, 1, p^{-n/3})$, then we have $xP(n)x^{-1} \subset TG_{n/3}$. We apply Lemma 3.4 to these groups, which gives

$$\dim \bar{L}^{P(n)} = \dim \bar{L}^{xP(n)x^{-1}} \leq |TG_{n/3} : xP(n)x^{-1}| \dim \bar{L}^{TG_{n/3}}.$$

We let $M = E^d$, so that $L \simeq \text{Hom}_{\text{cts}}(M, \mathbb{Z}_p)$. The isomorphism $M \simeq \prod_{i \in I} \mathbb{Z}_p$ from (2) implies that

$$L \simeq c_0(I, \mathbb{Z}_p) := \{f : I \rightarrow \mathbb{Z}_p \mid \text{for all } c > 0, |f(i)| > c \text{ for only finitely many } i\},$$

and that the vector spaces $\bar{L}$ and $\bar{M} := M/pM$ satisfy $\bar{M} = \bar{L}^*$ (where the quotient topology on $\bar{M}$ is the same as the pointwise topology). We know that M is a finitely generated and torsion $\Lambda_{\mathbb{Z}_p}$ -module, and because it has no $\mathbb{Z}_p$ -torsion this implies that $\bar{M}$ is a finitely generated and torsion Λ -module. Proposition 2.3 then gives

$$\begin{aligned} |TG_{n/3} : xP(n)x^{-1}| \dim \bar{L}^{TG_{n/3}} &\ll (100p^{-4/9})^{n/3} |G : xP(n)x^{-1}| \\ &= (100p^{-4/9})^{n/3} p^{3n}. \end{aligned}$$

By our choice of n , $(100p^{-4/9})^{n/3} p^{3n} \ll \lambda^{3-4/27} 10^n$, which completes the proof after choosing p sufficiently large.

2.5 Extension to other groups

The fact that we cannot presently obtain Theorem 1.1 for extensions of $\mathbb{Q}$, or higher GL_d , is due to the way we obtained the bound $h^2(\Gamma, V_\lambda) \leq h^0(G, \tilde{H}_{\mathbb{Q}_p}^2 \otimes_{\mathbb{Q}_p} V_\lambda)$ above. We did this by using the congruence subgroup property to show that the contributions of $\tilde{H}_{\mathbb{Q}_p}^0$ and $\tilde{H}_{\mathbb{Q}_p}^1$ to the sum (3) are trivial, so we are left with $h^0(G, \tilde{H}_{\mathbb{Q}_p}^2 \otimes_{\mathbb{Q}_p} V_\lambda)$. The fact that only an h^0 remains is essential to our argument, as it lets us use the bound (4) above (which isn't necessarily true for higher h^i).

To generalize our argument, we would need to bound the growth of $h^a(\Gamma, V)$ in the lowest degree a that cusp forms contribute (where V is a varying local system). For a group GL_d/F with $d \geq 4$, or $d = 3$ and $F \neq \mathbb{Q}$, we have $a \geq 3$. Therefore, to control the right hand side of the inequality

$$h^a(\Gamma, V) \leq \sum_{i+j=a} h^i(G, \tilde{H}_{\mathbb{Q}_p}^j \otimes_{\mathbb{Q}_p} V)$$

we would need to control a term $h^i(G, \tilde{H}_{\mathbb{Q}_p}^j \otimes_{\mathbb{Q}_p} V)$ with $i \geq 1$ and $j \geq 2$, and we currently do not know how to do this. In [17], we overcame this obstacle by showing that $\mathcal{V}_\lambda/p$ had an efficient filtration by modules isomorphic to $\mathbb{F}_p[G/P(n)]$ for varying n (or rather, the analogous statement in the GL_2 case). In the case of GL_3 , it would suffice to solve the following:

Problem 2.4. There is $\delta > 0$ such that for any λ , $\mathcal{V}_\lambda/p$ has a filtration by modules $\mathbb{F}_p[G/P(n_i)]$ such that

$$\sum_i |G : P(n_i)|^{1-4/81} \ll \lambda^{3-\delta}.$$

□

Note that the exponent $1 - 4/81$ comes from the bound $\dim \bar{L}^{P(n)} \ll |G : P(n)|^{1-4/81}$, which we implicitly deduced from Proposition 2.3 above.

3 Invariants of Λ modules

This section contains the proof of Proposition 2.3. We in fact prove a general version for any $SL_d(\mathbb{Z}_p)$, stated as Corollary 3.3 below. For $d \geq 2$, let $G_d(n) = \{g \in SL_d(\mathbb{Z}_p) : g \equiv 1(p^n)\}$ and $G_d = G_d(1)$. As d will be fixed for most of the proof, we shall usually omit it and simply write G and $G(n)$. We define T and Λ in the analogous way to Section 2.1. We shall deduce Corollary 3.3 from the following two results. The first uses only elementary representation theory, and the second follows from a theorem of M. Harris [13].

Theorem 3.1. Assume that $p > d$. Let L be a representation of G over $\mathbb{F}_p$ such that $\dim L^{G(n)} \ll_L p^{-n} |G : G(n)|$ for all n . We then have

$$\dim L^{TG(n)} \ll_L 10^{(d-1)n} p^{-(2/3)^{d-1}n} |G : TG(n)|.$$

□

Theorem 3.2 (M. Harris). Assume that $p > 2$. If L is a smooth admissible representation of G over $\mathbb{F}_p$ such that L^* is a finitely generated torsion Λ module, then $\dim L^{G(n)} \ll_L p^{-n} |G : G(n)|$ for all n .

□

See [10, Prop 2.17] for a strengthening of Theorem 3.2. Note that the theorem still holds if $p = 2$, provided we change G to the depth 4 principal congruence subgroup $G(2)$ (so that G remains a uniformly powerful pro- p group). Combining these gives:

Corollary 3.3. Assume that $p > d$. Let L be a smooth admissible representation of G over $\mathbb{F}_p$ such that L^* is a finitely generated torsion Λ module. Then

$$\dim L^{TG(n)} \ll_L 10^{(d-1)n} p^{-(2/3)^{d-1}n} |G : TG(n)|.$$

□

Note that if L^* is a finitely generated Λ module, then one has the trivial bounds $\dim L^{G(n)} \ll |G : G(n)|$ and $\dim L^{TG(n)} \ll |G : TG(n)|$, and Theorem 3.2 and Corollary 3.3 respectively represent power savings over these bounds under the assumption that L^* is torsion. To prove Theorem 3.1, we shall show that one may stretch the subgroup $G(n)$ into $TG(n)$ while maintaining control of the invariants, using the basic method of [17, Prop 7].

3.1 A lemma on passage to subgroups

The following lemma will let us pass bounds for invariants between subgroups of G .

Lemma 3.4. Let V be a representation of G over $\mathbb{F}_p$, and let $G \geq H_1 \geq H_2$ be open subgroups of G . We have $\dim V^{H_2} \leq |H_1 : H_2| \dim V^{H_1}$.

□

Proof. By Lemma 3.5, it suffices to find a chain of normal subgroups $H_1 = J_1 \triangleright J_2 \triangleright \dots \triangleright J_i = H_2$. We claim that the groups $J_k = (H_1 \cap G(k))H_2$ (which stabilize at H_2 for k large) suffice. First, one observes that these are in fact groups, as H_2 normalizes H_1 and $G(k)$. Next, we wish to show that J_k is normal in J_{k-1} . To do this, it suffices to check that H_2 and $H_1 \cap G(k-1)$ each normalize J_k . This is clear for H_2 , as it normalizes H_1 , H_2 , and $G(k)$. Moreover, $H_1 \cap G(k-1)$ normalizes $H_1 \cap G(k)$, and so it suffices to show that for $g \in H_1 \cap G(k-1)$ and $h \in H_2$ we have $ghg^{-1} \in J_k$. We have $[g, h] \in [G(k-1), G] \subset G(k)$ and $[g, h] \in [H_1, H_2] \subset H_1$, hence $ghg^{-1} \in (H_1 \cap G(k))h \subset J_k$ as required. ■

Lemma 3.5. Let $J_1 \triangleright J_2$ be two groups, with J_1/J_2 of order p . Let V be a representation of J_1 over $\mathbb{F}_p$. Then $\dim V^{J_2} \leq p \dim V^{J_1}$.

□

Proof. The space V^{J_2} carries a representation of J_1/J_2 . If we let $j \in J_1/J_2$ be nontrivial, then on V^{J_2} we have $\ker(1-j) = V^{J_1}$ and $(1-j)^p = 0$. The lemma follows. ■

3.2 Proof of Theorem 3.1.

Let $S \subset T$ be the torus

$$S = \left\{ \begin{pmatrix} x & & & \\ & \ddots & & \\ & & x & \\ & & & x^{1-d} \end{pmatrix} : x \in 1 + p\mathbb{Z}_p \right\}.$$

We first prove a version of Theorem 3.1 with T replaced by S .

Proposition 3.6. Assume that $p > d$. Let V be a representation of G over $\mathbb{F}_p$, and suppose that there exist $C, N > 0$ and $R < p^{d^2-1}$ such that $\dim V^{G(n)} \leq CR^n$ for all $n \leq N$. Let $\rho = R^{1/3}p^{(4-d^2)/3}$. Then

$$\dim V^{SG(n)} \leq C'(10\rho^{-1})^n R^n = C'10^n R^{2n/3} p^{(d^2-4)n/3}$$

for all $n \leq N$, where $C' = C \max\{1, (10\rho^{-1})^{-2}\}$.

□

We could prove Proposition 3.6 with any one dimensional torus, but the reason we have chosen S is that it commutes with the copy of G_{d-1} in the upper left hand block of G . It follows that $V^{SG(n)}$ is a G_{d-1} module, which lets us apply Proposition 3.6 inductively to bound $\dim V^{TG(n)}$ in Proposition 3.9 below.

Proof. If $\rho < 10$ then the bound we wish to prove is weaker than the trivial bound $\dim V^{SG(n)} \leq CR^n$. We may therefore assume that $\rho \geq 10$, in which case the bound we must prove is $\dim V^{SG(n)} \leq C(10\rho^{-1})^{n-2}R^n$. For any $n-1 \geq k \geq 0$, we define $S(n, k) = (S \cap G(n-k))G(n)$. One may think of $S(n, k)$ as the subgroup of G obtained by stretching $G(n)$ by k steps in the S direction. We shall prove by induction that

$$\dim V^{S(n, k)} \leq C(10\rho^{-1})^{k-1}R^n \tag{5}$$

for all $1 \leq n \leq N$ and $0 \leq k \leq n-1$. As $S(n, n-1) = SG(n)$, this gives the proposition. Note that (5) follows from the conditions of the proposition when $k = 0, 1$.

Fix (n, k) , and suppose that (5) holds for all (n', k') less than (n, k) in the lexicographic ordering. We may assume that $k \geq 2$, and hence that $n \geq 3$. As in [17, Prop 7], we shall deduce (5) for (n, k) from the cases $(n-1, k-1)$ and $(n, k-1)$, by applying inclusion-exclusion counting to the invariants under certain subgroups lying between $S(n-1, k-1)$ and $S(n, k-1)$.

It may be seen that $S(n, k-1)$ is normal in $S(n-1, k-1)$, and that the quotient $X = S(n-1, k-1)/S(n, k-1)$ is Abelian and isomorphic to the vector space $\mathbb{F}_p^{d^2-1}$. The image of $S(n, k)$ in X is a line, which we denote by ℓ . We define

$$N = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & p^{n-1} \\ & & & 1 \end{pmatrix}, \quad \overline{N} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & p^{n-1} & 1 \end{pmatrix},$$

and define $W \subset X$ to be the subspace spanned by ℓ , N , and $\overline{N}$. Define $U \subset W$ to be the subspace spanned by N and $\overline{N}$. If $Y \subset X$ is any subspace (which we may identify with a subgroup of G), we let V^Y be the vectors in V fixed by Y . The argument on [17, p. 1638] gives

$$\left(\frac{2m}{m-1} - 1\right) \dim V^\ell \leq \frac{2}{m-1} \dim V^0 + \frac{m(m-1)}{2} \dim V^W, \quad (6)$$

where $m = \lfloor \rho \rfloor$, and we briefly recall how this works. First, the following lemma implies that $\dim V^{\ell'} = \dim V^\ell$ for any line $\ell' \subset W$ not contained in U .

Lemma 3.7. If $\ell' \subset W$ is a line not contained in U , then there is $g \in G$ whose action by conjugation descends to X , and such that $g\ell'g^{-1} = \ell$.

□

Proof. This follows in the same way as [17, Lemma 8]. If we define

$$N' = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & p^{k-1} \\ & & & 1 \end{pmatrix}, \quad \overline{N}' = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & p^{k-1} & 1 \end{pmatrix},$$

it may be checked that N' and $\overline{N}'$ normalize $S(n-1, k-1)$ and $S(n, k-1)$, and that conjugation by N' or $\overline{N}'$ acts on W by shearings that fix U pointwise and translate in the directions of N and $\overline{N}$ respectively. ■

Next, if $P \subset W$ is a plane different from U , and $\ell_1, \dots, \ell_j \subset P$ are distinct lines that do not lie in U , then [17, Lemma 9] gives

$$\dim V^{\ell_1} + \dim \sum_{i=2}^j V^{\ell_i} \leq \dim \sum_{i=1}^j V^{\ell_i} + (j-1) \dim V^P. \quad (7)$$

The assumption $R < p^{d^2-1}$ implies that $m \leq \rho < p$, so that we may choose m lines $\ell_1, \dots, \ell_m$ satisfying these conditions. We may apply (7) successively to the collections $\{\ell_1, \dots, \ell_m\}, \{\ell_2, \dots, \ell_m\}, \dots, \{\ell_{m-1}, \ell_m\}$ to obtain

$$\begin{aligned} \dim V^{\ell_1} + \dim \sum_{i=2}^m V^{\ell_i} &\leq \dim \sum_{i=1}^m V^{\ell_i} + (m-1) \dim V^P \\ \dim V^{\ell_2} + \dim \sum_{i=3}^m V^{\ell_i} &\leq \dim \sum_{i=2}^m V^{\ell_i} + (m-2) \dim V^P \\ &\vdots \\ \dim V^{\ell_{m-1}} + \dim V^{\ell_m} &\leq \dim(V^{\ell_{m-1}} + V^{\ell_m}) + \dim V^P. \end{aligned}$$

Adding these and simplifying gives

$$\sum_{i=1}^m \dim V^{\ell_i} \leq \dim \sum_{i=1}^m V^{\ell_i} + \frac{m(m-1)}{2} \dim V^P \leq \dim V^0 + \frac{m(m-1)}{2} \dim V^P. \quad (8)$$

When combined with $\dim V^{\ell_i} = \dim V^\ell$, this becomes

$$m \dim V^\ell \leq \dim V^0 + \frac{m(m-1)}{2} \dim V^P. \quad (9)$$

If $P_1, \dots, P_m \subset W$ are planes containing ℓ , we may apply the argument from [17, Lemma 9] to the lines $P_1/\ell, \dots, P_m/\ell$ in W/ℓ to obtain the analog of (7), and hence of (8), which is

$$\sum_{i=1}^m \dim V^{P_i} \leq \dim V^\ell + \frac{m(m-1)}{2} \dim V^W.$$

Bounding each $\dim V^{P_i}$ from below using (9) and rearranging gives (6).

Our inductive hypothesis (5) for $(n-1, k-1)$ gives $\dim V^X \leq C(10\rho^{-1})^{k-2}R^{n-1}$, and combining this with Lemma 3.4 we have

$$\dim V^W \leq p^{d^2-4} \dim V^X \leq C(10\rho^{-1})^{k-2}R^{n-1}p^{d^2-4}.$$

The inductive hypothesis for $(n, k-1)$ gives $\dim V^0 \leq C(10\rho^{-1})^{k-2}R^n$, and substituting these into (6) gives

$$\left(\frac{2m}{m-1} - 1\right) \dim V^\ell \leq C(10\rho^{-1})^{k-2}R^n \left(\frac{2}{m-1} + \frac{m(m-1)}{2}R^{-1}p^{d^2-4}\right).$$

By our choice of m , we have

$$\begin{aligned} \left(\frac{2m}{m-1} - 1\right)^{-1} \left(\frac{2}{m-1} + \frac{m(m-1)}{2}R^{-1}p^{d^2-4}\right) \\ = \left(\frac{2m}{m-1} - 1\right)^{-1} \left(\frac{2}{m-1} + \frac{m(m-1)}{2}\rho^{-3}\right) \leq 10\rho^{-1}. \end{aligned}$$

This completes the inductive step, and hence the proof. ■

Remark 3.8. It may be possible to improve the bound of Proposition 3.6 by working with more conjugates of S than we do here. This amounts to conjugating ℓ within a $2d-1$ -dimensional subspace of X , rather than the three-dimensional space W . □

By applying Proposition 3.6 inductively, we may prove the following.

Proposition 3.9. Assume that $p > d$. Let V be a representation of G over $\mathbb{F}_p$, and suppose that there exist $C, N > 0$ and $R < p^{d^2-1}$ such that $\dim V^{G(n)} \leq CR^n$ for all $n \leq N$. Then

$$\dim V^{TG(n)} \leq C\kappa 10^{(d-1)n} R^{(2/3)^{d-1}n} p^{\sigma(d)n}$$

for all $n \leq N$, where $\sigma(d) = d(d-1) - (2/3)^{d-1}(d^2-1)$, and $\kappa > 0$ depends only on R, p , and d .

□

Proof. We proceed by induction on d . We let $\kappa > 0$ denote a constant depending only on R, p , and d that may vary from line to line. The base case of $d = 2$ is exactly the statement of Proposition 3.6. We next prove it for a given $d \geq 3$, assuming it holds for $d-1$. Let V, N, R , and C be as in the statement of the proposition. We think of the groups $G_{d-1}(n)$ as embedded in G_d in the upper left block. For any $k \leq n \leq N$, we may apply Lemma 3.4 to the groups $SG_d(k) \geq SG_d(n)G_{d-1}(k)$ to obtain

$$\dim V^{SG_d(n)G_{d-1}(k)} \leq |SG_d(k) : SG_d(n)G_{d-1}(k)| \dim V^{SG_d(k)};$$

note that $SG_d(n)G_{d-1}(k)$ is in fact a group, because S and $G_{d-1}(k)$ commute and they both normalize $G_d(n)$. Combining this with

$$|SG_d(k) : SG_d(n)G_{d-1}(k)| = p^{\dim(G_d/SG_{d-1})(n-k)} = p^{(2d-2)(n-k)}$$

and the bound for $\dim V^{SG_d(k)}$ from Proposition 3.6 gives

$$\dim V^{SG_d(n)G_{d-1}(k)} \leq p^{(2d-2)(n-k)} C\kappa 10^k R^{2k/3} p^{(d^2-4)k/3} \leq C\kappa 10^n p^{(2d-2)n} R^{2k/3} p^{(d^2-6d+2)k/3}.$$

This implies that we may invoke the induction hypothesis for the representation of G_{d-1} on $V_0 = V^{SG_d(n)}$ with data $C_0 = C\kappa 10^n p^{(2d-2)n}$, $R_0 = R^{2/3} p^{(d^2-6d+2)/3}$, and $N_0 = n$. (It may be checked that $R_0 < p^{(d-1)^2-1}$.) If T_0 is the diagonal subgroup of G_{d-1} , this gives

$$\begin{aligned} \dim V^{TG_d(n)} &= \dim V_0^{T_0 G_{d-1}(n)} \\ &\leq C_0 \kappa 10^{(d-2)n} R_0^{(2/3)^{d-2}n} p^{\sigma(d-1)n} \\ &= C\kappa 10^{(d-1)n} p^{(2d-2)n} R_0^{(2/3)^{d-2}n} p^{\sigma(d-1)n} \\ &= C\kappa 10^{(d-1)n} p^{(2d-2)n} R^{(2/3)^{d-1}n} p^{(2/3)^{d-2}n(d^2-6d+2)/3} p^{\sigma(d-1)n}. \end{aligned}$$

The proposition now follows after checking that the exponent of p satisfies

$$(2d-2) + (2/3)^{d-2}(d^2-6d+2)/3 + \sigma(d-1) = \sigma(d).$$

■

Finally, we deduce Theorem 3.1 from Proposition 3.9. We are given that the conditions of the proposition hold with $R = p^{d^2-2}$, some $C > 0$, and any N , and it may be checked that in this case the proposition gives exactly the conclusion of Theorem 3.1.

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