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A note on weak factorization of a Meyer-type Hardy space via a Cauchy integral operator

by

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Abstract. This paper provides a weak factorization for the Meyer-type Hardy space $H^1_b(\mathbb{R})$, and characterizations of its dual $\mathrm{BMO}_b(\mathbb{R})$ and its predual $\mathrm{VMO}_b(\mathbb{R})$ via boundedness and compactness of a suitable commutator with the Cauchy integral $\mathscr{C}_\Gamma$, respectively. Here $b(x) = 1 + iA'(x)$ where $A' \in L^\infty(\mathbb{R})$, and the Cauchy integral $\mathscr{C}_\Gamma$ is associated to the Lipschitz curve $\Gamma = \{x + iA(x) : x \in \mathbb{R}\}$.

1. Introduction and statement of main results. Given a bounded function $b : \mathbb{R}^n \to \mathbb{C}$ such that $\operatorname{Re} b(x) \geq 1$ for all $x \in \mathbb{R}^n$, the Meyer-type Hardy space $H^1_b(\mathbb{R}^n)$ consists of those functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the product bf belongs to the real Hardy space $H^1(\mathbb{R}^n)$. The Meyer-type space of bounded mean oscillation, denoted $\mathrm{BMO}_b(\mathbb{R}^n)$, consists of all functions $\mathfrak{A} : \mathbb{R}^n \to \mathbb{C}$ such that the function $\mathfrak{A}/b$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$, and is the dual of $H^1_b(\mathbb{R}^n)$. These spaces were introduced by Yves Meyer [Me, Chapter XI, Section 10, p. 358] in dimension one in connection with the study of the Cauchy integral associated with a Lipschitz curve and the $T(b)$ theorem. Observe that both the real Hardy space $H^1(\mathbb{R}^n)$ and its dual $\mathrm{BMO}(\mathbb{R}^n)$ consist of real-valued functions.

In this note we study the Meyer-type Hardy space $H^1_b(\mathbb{R})$ and its dual $\mathrm{BMO}_b(\mathbb{R})$ for $b(x) = 1 + iA'(x)$ where $A' \in L^\infty(\mathbb{R})$, via the Cauchy integral $\mathscr{C}_\Gamma$ associated to the Lipschitz curve $\Gamma = \{x + iA(x) : x \in \mathbb{R}\}$. We present a weak factorization of $H^1_b(\mathbb{R})$ in terms of the Cauchy integral $\mathscr{C}_\Gamma$. We also obtain a characterization of $\mathrm{BMO}_b(\mathbb{R})$ and of $\mathrm{VMO}_b(\mathbb{R})$, the Meyer-type

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space of vanishing mean oscillation, via boundedness and compactness of a suitable commutator with the Cauchy integral respectively.

The *Cauchy integral associated with the Lipschitz curve* Γ is the integral operator $\mathscr{C}_\Gamma$ given by

$$\mathscr{C}_\Gamma(f)(x) := \text{p.v.}\ \frac{1}{\pi i}\int_{\mathbb{R}} \frac{(1+iA'(y))f(y)}{y-x+i(A(y)-A(x))}\,dy,$$

where $f \in C_c^\infty(\mathbb{R})$. Note that it is not a standard Calderón–Zygmund operator because it lacks smoothness. The L^p-boundedness of $\mathscr{C}_\Gamma$ is equivalent to that of the related operator $\widetilde{\mathscr{C}_\Gamma}$ defined by

$$\widetilde{\mathscr{C}_\Gamma}(f)(x) := \text{p.v.}\ \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{y-x+i(A(y)-A(x))}\,dy.$$

Moreover, the kernel of $\widetilde{\mathscr{C}_\Gamma}$ satisfies standard size and smoothness estimates [LN+] and is therefore bounded on $L^p(\mathbb{R})$ for $p \in (1,\infty)$. Note that in the cited article the operator $\mathscr{C}_\Gamma$ was denoted $\widetilde{\mathscr{C}_\Gamma}$ and vice versa. Hence while $\mathscr{C}_\Gamma(f)$ is initially defined for $f \in C_c^\infty(\mathbb{R})$, the operator $\mathscr{C}_\Gamma$ can be extended to all $f \in L^p(\mathbb{R})$, for each $p \in (1,\infty)$.

The operator $\widetilde{\mathscr{C}_\Gamma}$ and its commutator with functions in BMO($\mathbb{R}$) were studied by Li, Nguyen, Ward, and Wick [LN+]. In this setting, one could appeal to a weak factorization for $H^1(\mathbb{R}^n)$ in terms of multilinear Calderón–Zygmund operators, due to Li and Wick [LW1, Theorem 1.3], to obtain the desired characterization of BMO($\mathbb{R}$) via boundedness of the commutator, and of VMO($\mathbb{R}$) via compactness of the commutator.

We want to study the Meyer-type Hardy space, bounded mean oscillation space, and vanishing mean oscillation space: $H^1_b(\mathbb{R})$, $\mathrm{BMO}_b(\mathbb{R})$, and $\mathrm{VMO}_b(\mathbb{R})$, via the rougher operator $\mathscr{C}_\Gamma$. As it turns out, we can derive these results from the results for the related Cauchy integral operator [LN+]. Nevertheless we also present a direct constructive proof of the weak factorization valid for $H^1_b(\mathbb{R})$ that may be of independent interest.

We now state our main results. For $b(x) = 1 + iA'(x)$, we introduce the associated bilinear form as follows:

$$\Pi_b(g,h)(x) := \frac{1}{b(x)}\big(g(x)\cdot \mathscr{C}_\Gamma(h)(x) - h(x)\cdot \mathscr{C}^*_\Gamma(g)(x)\big), \tag{1.1}$$

where $\mathscr{C}^*_\Gamma$ is the adjoint operator to $\mathscr{C}_\Gamma$.

THEOREM 1.1. *For any* $f \in H^1_b(\mathbb{R})$ *there exist a sequence* $\{\lambda_j^k\}_{j,k\ge 1} \in \ell^1$ *and functions* $g_j^k, h_j^k \in L^\infty(\mathbb{R})$ *for integers* $j,k \ge 1$ *with compact supports such that*

$$f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_b(g_j^k, h_j^k).$$

Moreover,

$$\|f\|_{H^1_b(\mathbb{R})} \approx \inf\Big\{\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\lambda_j^k|\, \|g_j^k\|_{L^2(\mathbb{R})}\|h_j^k\|_{L^2(\mathbb{R})} : f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k)\Big\}.$$

The double index notation, used for both the sequence and the functions appearing in the theorem, is a reflection of the Uchiyama construction which is executed in two layers, one indexed by integers $j \geq 1$ and the other by integers $k \geq 1$. We could have stated the theorem with a single index.

The commutator $[g, T]$ of a function g and an operator T is the new operator acting on suitable functions f, defined by $[g, T](f) := gT(f) - T(gf)$. It is well known that $a \in \mathrm{BMO}(\mathbb{R})$ (respectively $a \in \mathrm{VMO}(\mathbb{R})$) if and only if the commutator $[a, H]$ of a with the Hilbert transform is a bounded operator on $L^p(\mathbb{R})$ [CRW] (respectively is a compact operator [U1]). In [LN+], functions a in $\mathrm{BMO}_b(\mathbb{R})$ (respectively in $\mathrm{VMO}_b(\mathbb{R})$) were characterized via boundedness (respectively compactness) of $[a, \widetilde{\mathscr{C}_\Gamma}]$, the commutator with the related Cauchy operator. To characterize $\mathrm{BMO}_b(\mathbb{R})$ and $\mathrm{VMO}_b(\mathbb{R})$ we will consider the commutator of the Cauchy integral not with functions in $\mathrm{BMO}_b(\mathbb{R})$ or $\mathrm{VMO}_b(\mathbb{R})$ but with those functions divided by the accretive function b. In other words, we will consider for the next theorems the commutator $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ where $\mathfrak{A}$ is in $\mathrm{BMO}_b(\mathbb{R})$ or in $\mathrm{VMO}_b(\mathbb{R})$.

THEOREM 1.2. *Let $b(x) = 1 + iA'(x)$ and $p \in (1, \infty)$. If $\mathfrak{A} \in \mathrm{BMO}_b(\mathbb{R})$, then the commutator $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ is a bounded operator, and*

$$\|[\mathfrak{A}/b, \mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \lesssim \|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}.$$

Conversely, for any complex function $\mathfrak{A}$ such that $\mathfrak{A}/b$ is a real-valued function and $\mathfrak{A}/b \in L^1_{\mathrm{loc}}(\mathbb{R})$, if the commutator $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ is a bounded operator then $\mathfrak{A} \in \mathrm{BMO}_b(\mathbb{R})$, and

$$\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})} \lesssim \|[\mathfrak{A}/b, \mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}.$$

THEOREM 1.3. *Let $b(x) = 1 + iA'(x)$ and $p \in (1, \infty)$. If $\mathfrak{A} \in \mathrm{VMO}_b(\mathbb{R})$, then $[\mathfrak{A}/b, C_\Gamma]$ is compact on $L^p(\mathbb{R})$. Conversely, for any complex function $\mathfrak{A} \in \mathrm{BMO}_b(\mathbb{R})$ such that $\mathfrak{A}/b$ is real-valued and $\mathfrak{A}/b \in L^1_{\mathrm{loc}}(\mathbb{R})$, if $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ is compact on $L^p(\mathbb{R})$, then $\mathfrak{A} \in \mathrm{VMO}_b(\mathbb{R})$.*

Note that one can deduce these three theorems from the results in [LN+] and [LW1] directly, as we will show in Section 3.

The paper is organized as follows. In Section 2 we collect the necessary preliminaries needed to explain the result. In Section 3 we provide a connection between the classical Hardy and BMO spaces and the spaces introduced by Meyer. In Section 4 we provide another proof of Theorem 1.1 using a clever construction due to Uchiyama [U2].

We use the standard notation $A \lesssim B$ or $B \gtrsim A$ to mean that there exists an absolute constant C such that $A \leq CB$. Likewise, $A \approx B$ if and only if $A \lesssim B$ and $B \lesssim A$. We use $\langle f, g\rangle_{L^2(\mathbb{R})}$ to denote the L^2-pairing $\int_{\mathbb{R}} f(x)g(x)\,dx$. We denote by $C_c^\infty(\mathbb{R})$ the space of compactly supported infinitely differentiable functions on $\mathbb{R}$. Finally, χ_I is the characteristic function of the set $I \subset \mathbb{R}$, defined by $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise.

2. Preliminaries. In this section we introduce basic notions of accretive functions; the classical spaces: the Hardy space $H^1(\mathbb{R})$, the space $\mathrm{BMO}(\mathbb{R})$ of bounded mean oscillation functions, and the space $\mathrm{VMO}(\mathbb{R})$ of vanishing mean oscillation as well as their counterparts, the Meyer-type Hardy spaces: $H_b^1(\mathbb{R})$, $\mathrm{BMO}_b(\mathbb{R})$, and $\mathrm{VMO}_b(\mathbb{R})$, for b an accretive function. We also introduce the Cauchy integral operator $\mathscr{C}_\Gamma$ associated to a Lipschitz curve Γ and the related Cauchy integral operator $\widetilde{\mathscr{C}_\Gamma}$.

A function $b : \mathbb{R} \to \mathbb{C}$ is *accretive* if $b \in L^\infty(\mathbb{R})$ and there exists $\delta > 0$ such that $\operatorname{Re} b(x) \geq \delta$ for all $x \in \mathbb{R}$.

A locally integrable function $f : \mathbb{R} \to \mathbb{R}$ is said to be of *bounded mean oscillation*, written $f \in \mathrm{BMO}$ or $f \in \mathrm{BMO}(\mathbb{R})$, if

$$\|f\|_{\mathrm{BMO}} := \sup_I \frac{1}{|I|} \int_I |f(x) - f_I|\,dx < \infty.$$

Here the supremum is taken over all intervals I in $\mathbb{R}$ and $f_I := \frac{1}{|I|}\int_I f(y)\,dy$ is the average of the function f over the interval I.

A BMO function $f : \mathbb{R} \to \mathbb{R}$ is said to be of *vanishing mean oscillation*, written $f \in \mathrm{VMO}$ or $f \in \mathrm{VMO}(\mathbb{R})$, if the following three behaviors occur for small, large and far-from-the-origin intervals respectively:

$$\text{(i)} \qquad \lim_{\delta\to 0} \sup_{I:\,|I|<\delta} \frac{1}{|I|}\int_I |f(x) - f_I|\,dx = 0,$$

$$\text{(ii)} \qquad \lim_{R\to\infty} \sup_{I:\,|I|>R} \frac{1}{|I|}\int_I |f(x) - f_I|\,dx = 0,$$

$$\text{(iii)} \qquad \lim_{R\to\infty} \sup_{I:\,I\cap(-R,R)=\emptyset} \frac{1}{|I|}\int_I |f(x) - f_I|\,dx = 0.$$

The *Hardy space* $H^1(\mathbb{R})$ consists of those integrable functions $f : \mathbb{R} \to \mathbb{R}$ that admit an atomic decomposition $f(x) = \sum_{j=1}^\infty \lambda_j a_j(x)$ where $\sum_{j=1}^\infty |\lambda_j| < \infty$ and the functions a_j are L^∞-*atoms* (respectively L^2-*atoms*) in the sense that each a_j is supported on an interval I_j and satisfies the L^∞-size condition $\|a_j\|_{L^\infty(\mathbb{R})} \leq 1/|I_j|$ (respectively, L^2-size condition $\|a_j\|_{L^2(\mathbb{R})} \leq C|I_j|^{-1/2}$) and the cancellation condition $\int_{\mathbb{R}} a_j(x)\,dx = 0$. The H^1-norm can be defined using either type of atoms, for example

$$\|f\|_{H^1(\mathbb{R})} := \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j,\ a_j \text{ are } L^2\text{-atoms}\Big\}.$$

If instead we use L^∞-atoms we will get an equivalent norm [Gra, Section 6.6.4]. It is well known that $\mathrm{BMO}(\mathbb{R})$ is the dual of $H^1(\mathbb{R})$ [FS].

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be in $H^1_b(\mathbb{R})$, the *Meyer-type Hardy space associated with the accretive function* b, if $bf \in H^1(\mathbb{R})$; moreover

$$\|f\|_{H^1_b(\mathbb{R})} := \|bf\|_{H^1(\mathbb{R})}.$$

In other words, $f \in H^1_b(\mathbb{R})$ admits an atomic decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$ where $\sum_{j=1}^\infty |\lambda_j| < \infty$ and the functions a_j are L^∞-*atoms* (respectively L^2-atoms) in the sense that each a_j is supported on an interval I_j and satisfies the L^∞-size condition $\|a_j\|_{L^\infty(\mathbb{R})} \le 1/|I_j|$ (respectively, the L^2-size condition $\|a_j\|_{L^2(\mathbb{R})} \approx \|ba_j\|_{L^2(\mathbb{R})} \le C|I_j|^{-1/2}$) and the cancellation condition $\int_{\mathbb{R}} a_j(x)b(x)\,dx = 0$.

A locally integrable function $\mathfrak{A} : \mathbb{R} \to \mathbb{C}$ is said to be in $\mathrm{BMO}_b(\mathbb{R})$, the *Meyer-type* BMO *space associated with the accretive function* b, if

$$\mathfrak{A}/b \in \mathrm{BMO}(\mathbb{R}),$$

and we define its norm naturally to be $\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})} := \|\mathfrak{A}/b\|_{\mathrm{BMO}(\mathbb{R})}$. As a consequence of the H^1-BMO duality, $\mathrm{BMO}_b(\mathbb{R})$ is the dual of $H^1_b(\mathbb{R})$ [Me].

A locally integrable function $\mathfrak{A} : \mathbb{R} \to \mathbb{C}$ is said to be in $\mathrm{VMO}_b(\mathbb{R})$, the *Meyer-type* VMO *space associated with the accretive function* b, if

$$\mathfrak{A}/b \in \mathrm{VMO}(\mathbb{R}).$$

Suppose Γ is a curve in the complex plane $\mathbb{C}$ and f is a function defined on Γ. The *Cauchy integral* of f is the operator $\mathcal{C}_\Gamma$ defined on the complex plane for $z \notin \Gamma$ by

$$\mathcal{C}_\Gamma(f)(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{z-\zeta}\,d\zeta. \tag{2.1}$$

A curve Γ is said to be a *Lipschitz curve* if it can be written in the form $\Gamma = \{x + iA(x) : x \in \mathbb{R}\}$ where $A : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition

$$|A(x_1) - A(x_2)| \le L|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}. \tag{2.2}$$

The best constant L in (2.2) is referred to as the *Lipschitz constant* of Γ or of $A(x)$. One can show that A satisfies a Lipschitz condition if and only if A is differentiable almost everywhere on $\mathbb{R}$ and $A' \in L^\infty(\mathbb{R})$. The Lipschitz constant is $L = \|A'\|_\infty$.

The *Cauchy integral associated with the Lipschitz curve* Γ is the singular integral operator $\mathscr{C}_\Gamma$ acting on functions $f \in C_c^\infty(\mathbb{R})$ by

$$\mathscr{C}_\Gamma(f)(x) := \text{p.v.}\,\frac{1}{\pi i}\int_{\mathbb{R}} \frac{(1+iA'(y))f(y)}{y - x + i(A(y)-A(x))}\,dy \tag{2.3}$$

for $x \in \mathbb{R}$. The kernel of $\mathscr{C}_\Gamma$ is given by

$$C_\Gamma(x,y) = \frac{1}{\pi i}\,\frac{1+iA'(y)}{y-x+i(A(y)-A(x))}.$$

Note that this is not a standard Calderón–Zygmund kernel because the function $1+iA'$ does not necessarily possess any smoothness. As noted in [Gra, p. 289], the L^p-boundedness of $\mathscr{C}_\Gamma$ is equivalent to that of the related operator $\widetilde{\mathscr{C}_\Gamma}$ defined by

$$\widetilde{\mathscr{C}_\Gamma}(f)(x) := \text{p.v.}\,\frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{y-x+i(A(y)-A(x))}\,dy. \tag{2.4}$$

Moreover, the kernel of $\widetilde{\mathscr{C}_\Gamma}$ is given by

$$\widetilde{C}_\Gamma(x,y) = \frac{1}{\pi i}\,\frac{1}{y-x+i(A(y)-A(x))}. \tag{2.5}$$

The kernel $\widetilde{C}_\Gamma(x,y)$ satisfies standard size and smoothness ([1]) estimates [LN$^+$, Lemma 3.3] and is therefore bounded on $L^p(\mathbb{R})$ for $p \in (1,\infty)$. Therefore, while the operator $\mathscr{C}_\Gamma(f)$ is initially defined for $f \in C_c^\infty(\mathbb{R})$, it can be extended to all $f \in L^p(\mathbb{R})$, for each $p \in (1,\infty)$.

An operator T defined on $L^p(\mathbb{R})$ is *compact* on $L^p(\mathbb{R})$ if T maps bounded subsets of $L^p(\mathbb{R})$ into precompact sets. In other words, for all bounded sets $E \subset L^p(\mathbb{R})$, $T(E)$ is precompact. A set S is *precompact* if its closure is compact.

3. From classical spaces to Meyer Hardy spaces. In this section we take advantage of the known weak factorization result for $H^1(\mathbb{R})$ in terms of the Calderón–Zygmund singular integral operator $\widetilde{\mathscr{C}_\Gamma}$ as well as the characterization of $\mathrm{BMO}(\mathbb{R})$ via the boundedness of the commutator with $\widetilde{\mathscr{C}_\Gamma}$, and of $\mathrm{VMO}(\mathbb{R})$ via the compactness of the same commutator [LN$^+$] to deduce Theorems 1.1, 1.2, and 1.3.

We first consider the adjoint operator $\mathscr{C}_\Gamma^*(g)$. By a direct calculation, we can verify that for $f,g \in L^2(\mathbb{R})$,

$$\begin{aligned}\langle \mathscr{C}_\Gamma(f), g\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \text{p.v.}\,\frac{1}{\pi i}\int_{\mathbb{R}} \frac{(1+iA'(y))f(y)}{y-x+i(A(y)-A(x))}\,dy\ g(x)\,dx\\ &= \text{p.v.}\,\frac{1}{\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1+iA'(y))f(y)}{y-x+i(A(y)-A(x))}\,dy\ g(x)\,dx\end{aligned}$$

([1]) Namely: (size) $|\widetilde{C}_\Gamma(x,y)| \lesssim 1/|x-y|$ for all $x,y \in \mathbb{R}$ and (smoothness) $|\widetilde{C}_\Gamma(x,y) - \widetilde{C}_\Gamma(x_0,y)| + |\widetilde{C}_\Gamma(y,x) - \widetilde{C}_\Gamma(y,x_0)| \lesssim |x-x_0|/|x-y|^2$ for all $x,x_0,y \in \mathbb{R}$ such that $|x-x_0| \le |y-x|/2$.

$$= \text{p.v.}\, \frac{1}{\pi i} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{y - x + i(A(y) - A(x))} g(x)\, dx\, (1 + iA'(y)) f(y)\, dy$$
$$= \int_{\mathbb{R}} \text{p.v.}\, \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1}{y - x + i(A(y) - A(x))} g(x)\, dx\, (1 + iA'(y)) f(y)\, dy$$
$$= \int_{\mathbb{R}} b(y) (\widetilde{\mathscr{C}_\Gamma})^*(g)(y) f(y)\, dy = \langle f, \mathscr{C}_\Gamma^*(g) \rangle_{L^2(\mathbb{R})}.$$

In the third equality we used Fubini's Theorem to interchange integrals, while in the second and fourth equalities we used the Lebesgue Dominated Convergence Theorem to interchange the limit introduced by the principal value and the integration.

We therefore conclude that

$$\mathscr{C}_\Gamma^*(g)(x) = b(x) \cdot (\widetilde{\mathscr{C}_\Gamma})^*(g)(x). \tag{3.1}$$

Note that $(\widetilde{\mathscr{C}_\Gamma})^* = -\widetilde{\mathscr{C}_\Gamma}$.

We now use the weak factorization for $H^1(\mathbb{R})$ (valid for m-linear Calderón–Zygmund operators [LW1, Theorem 1.3]) for the Calderón–Zygmund operator $\widetilde{\mathscr{C}_\Gamma}$ [LN$^+$] to obtain the desired weak factorization for the Meyer-type Hardy space $H^1_b(\mathbb{R})$.

First proof of Theorem 1.1. The function f is in $H^1_b(\mathbb{R})$ if and only if bf is in $H^1(\mathbb{R})$ but by weak factorization of $H^1(\mathbb{R})$ there are a sequence $\{\lambda_{s,k}\}_{s,k\geq 1}$ and compactly supported bounded functions G^k_s and H^k_s such that $bf = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_{s,k} \Pi(G^k_s, H^k_s)$, where the bilinear form $\Pi(G, H)$ is defined by

$$\Pi(G, H)(x) = G(x) \cdot \widetilde{\mathscr{C}_\Gamma}(H)(x) - H(x) \cdot (\widetilde{\mathscr{C}_\Gamma})^*(G)(x).$$

Moreover,

$$\|bf\|_{H^1(\mathbb{R})} \approx \inf\Big\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_{s,k}| \|G^k_s\|_{L^2(\mathbb{R})} \|H^k_s\|_{L^2(\mathbb{R})} : bf = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_{s,k} \Pi(G^k_s, H^k_s) \Big\}.$$

Therefore

$$f(x) = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_{s,k} \frac{1}{b} \Pi(G^k_s, H^k_s)(x) = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_{s,k} \Pi_b(G^k_s, H^k_s/b)(x).$$

The last identity follows since by definition (1.1) of the bilinear form $\Pi_b(g, h)$, the fact that $\mathscr{C}_\Gamma(f) = \widetilde{\mathscr{C}_\Gamma}(bf)$, and identity (3.1), we have

$$\frac{1}{b} \Pi(G, H)(x) = \Pi_b(G, H/b)(x). \tag{3.2}$$

Let $g^k_s := G^k_s$ and $h^k_s := H^k_s/b$; both are compactly supported bounded functions and

$$f(x)=\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_{s,k}\Pi_b\big(g_s^k,h_s^k\big)(x).$$

Moreover $\|f\|_{H^1_b(\mathbb{R})}=\|bf\|_{H^1(\mathbb{R})}$, therefore

$$\|f\|_{H^1_b(\mathbb{R})}\approx\inf\Big\{\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_{s,k}|\|g_s^k\|_{L^2(\mathbb{R})}\|h_s^k\|_{L^2(\mathbb{R})}:f=\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_{s,k}\Pi_b(g_s^k,h_s^k)\Big\}$$

because $g_s^k=G_s^k$ and $\|h_s^k\|_{L^2(\mathbb{R})}\approx\|bh_s^k\|_{L^2(\mathbb{R})}=\|H_s^k\|_{L^2(\mathbb{R})}$ since b is an accretive function. This proves Theorem 1.1. ∎

If we know how to construct G_s^k and H_s^k then we know how to construct g_s^k and h_s^k, and vice versa. In the next section we provide an explicit construction of g_s^k and h_s^k, following Uchiyama's blueprint directly in our setting.

Before proceeding, we provide the proofs of Theorems 1.2 and 1.3 relying on the corresponding results for the related Cauchy integral operator $\widetilde{\mathscr{C}_\Gamma}$. Namely, a is in BMO (respectively in VMO) if and only if $[a,\widetilde{\mathscr{C}_\Gamma}]$ is bounded on $L^p(\mathbb{R})$ (respectively, is compact on $L^p(\mathbb{R})$) for $p\in(1,\infty)$. Furthermore,

$$\|a\|_{\mathrm{BMO}}\approx\|[a,\widetilde{\mathscr{C}_\Gamma}]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}.$$

Proof of Theorem 1.2. For $b(x)=1+iA'(x)$, suppose $\mathfrak{A}$ is in $\mathrm{BMO}_b(\mathbb{R})$, that is, $\mathfrak{A}/b\in\mathrm{BMO}(\mathbb{R})$; a direct calculation, using $\mathscr{C}_\Gamma(g)=\widetilde{\mathscr{C}_\Gamma}(bg)$, shows that

$$[\mathfrak{A}/b,\mathscr{C}_\Gamma](f)(x)=[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}](bf)(x).$$

Thus, since by [LN$^+$, Theorem 1.1] the commutator $[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}]$ is bounded on $L^p(\mathbb{R})$, we get

$$\begin{aligned}\|[\mathfrak{A}/b,\mathscr{C}_\Gamma](f)\|_{L^p(\mathbb{R})}&=\|[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}](bf)\|_{L^p(\mathbb{R})}\\&\le\|[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\|bf\|_{L^p(\mathbb{R})}\\&\lesssim\|\mathfrak{A}/b\|_{\mathrm{BMO}(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}=\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}.\end{aligned}$$

Conversely, for any given complex function $\mathfrak{A}$ such that $\mathfrak{A}/b$ is real-valued, $\mathfrak{A}/b\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\|[\mathfrak{A}/b,\mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}<\infty$, we see that

$$\begin{aligned}\|[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}](f)\|_{L^p(\mathbb{R})}&=\|[\mathfrak{A}/b,\mathscr{C}_\Gamma](f/b)\|_{L^p(\mathbb{R})}\\&\le\|[\mathfrak{A}/b,\mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\|f/b\|_{L^p(\mathbb{R})}\\&\lesssim\|[\mathfrak{A}/b,\mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}.\end{aligned}$$

Hence, $[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}]$ is bounded on $L^p(\mathbb{R})$ and by [LN$^+$, Theorem 1.1] we conclude that $\mathfrak{A}/b$ is in $\mathrm{BMO}(\mathbb{R})$ and

$$\|\mathfrak{A}/b\|_{\mathrm{BMO}(\mathbb{R})}\lesssim\|[\mathfrak{A}/b,\widetilde{\mathscr{C}_\Gamma}]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\lesssim\|[\mathfrak{A}/b,\mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}.$$

Hence, we conclude that $\mathfrak{A}$ is in $\mathrm{BMO}_b(\mathbb{R})$ and

$$\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})} \lesssim \|[\mathfrak{A}/b, \mathscr{C}_\Gamma]\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}.$$

This finishes the proof of Theorem 1.2. ∎

Similar considerations yield the proof of Theorem 1.3 from the knowledge that $a \in \mathrm{VMO}$ if and only if $[a, \widetilde{\mathscr{C}_\Gamma}]$ is a compact operator on $L^p(\mathbb{R})$ [LN+, Theorem 1.2].

Proof Theorem 1.3. For $b(x) = 1 + iA'(x) \in L^\infty(\mathbb{R})$, suppose $\mathfrak{A}$ is in $\mathrm{VMO}_b(\mathbb{R})$, that is, $\mathfrak{A}/b \in \mathrm{VMO}(\mathbb{R})$. Therefore by [LN+, Theorem 1.2] the commutator $[\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}]$ is compact. Let E be a bounded subset of $L^p(\mathbb{R})$. Then bE is a bounded subset of $L^p(\mathbb{R})$ since

$$\sup_{g\in bE} \|g\|_{L^p(\mathbb{R})} = \sup_{f\in E} \|bf\|_{L^p(\mathbb{R})} \le \|b\|_{L^\infty} \sup_{f\in E} \|f\|_{L^p(\mathbb{R})} < \infty.$$

Therefore $[\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}](bE)$ is a precompact set. Recall that $[\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}](bf)(x) = [\mathfrak{A}/b, \mathscr{C}_\Gamma](f)(x)$ for all $f \in L^p(\mathbb{R})$. Thus

$$[\mathfrak{A}/b, \mathscr{C}_\Gamma](E) = [\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}](bE).$$

Hence $[\mathfrak{A}/b, \mathscr{C}_\Gamma](E)$ is a precompact set for all bounded subsets E of $L^p(\mathbb{R})$. By definition $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ is compact.

Conversely, suppose $[\mathfrak{A}/b, \mathscr{C}_\Gamma]$ is compact. Then given a bounded subset F of $L^p(\mathbb{R})$, F/b is also a bounded subset of $L^p(\mathbb{R})$ since $\|b\|_{L^\infty} \ge 1$. Therefore $[\mathfrak{A}/b, \mathscr{C}_\Gamma](F/b)$ is precompact; but as before,

$$[\mathfrak{A}/b, \mathscr{C}_\Gamma](F/b) = [\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}](F),$$

so $[\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}](F)$ is a precompact set for all bounded subsets F of $L^p(\mathbb{R})$. By definition $[\mathfrak{A}/b, \widetilde{\mathscr{C}_\Gamma}]$ is a compact operator in $L^p(\mathbb{R})$ and by [LN+, Theorem 1.2] we conclude that $\mathfrak{A}/b \in \mathrm{VMO}(\mathbb{R})$, and therefore $\mathfrak{A} \in \mathrm{VMO}_b(\mathbb{R})$. This finishes the proof of Theorem 1.3. ∎

4. Weak factorization of the Meyer Hardy space—Uchiyama's construction. In this section we present a constructive proof of the functions g_s^k and h_s^k for $k, s \ge 1$, appearing in the weak factorization of $H_b^1(\mathbb{R})$. This argument closely follows Uchiyama's procedure [U2].

4.1. The upper bound in Theorem 1.1. Given $f \in H_b^1(\mathbb{R})$, suppose we have a factorization $f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_{s,k} \Pi_b(g_s^k, h_s^k)$ with $\{\lambda_{s,k}\} \in \ell^1$ and g_s^k and h_s^k compactly supported and bounded functions, as claimed in Theorem 1.1. Then Lemma 4.1 below implies that

$$\|f\|_{H_b^1(\mathbb{R})} \le \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_{s,k}|\, \|g_s^k\|_{L^2(\mathbb{R})} \|h_s^k\|_{L^2(\mathbb{R})}.$$

LEMMA 4.1. *Let $g, h \in L^\infty(\mathbb{R})$ with compact supports. Then $\Pi_b(g,h)$ is in $H^1_b(\mathbb{R})$ with*

$$\|\Pi_b(g,h)\|_{H^1_b(\mathbb{R})} \lesssim \|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}.$$

Proof. We first point out that for any $g, h \in L^\infty(\mathbb{R})$ with compact supports, $\Pi_b(g,h)$ is compactly supported in $\mathrm{supp}(g) \cup \mathrm{supp}(h)$. Next, it is easy to see that $\Pi_b(g,h) \in L^2(\mathbb{R})$, using the fact that $\mathscr{C}_\Gamma$ is a bounded operator in $L^2(\mathbb{R})$. Indeed,

$$\|\Pi_b(g,h)\|_{L^2(\mathbb{R})} \lesssim \|g\|_{L^\infty(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} + \|h\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$

Moreover, since by definition of adjoint, $\langle h, \mathscr{C}^*_\Gamma(g)\rangle_{L^2(\mathbb{R})} = \langle \mathscr{C}_\Gamma(h), g\rangle_{L^2(\mathbb{R})}$, the following cancellation holds:

$$\int_{\mathbb{R}} \Pi_b(g,h)(x)b(x)\,dx = \int_{\mathbb{R}} \big(g(x)\cdot \mathscr{C}_\Gamma(h)(x) - h(x)\cdot \mathscr{C}^*_\Gamma(g)(x)\big)\,dx = 0.$$

Hence, it is clear that up to multiplication by a certain constant, the bilinear form $\Pi_b(g,h)(x)$ is an L^2-atom of $H^1_b(\mathbb{R})$, that is, $\Pi_b(g,h) \in H^1_b(\mathbb{R})$.

Now it suffices to verify that the $H^1_b(\mathbb{R})$ norm of $\Pi_b(g,h)$ is controlled by an absolute multiple of $\|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}$. A simple duality computation shows that for $\mathfrak{A} \in \mathrm{BMO}_b(\mathbb{R})$ and for any $g, h \in L^\infty(\mathbb{R}^n)$ with compact supports,

$$\begin{aligned}\langle \mathfrak{A}, \Pi_b(g,h)\rangle_{L^2(\mathbb{R})} &= \langle \mathfrak{A}/b, g\cdot \mathscr{C}_\Gamma(h) - h\cdot \mathscr{C}^*_\Gamma(g)\rangle_{L^2(\mathbb{R})}\\ &= \langle g, [\mathfrak{A}/b, \mathscr{C}_\Gamma](h)\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

Remember that $\langle f, g\rangle_{L^2(\mathbb{R})}$ denotes the L^2 pairing $\int_{\mathbb{R}} f(x)g(x)\,dx$, not the L^2 inner product. Thus, from the upper bound as in Theorem 1.2, we get

$$\begin{aligned}|\langle \mathfrak{A}, \Pi_b(g,h)\rangle_{L^2(\mathbb{R})}| &= |\langle g, [\mathfrak{A}/b, \mathscr{C}_\Gamma](h)\rangle_{L^2(\mathbb{R})}|\\ &\lesssim \|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}.\end{aligned}$$

This together with the duality result of [Me], $H^1_b(\mathbb{R})^* = \mathrm{BMO}_b(\mathbb{R})$, gives

$$\begin{aligned}\|\Pi_b(g,h)\|_{H^1_b(\mathbb{R})} &\approx \sup_{\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}\le 1} |\langle \mathfrak{A}, \Pi_b(g,h)\rangle_{L^2(\mathbb{R})}|\\ &\lesssim \|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} \sup_{\|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}\le 1} \|\mathfrak{A}\|_{\mathrm{BMO}_b(\mathbb{R})}\\ &\lesssim \|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}. \ \blacksquare\end{aligned}$$

4.2. The factorization and the lower bound in Theorem 1.1. The proof of the factorization and of the lower bound in Theorem 1.1 is more algorithmic in nature and follows a proof strategy pioneered by Uchiyama [U2]. We begin with a fact that will play a prominent role in the algorithm below. It is a modification of a related fact for the standard Hardy space $H^1(\mathbb{R})$.

LEMMA 4.2. *Let $b(x) = 1 + iA'(x)$ with $A' \in L^\infty(\mathbb{R})$. Suppose f is a function satisfying $\int_{\mathbb{R}} f(x)b(x)\,dx = 0$, and $|f(x)| \le \chi_{I(x_0,1)}(x) + \chi_{I(y_0,1)}(x)$, where $|x_0 - y_0| =: M > 100$ and $I(z_0, L) := \{z \in \mathbb{R} : |z - z_0| < L\}$. Then $f \in H^1_b(\mathbb{R})$ and*

$$\|f\|_{H^1_b(\mathbb{R})} \lesssim \log M. \tag{4.1}$$

The lemma when $b \equiv 1$ is stated in [LW1, Lemma 2.2] without proof, the authors refer the reader to [DL$^+$, Lemma 3.1] and [LW2, Lemma 4.3] where the corresponding lemma, in the Bessel and Neumann Laplacian settings respectively, is stated and proved. We cannot apply [LW1, Lemma 2.2] directly because although $F = bf$ will satisfy $\int_{\mathbb{R}} F(x)\,dx = 0$ by hypothesis, it will not satisfy $|F(x)| \le \chi_{I(x_0,1)}(x) + \chi_{I(y_0,1)}(x)$, instead it satisfies $|F(x)| \le |b(x)|(\chi_{I(x_0,1)}(x) + \chi_{I(y_0,1)}(x))$. But we can apply it to $F_0 = bf/\|b\|_{L^\infty(\mathbb{R})}$, since $|F_0(x)| \le \chi_{I(x_0,1)}(x) + \chi_{I(y_0,1)}(x)$, to conclude that $F_0 \in H^1(\mathbb{R})$ and $\|F_0\|_{H^1(\mathbb{R})} \lesssim \log M$. Finally since $\|F_0\|_{H^1(\mathbb{R})} = \|bf\|_{H^1(\mathbb{R})}/\|b\|_{L^\infty(\mathbb{R})}$ we conclude that $f \in H^1_b(\mathbb{R})$ and

$$\|f\|_{H^1_b(\mathbb{R})} = \|bf\|_{H^1(\mathbb{R})} \lesssim \|b\|_{L^\infty(\mathbb{R})} \log M \lesssim \log M.$$

Nevertheless, for completeness, we present here a direct construction of an atomic decomposition in $H^1_b(\mathbb{R})$ for f that yields the estimate claimed in Lemma 4.2, which could have an interest in itself; it also provides a proof for [LW1, Lemma 2.2] by setting $b \equiv 1$. This construction yields an atomic decomposition for $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j$. However the H^1_b L^∞-atoms a_j built in the proof of Lemma 4.2 for the specific given f are not the H^1 L^∞-atoms one would get by multiplying by $\|b\|_\infty/b$ the H^1 L^∞-atoms A_j obtained by the same procedure applied to F_0 when $b \equiv 1$.

Proof of Lemma 4.2. We will show by construction that f has an atomic decomposition into $H^1_b(\mathbb{R})$ L^∞-atoms, using an idea of Coifman [CW]. We first define

$$f_1(x) = f(x)\chi_{I(x_0,1)}(x) \quad \text{and} \quad f_2(x) = f(x)\chi_{I(y_0,1)}(x).$$

Then $f = f_1 + f_2$ and

$$|f_1(x)| \lesssim \chi_{I(x_0,1)}(x) \quad \text{and} \quad |f_2(x)| \lesssim \chi_{I(y_0,1)}(x).$$

Define

$$g^1_1(x) := \frac{\chi_{I(x_0,2)}(x)}{\int_{I(x_0,2)} b(z)\,dz} \int_{\mathbb{R}} f_1(y)b(y)\,dy,$$

$$f^1_1(x) := f_1(x) - g^1_1(x), \qquad \alpha^1_1 := \|f^1_1\|_\infty |I(x_0, 2)|.$$

We claim that $a_1^1 := (\alpha_1^1)^{-1} f_1^1$ is an $H_b^1(\mathbb{R})$ L^∞-atom. First, by definition, a_1^1 is supported on $I(x_0, 2)$. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} a_1^1(x) b(x)\, dx &= (\alpha_1^1)^{-1} \int_{\mathbb{R}} (f_1(x) - g_1^1(x)) b(x)\, dx \\
&= (\alpha_1^1)^{-1} \Big(\int_{\mathbb{R}} f_1(x) b(x)\, dx - \int_{\mathbb{R}} \frac{\chi_{I(x_0,2)}(x)}{\int_{I(x_0,2)} b(z)\, dz} b(x)\, dx \int_{\mathbb{R}} f_1(y) b(y)\, dy \Big) \\
&= (\alpha_1^1)^{-1} \Big(\int_{\mathbb{R}} f_1(x) b(x)\, dx - \int_{\mathbb{R}} f_1(y) b(y)\, dy \Big) = 0
\end{aligned}$$

and

$$\|a_1^1\|_\infty \le |(\alpha_1^1)^{-1}|\, \|f_1^1\|_\infty = \frac{1}{|I(x_0, 2)|}.$$

Thus, a_1^1 is an $H_b^1(\mathbb{R})$ L^∞-atom. It also has the following estimate:

$$\begin{aligned}
|\alpha_1^1| &= \|f_1^1\|_\infty |I(x_0,2)| \le \|f_1\|_\infty |I(x_0,2)| + \|g_1^1\|_\infty |I(x_0,2)| \\
&\le |I(x_0,2)| + \frac{|I(x_0,2)|}{\big|\int_{I(x_0,2)} b(z)\, dz\big|} \int_{\mathbb{R}} |f_1(y)|\, |b(y)|\, dy \\
&\le 4 + 2\|b\|_{L^\infty(\mathbb{R})} \le 6\|b\|_{L^\infty(\mathbb{R})} \lesssim 1.
\end{aligned}$$

Here we have used $\|b\|_{L^\infty(\mathbb{R})} < \infty$, $|f_1| \le \chi_{I(x_0,1)}$, $|I(x_0, L)| = 2L$, and

$$\Big| \int_{I(x_0,2)} b(z)\, dz \Big| \ge \Big| \int_{I(x_0,2)} \operatorname{Re} b(z)\, dz \Big| \ge |I(x_0,2)|.$$

Moreover, we see that

$$f_1(x) = f_1^1(x) + g_1^1(x) = \alpha_1^1 a_1^1(x) + g_1^1(x).$$

We further write $g_1^1(x)$ as

$$g_1^1(x) = (g_1^1(x) - g_1^2(x)) + g_1^2(x) =: f_1^2(x) + g_1^2(x)$$

with

$$g_1^2(x) := \frac{\chi_{I(x_0,4)}(x)}{\int_{I(x_0,4)} b(z)\, dz} \int_{\mathbb{R}} f_1(y) b(y)\, dy.$$

Again, we define

$$\alpha_1^2 := \|f_1^2\|_\infty |I(x_0,4)| \quad \text{and} \quad a_1^2 := (\alpha_1^2)^{-1} f_1^2,$$

and a similar reasoning shows that a_1^2 satisfies the compact support condition and the size condition $\|a_1^2\|_\infty \le 1/|I(x_0,4)|$. Hence, it suffices to see that it also satisfies the cancellation condition with respect to b. In fact,

$$\begin{aligned}\int_{\mathbb{R}} a_1^2(x)b(x)\,dx &= (\alpha_1^2)^{-1}\int_{\mathbb{R}}(g_1^1(x)-g_1^2(x))b(x)\,dx\\ &= (\alpha_1^2)^{-1}\Big(\int_{\mathbb{R}}\frac{\chi_{I(x_0,2)}(x)}{\int_{I(x_0,2)} b(z)\,dz}b(x)\,dx\int_{\mathbb{R}} f_1(y)b(y)\,dy\\ &\qquad - \int_{\mathbb{R}}\frac{\chi_{I(x_0,4)}(x)}{\int_{I(x_0,4)} b(z)\,dz}b(x)\,dx\int_{\mathbb{R}} f_1(y)b(y)\,dy\Big)\\ &= (\alpha_1^2)^{-1}\Big(\int_{\mathbb{R}} f_1(y)b(y)\,dy - \int_{\mathbb{R}} f_1(y)b(y)\,dy\Big) = 0.\end{aligned}$$

Thus, a_1^2 is an $H_b^1(\mathbb{R})$ L^∞-atom. Moreover, it satisfies the following estimate:

$$\begin{aligned}|\alpha_1^2| &= \|f_1^2\|_\infty |I(x_0,4)| \le \|g_1^1\|_\infty |I(x_0,4)| + \|g_1^2\|_\infty |I(x_0,4)|\\ &\le \frac{|I(x_0,4)|}{|\int_{I(x_0,2)} b(z)\,dz|}\int_{\mathbb{R}} |f_1(y)|\,|b(y)|\,dy + \frac{|I(x_0,4)|}{|\int_{I(x_0,4)} b(z)\,dz|}\int_{\mathbb{R}} |f_1(y)|\,|b(y)|\,dy\\ &\le 4\|b\|_{L^\infty(\mathbb{R})} + 2\|b\|_{L^\infty(\mathbb{R})} = \|b\|_{L^\infty(\mathbb{R})} \lesssim 1.\end{aligned}$$

Here again we use the fact that for every $L > 0$,

$$\Big|\int_{I(x_0,L)} b(z)\,dz\Big| \ge \Big|\int_{I(x_0,L)} \operatorname{Re} b(z)\,dz\Big| \ge |I(x_0,L)|.$$

Thus we have

$$f_1(x) = \sum_{i=1}^{2} \alpha_1^i a_1^i(x) + g_1^2(x).$$

Continuing in this fashion we see that for each $i_0 \ge 1$,

$$f_1(x) = \sum_{i=1}^{i_0} \alpha_1^i a_1^i(x) + g_1^{i_0}(x),$$

where for $i \in \{2,\dots,i_0\}$,

$$\begin{aligned}g_1^i(x) &:= \frac{\chi_{I(x_0,2^i)}(x)}{\int_{I(x_0,2^i)} b(z)dz}\int_{\mathbb{R}} f_1(y)b(y)\,dy,\\ f_1^i(x) &:= g_1^{i-1}(x) - g_1^i(x),\\ \alpha_1^i &:= \|f_1^i\|_\infty |I(x_0,2^i)|, \qquad a_1^i(x) := (\alpha_1^i)^{-1} f_1^i(x).\end{aligned}$$

Here we choose i_0 to be the smallest positive integer such that $I(y_0,1) \subset I(x_0,2^{i_0})$. Then from the condition $|x_0-y_0| = M$, we obtain

$$i_0 \approx \log_2 M.$$

Moreover, for $i \in \{1, \dots, i_0\}$,

$$|\alpha_1^i| \le 6\|b\|_{L^\infty(\mathbb{R})} \lesssim 1.$$

Following the same steps, we also obtain, for the same $i_0 \ge 1$,

$$f_2(x) = \sum_{i=1}^{i_0} \alpha_2^i a_2^i(x) + g_2^{i_0}(x),$$

where for $i \in \{2, \dots, i_0\}$,

$$\begin{aligned} g_2^i(x) &:= \frac{\chi_{I(y_0,2^i)}(x)}{\int_{I(y_0,2^i)} b(z)\,dz} \int_{\mathbb{R}} f_2(y)b(y)\,dy, \\ f_2^i(x) &:= g_2^{i-1}(x) - g_2^i(x), \\ \alpha_2^i &:= \|f_2^i\|_\infty |I(y_0,2^i)|, \qquad a_2^i(x) := (\alpha_2^i)^{-1} f_2^i(x). \end{aligned}$$

Similarly, for $i \in \{1, \dots, i_0\}$, we can verify that each a_2^i is an $H_b^1(\mathbb{R})$ L^∞-atom and $|\alpha_2^i| \lesssim 1$.

Combining the decompositions above, we obtain

$$f(x) = \sum_{j=1}^{2} \Big(\sum_{i=1}^{i_0} \alpha_j^i a_j^i(x) + g_j^{i_0}(x) \Big).$$

We now analyse the tail $g_1^{i_0}(x) + g_2^{i_0}(x)$. Consider the interval $\overline{I}$ centered at $(x_0 + y_0)/2$ with length 2^{i_0+1}. Then $I(x_0, 1) \cup I(y_0, 1) \subset \overline{I}$, and $I(x_0, 2^{i_0})$ and $I(y_0, 2^{i_0})$ are both subsets of $\overline{I}$. Thus, since by hypothesis $\int_{\mathbb{R}} f(y)b(y)\,dy = 0$, we get

$$\frac{\chi_{\overline{I}}(x)}{\int_{\overline{I}} b(z)\,dz} \int_{I(x_0,1)} f_1(y)b(y)\,dy + \frac{\chi_{\overline{I}}(x)}{\int_{\overline{I}} b(z)\,dz} \int_{I(y_0,1)} f_2(y)b(y)\,dy = 0.$$

Hence, we can write

$$\begin{aligned} g_1^{i_0}(x) + g_2^{i_0}(x) &= \bigg(g_1^{i_0}(x) - \frac{\chi_{\overline{I}}(x)}{\int_{\overline{I}} b(z)\,dz} \int_{I(x_0,1)} f_1(y)b(y)\,dy \bigg) \\ &\quad + \bigg(g_2^{i_0}(x) - \frac{\chi_{\overline{I}}(x)}{\int_{\overline{I}} b(z)\,dz} \int_{I(y_0,1)} f_2(y)b(y)\,dy \bigg) \\ &=: f_1^{i_0+1}(x) + f_2^{i_0+1}(x). \end{aligned}$$

For $j = 1, 2$, we now define

$$\alpha_j^{i_0+1} := \|f_j^{i_0+1}\|_\infty |\overline{I}|, \qquad a_j^{i_0+1}(x) := (\alpha_j^{i_0+1})^{-1} f_j^{i_0+1}(x).$$

Again we can verify that $a_j^{i_0+1}$ for $j = 1, 2$ is an $H_b^1(\mathbb{R})$ L^∞-atom supported on I with the appropriate size and cancellation conditions

$$\|a_j^{i_0+1}\|_\infty \le 1/|\bar{I}| \quad \text{and} \quad \int_{\mathbb{R}} a_j^{i_0+1}(x) b(x)\, dx = 0.$$

Moreover, $|\alpha_j^{i_0+1}| \lesssim 1$.

Thus, we obtain an atomic decomposition

$$f(x) = \sum_{j=1}^{2} \sum_{i=1}^{i_0+1} \alpha_j^i a_j^i(x),$$

which implies that $f \in H_b^1(\mathbb{R})$ and

$$\|f\|_{H_b^1(\mathbb{R})} \le \sum_{j=1}^{2} \sum_{i=1}^{i_0+1} |\alpha_j^i| \lesssim \sum_{j=1}^{2} \sum_{i=1}^{i_0+1} 1 \lesssim \log M.$$

This finishes the proof of Lemma 4.2. ∎

Repeating the proof we find that if $r > 0$, $\int_{\mathbb{R}} f(x)b(x)\, dx = 0$ and $|f(x)| \le \chi_{I(x_0,r)}(x) + \chi_{I(y_0,r)}(x)$ where $|x_0 - y_0| \ge rM$, then $f \in H_b^1(\mathbb{R})$ and

$$\|f\|_{H_b^1(\mathbb{R})} \lesssim r \log M. \tag{4.2}$$

The additional r comes from the estimates of the coefficients $|\alpha_j^i| \lesssim r$ for $i = 1, \ldots, i_0 + 1$ and $j = 1, 2$ where $i_0 \sim \log M$.

Ideally, given an $H_b^1(\mathbb{R})$-atom a, we would like to find $g, h \in L^2(\mathbb{R})$ such that $\Pi_b(g, h) = a$ pointwise. While this cannot be accomplished in general, the theorem below shows that it is "almost" true.

THEOREM 4.3. *For every $H_b^1(\mathbb{R})$ L^∞-atom $a(x)$ and for all $\varepsilon > 0$ there exist a large positive number M and $g, h \in L^\infty(\mathbb{R})$ with compact supports such that*

$$\|a - \Pi_b(h, g)\|_{H_b^1(\mathbb{R})} < \varepsilon \quad \textit{and} \quad \|g\|_{L^2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})} \lesssim M.$$

Proof. Let $a(x)$ be an $H_b^1(\mathbb{R})$ L^∞-atom, supported in $I(x_0, r)$, the interval centered at x_0 with radius r. We first consider the construction of the explicit bilinear form $\Pi_b(h, g)$ and the approximation to $a(x)$. To begin, fix $\varepsilon > 0$. Choose $M \in [100, \infty)$ so large that

$$M^{-1} \log M < \varepsilon.$$

Now select $y_0 \in \mathbb{R}$ such that $y_0 - x_0 = Mr$. For any $y \in I(y_0, r)$ and any $x \in I(x_0, r)$, we have $|x - y| > Mr/2$. We set

$$g(x) := \chi_{I(y_0,r)}(x) \quad \text{and} \quad h(x) := -\frac{a(x)}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)}. \tag{4.3}$$

We first note that

$$|(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)| \gtrsim M^{-1}. \tag{4.4}$$

In fact, from the expression of $(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0) = -\widetilde{\mathscr{C}}_\Gamma(g)(x_0)$ we have

$$|(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)| = \left| \frac{1}{\pi i} \int\limits_{I(y_0,r)} \frac{1}{y - x_0 + i(A(y) - A(x_0))}\, dy \right| \gtrsim M^{-1}.$$

From the definitions of g and h, we obtain $\operatorname{supp}(g) = I(y_0, r)$ and $\operatorname{supp}(h) = I(x_0, r)$. Moreover, from (4.4) and the size estimate for the atom, we obtain

$$\|g\|_{L^\infty(\mathbb{R})} \approx 1 \quad \text{and} \quad \|h\|_{L^\infty(\mathbb{R})} = \frac{1}{|(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)|}\|a\|_{L^\infty(\mathbb{R})} \lesssim Mr^{-1}.$$

We also get

$$\|g\|_{L^2(\mathbb{R})} \approx r^{1/2} \quad \text{and} \quad \|h\|_{L^2(\mathbb{R})} = \frac{1}{|(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)|}\|a\|_{L^2(\mathbb{R})} \lesssim Mr^{-1/2}.$$

Hence $\|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} \lesssim M$. Now write

$$\begin{aligned} a(x) - \Pi_b(h, g)(x) &= a(x) - \frac{1}{b(x)}\big(g(x) \cdot \mathscr{C}_\Gamma(h)(x) - h(x) \cdot \mathscr{C}_\Gamma^*(g)(x)\big) \\ &= \left(a(x) + \frac{h(x)}{b(x)} \cdot \mathscr{C}_\Gamma^*(g)(x)\right) - \frac{g(x)}{b(x)} \cdot \mathscr{C}_\Gamma(h)(x) \\ &=: W_1(x) + W_2(x). \end{aligned}$$

By definition and using (3.1), we have

$$\begin{aligned} W_1(x) &= a(x) + \frac{1}{b(x)}\left(-\frac{a(x)}{(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)}\right) \cdot \big(b(x) \cdot (\widetilde{\mathscr{C}}_\Gamma)^*(g)(x)\big) \\ &= a(x)\left[1 - \frac{(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x)}{(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)}\right] = a(x) \cdot \frac{(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0) - (\widetilde{\mathscr{C}}_\Gamma)^*(g)(x)}{(\widetilde{\mathscr{C}}_\Gamma)^*(g)(x_0)}. \end{aligned}$$

Thus, since $(\widetilde{\mathscr{C}}_\Gamma)^* = -\widetilde{\mathscr{C}}_\Gamma$, for every $x \in I(x_0, r)$ we get

$$\begin{aligned} |W_1(x)| &= |a(x)| \cdot \frac{|\widetilde{\mathscr{C}}_\Gamma(g)(x_0) - \widetilde{\mathscr{C}}_\Gamma(g)(x)|}{|\widetilde{\mathscr{C}}_\Gamma(g)(x_0)|} \\ &\le CM\|a\|_{L^\infty(\mathbb{R})} \int\limits_{I(y_0,r)} \frac{|x - x_0|}{|x - y|^2}\, dy \\ &\le CMr^{-1}rr(Mr)^{-2} = C(Mr)^{-1}. \end{aligned}$$

Here we use the standard smoothness estimate for the Calderón–Zygmund kernel $\widetilde{C}_\Gamma(x, y)$ of $\widetilde{\mathscr{C}}_\Gamma$ (see [LN$^+$, Lemma 3.3] or [Gra, Example 4.1.6]). Since it is clear that $W_1(x)$ is supported on $I(x_0, r)$, we obtain

$$|W_1(x)| \le C(Mr)^{-1}\chi_{I(x_0,r)}(x).$$

We next estimate $W_2(x)$. By definition, $W_2(x)$ is supported on $I(y_0, r)$, and

$$\begin{aligned}
W_2(x) &= \frac{\chi_{I(y_0,r)}(x)}{b(x)} \cdot \mathscr{C}_\Gamma\Big(-\frac{a(\cdot)}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)}\Big)(x) \\
&= -\frac{\chi_{I(y_0,r)}(x)}{b(x)} \frac{1}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)} \cdot \mathscr{C}_\Gamma(a(\cdot))(x) \\
&= -\frac{\chi_{I(y_0,r)}(x)}{b(x)} \frac{1}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)} \frac{1}{\pi i} \int\limits_{I(x_0,r)} \frac{(1+iA'(y))a(y)}{y-x+i(A(y)-A(x))}\,dy \\
&= -\frac{1}{b(x)} \frac{1}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)} \frac{1}{\pi i} \int\limits_{I(x_0,r)} \widetilde{\mathscr{C}_\Gamma}(x,y) b(y) a(y)\,dy \\
&= -\frac{\chi_{I(y_0,r)}(x)}{b(x)} \frac{1}{(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)} \frac{1}{\pi i} \int\limits_{I(x_0,r)} \big(\widetilde{\mathscr{C}_\Gamma}(x,y) - \widetilde{\mathscr{C}_\Gamma}(x,x_0)\big) b(y) a(y)\,dy.
\end{aligned}$$

Here the last equality follows from the cancellation condition for the $H^1_b(\mathbb{R})$ L^∞-atom $a(x)$. Hence, the following estimate holds for $W_2(x)$ whenever $x \in I(y_0,r)$ (note that otherwise $W_2(x) = 0$ and any estimate will hold):

$$\begin{aligned}
|W_2(x)| &\le \frac{\chi_{I(y_0,r)}(x)}{|b(x)|} \frac{1}{|(\widetilde{\mathscr{C}_\Gamma})^*(g)(x_0)|} \\
&\qquad \times \frac{1}{\pi} \int\limits_{I(x_0,r)} |\widetilde{C}_\Gamma(x,y) - \widetilde{C}_\Gamma(x,x_0)|\,|b(y)|\,|a(y)|\,dy \\
&\qquad \lesssim \chi_{I(y_0,r)}(x) M \int\limits_{I(x_0,r)} \|a\|_{L^\infty(\mathbb{R})} \frac{|y-x_0|}{|x-x_0|^2}\,dy \lesssim (Mr)^{-1} \chi_{I(y_0,r)}(x),
\end{aligned}$$

once again by the smoothness of the kernel $\widetilde{C}_\Gamma(x,y)$ of $\widetilde{\mathscr{C}_\Gamma}$.

Combining the estimates of W_1 and W_2, we obtain

$$|a(x) - \Pi_b(g,h)(x)| \lesssim (Mr)^{-1}\big(\chi_{I(x_0,r)}(x) + \chi_{I(y_0,r)}(x)\big). \tag{4.5}$$

Next we point out that

$$\int\limits_{\mathbb{R}} [a(x) - \Pi_b(g,h)(x)] b(x)\,dx = 0, \tag{4.6}$$

since $a(x)$ has cancellation with respect to $b(x)$ and the same holds for $\Pi_b(g,h)(x)$.

Then from the size estimate (4.5) and the cancellation (4.6), together with the result in Lemma 4.2, more specifically the estimate (4.2), we infer that $a - \Pi_b(g,h) \in H^1_b(\mathbb{R})$ and

$$\big\|a - \Pi_b(g,h)\big\|_{H^1_b(\mathbb{R})} \lesssim M^{-1} \log M < C\epsilon.$$

This proves Theorem 4.3. ■

We deduce from Theorem 4.3 the following corollary concerning $H^1(\mathbb{R})$ L^∞-atoms.

COROLLARY 4.4. *For every $H^1(\mathbb{R})$ L^∞-atom A and for all $\varepsilon > 0$ there exist $M > 0$ and compactly supported L^∞ functions G and H such that $\|A - \Pi(H,G)\|_{H^1(\mathbb{R})} < \varepsilon$ and $\|G\|_{L^2(\mathbb{R})}\|H\|_{L^2(\mathbb{R})} \lesssim M$.*

Proof. Note that if A is an $H^1(\mathbb{R})$ L^∞-atom then A/b is an $H^1_b(\mathbb{R})$ L^∞-atom, hence by Theorem 4.3 for all $\varepsilon > 0$ there are $M > 0$ and compactly supported L^∞ functions g, h such that $\|A/b - \Pi_b(g,h)\|_{H^1_b(\mathbb{R})} \lesssim \varepsilon$. By (3.2), $\Pi_b(g,h) = b\Pi(g,bh)$; hence $\|A - b\Pi_b(g,h)\|_{H^1(\mathbb{R})} = \|A - \Pi(g,bh)\| \lesssim \varepsilon$. Let $G = g$ and $H = bh$; these are compactly supported L^∞ functions, and furthermore $\|G\|_{L^2(\mathbb{R})}\|H\|_{L^2(\mathbb{R})} \approx \|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} \lesssim M$. ∎

With this approximation result, we can now prove the main result.

Constructive proof of Theorem 1.1. By Theorem 4.1 we have

$$\|\Pi_b(g,h)\|_{H^1_b(\mathbb{R})} \lesssim \|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}.$$

It follows that if $f \in H^1_b(\mathbb{R})$ then for any representation

$$f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k)$$

we have

$$\|f\|_{H^1_b(\mathbb{R})} \lesssim \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\lambda_j^k|\, \|g_j^k\|_{L^2(\mathbb{R})}\|h_j^k\|_{L^2(\mathbb{R})}.$$

Consequently,

$$\|f\|_{H^1_b(\mathbb{R})} \lesssim \inf\Big\{\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\lambda_j^k|\, \|g_j^k\|_{L^2(\mathbb{R})}\|h_j^k\|_{L^2(\mathbb{R})} : f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k)\Big\}.$$

We turn to showing that the other inequality holds and that it is possible to obtain such a decomposition for any $f \in H^1_b(\mathbb{R})$. By the definition of $H^1_b(\mathbb{R})$, for any $f \in H^1_b(\mathbb{R})$ we can find a sequence $\{\lambda_j^1\} \in \ell^1$ and sequence of $H^1_b(\mathbb{R})$ L^∞-atoms a_j^1 such that $f = \sum_{j=1}^\infty \lambda_j^1 a_j^1$ and $\sum_{j=1}^\infty |\lambda_j^1| \le C_0\|f\|_{H^1_b(\mathbb{R})}$.

We explicitly track the implied absolute constant C_0 coming from the atomic decomposition since it will play a role in the convergence of the approach. Fix $\varepsilon > 0$ such that $\varepsilon C_0 < 1$. Then we also have a large positive number M with $M^{-1}\log M < \epsilon$. We apply Theorem 4.3 to each atom a_j^1. So there exist $g_j^1, h_j^1 \in L^\infty(\mathbb{R}^n)$ with compact supports and satisfying $\|g_j^1\|_{L^2(\mathbb{R})}\|h_j^1\|_{L^2(\mathbb{R})} \lesssim M$ and

$$\|a_j^1 - \Pi_b(g_j^1, h_j^1)\|_{H^1_b(\mathbb{R})} < \varepsilon \quad \text{for all } j > 0.$$

Now note that

$$f = \sum_{j=1}^{\infty} \lambda_j^1 a_j^1 = \sum_{j=1}^{\infty} \lambda_j^1 \Pi_b(g_j^1, h_j^1) + \sum_{j=1}^{\infty} \lambda_j^1 \big(a_j^1 - \Pi_b(g_j^1, h_j^1)\big) =: M_1 + E_1.$$

Observe that

$$\|E_1\|_{H_b^1(\mathbb{R})} \le \sum_{j=1}^{\infty} |\lambda_j^1| \, \|a_j^1 - \Pi_b(g_j^1, h_j^1)\|_{H_b^1(\mathbb{R})} \le \varepsilon \sum_{j=1}^{\infty} |\lambda_j^1| \le \varepsilon C_0 \|f\|_{H_b^1(\mathbb{R})}.$$

We now iterate the construction on the function E_1. Since $E_1 \in H_b^1(\mathbb{R})$, we can apply the atomic decomposition in $H_b^1(\mathbb{R})$ to find a sequence $\{\lambda_j^2\} \in \ell^1$ and a sequence $\{a_j^2\}$ of $H_b^1(\mathbb{R})$ L^∞-atoms such that $E_1 = \sum_{j=1}^{\infty} \lambda_j^2 a_j^2$ and

$$\sum_{j=1}^{\infty} |\lambda_j^2| \le C_0 \|E_1\|_{H_b^1(\mathbb{R})} \le \varepsilon C_0^2 \|f\|_{H_b^1(\mathbb{R})}.$$

Again, we will apply Theorem 4.3 to each L^∞-atom a_j^2. So there exist $g_j^2, h_j^2 \in L^\infty(\mathbb{R})$ with compact supports and satisfying $\|g_j^2\|_{L^2(\mathbb{R})} \|h_j^2\|_{L^2(\mathbb{R})} \lesssim M$ and

$$\|a_j^2 - \Pi_b(g_j^2, h_j^2)\|_{H_b^1(\mathbb{R})} < \varepsilon \quad \text{for all } j > 0.$$

We then have

$$E_1 = \sum_{j=1}^{\infty} \lambda_j^2 a_j^2 = \sum_{j=1}^{\infty} \lambda_j^2 \Pi_b(g_j^2, h_j^2) + \sum_{j=1}^{\infty} \lambda_j^2 (a_j^2 - \Pi_b(g_j^2, h_j^2)) =: M_2 + E_2.$$

But, as before,

$$\|E_2\|_{H_b^1(\mathbb{R})} \le \sum_{j=1}^{\infty} |\lambda_j^2| \, \|a_j^2 - \Pi_b(g_j^2, h_j^2)\|_{H_b^1(\mathbb{R})} \le \varepsilon \sum_{j=1}^{\infty} |\lambda_j^2| \le (\varepsilon C_0)^2 \|f\|_{H_b^1(\mathbb{R})}.$$

This implies that

$$\begin{aligned} f &= \sum_{j=1}^{\infty} \lambda_j^1 a_j^1 = \sum_{j=1}^{\infty} \lambda_j^1 \Pi_b(g_j^1, h_j^1) + \sum_{j=1}^{\infty} \lambda_j^1 (a_j^1 - \Pi_b(g_j^1, h_j^1)) \\ &= M_1 + E_1 = M_1 + M_2 + E_2 = \sum_{k=1}^{2} \sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k) + E_2. \end{aligned}$$

Repeating this construction for each $1 \le k \le K$ produces functions $g_j^k, h_j^k \in L^\infty(\mathbb{R})$ with compact supports and with $\|g_j^k\|_{L^2(\mathbb{R})} \|h_j^k\|_{L^2(\mathbb{R})} \lesssim M$ for all $j > 0$, sequences $\{\lambda_j^k\}_{j>0} \in \ell^1$ with $\|\{\lambda_j^k\}_{j>0}\|_{\ell^1} \le \varepsilon^{k-1} C_0^k \|f\|_{H_b^1(\mathbb{R})}$, and a function $E_K \in H_b^1(\mathbb{R})$ with $\|E_K\|_{H_b^1(\mathbb{R})} \le (\varepsilon C_0)^K \|f\|_{H_b^1(\mathbb{R})}$ such that

$$f = \sum_{k=1}^{K} \sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k) + E_K.$$

Letting $K \to \infty$ gives the desired decomposition

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_b(g_j^k, h_j^k).$$

We also have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k| \le \sum_{k=1}^{\infty} \varepsilon^{-1} (\varepsilon C_0)^k \|f\|_{H_b^1(\mathbb{R})} = \frac{C_0}{1 - \varepsilon C_0} \|f\|_{H_b^1(\mathbb{R})}.$$

Therefore $\{\lambda_j^k\}_{j,k \in \mathbb{Z}}$ is in ℓ^1 as claimed. This finishes the proof of Theorem 1.1. ∎

The weak factorization given by Theorem 1.1 can be used to prove the lower bound of Theorem 1.2, the same way as is done for example in [LW1]. However, we used the upper bound of Theorem 1.2 to prove Lemma 4.1 responsible for the upper bound in Theorem 1.1.

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