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A Legendre-based computational method for solving a class of Itô stochastic delay differential equations

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Abstract This paper provides a numerical method for solving a class of Itô stochastic delay differential equations (SDDEs). The method's novelty is its use of the spectral collocation approach using Legendre polynomials for solving SDDEs. We prove that the method is strongly convergent in L^2 and proceed to demonstrate its computational efficiency and superior accuracy.

Keywords Legendre collocation method · Stochastic delay differential equations · Strong solution · Lamperti transformation · Wiener process

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1 Introduction and motivation

A delay differential equation (DDE) is a type of differential equation in which the time derivatives at the present time depend on both the solution and the derivatives

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at previous points in time [25, pp. 440–441]. In the present work, we focus on the following differential equation of neutral type

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau_1), \cdots, x(t-\tau_n), x'(t-\sigma_1), \cdots, x'(t-\sigma_m)), & t \geq t_0, \\ x(t) = \phi(t), \qquad t \leq t_0, \end{cases} \tag{1}$$

where the quantities $\tau_i \geq 0$, $1 \leq i \leq n$ and $\sigma_j \geq 0$, $1 \leq j \leq m$ model the "delays" and where $\phi(t)$ is a function of initial history. The delays may be either constants or functions. Note that the coefficient on the right-hand side of (1) depends not only on delayed values of the solution x, but also on derivatives of the delayed values of the solution of x.

Ordinary DDEs are often used in lieu of non-delayed partial differential equations ([13]). The former are commonly solved in a stepwise fashion via the "method of steps" (see [33] for more comprehensive background on this method). When modeling systems with stochastic variables, stochastic delay differential equations (SDDEs) are often employed ([6, 9]). SDDEs have several applications in applied sciences. A compelling example from financial mathematics involves the use of SDDEs to model delays in commodity markets attributed to transportation and production ([9]). The advantage of modeling a system via SDDEs is that SDDEs are often more suitable stochastic models than their instantaneous counterparts for modeling phenomena that display time lags ([5, 6]). Several applications to option pricing in markets with memory have been provided in [3], since delays naturally arise with financial instruments such as Asian options or lookback options.

The general form of the stochastic version of (1) as an Itô SDDE is a stochastic differential equation ([8, 32])

$$\begin{cases} dx(t) = f(t, x(t), x(t-\tau))dt + g(t, x(t), x(t-\tau))dW(t), & t \geq t_0, \\ x(t) = \phi(t), \qquad t \in [-\tau, t_0], \end{cases} \tag{2}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the drift term and $g : \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ is the diffusion term. $W(t)$ is a d-dimensional standard Wiener process, which is a vector-valued stochastic process

$$W(t) = (W_1(t), W_2(t), \dots, W_d(t)), \tag{3}$$

with components $W_i(t)$ independent and standard one-dimensional Wiener processes. It is assumed that $\mathbb{E}\|\phi\|^2 < \infty$ and that the functions f and g are measurable and satisfy both Lipschitz and linear growth bound conditions in x.

The present paper develops a Legendre collocation method to solve a specific class of SDDEs. This particular class, given explicitly in [32] and motivated by [8, p.2], is

$$\begin{cases} dx(t) = f(t, x(t), x(t-\tau))dt + g(x(t))dW(t), & t \in [0, T], \\ x(t) = \phi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{4}$$

where $0 < T < \infty$. The class of SDDEs given in (4) is amenable to the Lamperti transformation ([18]), which will be discussed in Section 2.

Analytic solutions of SDDEs are often intractable, making numerically efficient schemes requisite. Existing numerical methods for solving SDDEs primarily rely on stochastic Taylor expansions (see [5, 34]). As the convergence order increases, the complexity of implementing numerical algorithms also increases. This is because one must compute more partial derivatives of the drift and diffusion functions ([8]).

We briefly survey related literature. In 2010, the authors of [29] proposed "an efficient Legendre-Gauss collocation method for solving nonlinear delay differential equations with variable delay." The authors showed that their method has high-order accuracy and can be implemented in a stable and efficient manner. The approach relies on using shifted Legendre polynomials to obtain a discrete system. The solutions are then obtained via a stable approach by determining the unknown coefficients of the collocation scheme. In 2015, the authors of [32] presented a Chebyshev spectral method for solving a class of SDDEs. They discussed the application of an interpolation polynomial "interpolated by choosing the first kind of Chebyshev-Gauss-Lobatto points" ([32]). Their work considers the Lamperti transformation. Like [32], we also arrive at a more stable form of the SDDE using the Lamperti transformation. However, unlike [32], we then construct the differentiation matrix corresponding to *Legendre collocation nodes*.

In addition to the SDDE literature, the present work is motivated by the SDE literature, and in particular, the recent work of [27]. The authors of [27] consider a collocation solution of SDEs without delays. The present work provides an extension to SDDEs. Further, and to the best of our knowledge, we are the first to offer formal convergence analysis for the *collocation-type* methods in the SDDE setting given in (4).

Our proposed algorithm for solving SDDEs differs from the aforementioned works of [29] and [32] in several respects. Firstly, we provide an algorithm which solves Itô SDDEs in matrix notation, which makes the problem more tractable and also requires less computational effort. Secondly, our method is explicit, whereas the method presented in [29] is implicit. Thirdly, our proposed algorithm is strongly convergent in L^2 (there is no convergence discussion provided in [27] and [32]). Finally, our Legendre collocation algorithm is computationally faster and more accurate than the Chebyshev-type method proposed in [32].

The remainder of this work is structured as follows. Section 2 discusses the strengths of the Lamperti transformation and shows how to construct the corresponding differentiation matrices of Legendre nodes. Section 3 gives the proposed Legendre collocation method for solving the class of SDDEs given in (4). Section 4 proves that our method is strongly convergent in L^2. Section 5 compares the applicability and efficiency of our Legendre collocation scheme with well-known discretized schemes in the literature. Section 6 offers a conclusion and an outline for future research.

2 Legendre collocation method and Lamperti transformation

Consider a standard probability space $(\Omega, \mathcal{F}_t, P)$. The filtration $\mathcal{F}_{t\geq 0}$ is assumed to be complete and right-continuous.

Consider a grid of $n+1$ points, $\{t_0, t_1, \ldots, t_n\}$. Usually, this set of nodes is considered to be equally spaced; however, polynomials of high degree along with an equally spaced grid may pose non-trivial difficulties [12, p. 29]. One well-studied example of these difficulties is the Runge phenomenon.

We wish to obtain acceptable accuracy with as few nodes as possible. To this end, we consider the Legendre polynomials on the shifted interval $[0, T]$. On the interval $[-1, 1]$, the Legendre polynomials $L_n(x)$ ([27] and [30, pp. 304–305]) are solutions to the following Legendre ordinary differential equation

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0, \tag{5}$$

with orthogonality

$$\int_{-1}^{1} L_m(x)L_n(x)dx=0, \tag{6}$$

for $m \neq n$.

If the function $f(x)$ belongs to a Lipschitz class of order greater than or equal to $1/2$ on $[-1, 1]$, then, it has the following uniformly convergent Legendre series expansion

$$f(x)=\sum_{n=0}^{\infty} a_n L_n(x), \tag{7}$$

where

$$a_n=\left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)L_n(x)dx, \qquad n=0,1,\cdots. \tag{8}$$

Furthermore, if f, f', $\ldots$, $f^{(k-1)}$ are absolutely continuous on $[-1, 1]$ and $\|f^{(k)}\|_T = V_k < \infty$ for some $k > 1$, then for each $n > k+1$ (see [28]), there is the error bound

$$|f(x)-f_n(x)| \leq \frac{V_k}{(k-1)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2k-3}{2}\right)}\sqrt{\frac{\pi}{2(n-k)}}, \quad x\in[-1,1], \tag{9}$$

where $\|\cdot\|_T$ is the Chebyshev-weighted seminorm and

$$f_n(x)=\sum_{j=0}^{n} a_j L_j(x). \tag{10}$$

(For further background on Legendre orthogonal polynomials, we refer the reader to [11, pp. 35–38]).

The approach for solving the class of SDDEs in (4) with Legendre collocation points is to choose a finite-dimensional space of candidate solutions (via Legendre polynomials) and the corresponding number of points in the domain, commonly referred to as "collocation points." We then use the Lamperti transformation ([18]) to transfigure the original SDDE into an SDDE with constant diffusion coefficient.

The Lamperti transformation is given by

$$y(t)=l(x(t))=\int_{z}^{x(t)}\frac{1}{g(u)}du, \tag{11}$$

where z lies in the state space of $x(t)$ and the integral is a deterministic Riemann-Stieltjes integral. It is important to note that the Lamperti transformation $y(t) =$

$l\,(x(t))$ often reduces the numerical instability of the simulation processes in the path-wise sense [15, 27]. Following such a procedure leads to the corresponding SDDE version of (4) as

$$dy(t) = \overline{f}(t, y(t), y(t-\tau))dt + dW(t), \tag{12}$$

with initial condition

$$y(0) = l(x(0)) = \int_z^{\phi(0)} \frac{1}{g(u)}du, \tag{13}$$

where

$$\overline{f}(t, y(t), y(t-\tau)) = \frac{f(t, l^{-1}(y(t)), l^{-1}(y(t-\tau)))}{g(l^{-1}(y(t)))} - \frac{1}{2}g_x(l^{-1}(y(t))). \tag{14}$$

In Section 3, we construct the Legendre spectral collocation method for the SDDE given in (12). The orthogonal basis allows the problem to be reduced to a system of linear or nonlinear equations.

3 Method of solution

We begin by considering differentiation matrices. Given the Legendre set of collocation nodes $\{t_0, t_1, \ldots, t_n\}$ in the interval $[0, T]$, the corresponding Legendre differentiation matrix may be constructed. Once the weights (here, denoted by d_{ij}) used to approximate the m-th derivatives (of u) at each grid point are determined, they may be evaluated as [12, chapter 3]

$$u_j^{(m)} = \sum_{j=0}^{m} d_{ij}^{(m)} u_j. \tag{15}$$

Sometimes, a few coefficients are different from zero (i.e., for each node i, those $n+1$ nodes that belong to its stencil). The right hand-side of the (15) is

$$\mathbf{u}^{(m)} \simeq \mathbf{D}^{(m)}\mathbf{u}, \tag{16}$$

where $\mathbf{D}^{(m)}$ is the differentiation matrix [26]. The formulation in (16) enables us to build a rapid numerical method.

In the present work, we only require the simplest case ($m = 1$) in producing the differentiation matrices based on the Legendre nodes. Upon implementation of this procedure, orthogonal Legendre polynomials of degree $n-1$ are considered including $n-1$ real *zeros* in the shifted interval $[0, T]$. Then, to build a non-equidistant grid of $n+1$ nodes, the boundary nodes are considered as the first and the last points.

Generally speaking, the differentiation matrices are singular [10, p. 479]. The SDDEs are initial value problems and the value of the exact solution at the initial time is given. For this reason, if t_0 is fixed to be zero in the initial SDDE, the first column and row of the differentiation matrices have no effect (multiplication by zeros). Henceforth, we consider the differentiation matrix $\bar{D}$ by removing the first row and column of $\mathbf{D}^{(1)}$, which is *non-singular*. If the initial value is not zero, then it can be incorporated into the SDDE by a transformation, resulting in a SDDE with an initial value of zero.

The differentiation matrices constructed in this spectral approach are all dense since the unique interpolation polynomial of degree n (the highest order possible) passes through all $n+1$ Legendre collocation points.

The differentiation matrices are built with the collocation points. As an example, considering five Legendre nodes, $\{0, 0.1127016653792583, 0.5, 0.8872983346207417, 1\}$ in the time interval $[0, 1]$, gives the differentiation matrix

$$\mathbf{D}_{5\times 5} = \begin{pmatrix} -13. & 14.7883 & -2.66667 & 1.87836 & -1. \\ -5.32379 & 3.87298 & 2.06559 & -1.29099 & 0.67621 \\ 1.5 & -3.22749 & 0 & 3.22749 & -1.5 \\ -0.67621 & 1.29099 & -2.06559 & -3.87298 & 5.32379 \\ 1. & -1.87836 & 2.66667 & -14.7883 & 13. \end{pmatrix}. \tag{17}$$

We proceed to write the SDDE in (12) in the following non-canonical form [20, pp. 152–153]

$$\frac{d}{dt} y(t) = \overline{f}(t, y(t), y(t-\tau)) + \frac{d}{dt} W(t). \tag{18}$$

Replacing the operator $\frac{d}{dt}$ with the Legendre differentiation matrix (for finding the strong solution) yields

$$\bar{D} y(t) = \overline{f}(t, y(t), y(t-\tau)) + \bar{D} W(t), \tag{19}$$

which further simplifies to

$$y - \bar{D}^{-1}\overline{f} - W = 0, \tag{20}$$

where W is the vector of Wiener increments. The formulation in (20) is always a linear or nonlinear system of algebraic equations with dimension $n \times n$, when there are $n+1$ collocation points in the time interval.

In (20), the function $\overline{f}(t, y(t), y(t-\tau))$ determines whether the final algebraic system for path-wise approximations are linear or nonlinear. If the discretized system of algebraic equations is linear, then linear solvers (such as LU-based decompositions) which are quite good for dense systems might be applied. Otherwise, a nonlinear solver such as Newton's method may solve the nonlinear system ([1]).

In numerical implementations, the vector y consisting of n entries is calculated via the algebraic system of (20). The inverse of the Lamperti transformation is $x = l^{-1}(y)$, which is the final strong solution (in the path-wise sense) of (4).

Although the discussion given above is done for the one-dimensional SDDEs for the sake of simplicity in notations only, the use of the method for system of SDDEs is quite straightforward by applying the transform (11) for each equation in a system of SDDEs. When an SDE is a d-dimensional system and the time interval is discretized into n partitions, the resulting algebraic system from the Legendre collocation method for SDDEs has dimension $(dn) \times (dn)$. We illustrate this at the end of Section 5.

Furthermore, according to [19], an expansion of the drift term in (18) $\overline{f}(t, y(t), y(t-\tau))$ "in powers of τ using a Taylor expansion around $y(t)$ is valid to quadratic order in τ." When following and imposing such an expansion to transform a SDDE into a SDE, it is requisite to collect all terms of order $\sqrt{dt}$ and dt [19, p.

181]. This approach is useful when tackling a system of SDDEs in which the delay is small.

4 Convergence analysis

The solution of a discretized problem converges to the solution of a continuous problem as the size of the mesh approaches zero. Correspondingly, convergence depends on the number of grid points (denoted by n) and grid spacing. In our method, the number of grid points in the discretization process (constructing the differentiation matrix) is inversely proportional to the grid spacing. We first recall a few definitions and then prove in Theorem 1 that our method is strongly convergent in L^2.

Consider a random variable X with distribution f_X and finite expectation. Suppose $p \geq 2$ and denote $L^p(\Omega, H)$ the collection of all strongly measurable, pth integrable H-valued random variables. $L^p(\Omega, H)$ is a Banach space with

$$\|V\|_{L^p(\Omega,H)} := (\mathbb{E}[\|V\|^p])^{1/p} \tag{21}$$

for each $V \in L^p(\Omega, H)$. We will henceforth work with the space $L^2(\Omega, H)$.

An $\mathbb{R}$-valued stochastic process

$$y(t) : [-\tau, T] \times \Phi \to \mathbb{R}, \tag{22}$$

is called a "strong solution" of the systems of equations in (4) if it is a measurable sample-continuous process such that $y(t)|_{[0,T]}$ is $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted, f and g are continuous functions, and y satisfies the system of equations in (4) almost surely with initial condition $y(t) = \phi(t)$ for $t \in [-\tau, 0]$. A solution $y(t)$ is considered path-wise unique if indistinguishable from any other solution $y(t)$.

Definition 1 (Grönwall's inequality [2]) Let $\xi, \varpi \in [t_0, T] \to \mathbb{R}$ satisfy the following relation

$$0 \leq \xi(t) \leq \varpi(t) + L \int_{t_0}^{t} \xi(s)ds, \tag{23}$$

for any $t \in [t_0, T]$, where $L > 0$. Then,

$$\xi(t) \leq \varpi(t) + L \int_{t_0}^{t} \exp(L(t-s))\varpi(s)ds. \tag{24}$$

Definition 2 The sequence $\{y_n\}$ converges to y in L^2 if for each n, $\mathbb{E}(\|y_n\|^2) < \infty$ and $\mathbb{E}(\|y_n - y\|^2) \to 0$ as $n \to \infty$ [17, p. 39].

Let $\|\cdot\|$ be the L^2 norm and

$$e_n(t) = y(t) - y_n(t) \tag{25}$$

be an error function of the (path-wise) approximate solution $y_n(t)$ to the exact solution $y(t)$ (in the strong sense [24, p.74]). The subscript n refers to the number of collocation points in the integration interval. Note that e_n is $\mathcal{F}_{tn}$-measurable since both $y(t)$, $y_n(t)$ are $\mathcal{F}_{t n}$-measurable random variables and that $(\mathbb{E}[\|e_n\|^2])^{1/2}$ is the L^2 norm of (25) [5].

Theorem 1 *Let $y(t)$ be the exact solution and $y_n(t)$ be the Legendre spectral collocation approximate solution of (18). Let us further assume 1–5 below.*

1. *f and g are sufficiently smooth with uniformly bounded derivatives.*
2. *For every T and n, there are positive constants $\lambda_1, \ldots, \lambda_4$ depending only on T and n such that for all $0 \le t \le T$, we have*

$$|f(\theta,\alpha) - f(\vartheta,\beta)| \le \lambda_1|\theta - \vartheta| + \lambda_2|\alpha - \beta| \tag{26}$$

and

$$|g(\theta,\alpha) - g(\vartheta,\beta)| \le \lambda_3|\theta - \vartheta| + \lambda_4|\alpha - \beta|, \tag{27}$$

for all $\alpha, \beta, \vartheta, \theta \in \mathbb{R}$,

3. *The coefficients satisfy the linear growth conditions*

$$|f(\varrho,\sigma)|^2 \le A_1(1 + |\varrho|^2 + |\sigma|^2) \tag{28}$$

and also

$$|g(\varrho,\sigma)|^2 \le A_2(1 + |\varrho|^2 + |\sigma|^2), \tag{29}$$

for constants $A_1, A_2 \ge 0$,

4. $\mathbb{E}(\|y(t)\|^2) < \infty$.
5. *The function $\phi(t)$ is Hölder-continuous with exponent γ* [5, p. 318].

Then, $y_n(t)$ converges to $y(t)$ in L^2.

Proof After employing the Lamperti transformation, we can express $y_n(t)$ in the differential form

$$dy_n(t) = \bar{f}(t, y_n(t), y_n(t-\tau))dt + dW_n(t), \tag{30}$$

wherein $W_n(t)$ is an n-component vector of Wiener increments. The corresponding integral form is

$$y_n(t) = y(0) + \int_0^t \bar{f}(s, y_n(s), y_n(s-\tau))ds + \int_0^t dW_n(s). \tag{31}$$

We now employ logic in the spirit of [4, 14, 21]. Using the error function $e_n(t) = y(t) - y_n(t)$, (31), and the following exact solution

$$y(t) = y(0) + \int_0^t \bar{f}(s, y(s), y(s-\tau))ds + \int_0^t dW(s), \tag{32}$$

we obtain

$$e_n(t) = \int_0^t [\bar{f}(s, y(s), y(s-\tau)) - \bar{f}(s, y_n(s), y_n(s-\tau))]ds + \int_0^t [dW(s) - dW_n(s)]. \tag{33}$$

Applying the expectation operator yields

$$\mathbb{E}[\|e_n(t)\|^2] \leq 2\mathbb{E}\left[\left\|\int_0^t \bar{f}(s, y(s), y(s-\tau)) - \bar{f}(s, y_n(s), y_n(s-\tau))ds\right\|^2\right] + 2\mathbb{E}\left[\left\|\int_0^t (dW(s) - dW_n(s))\right\|^2\right], \tag{34}$$

which follows from the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. Further, applying the Itô isometry [24, p.26] and the Hölder's inequality, we have

$$\mathbb{E}[\|e_n(t)\|^2] \leq 2\eta\left(\int_0^t \mathbb{E}[\|\bar{f}(s, y(s), y(s-\tau)) - \bar{f}(s, y_n(s), y_n(s-\tau))\|^2]ds\right) + 2\eta\mathbb{E}\left[\|W(t) - W_n(t)\|^2\right], \tag{35}$$

wherein η is a positive constant. From the Lipschitz condition in (26), we obtain

$$\mathbb{E}[\|f(\theta,\alpha) - f(\vartheta,\beta)\|^2] + \mathbb{E}[\|g(\theta,\alpha) - g(\vartheta,\beta)\|^2] \leq 4\lambda^2\left(\mathbb{E}[\|\theta-\vartheta\|^2] + \mathbb{E}[\|\alpha-\beta\|^2]\right), \tag{36}$$

where λ is a positive constant and subsequently

$$\mathbb{E}[\|e_n(t)\|^2] \leq 8\eta\lambda^2\int_0^t \left(\mathbb{E}[\|e_n(s)\|^2] + \mathbb{E}[\|e_n(s-\tau)\|^2]\right)ds + 2\eta\mathbb{E}\left[\|e_n^W(t)\|^2\right], \tag{37}$$

where

$$e_n^W(t) = W(t) - W_n(t). \tag{38}$$

Applying Grönwall's inequality on the right side of the inequality in (37) with $\xi(s) = \left(\mathbb{E}[\|e_n(s)\|^2] + \mathbb{E}[\|e_n(s-\tau)\|^2]\right)$, we obtain

$$\mathbb{E}[\|e_n(t)\|^2] \leq 2\eta\mathbb{E}\left[\|e_n^W(t)\|^2\right] + 16\eta\lambda^2 L\int_0^t \exp(L(t-s))\mathbb{E}[\|e_n^W(s)\|^2]ds. \tag{39}$$

Here, L is a positive suitable constant. Letting $n \to \infty$, we obtain

$$\mathbb{E}[\|e_n(t)\|^2] \to 0. \tag{40}$$

This proves that our method is strongly convergent in L^2. □

5 Illustrative test problems

In this section, we examine the numerical precision of the new computational scheme for SDDEs using Mathematica 10 [31] with machine precision arithmetic.[1]

[1] http://reference.wolfram.com/language/tutorial/MachinePrecisionNumbers.html.en

For purposes of comparison, we employ Milstein's method [22], henceforth denoted as "MM," and which is given by

$$\begin{aligned} x_{i+1} = x_i + f(t_i, x_i, h_i)\Delta t_i + g(t_i, x_i, h_i)\Delta w_i \\ +\frac{1}{2} g(t_i, x_i, h_i)\frac{\partial g}{\partial x}(t_i, x_i, h_i)\left(\Delta w_i^2 - \Delta t_i\right). \end{aligned} \tag{41}$$

Here,

$$\Delta t_i = t_{i+1} - t_i, \qquad \Delta w_i = w_{t_{i+1}} - w_{t_i}, \tag{42}$$

and h_i includes the approximation for the time lag. We denote the Chebyshev method for Itô SDDEs developed in [32] as "CM." We denote our proposed Legendre collocation scheme given in (20) by "LM." The computer specifications are Microsoft Windows 7 Intel(R), CPU 3.10GHz, 64-bit operating system, with 8GB of RAM.

Now, we consider an example from biological sciences and provide computational results under varied parameter settings. It is commonly assumed that biological systems function in an environment with an overall noise rate modeled as $\beta dW(t)$ 9 ([8]). As in both [8, p.303] and [32, p.8], we consider a population $x(t)$ and the dyanmics of the SDDE below

$$\begin{aligned} dx(t) &= [ax(t) + bx(t-1)]dt \\ &\quad +[\beta_1 + \beta_2 x(t) + \beta_3 x(t-1)]dW(t), \quad 0 \le t \le 1, \\ \phi(t) &= 1 + t, \qquad -1 \le t \le 0, \end{aligned} \tag{43}$$

wherein $W(t)$ is the standard Wiener process. Delays are common in various biological systems; the author of [19] writes that "delays arise from finite maturation or division time of various cellular species, such as blood cell lines, or the synthesis of various molecular species, as in the immunological system or genetic control systems." Employing the method of steps, we calculate an explicit solution on the first interval [0, 1]. We use $\phi(t)$ for $-1 \le t \le 0$ as an initial function and let $\beta_2 = \beta_3 = 0$. This solution on $0 \le t \le 1$ is given by

$$x(t) = \exp(at)\left(1 + \frac{b}{a^2}\right) - \frac{b}{a}t - \frac{b}{a^2} + \beta_1 \exp(at)\int_0^t \exp(-as)dW(s). \tag{44}$$

We note that MM is very similar to Euler-Maruyama's method (see [8]) since the noise is additive and and both of the methods converge strongly with the same convergence order, i.e., one. The methods can be compared in the strong sense using the absolute errors $e(T) = \|x(T) - x_n(T)\|$ at the final temporal node $T = 1$.

In the simulation procedures, a high number of steps, i.e., 2^8, are used to simulate the involved Wiener processes. Numerical results and comparison for different values of a, b, and β_1 are provided in Tables 1, 2, 3, 4, and 5. In Tables 1 and 2, the results are given for the ODE case and the delayed ODE case, respectively. These tables show that the LM method is overall more accurate than either the MM or CM method. Tables 3–5 report the numerical comparisons for different cases of the SDDE given in (43) over $M = 500$ paths. These tables too show that the LM method is overall more accurate than either of the MM or CM method.

In Fig. 1, a sample solution corresponding to the path generated by fixing the randomly generated numbers via `SeedRandom[12345]` is illustrated alongside the LM numerical solution applying $a = 0.1$, $b = 2$, $\beta_1 = 1$ ($n = 128$). The figure

Table 1 Results of absolute error comparisons for $a = 1$, $b = 0$ and $\beta_1 = 0$

n	MM	CM	LM
2	0.468282	0.218282	0.218282
4	0.276876	0.000460046	0.0000278602
8	0.152497	4.64034×10^{-10}	8.88178×10^{-16}
16	0.0803533	7.10542×10^{-15}	4.44089×10^{-16}

Table 2 Results of absolute error comparisons for $a = 0.5$, $b = 0.5$, and $\beta_1 = 0$

n	MM	CM	LM
2	0.258664	0.0370729	0.0370729
4	0.140744	0.0000185607	3.35083×10^{-7}
8	0.0736535	1.23367×10^{-12}	0.
16	0.0377108	3.55271×10^{-15}	0.

Table 3 Results of absolute error comparisons for $a = 0.1$, $b = 2$, and $\beta_1 = 1$

n	MM	CM	LM
2	0.475183	4.81090×10^{-2}	4.81193×10^{-2}
4	0.223064	1.60297×10^{-2}	6.24754×10^{-3}
8	0.114612	9.43396×10^{-4}	7.99978×10^{-4}
16	5.62998×10^{-2}	3.39393×10^{-3}	2.99805×10^{-3}
32	2.79259×10^{-2}	1.52520×10^{-3}	1.38088×10^{-3}
64	1.35799×10^{-2}	1.26730×10^{-3}	1.20033×10^{-3}
128	6.45190×10^{-3}	8.85651×10^{-4}	8.63662×10^{-4}
256	3.11027×10^{-3}	7.00025×10^{-4}	7.01795×10^{-4}

Table 4 Results of absolute error comparisons for $a = 0.01$, $b = 1$, and $\beta_1 = 2$

n	MM	CM	LM
2	2.40221×10^{-1}	7.83156×10^{-3}	7.83448×10^{-3}
4	1.16681×10^{-1}	2.95138×10^{-3}	1.21323×10^{-3}
8	5.88136×10^{-2}	1.54485×10^{-4}	1.34849×10^{-4}
16	2.91794×10^{-2}	6.44773×10^{-4}	5.78715×10^{-4}
32	1.45386×10^{-2}	2.86168×10^{-4}	2.62367×10^{-4}
64	7.19224×10^{-3}	2.42455×10^{-4}	2.31399×10^{-4}
128	3.53203×10^{-3}	1.67827×10^{-4}	1.64194×10^{-4}
256	1.74336×10^{-3}	1.33146×10^{-4}	1.33469×10^{-4}

Table 5 Results of absolute error comparisons for $a = -0.001$, $b = 2$, and $\beta_1 = 2$

n	MM	CM	LM
2	5.00804×10^{-1}	8.20984×10^{-4}	8.21285×10^{-4}
4	2.50722×10^{-1}	2.92190×10^{-4}	1.20909×10^{-4}
8	1.25308×10^{-1}	1.50502×10^{-5}	1.31956×10^{-5}
16	6.26754×10^{-2}	6.40767×10^{-5}	5.76247×10^{-5}
32	3.13424×10^{-2}	2.83959×10^{-5}	2.60753×10^{-5}
64	1.56788×10^{-2}	2.41157×10^{-5}	2.30375×10^{-5}
128	7.84575×10^{-3}	1.66734×10^{-5}	1.63190×10^{-5}
256	3.92513×10^{-3}	1.32338×10^{-5}	1.32658×10^{-5}

shows our computational method's superiority in finding the strong solution of the SDDEs.

The numerical stability of the schemes is also tested using time domains and toward this goal, for the settings $a = 0.5$, $b = 5$, $\beta_1 = -10$, and `SeedRandom[1234]` using different numbers of computational nodes. As illustrated in Figs. 2 and 3, the proposed LM scheme for SDDEs provides stable and accurate behavior even with very low number of nodes, and the results are stable regardless of the working domain for time. Note that for larger intervals, smaller step sizes are required for the EM and the MM, whereas our method remains convergent and stable.

The Legendre collocation approach for deterministic DDEs produces the error bound $\mathbb{E}(\|y(t) - y_n(t)\|^2) \sim \exp(-\alpha n)$, $\alpha > 0$, ([12, chapter 4]) where n is the degree of the interpolating Legendre polynomial. Due to noise in the SDEE case, the

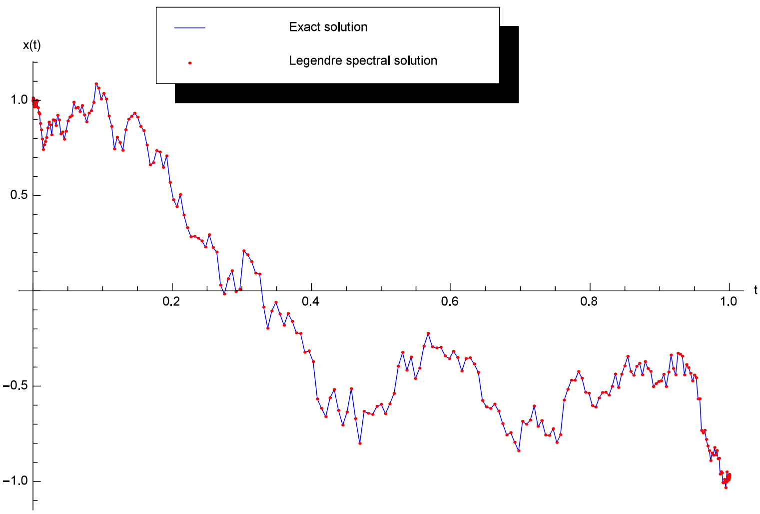

Fig. 1 The exact realization (blue) and the approximate solution (red) by LM for one sample path

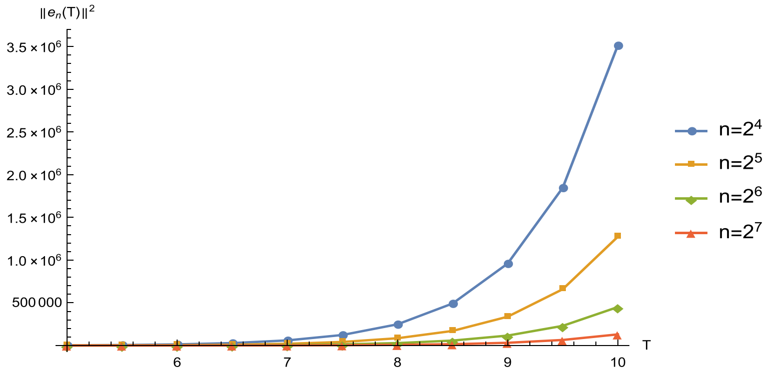

Fig. 2 Unstable convergence history of MM in a larger domain

exponential convergence rate cannot be observed by increasing the number of nodes. For SDDEs, a low number of (shifted) Legendre zeros are sufficient to obtain results with high accuracy.

We now discuss the applicably of our method to systems of SDEs by considering the following system of two-dimensional Itô SDEs given in [16]:

$$d\begin{pmatrix} x^{(1)}(t) \\ x^{(2}(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x^{(1)}(t) \\ x^{(2}(t) \end{pmatrix} dt + \frac{3}{2} \begin{pmatrix} \sin(x^{(1)}(t)) \\ \sin(x^{(2)}(t)) \end{pmatrix} dW(t), \tag{45}$$

where $x(t) = (x^{(1)}(t), x^{(2)}(t))^*$, $0 \le t \le 1$ and $x(0) = (x^{(1)}(0), x^{(2)}(0))^* = (1/2, 1/2)^*$. For this system of SDEs, the Lamperti transformation (in the domain of validity) is defined by

$$y^{(1)}(t) = \frac{2}{3} \ln\left(\cot\left(\frac{1}{4}\right) \tan\left(\frac{x^{(1)}(t)}{2}\right)\right), \tag{46}$$

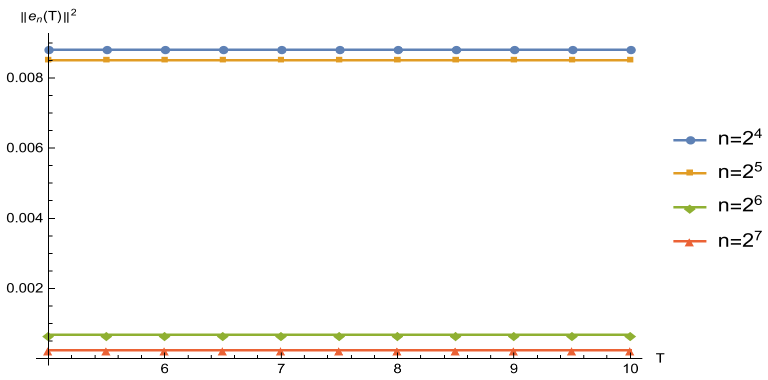

Fig. 3 Stable convergence history of LM in a larger domain

The inverse Lamperti transformation is given as

$$x^{(1)}(t) = 2\tan^{-1}\left(\tan\left(\frac{1}{4}\right)\exp\left(\frac{3y^{(1)}(t)}{2}\right)\right). \tag{47}$$

The Lamperti transformation for $y^{(2)}(t)$ and its inverse can be similarly constructed.

After applying the above Lamperti transformations, we obtain the following nonlinear system of SDEs

$$\begin{aligned} dy^{(1)}(t) &= \overline{f}_1(t, y^{(1)}(t), y^{(2)}(t))dt + dW(t), \\ dy^{(2)}(t) &= \overline{f}_2(t, y^{(1)}(t), y^{(2)}(t))dt + dW(t), \end{aligned} \tag{48}$$

wherein

$$\begin{aligned} \overline{f}_1(t, y^{(1)}(t), y^{(2)}(t)) = &-\tfrac{1}{24}\csc\left(2\tan^{-1}\left(\exp\left(\tfrac{3y^{(1)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right) \\ &\times\left(96\tan^{-1}\left(\exp\left(\tfrac{3y^{(1)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right. \\ &-32\tan^{-1}\left(\exp\left(\tfrac{3y^{(2)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right) \\ &\left.+9\sin\left(4\tan^{-1}\left(\exp\left(\tfrac{3y^{(1)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right)\right), \end{aligned} \tag{49}$$

and

$$\begin{aligned} \overline{f}_2(t, y^{(1)}(t), y^{(2)}(t)) = &-\tfrac{1}{24}\csc\left(2\tan^{-1}\left(\exp\left(\tfrac{3y^{(2)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right) \\ &\times\left(64\tan^{-1}\left(\exp\left(\tfrac{3y^{(2)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right. \\ &-32\tan^{-1}\left(\exp\left(\tfrac{3y^{(1)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right) \\ &\left.+9\sin\left(4\tan^{-1}\left(\exp\left(\tfrac{3y^{(2)}(t)}{2}\right)\tan\left(\tfrac{1}{4}\right)\right)\right)\right). \end{aligned} \tag{50}$$

As noted by [23], the Lamperti transformation removes the super-linearity in the diffusion coefficient as long as the diffusion coefficient of the original SDE is strictly positive on the domain. It can then be used to approximate the transformed process with the backward Euler scheme. Further, as noted by [14], one can use the aforementioned framework to obtain "strong convergence results for several SDEs with non-Lipschitz coefficients" ([14]).

We now consider the SDDEs with non-global Lipschitz coefficients using our proposed model by considering the Cox-Ingersoll-Ross (CIR) model (see [7, 23]). The CIR model is

$$dx(t) = \kappa(\theta - x(t))dt + \sigma\sqrt{x(t)}dW(t). \tag{51}$$

The Feller condition

$$2\kappa\theta \geq \sigma^2, \tag{52}$$

guarantees that its solution is strictly positive when $x(0) > 0$. The classical discretization methods encounter with several difficulties in the simulation process of such financial models (see [23]). To solve such SDEs using our Legendre collocation scheme, we invoke the Lamperti transformation

$$y(t) = \sqrt{x(t)}, \tag{53}$$

and obtain the following Itô SDE [23]:

$$dy(t) = \frac{1}{2}\kappa \left(\left(\theta - \frac{\sigma^2}{4\kappa} \right) y(t)^{-1} - y(t) \right) dt + \frac{1}{2}\sigma dW(t). \tag{54}$$

Applying (20) to (54) yields a nonlinear system of algebraic equation for each realization that can be solved like the backward Euler scheme on (54).

6 Summary

This work proposes the Legendre spectral collocation scheme to solve the class of SDDEs given by the system of equations in (4). The Lamperti transformation enables us to reduce the instability of the underlying problem.

Utilizing the Legendre collocation method, we construct the nth degree interpolating polynomials to obtain the differentiation matrices. The convergence of the proposed scheme is theoretically studied and the numerical results confirm both its accuracy and its computationally efficient performance.

Our future research will concern applying some other general shifted orthogonal polynomials (or perhaps even non-orthogonal polynomials) for constructing the corresponding differentiation matrices. In addition, we plan to explore computational methods that may benefit from transformations other than the Lamperti transformation.

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