Rapid Communications

Parton construction of particle-hole-conjugate Read-Rezayi parafermion fractional quantum Hall states and beyond

Ajit C. Balram,[1] Maissam Barkeshli,[2] and Mark S. Rudner[1]
[1]*Niels Bohr International Academy and the Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark*
[2]*Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, Maryland 20472, USA*

(Received 11 February 2019; published 19 June 2019)

The Read-Rezayi (RR) parafermion states form a series of exotic non-Abelian fractional quantum Hall (FQH) states at filling $\nu = k/(k+2)$. Computationally, the wave functions of these states are prohibitively expensive to generate for large systems. We introduce a series of parton states, denoted $\bar{2}^k 1^{k+1}$, and show that they lie in the same universality classes as the particle-hole-conjugate RR ("anti-RR") states. Our analytical results imply that a $[U(1)_{k+1} \times U(2k)_{-1}]/[SU(k)_{-2} \times U(1)_{-1}]$ coset conformal field theory describes the edge excitations of the $\bar{2}^k 1^{k+1}$ state, suggesting nontrivial dualities with respect to previously known descriptions. The parton construction allows wave functions in anti-RR phases to be generated for hundreds of particles. We further propose the parton sequence $\bar{n}\bar{2}^2 1^4$, with $n = 1, 2, 3$, to describe the FQH states observed at $\nu = 2 + 1/2$, $2 + 2/5$, and $2 + 3/8$.

DOI: 10.1103/PhysRevB.99.241108

The fractional quantum Hall effect (FQHE) [1] has revealed a variety of emergent many-body quantum phases that host exotic topological excitations. An important development in the field came about by a proposal of Moore and Read that the 5/2 state observed in the half-filled second Landau level (SLL) of GaAs [2] could be described by a Pfaffian wave function [3]. The excitations of the Pfaffian state are Majorana quasiparticles that feature non-Abelian braiding statistics [4]. Subsequently, Read and Rezayi (RR) proposed a class of FQH states hosting more general parafermionic excitations, including exotic Fibonacci anyons [5,6]. Intriguingly, systems hosting such non-Abelian excitations may be utilized for fault-tolerant quantum computation [7–10].

Here, we are motivated by the FQHEs observed in GaAs at filling factors $\nu = 2 + 2/3$, $2 + 1/2$, $2 + 2/5$, and $2 + 3/8$ (see Refs. [2,11–15]). Numerical studies have produced strong evidence that the first three members of this sequence may be described by the particle-hole conjugates of "k-cluster" RR wave functions [5,6] (abbreviated as aRRk, where aRR stands for "anti-Read-Rezayi"), with $k = 1$, 2, and 3, respectively [16–25]. In particular, the results in Refs. [22–25] indicate that the ground state of the experimentally observed [11–15] FQHE at filling factor $2 + 2/5$ is well described by an aRR state that hosts Fibonacci anyons. This suggests that the 12/5 FQHE may provide a solid state platform for universal fault-tolerant quantum computation.

The wave function of a k-cluster RR state is obtained by symmetrizing over partitions of N particles into k clusters, where each cluster forms a Laughlin state [26]. (Here, N and k are positive integers with N divisible by k.) Importantly, the operation of symmetrization is computationally expensive, making it difficult to numerically evaluate these states and study their properties for large systems. The RR states can alternatively be obtained by exact diagonalization of model Hamiltonians [5] or using Jack polynomials [27], but these procedures are also limited to small sizes ($N \lesssim 30$). Thus there is great impetus to find more efficient representations of wave functions [28–31] in these exotic phases, to enable their further study.

In this Rapid Communication we introduce the $\bar{2}^k 1^{k+1}$ family of parton wave functions [32], which for each $k = \{1, 2, 3, \ldots\}$ provides a state at filling factor $\nu = 2/(k+2)$ within the same universality class as the aRRk state. These parton wave functions can be evaluated for hundreds of particles, and thus provide the means to numerically investigate the properties of parafermions in large systems. The $k = 1$ and $k = 2$ members of this parton family map onto states that were previously shown to lie in the same phases as the particle-hole conjugates of the 1/3 Laughlin state [33,34], and of the 1/2 Pfaffian state [35] (i.e., the "anti-Pfaffian" state [36,37]), respectively. Below, we give numerical evidence, based on wave-function overlaps and entanglement spectra, that the $\bar{2}^k 1^{k+1}$ state with $k = 3$ is topologically equivalent to the aRR3 state. Using the effective field theory that arises from the parton mean-field ansatz, we compute several topological properties of $\bar{2}^k 1^{k+1}$, including its chiral central charge, ground-state degeneracy on the torus, and anyon content, and show that they match those of the aRRk state.

Background. Throughout this Rapid Communication we assume a single-component system, and consider an ideal setting with zero width, no LL mixing, and zero disorder. The problem of interacting electrons confined to a given LL can be equivalently treated as a problem of electrons residing in the lowest Landau level (LLL), interacting via an effective interaction [38]. Thus we employ wave functions that reside

 ©2019 American Physical Society

in the LLL, keeping in mind that they can describe the FQHE in any LL (in particular, the SLL).

The wave function of the N-particle, k-cluster RR state $\Psi^{\mathrm{RR}k}_{k/(k+2)}$ at filling factor $\nu = k/(k+2)$ is [5,6,39]

$$\Psi^{\mathrm{RR}k}_{k/(k+2)} = \mathbb{S}\left[\prod_{i_1<j_1}(z_{i_1}-z_{j_1})^2\cdots\prod_{i_k<j_k}(z_{i_k}-z_{j_k})^2\right] \times\prod_{i<j}(z_i-z_j)\exp\left[-\sum_i\frac{|z_i|^2}{4\ell^2}\right], \quad (1)$$

where z_i, with $i = \{1,\dots,N\}$, is the two-dimensional coordinate of the ith electron, written as a complex number. (For ease of notation, below we suppress the ubiquitous Gaussian factors from all wave functions.) The N particles are partitioned into k internally correlated "clusters" of N/k particles, with the product $\prod_{i_l<j_l}(z_{i_l}-z_{j_l})^2$ describing the correlations within a given cluster, l. The symbol $\mathbb{S}$ denotes symmetrization over all such partitions. The corresponding k-cluster anti-RR state, $\Psi^{\mathrm{aRR}k}_{2/(k+2)}$, is described by the wave function

$$\Psi^{\mathrm{aRR}k}_{2/(k+2)} = \mathcal{P}_{\mathrm{ph}}\left[\Psi^{\mathrm{RR}k}_{k/(k+2)}\right], \quad (2)$$

where $\mathcal{P}_{\mathrm{ph}}$ denotes the operation of particle-hole conjugation. Due to particle-hole conjugation, $\Psi^{\mathrm{aRR}k}_{2/(k+2)}$ occurs at filling factor $\nu = 1 - k/(k+2) = 2/(k+2)$.

For numerical work we employ the compact spherical geometry introduced by Haldane [38]. In this geometry, N electrons move on the surface of a sphere in the presence of a radial magnetic field B, the source of which is a Dirac monopole of strength $2Q$ sitting at the center of the sphere [40]. The total magnetic flux through the sphere of radius R is $4\pi R^2 B = 2Q(h/e)$. The radius of the sphere is thus related to the magnetic length, $\ell = \sqrt{\hbar/(eB)}$, via $R = \sqrt{Q}\ell$. Due to the spherical symmetry, the total orbital angular momentum L and its z component L_z are good quantum numbers in this geometry.

Gapped quantum Hall ground states are rotationally invariant, i.e., they are uniform on the sphere and have $L_z = L = 0$. At a given filling factor ν, one may find a variety of candidate ground states featuring distinct types of topological order [41]. Each candidate ground state is realized at a specific value of the total magnetic flux through the sphere, $2Q = \nu^{-1}N - \mathcal{S}$, which is offset from its value in the plane, N/ν, by a rational number $\mathcal{S}$ called the shift [41]. If two states occur at different shifts, then they must describe different phases. Note that the converse, however, does not hold: Topologically distinct states may occur with the same shift.

Before moving on to our parton ansatz, for reference we summarize some of the key properties of the RRk and aRRk states defined in Eqs. (1) and (2). The k-cluster RR state in Eq. (1) occurs at monopole strength $2Q = [k/(k+2)]^{-1}N - 3$, corresponding to the shift $\mathcal{S}^{\mathrm{RR}k} = 3$. The topological order of $\Psi^{\mathrm{RR}k}_{k/(k+2)}$ is furthermore exhibited through the quantized thermal Hall conductance that it supports, $\kappa^{\mathrm{RR}k}_{xy} = 3k/(k+2)$, in units of $[\pi^2 k_{\mathrm{B}}^2/(3h)]T$, where k_{B} is Boltzmann's constant and T is the system's temperature [5,42]. In contrast, the aRRk states in Eq. (2) are characterized by the flux-particle relation $2Q = [(k+2)/2]N - (1-k)$, corresponding to shift $\mathcal{S}^{\mathrm{aRR}k} = 1 - k$. The thermal Hall conductance supported by $\Psi^{\mathrm{aRR}k}_{2/(k+2)}$ is given by $\kappa^{\mathrm{aRR}k}_{xy} = 1 - \kappa^{\mathrm{RR}k}_{xy} = -2(k-1)/(k+2)$, again in units of $[\pi^2 k_{\mathrm{B}}^2/(3h)]T$ [5,42].

TABLE I. Overlaps of N-particle FQH states on a sphere with magnetic flux $2Q$ corresponding to that of the particle-hole conjugate of the $k = 3$ Read-Rezayi state (aRR3). We compare the wave functions of the parton state $\Psi^{\bar{2}^3 1^4}_{2/5}$ [Eq. (3)], the aRR3 state $\Psi^{\mathrm{aRR3}}_{2/5}$ [Eq. (2)], and the ground state obtained by exact diagonalization using the SLL Coulomb pseudopotentials $\Psi^{\mathrm{SLL}}_{2/5}$. The numbers for $|\langle\Psi^{\mathrm{SLL}}_{2/5}|\Psi^{\mathrm{aRR3}}_{2/5}\rangle|$ were previously given in Refs. [5,6,20,23,51].

N	$2Q$	$\lvert\langle\Psi^{\mathrm{SLL}}_{2/5}\lvert\Psi^{\mathrm{aRR3}}_{2/5}\rangle\rvert$	$\lvert\langle\Psi^{\bar{2}^3 1^4}_{2/5}\lvert\Psi^{\mathrm{aRR3}}_{2/5}\rangle\rvert$	$\lvert\langle\Psi^{\mathrm{SLL}}_{2/5}\lvert\Psi^{\bar{2}^3 1^4}_{2/5}\rangle\rvert$
4	12	0.9854	0.9173	0.8362
6	17	0.9022	0.9107	0.6797
8	22	0.9836	0.8821	0.8252

Parton states. We now define a family of parton states, denoted $\bar{2}^k 1^{k+1}$, each of which lies in the same universality class as the corresponding aRRk state, $\Psi^{\mathrm{aRR}k}_{2/(k+2)}$. The $\bar{2}^k 1^{k+1}$ parton wave function $\Psi^{\bar{2}^k 1^{k+1}}_{2/(k+2)}$ is formed from a product of integer quantum Hall (IQH) states,

$$\Psi^{\bar{2}^k 1^{k+1}}_{2/(k+2)} = \mathcal{P}_{\mathrm{LLL}}[\Phi_2^*]^k\Phi_1^{k+1} \sim \frac{\left[\Psi^{\mathrm{CF}}_{2/3}\right]^k}{\Phi_1^{k-1}}, \quad (3)$$

where Φ_n is the $\nu = n$ IQH wave function of N particles, and $\mathcal{P}_{\mathrm{LLL}}$ denotes projection into the lowest Landau level. Here, $\Psi^{\mathrm{CF}}_{2/3} = \mathcal{P}_{\mathrm{LLL}}\Phi_2^*\Phi_1^2$ denotes the $\nu = 2/3$ composite fermion (CF) wave function [43]. The $\sim$ sign indicates that (for $k > 1$) the rightmost expression in Eq. (3) differs from that in the middle in the details of how the projection to the LLL is carried out. We do not expect such details of the projection to change the topological properties of the state [34,44].

Crucially, the wave function given on the right-hand side of Eq. (3) can be efficiently evaluated for large systems. This is so because the constituent CF wave function $\Psi^{\mathrm{CF}}_{2/3}$ can be evaluated for hundreds of electrons using the so-called Jain-Kamilla projection [45,46], the details of which can be found in the literature [47–50].

When mapped to the spherical geometry, the states given in Eq. (3) occur at monopole strength $2Q = [(k+2)/2]N - (1-k)$, corresponding to filling factor $\nu = 2/(k+2)$ and shift $\mathcal{S}^{\bar{2}^k 1^{k+1}} = 1 - k$. These fillings and shifts precisely match those of the aRRk states described by Eq. (2). This observation suggests that the wave functions given in Eq. (3) could lie in the same phases as the corresponding aRR states. For $k = 1$, the $\bar{2}1^2$ state described by Eq. (3) is precisely the $2/3$ CF state (see above); this state is almost identical to the particle-hole conjugate of the $1/3$ Laughlin state [33,34]. In Ref. [35] we studied the $\bar{2}^2 1^3$ state, and showed that it lies in the anti-Pfaffian [36,37] universality class. Below, we discuss the case for arbitrary values of k.

Numerical results. We first provide numerical evidence to show that $\Psi^{\bar{2}^k 1^{k+1}}_{2/5}$ with $k = 3$ lies in the same phase as $\Psi^{\mathrm{aRR3}}_{2/5}$ given in Eq. (2). In Table I we show overlaps of the parton wave function $\Psi^{\bar{2}^3 1^4}_{2/5}$ with $\Psi^{\mathrm{aRR3}}_{2/5}$, as well as with the numerically obtained exact ground state using the second Landau level Coulomb pseudopotentials $\Psi^{\mathrm{SLL}}_{2/5}$. We find that the parton

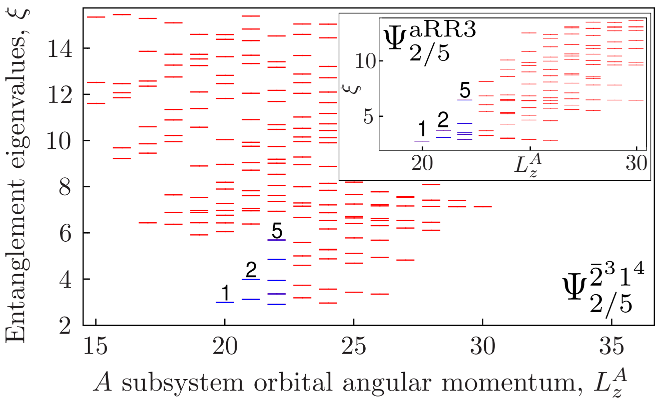

FIG. 1. Orbital entanglement spectrum of the $\Psi_{2/5}^{\bar{2}^3 1^4}$ parton state for $N = 8$ electrons at a flux $2Q = 22$ on the sphere. The entanglement spectrum is calculated with respect to two subsystems, A and B, with $N_A = N_B = 4$ electrons and $l_A = 12$ and $l_B = 11$ orbitals, respectively. The entanglement levels are labeled by the z component of the total orbital angular momentum of the A subsystem, L_z^A. For comparison, in the inset we show the corresponding entanglement spectrum for $\Psi_{2/5}^{\text{aRR3}}$. The multiplicities of low-lying levels (starting from $L_z^A = 20$, going from left to right) are given by $1, 2, 5, \ldots$ and are identical for the two states.

wave function has a good overlap with the corresponding anti-RR state. Furthermore, both $\Psi_{2/5}^{\text{aRR3}}$ and $\Psi_{2/5}^{\bar{2}^3 1^4}$ display a decent overlap with the SLL Coulomb ground state $\Psi_{2/5}^{\text{SLL}}$. Similar to the aRR3 state [5,6,22–25], the $k = 3$ parton state of Eq. (3) can thus serve as a good candidate to describe the quantum Hall liquid occurring at $\nu = 12/5$.

We provide further numerical evidence of the topological equivalence between $\Psi_{2/5}^{\bar{2}^3 1^4}$ and $\Psi_{2/5}^{\text{aRR3}}$ by comparing their entanglement spectra. The entanglement spectrum is a useful characterization tool, as it captures the structure of a FQH state's edge excitations [52]. The multiplicities of the low-lying entanglement levels carry a fingerprint of the topological order of the underlying state. Two states that lie in the same topological phase are expected to yield identical multiplicities. In Fig. 1 we show the orbital entanglement spectrum [53] of the $\bar{2}^3 1^4$ state obtained on the sphere for a system of $N = 8$ electrons at flux $2Q = 22$. The multiplicities of the low-lying entanglement levels of $\Psi_{2/5}^{\bar{2}^3 1^4}$ are identical to those of $\Psi_{2/5}^{\text{aRR3}}$. Thus we conclude that the $\bar{2}^3 1^4$ parton state likely lies in the same phase as the aRR3 state.

Field theory results. Next, we consider the effective field theory that describes the associated parton mean-field ansatz (focusing on $k \geqslant 2$). Consider the following parton decomposition of the electron operator, $\wp = b f_1 \cdots f_k$, where the f_i fields are fermions and b is a boson (fermion) for k odd (even). In the mean-field ansatz, b forms a $\nu = 1/(k+1)$ Laughlin FQH state [26], while each fermion species f_i forms a $\nu = -2$ IQH state. This ansatz has a $U(1) \times SU(k)$ gauge symmetry.

Integrating out the partons yields a non-Abelian Chern-Simons (CS) theory that we can use to explicitly compute the ground-state degeneracy on the torus [see Supplemental Material (SM) [54]]. Carrying out this calculation for $k = 2, 3, 4, 5, 6, 7$, we find a torus ground-state degeneracy of $(k+1)(k+2)/2$, which agrees with the expected results for the aRR states. Using the field theory in combination with general consistency conditions from topological quantum field theory, we further demonstrate [54] that the anyon content for the $k = 3$ parton state precisely matches that of aRR3. For $k > 3$ we derive a number of general properties for the anyon content of the parton states and show that they match with those of the corresponding aRR states [54].

Finally, we consider the edge theory. The parton mean-field state (before implementing the gauge projection) is described by a $U(1)_{k+1} \times U(2k)_{-1}$ Wess-Zumino-Witten (WZW) conformal field theory (CFT) [55]. This CFT comprises $2k$ upstream-moving chiral fermion modes and 1 downstream-moving chiral mode, giving a chiral central charge $c_{-,\text{MF}} = -2k + 1$. The gauge projection in the edge theory requires us to project out modes transforming nontrivially under the $U(1) \times SU(k)$ gauge symmetry, which leads to the $[U(1)_{k+1} \times U(2k)_{-1}]/[U(1)_{-1} \times SU(k)_{-2}]$ coset CFT [54–56]. The total central charge is $c_- = c_{-,\text{MF}} - c_{-,\text{gauge}}$, where $c_{-,\text{gauge}} = -1 - 2(k^2 - 1)/(k+2)$ is the chiral central charge of the gauge degrees of freedom [54]. We thus obtain $c_- = 1 - 3k/(k+2)$, which precisely matches the chiral central charge of the aRRk state [5,42].

We note that a number of field theories for RR states have been described previously, such as an $SU(2)_k \times U(1)$ CS theory and a $U(1) \times Sp(k)_1$ CS theory [57,58]. The equivalence between these two theories is related to level-rank duality [55,57]. Those results imply that the edge theory of the aRR states can be described by a $U(1)_1 \times SU(2)_{-k} \times U(1)$ WZW theory or, equivalently, by a dual $U(1)_1 \times Sp(k)_{-1} \times U(1)$ WZW theory. Our results imply that a $[U(1)_{k+1} \times U(2k)_{-1}]/[U(1)_{-1} \times SU(k)_{-2}]$ coset CFT can also describe the edge excitations of the aRR states, which suggests another nontrivial duality among these theories. We leave a detailed study of these dualities for future work.

Discussion. A major advantage of our parton wave functions is that they can be constructed for large systems. As a proof of principle, we numerically demonstrate that the smallest charge quasiparticle (QP) of the 2/5 parton state carries a charge $-e/5$, where $-e < 0$ is the charge of the electron [32]. To this end, we create a model state at filling factor $\nu = 2/5$ with two far-separated QPs, one located at each pole of the sphere [59],

$$\Psi_{2/5}^{\text{2-QPs}} = \mathcal{P}_{\text{LLL}} \big[\Phi_2^{(\text{2-h})}\big]^* \big[\Phi_2^2\big]^* \Phi_1^4 \sim \frac{\Psi_{2/3}^{\text{CF},(\text{2-QP})} \big[\Psi_{2/3}^{\text{CF}}\big]^2}{\Phi_1^2}, \quad (4)$$

where $\Phi_2^{(\text{2-h})}$ and $\Psi_{2/3}^{\text{CF},(\text{2-QP})}$ are the $\nu = 2$ IQH and $\nu = 2/3$ CF states with two holes or two QPs, respectively, located at opposite poles of the sphere. The CF wave functions are evaluated using the Jain-Kamilla method [45–50]. In Fig. 2(a) we show the density profile $\rho(\mathbf{r})$ of $\Psi_{2/5}^{\text{2-QPs}}$ for $N = 80$ electrons. Close to the equator, the density approaches the value ρ_0 of the uniform $\bar{2}^3 1^4$ state. To extract the QP charge, we integrate the deviation of the charge density from its uniform value, $\rho(\mathbf{r}) - \rho_0 \equiv \delta\rho(\mathbf{r})$, over the northern hemisphere. In Fig. 2(b) we plot the cumulative charge $q(r) = \int_0^r \delta\rho(\mathbf{r}') d^2\mathbf{r}'$ as a function latitude, parametrized by the arc distance r along the dashed contour shown in Fig. 2(a). From the limiting value of $q(r)$ at the equator we extract a charge of $-0.197e$,

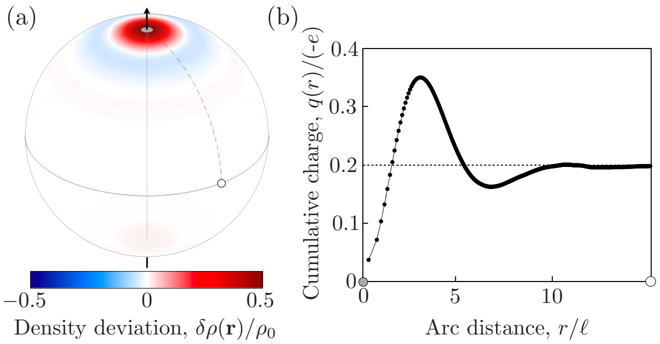

FIG. 2. (a) Density profile $\rho(\mathbf{r})$ of a state with two far-separated quasiparticles at $\nu = 2/5$, modeled by the parton wave function given in Eq. (4) for $N = 80$ electrons on the sphere. The two quasiparticles are located at the north and south pole of the sphere. The color represents the density deviation from its value ρ_0 in the uniform $\Psi_{2/5}^{\bar{2}^3 1^4}$ state: $\delta\rho(\mathbf{r})/\rho_0 = [\rho(\mathbf{r}) - \rho_0]/\rho_0$. (b) The integrated cumulative charge $q(r)$ (see text for the definition) as a function of latitude, parametrized by the distance r along the arc from the north pole to the equator (in units of magnetic length ℓ). The cumulative charge approaches the value $-0.2e$ near the equator.

which is close to the expected value of $-0.2e$ (attained in the thermodynamic limit when the QPs do not overlap).

Building on our results for the $k = 3$ case, we are led to consider a new $\bar{n}\bar{2}^2 1^4$ parton sequence described by the wave functions

$$\Psi_{n/(3n-1)}^{\bar{n}\bar{2}^2 1^4} = \mathcal{P}_{\rm LLL}[\Phi_n^*][\Phi_2^*]^2\Phi_1^4 \sim \frac{\Psi_{n/(2n-1)}^{\rm CF}\left[\Psi_{2/3}^{\rm CF}\right]^2}{\Phi_1^2}. \quad (5)$$

In the spherical geometry, $\Psi_{n/(3n-1)}^{\bar{n}\bar{2}^2 1^4}$ occurs at monopole strength $2Q = [(3n-1)/n]N + n$ and hence has a filling factor $\nu = n/(3n-1)$ and shift $\mathcal{S}^{\bar{n}\bar{2}^2 1^4} = -n$. We thus obtain states at filling factors $\nu = 1/2, 2/5, 3/8, \ldots$ for $n = 1, 2, 3, \ldots$, respectively. The $n = 1$ member of this sequence likely lies in the same universality class as the $\bar{2}^2 1^3$ state [60], which we showed in a previous work lies in the anti-Pfaffian phase [35]. We discussed the $n = 2$ case in detail in this Rapid Communication and concluded that it lies in the same phase as the aRR3 state. Intriguingly, the $n = 3$ state of Eq. (5) provides a candidate ground-state wave function that could possibly describe the FQHE at $2 + 3/8$ [11–15,61,62]. We thus speculate that the $\bar{n}\bar{2}^2 1^4$ family of parton states may capture the observed plateaus at 2/5 and 3/8 in the SLL of GaAs [11–15] that were not covered in the $\bar{n}\bar{2}1^3$ sequence of Ref. [63]. Generically, we find that the parton states of Eq. (5) are topologically different from other families of candidate states occurring at the same $n/(3n-1)$ sequence of filling factors [54].

Although we only considered states with a single component, our parton construction can be extended in a straightforward manner to build multicomponent states at the corresponding filling factors, where the different components could represent either the spin, valley, or orbital degrees of freedom. The properties of these states remain to be explored.

Taken together with our previous works [35,63], the results presented in this Rapid Communication suggest that almost all fractional quantum Hall states observed in the second Landau level of GaAs could be described by the $\bar{n}\bar{2}1^3$ or $\bar{n}\bar{2}^2 1^4$ parton ansatz, with $n = 1, 2, 3$, or their particle-hole conjugates. The states observed in the second LL that do not fall in these sequences or their particle-hole conjugates, e.g., at filling factor 1/5 and 2/7, are likely well described by composite fermion states [16,51] (which are also parton states). In all, except for the lowest Landau level states at $\nu = 4/11$ and 5/13 (see, e.g., Refs. [64–66]), it appears that all fractional quantum Hall states observed to date (or their particle-hole conjugates), including in graphene [67–73] and wide quantum wells [74–78], admit simple parton descriptions.

Acknowledgments. The Center for Quantum Devices is funded by the Danish National Research Foundation. This work was supported by the European Research Council (ERC) under the European Union Horizon 2020 Research and Innovation Programme, Grant Agreement No. 678862. A.C.B. and M.S.R. also thank the Villum Foundation for support. M.B. is supported by NSF CAREER (DMR-1753240), JQI-PFC-UMD, and an Alfred P. Sloan Research Fellowship. Some of the numerical calculations were performed using the DIAGHAM package, for which we are grateful to its authors.

[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[3] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[4] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[5] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[6] E. H. Rezayi and N. Read, Phys. Rev. B **79**, 075306 (2009).
[7] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[8] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[9] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[10] M. H. Freedman, M. Larsen, and Z. Wang, Commun. Math. Phys. **227**, 605 (2002).
[11] J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **93**, 176809 (2004).
[12] W. Pan, J. S. Xia, H. L. Stormer, D. C. Tsui, C. Vicente, E. D. Adams, N. S. Sullivan, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. B **77**, 075307 (2008).
[13] H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, Phys. Rev. B **77**, 081301(R) (2008).
[14] A. Kumar, G. A. Csáthy, M. J. Manfra, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **105**, 246808 (2010).
[15] C. Zhang, C. Huan, J. S. Xia, N. S. Sullivan, W. Pan, K. W. Baldwin, K. W. West, L. N. Pfeiffer, and D. C. Tsui, Phys. Rev. B **85**, 241302(R) (2012).

[16] N. d'Ambrumenil and A. M. Reynolds, J. Phys. C **21**, 119 (1988).
[17] A. C. Balram, Y.-H. Wu, G. J. Sreejith, A. Wójs, and J. K. Jain, Phys. Rev. Lett. **110**, 186801 (2013).
[18] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[19] V. W. Scarola, S.-Y. Lee, and J. K. Jain, Phys. Rev. B **66**, 155320 (2002).
[20] K. Pakrouski, M. R. Peterson, T. Jolicoeur, V. W. Scarola, C. Nayak, and M. Troyer, Phys. Rev. X **5**, 021004 (2015).
[21] E. H. Rezayi, Phys. Rev. Lett. **119**, 026801 (2017).
[22] A. Wójs, Phys. Rev. B **80**, 041104(R) (2009).
[23] W. Zhu, S. S. Gong, F. D. M. Haldane, and D. N. Sheng, Phys. Rev. Lett. **115**, 126805 (2015).
[24] R. S. K. Mong, M. P. Zaletel, F. Pollmann, and Z. Papić, Phys. Rev. B **95**, 115136 (2017).
[25] K. Pakrouski, M. Troyer, Y.-L. Wu, S. Das Sarma, and M. R. Peterson, Phys. Rev. B **94**, 075108 (2016).
[26] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[27] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[28] M. P. Zaletel and R. S. K. Mong, Phys. Rev. B **86**, 245305 (2012).
[29] C. H. Lee, Z. Papić, and R. Thomale, Phys. Rev. X **5**, 041003 (2015).
[30] C. Repellin, T. Neupert, B. A. Bernevig, and N. Regnault, Phys. Rev. B **92**, 115128 (2015).
[31] C. Fernández-González, R. S. K. Mong, O. Landon-Cardinal, D. Pérez-García, and N. Schuch, Phys. Rev. B **94**, 155106 (2016).
[32] J. K. Jain, Phys. Rev. B **40**, 8079 (1989).
[33] X. G. Wu, G. Dev, and J. K. Jain, Phys. Rev. Lett. **71**, 153 (1993).
[34] A. C. Balram and J. K. Jain, Phys. Rev. B **93**, 235152 (2016).
[35] A. C. Balram, M. Barkeshli, and M. S. Rudner, Phys. Rev. B **98**, 035127 (2018).
[36] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. **99**, 236806 (2007).
[37] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. **99**, 236807 (2007).
[38] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[39] A. Cappelli, L. S. Georgiev, and I. T. Todorov, Nucl. Phys. B **599**, 499 (2001).
[40] We follow the convention of writing wave functions in the planar disk geometry, as in Eqs. (1) and (2), which can be adapted to the spherical geometry using the stereographic projection.
[41] X. G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[42] W. Bishara, G. A. Fiete, and C. Nayak, Phys. Rev. B **77**, 241306(R) (2008).
[43] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[44] R. V. Mishmash, D. F. Mross, J. Alicea, and O. I. Motrunich, Phys. Rev. B **98**, 081107(R) (2018).
[45] J. K. Jain and R. K. Kamilla, Int. J. Mod. Phys. B **11**, 2621 (1997).
[46] J. K. Jain and R. K. Kamilla, Phys. Rev. B **55**, R4895 (1997).
[47] G. Möller and S. H. Simon, Phys. Rev. B **72**, 045344 (2005).
[48] J. K. Jain, *Composite Fermions* (Cambridge University Press, New York, 2007).
[49] S. C. Davenport and S. H. Simon, Phys. Rev. B **85**, 245303 (2012).
[50] A. C. Balram, C. Tőke, A. Wójs, and J. K. Jain, Phys. Rev. B **92**, 075410 (2015).
[51] B. Kuśmierz and A. Wójs, Phys. Rev. B **97**, 245125 (2018).
[52] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[53] M. Haque, O. Zozulya, and K. Schoutens, Phys. Rev. Lett. **98**, 060401 (2007).
[54] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.99.241108 for (i) a discussion of the topological properties of the $\bar{2}^k 1^{k+1}$ parton state derived from the low-energy effective theory of its edge, and (ii) a comparison of the $\bar{n}\bar{2}^2 1^4$ parton states with other known families of candidate states occurring at the same $\nu = n/(3n-1)$ sequence of filling factors, which includes Refs. [79–84].
[55] P. D. Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer, Berlin, 1997).
[56] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999).
[57] M. Barkeshli and X.-G. Wen, Phys. Rev. B **81**, 155302 (2010).
[58] Defining these theories properly requires incorporating some discrete gauge symmetries as well, which we do not describe here.
[59] The state Φ_2 can only be constructed for an even number of particles on the sphere, thus precluding the construction of a state with a single QP at $\nu = 2/5$ using our parton ansatz.
[60] The $\bar{1}\bar{2}^2 1^4$ and $\bar{2}^2 1^3$ parton states differ from each other only by a factor of $|\Phi_1|^2$. Such factors of $|\Phi_1|^2$ are expected to only weakly affect the resulting LLL-projected wave function [34,44].
[61] C. Tőke, C. Shi, and J. K. Jain, Phys. Rev. B **77**, 245305 (2008).
[62] J. A. Hutasoit, A. C. Balram, S. Mukherjee, Y.-H. Wu, S. S. Mandal, A. Wójs, V. Cheianov, and J. K. Jain, Phys. Rev. B **95**, 125302 (2017).
[63] A. C. Balram, S. Mukherjee, K. Park, M. Barkeshli, M. S. Rudner, and J. K. Jain, Phys. Rev. Lett. **121**, 186601 (2018).
[64] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **90**, 016801 (2003).
[65] W. Pan, K. W. Baldwin, K. W. West, L. N. Pfeiffer, and D. C. Tsui, Phys. Rev. B **91**, 041301(R) (2015).
[66] N. Samkharadze, I. Arnold, L. N. Pfeiffer, K. W. West, and G. A. Csáthy, Phys. Rev. B **91**, 081109(R) (2015).
[67] X. Du, I. Skachko, F. Duerr, A. Luican, and E. Y. Andrei, Nature (London) **462**, 192 (2009).
[68] K. Bolotin, F. Ghahari, M. D. Shulman, H. Stormer, and P. Kim, Nature (London) **462**, 196 (2009).
[69] B. E. Feldman, A. J. Levin, B. Krauss, D. A. Abanin, B. I. Halperin, J. H. Smet, and A. Yacoby, Phys. Rev. Lett. **111**, 076802 (2013).
[70] F. Amet, A. J. Bestwick, J. R. Williams, L. Balicas, K. Watanabe, T. Taniguchi, and D. Goldhaber-Gordon, Nat. Commun. **6**, 5838 (2015).
[71] A. C. Balram, C. Tőke, A. Wójs, and J. K. Jain, Phys. Rev. B **92**, 205120 (2015).
[72] Y. Zeng, J. I. A. Li, S. A. Dietrich, O. M. Ghosh, K. Watanabe, T. Taniguchi, J. Hone, and C. R. Dean, Phys. Rev. Lett. **122**, 137701 (2019).
[73] Y. Kim, A. C. Balram, T. Taniguchi, K. Watanabe, J. K. Jain, and J. H. Smet, Nat. Phys. **15**, 154 (2019).

[74] D. R. Luhman, W. Pan, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **101**, 266804 (2008).
[75] J. Shabani, T. Gokmen, and M. Shayegan, Phys. Rev. Lett. **103**, 046805 (2009).
[76] J. Shabani, T. Gokmen, Y. T. Chiu, and M. Shayegan, Phys. Rev. Lett. **103**, 256802 (2009).
[77] J. Shabani, Y. Liu, M. Shayegan, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, Phys. Rev. B **88**, 245413 (2013).
[78] W. N. Faugno, A. C. Balram, M. Barkeshli, and J. K. Jain, arXiv:1904.07164 [Phys. Rev. Lett. (to be published)].
[79] A. B. Zamolodchikov and V. A. Fateev, Zh. Eksp. Teor. Fiz. **89**, 380 (1985) [JETP **62**, 215 (1985)].
[80] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[81] P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 125323 (2008).
[82] T. Jolicoeur, Phys. Rev. Lett. **99**, 036805 (2007).
[83] E. Rowell, R. Stong, and Z. Wang, Commun. Math. Phys. **292**, 343 (2009).
[84] Z. Wang, *Topological Quantum Computation* (American Mathematical Society, Providence, RI, 2010).