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Calculus of Variations

Estimates and monotonicity for a heat flow of isometric G_2-structures

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Abstract

Given a 7-dimensional compact Riemannian manifold (M, g) that admits G_2-structure, all the G_2-structures that are compatible with the metric g are parametrized by unit sections of an octonion bundle over M. We define a natural energy functional on unit octonion sections and consider its associated heat flow. The critical points of this functional and flow precisely correspond to G_2-structures with divergence-free torsion. In this paper, we first derive estimates for derivatives of $V(t)$ along the flow and prove that the flow exists as long as the torsion remains bounded. We also prove a monotonicity formula and an ε-regularity result for this flow. Finally, we show that within a metric class of G_2-structures that contains a torsion-free G_2-structure, under certain conditions, the flow will converge to a torsion-free G_2-structure.

Mathematics Subject Classification 53C10 · 53C29 · 58E30 · 58E15

1 Introduction

A fundamental problem in the study of 7-dimensional manifolds with G_2 -structures is the question of general existence conditions for torsion-free G_2-structures, which are the ones that correspond to metrics with holonomy contained in G_2. One of the possible approaches is to try and construct a flow of G_2-structures which under certain conditions would converge to torsion-free G_2-structures. This approach was originally pioneered by Robert Bryant [6] when he introduced the Laplacian flow of closed G_2-structures, i.e. ones for which the defining 3-form φ is closed. Later, Karigiannis, McKay, and Tsui [33] introduced a similar flow, known as the *Laplacian coflow*, for co-closed G_2-structures. It has some similar properties to Bryant's flow - its stationary points are precisely torsion-free G_2-structures, and it may be interpreted

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as the gradient flow of the volume functional. However, as discovered in [20], it has a crucial deficiency in that it is non-parabolic. In that paper, the author attempted to rectify the coflow by introducing the *modified coflow*, which has an additional term that changes the sign of the term that made the coflow non-parabolic, but would still preserve the co-closed condition. However, the new flow lacks some of nicer features of the original coflow—in particular, it has additional non-torsion-free stationary points and it is not known if it can be written as a gradient flow of some functional. One of the advantages of working with co-closed G_2-structures is that they are generally more abundant than closed ones. An application of the h-principle in [11] shows that any compact manifold that admits G_2-structures will also admit co-closed G_2-structures. Therefore, it is very important to understand under which conditions it is possible to deform a co-closed G_2 -structure to a torsion-free one.

More specifically, given a co-closed G_2-structure, i.e. one where $\psi := *\varphi$ is a closed 4-form, the linearization at ψ of the corresponding Hodge Laplacian is an indefinite operator. This is in contrast to the closed case, i.e. when $d\varphi = 0$, where the linearization at φ of Δ_φ is a semi-definite operator, which can then made strongly elliptic by the addition of a Lie derivative term to take into account diffeomorphism invariance. In the co-closed case, the term that causes Δ_ψ to be indefinite is $\pi_7\left(\Delta_\psi \psi\right)$, which is the component of $\Delta_\psi \psi$ in the 7 -dimensional representation Λ^4_7 of G_2. This term is however determined by div T—the divergence of the torsion [20,24]. It should be noted that for closed G_2-structures, div T always vanishes [6], so that is why this issue doesn't arise in that case. Therefore, the condition div $T = 0$ may be thought of as another "gauge-fixing" condition to make $\Delta_\psi \psi$ elliptic. From the point of view of G_2-structure coflows, the condition div $T = 0$ makes the original and modified coflows equal to leading order. Therefore, these considerations make it very important to understand this divergence-free torsion property and in particular, under which conditions do G_2-structures with div $T = 0$ exist.

Another motivation for looking at divergence-free torsion comes from the following observation. As noted above, div T enters the Λ^4_7 part of $\Delta_\psi \psi$. However, it is known [31] that deformations of φ along Λ^3_7, and equivalently of ψ along Λ^4_7, keep the metric unchanged and simply deform the G_2-structure within a fixed metric class. Therefore, fixing div $T = 0$ essentially corresponds to taking particular a representatives of the metric class. Indeed, in an investigation of isometric G_2-structures (that is, ones that are compatible with the same metric) in [23], it was found that on a compact manifold, G_2-structures with div $T = 0$ are precisely the critical points of the L^2-norm of the torsion when restricted to a fixed metric class. In [23] this functional was also reformulated as an energy functional $\mathscr{E} = \int_M |DV|^2 \operatorname{vol}$ for unit octonion sections that parametrize isometric G_2-structures, were V is a unit octonion section and D is the octonion covariant derivative defined with respect to some fixed background G_2-structure. This allowed to rewrite the condition div $T = 0$ as a semilinear elliptic equation for octonion sections and similarly, the negative gradient flow of $\mathscr{E}$ then becomes a semilinear heat equation

$$\frac{\partial V}{\partial t} = \Delta_D V + |DV|^2 V \tag{1.1}$$

where $\Delta_D = -D^* D$ is the Laplacian operator corresponding to D.

Given that (1.1) precisely corresponds to the Λ^4_7 component of the Laplacian coflow, it is crucial to understand its properties in more detail. In particular, it is expected that at least under some conditions, it should converge to a G_2-structure with div $T = 0$. In future work, this may be used as a gauge-fixing condition that could relate the original coflow and the modified one.

It is noteworthy that this flow has remarkable similarities to the harmonic heat flow and the Yang–Mills flow. Just as these two classical flows, (1.1) appears as the gradient flow of an energy functional and in the analysis it becomes clear that many of the tools used for the harmonic heat flow and the Yang–Mills flow can be adapted in this setting as well. As such, it is another example of a flow that doesn't change the geometry of the underlying space (as opposed to the Ricci flow and aforementioned Laplacian flows of G_2-structures), but is still fundamentally related to the geometry.

In this paper, we first give a brief overview of G_2-structures and octonion bundles in Sects. 2 and 3. Then, in Sect. 4 we reintroduce the energy functional for octonions and consider some of its properties. In Sect. 5, we then work out estimates for the flow (1.1). For convenience, we introduce the quantity $\Lambda(x,t) = |DV(x,t)|^2 = \left|T^{(V)}\right|^2$, where $T^{(V)}$ is the torsion of the G_2-structure that corresponds to the octonion section V. We work out the evolution of derivatives of V and prove the following

1. If $V(t)$ is a smooth solution to (1.1) on a finite maximal time interval $[0, t_{\max})$, then for any $0 \le t < t_{\max}$,
$$\Lambda(t) \le \frac{2R_1}{\left(1 + \frac{2R_1}{\Lambda_0 + R_2}\right) e^{-4R_1 t} - 1} - R_2 \tag{1.2}$$
where $\Lambda(t) = \sup_{x \in M} \Lambda(x,t)$ and R_1, R_2 are some constants that depend on the curvature and the background G_2-structure.
2. If $V(t)$ is a smooth solution to (1.1) on a finite maximal time interval $[0, t_{\max})$, and $\Lambda(t)$ is bounded, then all derivatives of V also remain bounded.
3. As long as $\Lambda(t)$, and hence $\left|T^{(V)}\right|$, remains bounded, there will exist a smooth solution $V(t)$ to the flow (1.1).

The methods used here are similar to what Lotay and Wei [36] used for the Laplacian flow of G_2-structures, Weinkove [42] used for the Yang–Mills flow, Grayson and Hamilton [19] used for the harmonic map flow, and which originally Shi [39] introduced for the Ricci flow.

In Sect. 6, given a solution V of (1.1), we define the quantity
$$Z(t) = (t_0 - t) \int_M |DV|^2 k \,\mathrm{vol} \tag{1.3}$$
where k is a positive scalar solution of the backwards heat equation, evolving backwards in time from $t = t_0$, and in Theorem 6.1, we prove that $Z(t)$ satisfies an almost monotonicity formula. While $Z(t)$ is not strictly monotonic along the flow, it is well-behaved and can be controlled. In particular, we show that for $t \ge \tau$,
$$Z(t) \le CZ(\tau) + C(t - \tau)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \tag{1.4}$$
where C is some constant that depends on the geometry of the manifold and $\mathscr{E}_0$ is the initial value of the functional $\mathscr{E}$. This is similar to the monotonicity results obtained by Hamilton for the harmonic map heat flow and the Yang–Mills flow in [27]. Other versions of monotonicity results had been obtained for the harmonic map flow in [7,8,40] and for the Yang–Mills flow in [9,29,34].

In Sect. 7, we define the $\mathscr{F}$-functional, which essentially replaces k in Z by the heat kernel of the backwards heat equation. Applying the monotonicity formula allows us to prove an ε-regularity result for solutions of (1.1) in Theorem 7.1, which says that if $\mathscr{F} \le \varepsilon$ for some $\varepsilon > 0$, then the flow may always be smoothly extended. This then leads on to global existence of solutions for sufficiently small initial energy density $\Lambda_0 = |DV(0)|^2$. This again builds upon prior work on the harmonic map heat flow and the Yang–Mills flow. An elliptic version of ε-regularity for harmonic maps was originally introduced by Schoen and Uhlenbeck in [38], and parabolic versions were given by Struwe [40], Chen and Ding [7], Grayson and Hamilton [19]. For the Yang–Mills flow, ε-regularity results were given by Chen and Shen in [9] and Weinkove in [42].

In Sect. 8, we consider a special case when the flow takes place in the presence of a torsion-free G_2-structure, that is, the metric has holonomy contained in G_2. In that case, we can take the background G_2-structure to be torsion-free, and consider the flow (1.1) starting from an arbitrary octonion section. That particular torsion-free G_2-structure is then represented by constant octonion sections ± 1. Therefore, it makes sense to decompose the unit octonion $V(t)$ into real and imaginary parts and then the evolution of the real part $f(t) = Re\, V(t)$ is of particular interest. Indeed, we find that f^2 satisfies the Minimum Principle, so that pointwise, it is bounded below by its infimum at $t = 0$, and moreover, if initially $\inf |f(0)| > 0$, then the L^1-norm of f grows monotonically along the flow, with the time derivative bounded below by a constant multiple of $\mathscr{E}$. This is significant because clearly $|f(t)|$ is bounded above by 1, and hence if the flow exists for all $t \geq 0$ (such as under the condition from Sect. 7), then it must reach a torsion-free G_2-structure, i.e. a global minimum of $\mathscr{E}$. This is again very similar to the behavior of the harmonic map heat flow, where if the flow satisfies a small initial energy condition and the initial map is homotopic to a constant map, then the heat flow will converge to a constant map [7].

Note that two days after the initial version of this paper appeared on arXiv, a paper by Dwivedi, Gianniotis, and Karigiannis [14] that has a substantial but independent overlap with this paper has also been posted. However, while a number of conclusions and techniques are similar, the points of view on the flow (5.1) are different. In this paper, we regard this as a flow of octonion sections, while in [14] a more traditional geometric flows approach is used. Both approaches are valuable and complementary and provide different perspectives on the same phenomenon. Since the appearance of the initial versions of these two papers, there has been a very useful cross-pollination of ideas and in the final version of the present paper in some instances we allude to [14] for additional clarity and completeness.

Even more recently, a preprint by Loubeau and Sá Earp [37] appeared, where a similar flow is studied but from yet another point of view. In [37], a more general concept of a harmonic geometric structure is defined, which in the G_2 case reduces to critical points of the functional $\mathscr{E}$—that is, G_2 -structures with divergence-free torsion. Similarly, the harmonic flow of geometric structures then reduces to the flow (1.1) in the G_2 case.

2 G_2-structures

The 14-dimensional group G_2 is the smallest of the five exceptional Lie groups and is closely related to the octonions, which is the noncommutative, nonassociative, 8-dimensional normed division algebra. In particular, G_2 can be defined as the automorphism group of the octonion algebra. Given the octonion algebra $\mathbb{O}$, there exists a unique orthogonal decomposition into a real part, that is isomorphic to $\mathbb{R}$, and an *imaginary* (or *pure*) part, that is isomorphic to $\mathbb{R}^7$

$$\mathbb{O} \cong \mathbb{R} \oplus \mathbb{R}^7 \tag{2.1}$$

Correspondingly, given an octonion $a \in \mathbb{O}$, we can uniquely write

$$a = Re\, a + Im\, a$$

where $Re\, a \in \mathbb{R}$, and $Im\, a \in \mathbb{R}^7$. We can now use octonion multiplication to define a vector cross product $\times$ on $\mathbb{R}^7$. Given $u, v \in \mathbb{R}^7$, we regard them as octonions in $Im\, \mathbb{O}$, multiply them together using octonion multiplication, and then project the result to $Im\, \mathbb{O}$ to obtain a new vector in $\mathbb{R}^7$

$$u \times v = Im\,(uv)\,. \tag{2.2}$$

The subgroup of $GL\,(7, \mathbb{R})$ that preserves this vector cross product is then precisely the group G_2. A detailed account of the properties of the octonions and their relationship to exceptional Lie groups is given by John Baez in [2]. The structure constants of the vector cross product define a 3-form on $\mathbb{R}^7$, hence G_2 is alternatively defined as the subgroup of $GL\,(7, \mathbb{R})$ that preserves a particular 3-form φ_0 [30].

Definition 2.1 Let $\left(e^1, e^2, \ldots, e^7\right)$ be the standard basis for $\left(\mathbb{R}^7\right)^*$, and denote $e^i \wedge e^j \wedge e^k$ by e^{ijk}. Then define φ_0 to be the 3-form on $\mathbb{R}^7$ given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2.3}$$

Then G_2 is defined as the subgroup of $GL\,(7, \mathbb{R})$ that preserves φ_0.

In general, given a n-dimensional manifold M, a G-structure on M for some Lie subgroup G of $GL\,(n, \mathbb{R})$ is a reduction of the frame bundle F over M to a principal subbundle P with fibre G. A G_2-structure is then a reduction of the frame bundle on a 7 -dimensional manifold M to a G_2-principal subbundle. The obstructions for the existence of a G_2-structure are purely topological. It well-known [16–18] that a manifold admits a G_2-structure if and only if the Stiefel-Whitney classes w_1 and w_2 both vanish.

It turns out that there is a 1–1 correspondence between G_2 -structures on a 7-manifold and smooth 3-forms φ for which the 7-form-valued bilinear form B_φ as defined by (2.4) is positive definite (for more details, see [5] and the arXiv version of [28]).

$$B_\varphi\,(u, v) = \frac{1}{6}\,(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \tag{2.4}$$

Here the symbol $\lrcorner$ denotes contraction of a vector with the differential form, which can be written in local coordinates as

$$(u \lrcorner \varphi)_{mn} = u^a \varphi_{amn} \tag{2.5}$$

where we have also used the Einstein summation convention, which we will be using henceforth whenever dealing with expressions in local coordinates.

A smooth 3-form φ is said to be *positive* if B_φ is the tensor product of a positive-definite bilinear form and a nowhere-vanishing 7-form. In this case, it defines a unique Riemannian metric g_φ and volume form vol_φ such that for vectors u and v, the following holds

$$g_\varphi\,(u, v)\,\mathrm{vol}_\varphi = \frac{1}{6}\,(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \tag{2.6}$$

An equivalent way of defining a positive 3-form φ, is to say that at every point, φ is in the $GL\,(7, \mathbb{R})$-orbit of φ_0. It can be easily checked that the metric (2.6) for $\varphi = \varphi_0$ is in fact precisely the standard Euclidean metric g_0 on $\mathbb{R}^7$. Therefore, every φ that is in the $GL\,(7, \mathbb{R})$-orbit of φ_0 has an *associated* Riemannian metric g that is in the $GL\,(7, \mathbb{R})$-orbit of g_0. The only difference is that the stabilizer of g_0 (along with orientation) in this orbit is the group $SO\,(7)$, whereas the stabilizer of φ_0 is $G_2 \subset SO\,(7)$. This shows that positive 3-forms forms that correspond to the same metric, i.e., are *isometric*, are parametrized by $SO\,(7)\,/G_2 \cong \mathbb{RP}^7 \cong S^7/\mathbb{Z}_2$. Therefore, on a Riemannian manifold, metric-compatible

G_2-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an S^7-bundle, with antipodal points identified.

The *intrinsic torsion* of a G_2-structure is defined by $\nabla\varphi$, where ∇ is the Levi-Civita connection for the metric g that is defined by φ. Following [32], we have

$$\nabla_a\varphi_{bcd} = 2T_a{}^e\psi_{ebcd} \tag{2.7a}$$
$$\nabla_a\psi_{bcde} = -8T_{a[b}\varphi_{cde]} \tag{2.7b}$$

where T_{ab} is the *full torsion tensor*, note that an additional factor of 2 is for convenience, and $\psi = *\varphi$ is the 4-form that is the Hodge dual of φ with respect to the metric g. In general we can split T_{ab} according to irreducible representations **1**, **7**, **14**, and **27** of G_2 into *torsion components*

$$2T = \frac{1}{4}\tau_0 g - \tau_1 \lrcorner \varphi + \frac{1}{2}\tau_2 - \frac{1}{3}\tau_3 \tag{2.8}$$

where τ_0 is a function, and gives the **1** component of T. We also have τ_1, which is a 1-form and hence gives the **7** component, τ_2 is a 2-form in the **14** representation, and τ_3 is a traceless symmetric 2-tensor, giving the **27** component. As shown by Karigiannis in [32], the torsion components τ_i relate directly to the expression for $d\varphi$ and $d\psi$. In fact, in our notation,

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\mathrm{i}_\varphi(\tau_3) \tag{2.9a}$$
$$d\psi = 4\tau_1 \wedge \psi + *\tau_2. \tag{2.9b}$$

Here i_φ is a map that takes symmetric 2-tensors to 3 -forms and given a decomposable 2-tensor $\alpha \otimes \alpha$, where α is a 1-form,

$$\mathrm{i}_\varphi(\alpha \otimes \alpha) = \frac{1}{3}\alpha \wedge (\alpha \lrcorner \varphi).$$

Note that in [20–22,25] a different convention for the torsion is used is used: τ_1 in that convention corresponds to $\frac{1}{4}\tau_0$ here, τ_7 corresponds to $-\tau_1$ here, $\mathrm{i}_\varphi(\tau_{27})$ corresponds to $-\frac{1}{3}\tau_3$, and τ_{14} corresponds to $\frac{1}{2}\tau_2$. The notation used here is widely used elsewhere in the literature.

An important special case is when the G_2-structure is torsion-free, that is, $T = 0$. This is equivalent to $\nabla\varphi = 0$, and hence torsion-free G_2-structures are also called parallel G_2-structures. Also, by Fernández and Gray [16], this condition is also equivalent to $d\varphi = d\psi = 0$. Moreover, a G_2-structure is torsion-free if and only if the holonomy of the corresponding metric is contained in G_2 [30]. On a compact manifold, the holonomy group is then precisely equal to G_2 if and only if the fundamental group π_1 is finite. If $d\varphi = 0$, then we say φ defines a *closed* G_2-structure. In that case, $\tau_0 = \tau_1 = \tau_3 = 0$ and only τ_2 is in general non-zero. In this case, $T = \frac{1}{4}\tau_2$ and is hence skew-symmetric. If instead, $d\psi = 0$, then we say that we have a *co-closed* G_2-structure. In this case, τ_1 and τ_2 vanish in (2.9b) and we are left with τ_0 and τ_3 components. In particular, the torsion tensor T_{ab} is now symmetric. There are of course other, intermediate, torsion classes. For example, if τ_1 is the only non-zero torsion component, the G_2 structure is said to be locally conformally parallel, since it is known [10,21] that a conformal transformation can at least locally give a parallel G_2-structure. If τ_1 is exact, then a suitable conformal transformation gives a global parallel G_2-structure.

3 Octonion bundle

In [23], the author defined the octonion bundle on a manifold with a G_2 -structure.

Definition 3.1 Let M be a smooth 7-dimensional manifold with a G_2-structure (φ, g). The *octonion bundle* $\mathbb{O}M \cong \Lambda^0 \oplus TM$ on M is a rank 8 real vector bundle equipped with an octonion product of sections given by

$$A \circ_\varphi B = \begin{pmatrix} ab - g(\alpha, \beta) \\ a\beta + b\alpha + \alpha \times_\varphi \beta \end{pmatrix} \tag{3.1}$$

for any sections $A = (a, \alpha)$ and $B = (b, \beta)$. Here we define $\times_\varphi$ by $g\left(\alpha \times_\varphi \beta, \gamma\right) = \varphi(\alpha, \beta, \gamma)$ and given $A \in \Gamma(\mathbb{O}M)$, we write $A = (Re\, A, Im\, A)$. The metric on TM is extended to $\mathbb{O}M$ to give the octonion inner product $\langle A, B\rangle = ab + g(\alpha, \beta)$.

The product (3.1) is non-associative and the associator for $\circ_\varphi$ is given by

$$\begin{aligned}[A, B, C]_\varphi &= A \circ_\varphi \left(B \circ_\varphi C\right) - \left(A \circ_\varphi B\right) \circ_\varphi C \\ &= 2\left(\psi(\cdot, \alpha, \beta, \gamma)\right)^\sharp \end{aligned} \tag{3.2}$$

where α, β, γ are the imaginary parts of A, B, C and $(\psi(\cdot, \alpha, \beta, \gamma))^\sharp$ is the vector field obtained from the 1-form $\psi(\cdot, \alpha, \beta, \gamma)$ using the metric.

Given the octonion bundle $\mathbb{O}M$ with the octonion algebra defined by the G_2-structure φ with torsion tensor T, we can extend the Levi-Civita connection ∇ to sections of $\mathbb{O}M$. Let $A = (a, \alpha) \in \Gamma(\mathbb{O}M)$, then define the covariant derivative on $\mathbb{O}M$ as

$$\nabla_X A = (\nabla_X a, \nabla_X \alpha) \tag{3.3}$$

for any $X \in \Gamma(TM)$. Then, as shown in [23]

$$\nabla_X (AB) = (\nabla_X A) \circ_\varphi B + A \circ_\varphi (\nabla_X B) - [T_X, A, B] \tag{3.4}$$

where $T_X = (0, X \lrcorner T)$. We can then define an adapted octonion covariant derivative.

Definition 3.2 Define the octonion covariant derivative D such for any $X \in \Gamma(TM)$,

$$D_X : \Gamma(\mathbb{O}M) \longrightarrow \Gamma(\mathbb{O}M)$$

given by

$$D_X A = \nabla_X A - A \circ_\varphi T_X \tag{3.5}$$

for any $A \in \Gamma(\mathbb{O}M)$. As before, $T_X = (0, X \lrcorner T) \in \Gamma(Im\, \mathbb{O}M)$.

From now on, let us suppress $\circ_\varphi$ for octonion product defined by φ. As shown in [23], D satisfies a number of useful properties. In particular, it is metric-compatible, and satisfies a partial product rule

$$D_X(AB) = (\nabla_X A) B + A(D_X B). \tag{3.6}$$

We can also see that

$$D_X 1 = -T_X. \tag{3.7}$$

For a fixed vector field X, we have $T_X = (0, X \lrcorner T) \in \Gamma(Im\, \mathbb{O}M)$, so the full torsion tensor T may now be interpreted as a 1-form with values in $Im\, \mathbb{O}M$, that is, T is a map from $\Gamma(TM)$ to $\Gamma(Im\, \mathbb{O}M)$ that takes X to T_X. So as in [23], we will regard $T \in \Omega^1(Im\, \mathbb{O}M)$.

Recall from [23], that given a unit octonion section V on $\mathbb{O}M$ we may define a modified product on $\mathbb{O}M$

$$A \circ_V B = (AV)\left(V^{-1}B\right) = AB + [A, B, V]V^{-1} \tag{3.8}$$

This product then induces a new G_2-structure that is compatible with the same metric g as φ and is given by

$$\sigma_V(\varphi) = \left(v_0^2 - |v|^2\right)\varphi - 2v_0 v \lrcorner \psi + 2v \wedge (v \lrcorner \varphi) \tag{3.9}$$

where $V = (v_0, v)$. It was explained by Bryant in [6] that all G_2-structures that are isometric to φ are given by (3.9) for some V. In particular, this also gives an explicit parametrization of G_2-structures that are compatible with g as sections of an $S^7/\mathbb{Z}_2 \cong \mathbb{RP}^7$ -bundle over M. In [23] it was shown that given two unit octonion sections U and V,

$$\sigma_U(\sigma_V(\varphi)) = \sigma_{UV}(\varphi). \tag{3.10}$$

This allows to move easily between isometric G_2-structures. Moreover, it was also shown how the torsion and hence the octonion covariant derivative D depend on the choice of V.

Theorem 3.3 ([23]) *Let M be a smooth 7-dimensional manifold with a G_2 -structure (φ, g) with torsion $T \in \Omega^1(Im\,\mathbb{O}M)$ and corresponding octonion covariant derivative D. For a unit section $V \in \Gamma(\mathbb{O}M)$, consider the G_2-structure $\sigma_V(\varphi)$. Then, the torsion $T^{(V)}$ of $\sigma_V(\varphi)$ is given by*

$$T^{(V)} = -(DV)V^{-1}. \tag{3.11}$$

Also, let $D^{(V)}$ be the octonion covariant derivative corresponding to $\sigma_V(\varphi)$. Then, for any octonion section A, we have,

$$D^{(V)}A = (D(AV))V^{-1}. \tag{3.12}$$

We will refer to a particular choice of a G_2-structure on M as a *background* G_2-structure. Namely, given a background G_2 -structure φ, we will write any other isometric G_2-structure as $\sigma_V(\varphi)$, or will just refer to it as the G_2-structure defined by the octonion section V. Similarly, the octonion derivative D will be defined relative to φ and its torsion T. From (3.10) and (3.12) we see that we can easily change the background G_2-structure.

For some tensor bundle $\mathscr{T}$ on M, define $\mathscr{T} \otimes \mathbb{O}M$ to be the bundle of octonion-valued tensors. Then we can extend D to sections of $\mathscr{T} \otimes \mathbb{O}M$, and in particular we can also define the covariant exterior derivative on sections $\Omega^p(\mathbb{O}M)$ of the bundle of octonion-valued differential forms $(\Lambda^p T^*M) \otimes \mathbb{O}M$

$$d_D : \Omega^p(\mathbb{O}M) \longrightarrow \Omega^{p+1}(\mathbb{O}M). \tag{3.13}$$

such that

$$d_D Q = d_\nabla Q - (-1)^p\, Q \overset{\circ}{\wedge} T \tag{3.14}$$

where d_∇ is the skew-symmetrized ∇ and $\overset{\circ}{\wedge}$ is a combination of exterior product and octonion product. Also define the *divergence* of a p-form P with respect to D as the $(p-1)$-form Div P, given by

$$(\mathrm{Div}\, P)_{b_2 \ldots b_p} = D_{b_1} P^{b_1}{}_{b_2 .. b_p}. \tag{3.15}$$

In [23] we found the following properties of T as a $Im\,\mathbb{O}M$-valued 1-form

Proposition 3.4 *Suppose the octonion product on $\mathbb{O}M$ is defined by the G_2-structure φ with torsion T. Then,*

$$d_D T = \frac{1}{4}(\pi_7\,\mathrm{Riem}) \tag{3.16}$$

$$\mathrm{Div}\, T = |T|^2 + \mathrm{div}\, T \tag{3.17}$$

where π_7 Riem $\in \Omega^2(Im\,\mathbb{O}M) \cong \Omega^2(TM)$*- a vector-valued 2-form given by* $(\pi_7 \operatorname{Riem})_{ab}^{\ \ c} = (\operatorname{Riem})_{abmn}\varphi^{mnc}$. *Also,* $\operatorname{div} T \in \Omega^0(Im\,\mathbb{O}M)$ *is given by* $(\operatorname{div} T)^a = \nabla^b T_b^{\ a}$ *and* $|T|^2 \in \Omega^0(Re\,\mathbb{O}M)$ *is given by* $|T|^2 = T_{ab}T^{ab}$.

In particular, using Proposition 3.4, we can now work out the commutator $[D_a, D_b]$ on octonion-valued tensors.

Lemma 3.5 *Suppose* $P \in \Gamma(\mathscr{T} \otimes \mathbb{O}M)$. *Then,*

$$D_a D_b P - D_b D_a P = \operatorname{Riem}(P)_{ab} - \frac{1}{4}P(\pi_7 \operatorname{Riem})_{ab} \tag{3.18}$$

where Riem (P) *gives the action of the Riemann curvature endomorphism on* P *regarded as a section of* $\mathscr{T} \oplus (\mathscr{T} \otimes TM)$.

Proof From the definition of D (3.5) as well as the product rule property (3.6), we have

$$\begin{aligned} D_a D_b P &= D_a(\nabla_b P - PT_b) \\ &= \nabla_a \nabla_b P - (\nabla_b P)T_a - (\nabla_a P)T_b - P(D_a T_b) \end{aligned}$$

and hence,

$$\begin{aligned} D_a D_b P - D_b D_a P &= \nabla_a \nabla_b P - \nabla_b \nabla_a P - P(D_a T_b - D_b T_a) \\ &= \operatorname{Riem}(P)_{ab} - \frac{1}{4}P(\pi_7 \operatorname{Riem})_{ab} \end{aligned}$$

where we have also used (3.16). □

For convenience, we'll denote the curvature operator by F, so that

$$F_{ab}(P) = \operatorname{Riem}(P)_{ab} - \frac{1}{4}P(\pi_7 \operatorname{Riem})_{ab} \tag{3.19}$$

Define the Laplacian operator Δ_D on $\mathbb{O}M$-valued tensors as

$$\Delta_D P = D^a D_a P \tag{3.20}$$

where $P \in \Gamma(\mathscr{T} \otimes \mathbb{O}M)$. More explicitly, this is given by

$$\begin{aligned} \Delta_D P &= D^a(D_a P) \\ &= D^a(\nabla_a P - PT_a) \\ &= D^a(\nabla_a P) - \left(\nabla^a P\right)T_a - P\left(D^a T_a\right) \\ &= \Delta P - 2(\nabla_a P)T^a - P(\operatorname{Div} T) \end{aligned} \tag{3.21}$$

For a tensor product of two $\mathbb{O}M$-valued tensors, we find

$$\begin{aligned} \Delta_D(P \otimes Q) &= D^a((\nabla_a P) \otimes Q + P \otimes (D_a Q)) \\ &= (\Delta P) \otimes Q + 2(\nabla_a P) \otimes \left(D^a Q\right) + P \otimes (\Delta_D Q) \end{aligned} \tag{3.22}$$

We will also need to know how to commute Δ_D and D.

Lemma 3.6 *Suppose* $P \in \Gamma(\mathscr{T} \otimes \mathbb{O}M)$. *Then,*

$$\begin{aligned} D_b(\Delta_D P) - \Delta_D(D_b P) = &-2\left(\operatorname{Riem}_{ab}\nabla^a\right)(P) + \frac{1}{4}\left(\nabla^a P\right)(\pi_7 \operatorname{Riem})_{ab} \\ &- \operatorname{Ric}_{bc}\nabla^c P - \operatorname{Riem}_b^{\ a}(PT_a) + \operatorname{Riem}_{ab}(P)T^a \end{aligned}$$

$$+\frac{1}{4}P\left(D^{a}\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}\right)-\left(\mathrm{div}\,\mathrm{Riem}\right)_{b}\left(P\right). \tag{3.23}$$

where Riem *is the Riemann curvature endomorphism on an appropriate tensor bundle.*

Proof Using (3.18) and (3.6) repeatedly, we have

$$\begin{aligned}D_{b}\left(\Delta_{D}P\right)&=D_{b}D^{a}D_{a}P=D^{a}D_{b}D_{a}P+F_{b}{}^{a}\left(D_{a}P\right)\\&=\Delta_{D}D_{b}P-D^{a}\left(F_{ab}\left(P\right)\right)+F_{b}{}^{a}\left(D_{a}P\right)\end{aligned}$$

More concretely,

$$\begin{aligned}D^{a}\left(F_{ab}\left(P\right)\right)&=D^{a}\left(\mathrm{Riem}_{ab}\left(P\right)-\frac{1}{4}P\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}\right)\\&=\left(\nabla^{a}\,\mathrm{Riem}_{ab}\right)\left(P\right)+\left(\mathrm{Riem}_{ab}\,\nabla^{a}\right)\left(P\right)-\mathrm{Riem}_{ab}\left(P\right)T^{a}\\&\quad-\frac{1}{4}\left(\nabla^{a}P\right)\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}-\frac{1}{4}P\left(D^{a}\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}\right)\end{aligned} \tag{3.24}$$

$$F_{b}{}^{a}\left(D_{a}P\right)=-\,\mathrm{Riem}\left(\nabla_{a}P\right)^{a}{}_{b}-\mathrm{Riem}_{b}^{\ a}\left(PT_{a}\right) \tag{3.25}$$

We also have

$$\mathrm{Riem}\left(\nabla_{a}P\right)^{a}{}_{b}=\mathrm{Ric}_{bc}\,\nabla^{c}P+\left(\mathrm{Riem}^{a}{}_{b}\,\nabla_{a}\right)P$$

where $\left(\mathrm{Riem}^{a}{}_{b}\,\nabla_{a}\right)P$ means a composition of operators ∇ and Riem, both acting on sections of the bundle $\mathscr{T}\oplus\left(\mathscr{T}\otimes TM\right)$, as opposed to $\mathrm{Riem}\left(\nabla_{a}P\right)^{a}{}_{b}$, where Riem acts on ∇P as a section of the bundle $T^{*}M\otimes\left(\mathscr{T}\oplus\left(\mathscr{T}\otimes TM\right)\right)$. Combining everything, we obtain (3.23). □

In (3.23), note that

$$\begin{aligned}D^{a}\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}&=\nabla^{a}\left(\mathrm{Riem}_{abcd}\,\varphi^{cdm}\delta_{m}\right)-\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}T^{a}\\&=\left(\mathrm{div}\,\mathrm{Riem}\right)_{bcd}\,\varphi^{cdm}\delta_{m}+2\,\mathrm{Riem}_{abcd}\,T^{ae}\psi_{e}{}^{cdm}\delta_{m}\\&\quad-\left(\pi_{7}\,\mathrm{Riem}\right)_{ab}T^{a}\end{aligned} \tag{3.26}$$

where δ is the canonical $Im\,\mathbb{O}M$-valued 1-form that gives the isomorphism from TM to $Im\,\mathbb{O}M$, so in local coordinates, for any value of the index m, δ_{m} is an imaginary octonion. We see that any terms in (3.23) that do not involve derivatives of P, either involve div Riem or a combination of Riem and T. Hence, we can schematically write

$$D\left(\Delta_{D}P\right)=\Delta_{D}\left(DP\right)+\mathrm{Riem}*DP+\left(\mathrm{div}\,\mathrm{Riem}+\mathrm{Riem}*T\right)*P \tag{3.27}$$

where $*$ denotes some contraction involving g and/or φ.

Consider $\left\langle\Delta_{D}P,P\right\rangle$

$$\begin{aligned}\left\langle\Delta_{D}P,P\right\rangle&=\left\langle D_{a}D^{a}P,P\right\rangle\\&=\nabla_{a}\left\langle D^{a}P,P\right\rangle-\left|DP\right|^{2}\\&=\nabla_{a}\left(\nabla^{a}\left|P\right|^{2}-\left\langle P,D^{a}P\right\rangle\right)-\left|DP\right|^{2}\end{aligned}$$

where we have used metric compatibility of D. Thus,

$$\mathrm{div}\left\langle DP,P\right\rangle=\frac{1}{2}\Delta\left|P\right|^{2} \tag{3.28}$$

and hence,

$$\left\langle\Delta_{D}P,P\right\rangle=\frac{1}{2}\Delta\left|P\right|^{2}-\left|DP\right|^{2}. \tag{3.29}$$

In particular, for a unit octonion section V,

$$\langle \Delta_D V, V \rangle = -|DV|^2 . \tag{3.30}$$

4 Energy functional

Given a 7-dimensional Riemannian manifold that admit G_2-structures, we have a choice of G_2 -structures that correspond to the given Riemannian metric g. As we have seen, after fixing an arbitrary G_2-structure φ in this metric class, all the other G_2-structures that are compatible with g are parametrized by unit octonion sections, up to a sign. Given a unit octonion section V, the corresponding G_2-structure $\sigma_V(\varphi)$ will have torsion $T^{(V)}$ given by $T^{(V)} = -(DV)V^{-1}$, where D is the octonion covariant derivative with respect to φ. The question is how to pick the "best" representative of this metric class. The choice of a particular G_2-structure in a fixed metric class is similar to choosing a gauge in gauge theory. Obviously, if the metric has holonomy contained in G_2, then the "best" representative should be a torsion-free G_2 -structure that corresponds to that metric. On compact manifolds, a reasonable approach would be to pick a gauge that minimizes some functional. The natural choice is the L^2-norm of the torsion. Suppose M is now compact, in [23] the author defined the functional $\mathscr{E} : \Gamma(S\mathbb{O}M) \longrightarrow \mathbb{R}$, where $S\mathbb{O}M$ is the unit sphere subbundle, by

$$\mathscr{E}(V) = \int_M \left|T^{(V)}\right|^2 \mathrm{vol} \tag{4.1}$$

$$= \int_M \left|(DV)V^{-1}\right|^2 \mathrm{vol} \tag{4.2}$$

$$= \int_M |DV|^2 \mathrm{vol}. \tag{4.3}$$

This is simply the energy functional for unit octonion sections. It should be noted that $\mathscr{E}(V)$ is independent of the choice of the background G_2-structure and thus really only depends on the G_2 -structure $\sigma_V(\varphi)$. So it may equivalently be considered as a functional on the space of G_2-structures that are compatible with the metric g. A similar energy functional for spinors has been studied by Ammann, Weiss and Witt [1], however in their case, the metric was unconstrained, and so the functional was both on spinors and metrics.

Using the properties of D, we easily obtain the critical points.

Proposition 4.1 ([23]) *The critical points of $\mathscr{E}$ satisfy*

$$\Delta_D V + |DV|^2 V = 0 \tag{4.4}$$

and equivalently

$$\mathrm{div}\, T^{(V)} = 0. \tag{4.5}$$

The condition (4.5) comes from the identity

$$\Delta_D V + |DV|^2 V = -\left(\mathrm{div}\, T^{(V)}\right) V. \tag{4.6}$$

We see from (4.5) that the critical points of $\mathscr{E}$ correspond to G_2-structures that have divergence-free torsion. This description fits very well with the interpretation of the G_2-structure torsion as a connection for a non-associative gauge theory. The condition $\mathrm{div}\, T = 0$ is then simply the analog of the Coulomb gauge. It is well-known (e.g. [12,13,41]) that in gauge theory, given some reference connection A_0, a connection $A = A_0 + a$ is said to be in the Coulomb gauge relative to A_0 if $d^*_{A_0} a = 0$ and A is gauge equivalent to A_0. Moreover, a then corresponds to critical points of the L^2-norm of $A - A_0$ within the gauge group orbit of A_0. In our situation, we have a very similar thing happening, where the Levi-Civita connection ∇ plays the role of the reference connection A_0 and T has the role of a. The divergence-free torsion condition can equivalently be written as $d^*_\nabla T = 0$.

In general, unless $DV = 0$ (and hence $T^{(V)} = 0$), critical points of $\mathscr{E}$ with $\operatorname{div} T^{(V)} = 0$ will not be local extrema of $\mathscr{E}$.

Proposition 4.2 *Suppose $V(s,t)$ is a two-parameter family of unit octonion sections, then the Hessian of $\mathscr{E}$ at a critical point is given by*

$$\frac{\partial^2 \mathscr{E}(V(s,t))}{\partial s \partial t} = 2\int_M \left(\left\langle D\dot{V}, DV' \right\rangle - |DV|^2 \left\langle \dot{V}, V' \right\rangle \right) \mathrm{vol} \tag{4.7}$$

where $\dot{V} = \frac{\partial}{\partial t} V(s,t)$ and $V' = \frac{\partial}{\partial s} V(s,t)$.

Proof To enforce the condition $|V|^2 = 1$, we may rewrite $\mathscr{E}$ as a functional on $\Gamma(\mathbb{O}M)$ with a Lagrange multiplier λ

$$\mathscr{E}(V) = \int_M \left(|DV|^2 - \lambda \left(|V|^2 - 1 \right) \right) \mathrm{vol}$$

where $\lambda = |DV|^2$ at a critical point. From [23] we know that the first variation is given by

$$\begin{aligned} \frac{\partial}{\partial t}\mathscr{E}(V(s,t)) &= \int_M \left(\frac{\partial}{\partial t} |DV(s,t)|^2 - \lambda \frac{\partial}{\partial t} |V(s,t)|^2 \right) \mathrm{vol} \\ &= 2\int_M \left(\left\langle D\frac{\partial}{\partial t} V(s,t), DV(s,t) \right\rangle - \lambda \left\langle V(s,t), \frac{\partial}{\partial t} V(s,t) \right\rangle \right) \mathrm{vol} \end{aligned}$$

and hence, the second variation is

$$\begin{aligned} \frac{\partial^2 \mathscr{E}(V(s,t))}{\partial s \partial t} &= 2\int_M \Bigg(\left\langle D\frac{\partial}{\partial t} V(s,t), D\frac{\partial}{\partial s} V(s,t) \right\rangle + \left\langle D\frac{\partial^2}{\partial s \partial t} V(s,t), DV(s,t) \right\rangle \\ &\quad - \lambda \left\langle \frac{\partial}{\partial s} V(s,t), \frac{\partial}{\partial t} V(s,t) \right\rangle - \lambda \left\langle V(s,t), \frac{\partial^2}{\partial s \partial t} V(s,t) \right\rangle \Bigg) \mathrm{vol} \\ &= 2\int_M \Bigg(\left\langle D\frac{\partial}{\partial t} V(s,t), D\frac{\partial}{\partial s} V(s,t) \right\rangle - \lambda \left\langle \frac{\partial}{\partial s} V(s,t), \frac{\partial}{\partial t} V(s,t) \right\rangle \\ &\quad - \left\langle \frac{\partial^2}{\partial s \partial t} V(s,t), \Delta_D V(s,t) + \lambda V(s,t) \right\rangle \Bigg) \mathrm{vol}, \end{aligned}$$

where we have integrated by parts. However, at a critical point $\lambda = |DV|^2$ and (4.4) is satisfied, hence the second derivative term vanishes and at a critical point obtain (4.7). □

The characterization of divergence-free torsion as corresponding to critical points of the functional $\mathscr{E}$ shows that G_2-structures with such torsion are in some sense special. On the other hand, it is quite a broad class of G_2-structures. In [23], a Dirac operator $\not{D}$ was defined on the octonion bundle. For an octonion section V, nn local coordinates it is given by $\not{D}V = \delta^a \circ_\varphi (D_a V)$, where δ is the canonical $\operatorname{Im} \mathbb{O}M$-valued 1-form as defined in Sect. 3 and $\circ_\varphi$ is the octonion product defined by the G_2-structure φ. This definition is analogous to the standard definition on spinors using Clifford multiplication. It was then shown that unit norm eigensections of $\not{D}$ are critical points of $\mathscr{E}$. These correspond to G_2-structures with torsion that have constant τ_0 and vanishing τ_1, but with arbitrary τ_2 and τ_3. An almost complementary set of G_2-structures also yields divergence-free torsion—these are locally conformally parallel G_2 -structures with $\tau_0 = \tau_2 = \tau_3 = 0$ and $\tau_1 \neq 0$. Overall, we have the following.

Theorem 4.3 *Suppose φ is a G_2-structure on a 7 -dimensional manifold, with torsion T and components of torsion $\tau_0, \tau_1, \tau_2, \tau_3$. Then,* $\operatorname{div} T = 0$ *if one of the following holds*

1. *τ_0 is constant and $\tau_1 = 0$ and arbitrary τ_2 and τ_3*
2. *$\tau_0 = \tau_2 = \tau_3 = 0$ and arbitrary τ_1*

Proof The condition 1 is proved in [23, Prop. 10.5]. For condition 2, recall from [16], that if $\tau_0 = \tau_2 = \tau_3 = 0$, then $d\tau_1 = 0$. Then, from (2.8) and (2.7a), we have

$$\begin{aligned}
(\operatorname{div} T)^b &= -\nabla_a \left(\tau_1^c \varphi^{ab}{}_c\right) = -\tau_1^c \nabla_a \varphi^{ab}{}_c \\
&= 2\tau_1^c T_{ad} \psi^{adb}{}_c = -\tau_1^c \tau_1^e \varphi_{ead} \psi^{adb}{}_c \\
&= -4\tau_1^c \tau_1^e \varphi_{ce}{}^b = 0.
\end{aligned}$$

□

5 Heat flow

In general, however, we don't know if the functional $\mathscr{E}$ has any critical points for a given metric. However, another approach, that has been successful in the study of harmonic maps and in Yang–Mills theory is to consider the negative gradient flow of $\mathscr{E}$. This gives the following initial value problem for a time-dependent unit octonion section $V(t) \in \Gamma(S\mathbb{O}M)$

$$\begin{cases} \frac{\partial V}{\partial t} = \Delta_D V + |DV|^2 V \\ V(0) = V_0, \end{cases} \tag{5.1}$$

which was introduced in [23]. Here D is defined with respect to some background G_2-structure φ with torsion T. This will be unambiguous because time-dependent G_2-structure along the flow will be denoted by $\varphi_V = \sigma_V(\varphi)$ with torsion $T^{(V)}$ and Hodge dual 4-form ψ_V. Although initially we have to make a choice of background φ, we find that the flow is actually invariant under a change of the background G_2-structure. Indeed, suppose $\tilde{\varphi} = \sigma_U(\varphi)$, for some unit octonion section U, then from (3.10),

$$\sigma_V(\varphi) = \sigma_{VU^{-1}}(\sigma_U(\varphi)) = \sigma_{VU^{-1}}(\tilde{\varphi}). \tag{5.2}$$

Moreover, from (3.12),

$$D^{(U)}\left(VU^{-1}\right) = (DV)\,U^{-1} \tag{5.3}$$

where $D^{(U)}$ is the covariant derivative defined with respect to $\tilde{\varphi}$. Now, consider the corresponding Laplacian $\Delta_{D^{(U)}}$

$$\begin{aligned}
\Delta_{D^{(U)}}\left(VU^{-1}\right) &= -\left(D^{(U)}\right)^* D^{(U)}\left(VU^{-1}\right) \\
&= -\left(D^{(U)}\right)^* \left((DV)\,U^{-1}\right) \\
&= (\Delta_D V)\,U^{-1}
\end{aligned} \tag{5.4}$$

where we have applied (5.3) twice. Hence, if we set $W = VU^{-1}$, we find that (5.1) is equivalent to

$$\begin{cases} \frac{\partial W}{\partial t} = \Delta_{D^{(U)}} W + \left|D^{(U)} W\right|^2 W \\ W(0) = V_0 U^{-1} \end{cases}. \tag{5.5}$$

Therefore, we can always change the background G_2-structure as convenient.

The flow (5.1) is clearly parabolic and by standard parabolic theory, therefore has short-time existence and uniqueness. In [3,14], this flow was reformulated explicitly in terms of the imaginary part of V and was explicitly shown to be parabolic as a PDE on vector fields.

Theorem 5.1 *There exists an $\varepsilon > 0$ such that there exists a unique solution of (5.1) on $M \times [0, \varepsilon)$.*

From (4.6), an equivalent way of writing the flow (5.1) is

$$\frac{\partial V}{\partial t} = -\left(\operatorname{div} T^{(V)}\right) V. \tag{5.6}$$

Moreover, as an evolution equation for $\varphi_V(t) = \sigma_{V(t)}(\varphi)$, it can also be rewritten as

$$\frac{\partial \varphi_V(t)}{\partial t} = 2\left(\operatorname{div} T(t)\right) \lrcorner \psi_V(t) \tag{5.7}$$

which we can obtain from the following simple lemma.

Lemma 5.2 *Suppose a one-parameter family of unit octonion sections $V(t)$ satisfies the evolution equation*

$$\frac{\partial V}{\partial t} = -QV \tag{5.8}$$

for some time-dependent sections $Q(t) \in \Gamma(\operatorname{Im} \mathbb{O}M)$. Then, the corresponding G_2-structure 3-forms $\varphi_V(t) = \sigma_{V(t)}(\varphi)$ satisfy the evolution equation

$$\frac{\partial \varphi_V(t)}{\partial t} = 2Q(t) \lrcorner \psi_V(t). \tag{5.9}$$

and the torsion $T^{(V)}$ satisfies

$$\frac{\partial T^{(V)}}{\partial t} = \nabla Q(t) + 2T^{(V)} \times_V Q(t) \tag{5.10}$$

where $\times_V$ is the cross-product defined by the G_2-structure $\varphi_V(t)$.

Proof We can extract $\frac{\partial \varphi_V(t)}{\partial t}$ by considering what happens to the modified product $\circ_{V(t)}$ (3.8). Let A and B be two fixed octonions, then

$$\begin{aligned}\frac{\partial}{\partial t}\left(A \circ_{V(t)} B\right) &= \frac{\partial}{\partial t}\left((AV)\left(\bar{V}B\right)\right) \\ &= \left(A\frac{\partial V}{\partial t}\right)\left(\bar{V}B\right) + (AV)\left(\frac{\partial \bar{V}}{\partial t}B\right)\end{aligned}$$

where we have used $V^{-1} = \bar{V}$ since V is a unit octonion. Using (5.8) and $\frac{\partial \bar{V}}{\partial t} = \bar{V}Q$, we then obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(A \circ_{V(t)} B\right) &= -\left(A(QV)\right)\left(\bar{V}B\right) + (AV)\left(\left(\bar{V}Q\right)B\right) \\ &= -\left((A \circ_V Q)V\right)\left(\bar{V}B\right) + (AV)\left(\bar{V}(Q \circ_V B)\right) \\ &= -(A \circ_V Q) \circ_V B + A \circ_V (Q \circ_V B) \\ &= [A, Q, B]_V\end{aligned}$$

where we have again used the definition (3.8) of $\circ_V$ and $[\cdot, \cdot, \cdot]_V$ is the associator with respect to $\circ_V$. Using the relationship (3.2) between the associator and ψ, we obtain (5.9).

Similarly,

$$\frac{\partial T^{(V)}}{\partial t} = -\frac{\partial\left((DV)\,V^{-1}\right)}{\partial t} = (D\,(QV))\,V^{-1} - (DV)\left(V^{-1}Q\right)$$

$$= D^{(V)}Q + T^{(V)} \circ_V Q \tag{5.11}$$

$$= \nabla Q + 2T^{(V)} \times_V Q. \tag{5.12}$$

□

By definition of the negative gradient flow, the energy functional $\mathscr{E}$ is decreasing along the flow (5.1) whenever $\operatorname{div} T \neq 0$. More precisely, $\mathscr{E}(t)$ satisfies the following equation, which follows immediately from (5.10) with $Q = \operatorname{div} T$.

Lemma 5.3 *Along the flow* (5.1)*, the functional* $\mathscr{E}$ *satisfies*

$$\frac{d\mathscr{E}}{dt} = -2\int_M \left|\operatorname{div} T^{(V)}\right|^2 \mathrm{vol} \tag{5.13a}$$

$$\frac{d^2\mathscr{E}}{dt^2} = 4\int_M \left(\left|D^{(V)}\left(\operatorname{div} T^{(V)}\right)\right|^2 - \left|T^{(V)}\right|^2\left|\operatorname{div} T^{(V)}\right|^2\right) \mathrm{vol} \tag{5.13b}$$

where we regard $\operatorname{div} T^{(V)}$ *as sections of* $\operatorname{Im}\mathbb{O}M$ *and* $T^{(V)} \in \Omega^1(\operatorname{Im}\mathbb{O}M)$. *The norm* $|\cdot|$ *is obtained by extending the metric to* $\Omega^1(\mathbb{O}M)$.

Proof Using (5.10) with $Q = \operatorname{div} T^{(V)}$, we have

$$\begin{aligned}
\frac{d\mathscr{E}}{dt} &= 2\int_M \left\langle T^{(V)}, \frac{\partial T^{(V)}}{\partial t}\right\rangle \mathrm{vol} \\
&= 2\int_M \left(\left\langle T^{(V)}, \nabla Q\right\rangle + 2\left\langle T^{(V)}, T^{(V)} \times_V Q\right\rangle\right) \mathrm{vol} \\
&= -2\int_M |Q|^2 \,\mathrm{vol}
\end{aligned}$$

where the second term in the second line vanishes by symmetry considerations and the first term is integrated by parts.

Using (5.11), and suppressing $\circ_V$, we have

$$\begin{aligned}
\frac{d^2\mathscr{E}}{dt^2} &= -4\int_M \left\langle Q, \operatorname{div}\left(\frac{\partial T^{(V)}}{\partial t}\right)\right\rangle \mathrm{vol} \\
&= -4\int_M \left\langle Q, \operatorname{div}\left(D^{(V)}Q + T^{(V)}Q\right)\right\rangle \mathrm{vol} \\
&= 4\int_M \left\langle \nabla Q, D^{(V)}Q + T^{(V)}Q\right\rangle \mathrm{vol} \\
&= 4\int_M \left\langle D^{(V)}Q + QT^{(V)}, D^{(V)}Q + T^{(V)}Q\right\rangle \mathrm{vol} \\
&= 4\int_M \left(\left|D^{(V)}Q\right|^2 + \left\langle QT^{(V)} + T^{(V)}Q, D^{(V)}Q\right\rangle\right. \\
&\qquad \left. + \left\langle QT^{(V)}, T^{(V)}Q\right\rangle\right) \mathrm{vol}\,.
\end{aligned} \tag{5.14}$$

Note that Q and $T^{(V)}$ are both imaginary octonions, so $QT^{(V)} + T^{(V)}Q$ only has a real part. On the other hand, in

$$D^{(V)}Q = \nabla Q - QT^{(V)},$$

the derivative term ∇Q is pure imaginary so the real part comes from $QT^{(V)}$. Hence

$$\left\langle QT^{(V)} + T^{(V)}Q, D^{(V)}Q\right\rangle = -\left\langle QT^{(V)} + T^{(V)}Q, QT^{(V)}\right\rangle.$$

Thus, overall,

$$\frac{d^2\mathscr{E}}{dt^2} = 4\int_M \left(\left|D^{(V)}Q\right|^2 - \left|QT^{(V)}\right|^2\right)\mathrm{vol}\,.$$

However, note that more explicitly, we can write

$$\begin{aligned}\left|QT^{(V)}\right|^2 &= g^{ab}\left\langle QT_a^{(V)}, QT_b^{(V)}\right\rangle \\ &= g^{ab}\left\langle \bar{Q}\left(QT_a^{(V)}\right), T_b^{(V)}\right\rangle = g^{ab}\left\langle \left(\bar{Q}Q\right)T_a^{(V)}, T_b^{(V)}\right\rangle \\ &= |Q|^2\left|T^{(V)}\right|^2\end{aligned}$$

and hence we obtain (5.13b). □

Remark 5.4 To work out the second derivative, we could alternatively use (5.10) to obtain

$$\begin{aligned}\frac{d^2\mathscr{E}}{dt^2} &= -4\int_M \left\langle Q, \mathrm{div}\left(\nabla Q + 2T^{(V)}\times_V Q\right)\right\rangle \mathrm{vol} \\ &= 4\int_M |\nabla Q|^2 + 2\left\langle \nabla Q, T^{(V)}\times_V Q\right\rangle \mathrm{vol}\,. \end{aligned}\tag{5.15}$$

This is then essentially the same expression that appears in Lemma 5.10 in [14].

In [14], the second derivative of $\mathscr{E}$ was estimated using the first non-zero eigenvalue of the Laplacian on vector fields as long as the pointwise norm square of the torsion was sufficiently small. From (5.13b), we can say that

$$\frac{d^2\mathscr{E}}{dt^2} \geq 4\lambda_1(V)\int_M \left|\mathrm{div}\, T^{(V)}\right|^2 \mathrm{vol}\tag{5.16}$$

where $\lambda_1(V)$ is lowest (non-zero) eigenvalue of the operator $H_V = -\Delta_{D^{(V)}} - \left|T^{(V)}\right|^2$. By compactness of M, this operator clearly has a discrete spectrum. Also, from (3.7) and (3.17), we see that $\mathrm{div}\, T^{(V)} = -H_V(1)$, and hence, $\mathrm{div}\, T^{(V)}$ is L^2-orthogonal to the kernel of H_V. The operator $-\Delta_{D^{(V)}}$ has a non-negative spectrum that is independent of V, which can be seen from the covariance property (5.4), however H_V will in general have a spectrum that depends on V, and does not have to be non-negative. On the other hand, if $\left|T^{(V)}\right|^2$ is less than first non-zero eigenvalue of $-\Delta_{D^{(V)}}$, then $\lambda_1(V)$ will be positive, and thus we obtain an analogue of the estimate from [14].

Corollary 5.5 *Let $\lambda > 0$ be first non-zero eigenvalue of the operator $-\Delta_D$. Then, whenever $\left|T^{(V)}\right|^2 = |DV|^2 \leq \frac{1}{2}\lambda$,*

$$\frac{d}{dt}\int_M \left|\mathrm{div}\, T^{(V)}\right|^2 \mathrm{vol} \leq -\lambda\int_M \left|\mathrm{div}\, T^{(V)}\right|^2 \mathrm{vol}\,.\tag{5.17}$$

We will adapt the techniques introduced by Shi for the Ricci flow [39], that were later used in [19] for the harmonic map heat flow and in [36] for the Laplacian flow of closed G_2-structures, to prove estimates for a finite time blow-up for the flow (5.1). Let us introduce the quantity

$$\Lambda(x,t) = |DV(x,t)|^2\,.\tag{5.18}$$

Of course, from (3.11), we see that $\Lambda(x,t) = \left|T^{(V)}(x,t)\right|^2$.

At every $t \in \mathbb{R}$ for which (5.1) is defined, let us also define

$$\Lambda(t) = \sup_{x \in M} \Lambda(x,t). \tag{5.19}$$

Let $\Lambda(0) = \Lambda_0$ be the maximal initial energy density, and equivalently the maximal initial pointwise norm squared of the torsion tensor $\sup_{x\in M} \left|T^{(V)}(x,0)\right|$.

Our main result in this section is the following

Theorem 5.6 *Suppose $V(t)$ is a solution to (5.1) on a finite maximal time interval $[0, t_{\max})$. Then*

$$\lim_{t \longrightarrow t_{\max}^-} \Lambda(t) = \infty \tag{5.20}$$

and moreover,

$$\Lambda(t) \geq \frac{1}{2(t_{\max} - t)} - C_0 \tag{5.21}$$

where $C_0 > 0$ depends on the curvature and torsion of the background G_2-structure.

The above theorem in particular shows that as long as $\Lambda(t)$, and equivalently $\left|T^{(V)}(x,t)\right|$, is bounded, a solution to (5.1) will exist. To prove Theorem 5.6, we will use the following strategy

1. We will work out the evolution of DV and hence $|DV|^2$ in Lemma 5.7. From the Maximum Principle, this will also give an upper bound for $\Lambda(t)$ in Theorem 5.9.
2. We will obtain the evolution of $\left|D^2V\right|^2$ and $\left|D^3V\right|^2$, and then in Theorem 5.10, by induction we will obtain bounds on $\left|D^kV\right|^2$ in terms of Λ.
3. These bounds will then be used to show that whenever $\Lambda(t)$ is finite, the flow $V(t)$ may be smoothly extended further. This will then prove (5.20).

In the estimates that follow, we will use $*$ to denote any multilinear contraction that involves g, g^{-1}, φ, ψ, and we will drop irrelevant constant factors. Sometimes we will generically use C for a constant, which may denote a different constant in different places.

Lemma 5.7 *Along the flow (5.1), $|DV|^2$ evolves as*

$$\begin{aligned}\frac{\partial\left(|DV|^2\right)}{\partial t} &= \Delta_D |DV|^2 - 2\left|D^2V\right|^2 + 2|DV|^4 - 4\,\mathrm{Riem}^{a\ \,m}_{\ b\ \,n}\left\langle\left(\nabla_a v^n\right)\delta_m, D^bV\right\rangle \\ &\quad - 2\,\mathrm{Ric}_{bc}\left\langle\nabla^bV, D^cV\right\rangle + \frac{1}{2}\left\langle\left(\nabla^aV\right)(\pi_7\,\mathrm{Riem})_{ab}, D^bV\right\rangle \\ &\quad - 2\,(\mathrm{div}\,\mathrm{Riem})^{\ \,m}_{b\ \,n}\, v^n\left\langle\delta_m, D^bV\right\rangle + \frac{1}{2}\left\langle V\,\mathrm{Div}\,(\pi_7\,\mathrm{Riem})_b, D^bV\right\rangle \\ &\quad - 2\left\langle\mathrm{Riem}_{ab}(V)\,T^a - \mathrm{Riem}_b^{\ a}(VT_a), D^bV\right\rangle\end{aligned} \tag{5.22}$$

where $v = \mathrm{Im}\,V$. Moreover, the evolution of $|DV|^2$ satisfies the following inequality

$$\frac{\partial|DV|^2}{\partial t} \leq \Delta|DV|^2 - 2\left|D^2V\right|^2 + 2\left(|DV|^4 + 2R_1|DV|^2 + R_2|DV|\right) \tag{5.23}$$

where R_1 is a constant multiple of $\sup_M |\mathrm{Riem}|$ *and R_2 is a linear combination of* $\sup_M |\mathrm{div}\,\mathrm{Riem}|$ *and* $\sup_M |T|\,|\mathrm{Riem}|$.

Proof We have V satisfying the flow

$$\frac{\partial V}{\partial t} = \Delta_D V + |DV|^2 V \tag{5.24}$$

and hence, using Lemma 3.6,

$$\begin{aligned}\frac{\partial (DV)}{\partial t} &= D\left(\frac{\partial V}{\partial t}\right) = D\left(\Delta_D V + |DV|^2 V\right) \\ &= D(\Delta_D V) + \left(\nabla |DV|^2\right) V + |DV|^2 DV \\ &= \Delta_D (DV) + \mathrm{Riem} * DV + (\mathrm{div}\,\mathrm{Riem} + \mathrm{Riem} * T) * V \\ &\quad + \left(\nabla |DV|^2\right) V + |DV|^2 DV.\end{aligned} \tag{5.25}$$

Moreover, now,

$$\begin{aligned}\frac{\partial |DV|^2}{\partial t} &= 2\left\langle \frac{\partial (DV)}{\partial t}, DV \right\rangle \\ &= 2\langle D(\Delta_D V), DV\rangle + 2|DV|^4.\end{aligned} \tag{5.26}$$

Using (3.29), we then obtain (5.22). The inequality (5.23) follows immediately using (3.27). □

Corollary 5.8 *If the background G_2-structure φ is torsion-free, then $|\nabla V|^2$ satisfies the following evolution equation*

$$\frac{\partial \left(|\nabla V|^2\right)}{\partial t} = \Delta |\nabla V|^2 - 2\left|\nabla^2 V\right|^2 + 2|\nabla V|^4 + 4\,\mathrm{Riem}(\nabla v, \nabla v) \tag{5.27}$$

where $\mathrm{Riem}(\nabla v, \nabla v) = \mathrm{Riem}_{abcd}(\nabla^a v^c)\left(\nabla^b v^d\right)$ *for* $v = \mathrm{Im}\, V$.

Proof If $T = 0$, then $D = \nabla$. Also then $\pi_7 \mathrm{Riem} = 0$, and hence $\mathrm{Ric} = 0$, and similarly $\mathrm{div}\,\mathrm{Riem} = 0$. Then, (5.27) follows immediately from (5.22). □

The expression (5.27) is similar to the evolution of the energy density of harmonic maps in [15], however we have the additional $|\nabla V|^4$ term that is quadratic in the dependent variable. As it is well-known in the theory of semilinear PDEs, such quadratic terms in general lead to blow-ups. We can however get an estimate on the maximal time for which the energy density is finite.

Theorem 5.9 *Suppose $V(t)$ is a solution to (5.1) on a finite maximal time interval $[0, t_{\max})$. Then for any $t \in [0, t_{\max})$,*

$$\Lambda(t) \leq \frac{2R_1}{\left(1 + \frac{2R_1}{\Lambda_0 + R_2}\right)e^{-4R_1 t} - 1} - R_2, \tag{5.28}$$

where R_1 is a constant multiple of $\sup_M |\mathrm{Riem}|$ *and R_2 is a linear combination of* $\sup_M |\mathrm{div}\,\mathrm{Riem}|$ *and* $\sup_M |T|\,|\mathrm{Riem}|$.

Proof From (5.23), using Young's inequality, we can say that for any $\varepsilon > 0$

$$\begin{aligned}\frac{\partial |DV|^2}{\partial t} &\leq \Delta |DV|^2 - 2\left|D^2 V\right|^2 + 2\left(|DV|^4 + 2(R_1 + \varepsilon R_2)|DV|^2 + \frac{1}{8\varepsilon}R_2\right) \\ &\leq \Delta |DV|^2 - 2\left|D^2 V\right|^2 + 2\left(\varepsilon R_2 + |DV|^2\right)^2 + 4R_1\left(\varepsilon R_2 + |DV|^2\right)\end{aligned}$$

$$- 2\varepsilon^2 R_2^2 - 4\varepsilon R_1 R_2 + \frac{1}{4\varepsilon} R_2. \tag{5.29}$$

Taking ε such that $4\varepsilon R_1 \geq \frac{1}{4\varepsilon}$, then redefining R_2 as εR_2, and using $h(x,t) = R_2 + |DV(x,t)|^2$, we get

$$\frac{\partial h}{\partial t} \leq \Delta h - 2\left|D^2 V\right|^2 + 2h^2 + 4R_1 h. \tag{5.30}$$

Note that in the torsion-free case, from (5.27), we can set $R_1 = \sup_M |\mathrm{Riem}|$. Now, h is a subsolution of the equation

$$\frac{\partial u}{\partial t} = \Delta u + 2u^2 + 4R_1 u. \tag{5.31}$$

By the Maximum Principle, $h(x,t)$ is dominated by solutions of (5.31) if $\Lambda(x,0) \leq u(x,0)$ for all x. Since for $t = 0$, $h(x,0) \leq h(0)$, we can take $u = u(t)$ with $u(0) = h(0) = \Lambda_0 + R_2$. Solving the ODE $\frac{du}{dt} = 2u^2 + 4R_1 u$ with these initial conditions, then gives us the bound (5.28). □

Given a solution V to (5.1), define $\Lambda^{(m)}(t) = \sup_{x\in M}\left(|D^m V(x,t)|^2\right)$. Then we have the following estimates.

Theorem 5.10 *For any positive integer $m \geq 2$ there exists a constant $C_m > 0$ that only depends on M and the background G_2-structure, such that if $V(t)$ is a solution to (5.1) for $t \in [0, t_0)$ with $\Lambda(t) \leq K$, with $K \geq 1$, then*

$$\Lambda^{(m)}(t) \leq C_m K^m \text{ for } t \in [0, t_0). \tag{5.32}$$

Proof Consider first the evolution of $D^2 V$. From (5.25) and (3.27), we can write schematically

$$\begin{aligned}\frac{\partial\left(D^2 V\right)}{\partial t} &= D\frac{\partial DV}{\partial t}\\ &= D\Delta_D(DV) + D^2\left(|DV|^2 V\right)\\ &\quad + D(\mathrm{Riem} * DV) + D((\mathrm{div}\,\mathrm{Riem} + \mathrm{Riem} * T) * V).\end{aligned}$$

Applying (3.27) again, we have

$$\begin{aligned}\frac{\partial\left(D^2 V\right)}{\partial t} = {} & \Delta_D\left(D^2 V\right) + \nabla^2\left(|DV|^2\right) V + 2\nabla\left(|DV|^2\right) DV + |DV|^2 D^2 V\\ & + \mathrm{Riem} * D^2 V + (\nabla\,\mathrm{Riem} + \mathrm{Riem} * T) * DV\\ & + (\nabla(\mathrm{div}\,\mathrm{Riem}) + \nabla\,\mathrm{Riem} * T + \mathrm{Riem} * T * T + \mathrm{Riem} * \nabla T) * V\end{aligned} \tag{5.33}$$

and thus,

$$\begin{aligned}\frac{\partial\left(\left|D^2 V\right|^2\right)}{\partial t} = {} & \Delta\left|D^2 V\right|^2 - 2\left|D^3 V\right|^2 + 2\left\langle\nabla^2\left(|DV|^2\right) V, D^2 V\right\rangle\\ & + 4\left\langle\nabla\left(|DV|^2\right) DV, D^2 V\right\rangle + 2|DV|^2\left|D^2 V\right|^2\\ & + \left\langle\mathrm{Riem} * D^2 V, D^2 V\right\rangle + \left\langle(\nabla\,\mathrm{Riem} + \mathrm{Riem} * T) * DV, D^2 V\right\rangle\\ & + \left\langle(\nabla(\mathrm{div}\,\mathrm{Riem}) + \nabla\,\mathrm{Riem} * T + \mathrm{Riem} * T * T + \mathrm{Riem} * \nabla T) * V, D^2 V\right\rangle.\end{aligned} \tag{5.34}$$

Also, note that

$$\langle V, D_a D_b V\rangle = \nabla_a \langle V, D_b V\rangle - \langle D_a V, D_b V\rangle = -\langle D_a V, D_b V\rangle \tag{5.35a}$$

$$\begin{aligned}\nabla_a \nabla_b |DV|^2 &= \nabla_a \left(\left\langle D_b D_c V, D^c V\right\rangle + \left\langle D_c V, D_b D^c V\right\rangle\right)\\ &= 2\left\langle D_a D_b D_c V, D^c V\right\rangle + 2\left\langle D_b D_c V, D_a D^c V\right\rangle\end{aligned} \tag{5.35b}$$

$$\nabla\left(|DV|^2\right) = 2\left\langle DV, D^2 V\right\rangle. \tag{5.35c}$$

Thus,

$$\begin{aligned}\left\langle \nabla^2\left(|DV|^2\right) V, D^2 V\right\rangle = 2\left\langle D_a D_b D_c V, D^c V\right\rangle \left\langle V, D^a D^b V\right\rangle\\ - 2\left\langle D_b D_c V, D_a D^c V\right\rangle \left\langle D^a V, D^b V\right\rangle\end{aligned}$$

and hence,

$$\begin{aligned}\left|\left\langle \nabla^2\left(|DV|^2\right) V, D^2 V\right\rangle\right| &\leq 2\left|D^3 V\right| |DV|^3 + 2\left|D^2 V\right|^2 |DV|^2\\ \left\langle \nabla\left(|DV|^2\right) DV, D^2 V\right\rangle &\leq 2|DV|^2 \left|D^2 V\right|^2.\end{aligned}$$

Overall, we then get

$$\begin{aligned}\frac{\partial\left(\left|D^2 V\right|^2\right)}{\partial t} \leq \Delta\left|D^2 V\right|^2 - 2\left|D^3 V\right|^2 + 4\left|D^3 V\right| |DV|^3 + C_1 |DV|^2 \left|D^2 V\right|^2\\ + C_2\left|D^2 V\right|^2 + C_3\left|D^2 V\right| |DV| + C_4\left|D^2 V\right|.\end{aligned} \tag{5.36}$$

Now, using Young's inequality for any $\varepsilon_1 > 1$ we have

$$\left|D^3 V\right| |DV|^3 \leq \frac{\varepsilon_1}{2}\left|D^3 V\right|^2 + \frac{1}{2\varepsilon_1}|DV|^6$$

and hence (5.36) becomes

$$\begin{aligned}\frac{\partial\left(\left|D^2 V\right|^2\right)}{\partial t} \leq \Delta\left|D^2 V\right|^2 - 2\left(1-\varepsilon_1\right)\left|D^3 V\right|^2 + C_1\left(\Lambda(x,t)+1\right)\left|D^2 V\right|^2\\ + C_2\left(\Lambda(x,t) + \Lambda(x,t)^3\right).\end{aligned}$$

Now, by hypothesis, $\Lambda(x,t) \leq K$, and $K \geq 1$, so we have

$$\frac{\partial\left(\left|D^2 V\right|^2\right)}{\partial t} \leq \Delta\left|D^2 V\right|^2 - 2\left(1-\varepsilon_1\right)\left|D^3 V\right|^2 + C_1 K\left|D^2 V\right|^2 + C_2 K^3, \tag{5.37}$$

where we assume $\varepsilon_1 < 1$. From (5.23), we also have

$$\frac{\partial \Lambda(x,t)}{\partial t} \leq \Delta\Lambda - 2\left|D^2 V\right|^2 + C_3 K^2. \tag{5.38}$$

Now let

$$h = \left(8K + \Lambda(x,t)\right)\left|D^2 V\right|^2. \tag{5.39}$$

Then,

$$\begin{aligned}\frac{\partial h}{\partial t} &= \frac{\partial \Lambda(x,t)}{\partial t}\left|D^2 V\right|^2 + \left(8K + \Lambda(x,t)\right)\frac{\partial\left(\left|D^2 V\right|^2\right)}{\partial t}\\ &\leq \left|D^2 V\right|^2 \Delta\Lambda(x,t) + \left(8K + \Lambda(x,t)\right)\Delta\left|D^2 V\right|^2\end{aligned}$$

$$-2\left|D^2V\right|^4-16\left(1-\varepsilon_1\right)K\left|D^3V\right|^2$$
$$+C_1K^2\left|D^2V\right|^2+C_2K^4 \tag{5.40}$$

for some new constants C_1 and C_2. On the other hand,

$$\Delta h=\left|D^2V\right|^2\Delta\Lambda\left(x,t\right)+\left(8K+\Lambda\left(x,t\right)\right)\Delta\left|D^2V\right|^2+2\nabla_a\Lambda\nabla^a\left|D^2V\right|^2$$

and

$$\begin{aligned}2\nabla_a\Lambda\nabla^a\left|D^2V\right|^2&\geq-2\left|\nabla\left|DV\right|^2\right|\left|\nabla\left|D^2V\right|^2\right|\\&\geq-8\left|DV\right|\left|D^3V\right|\left|D^2V\right|^2\\&\geq-16\varepsilon_2K\left|D^3V\right|^2-\frac{1}{\varepsilon_2}\left|D^2V\right|^4,\end{aligned}$$

where we have used Young's Inequality with $\varepsilon_2>0$. Thus, overall,

$$\begin{aligned}\left|D^2V\right|^2\Delta\Lambda\left(x,t\right)+\left(8K+\Lambda\left(x,t\right)\right)\Delta\left|D^2V\right|^2\leq\Delta h+16\varepsilon_2K\left|D^3V\right|^2\\+\frac{1}{\varepsilon_2}\left|D^2V\right|^4,\end{aligned} \tag{5.41}$$

and so we obtain

$$\begin{aligned}\frac{\partial h}{\partial t}&\leq\Delta h+16K\left(\varepsilon_1+\varepsilon_2-1\right)K\left|D^3V\right|^2+\left(\frac{1}{\varepsilon_2}-2\right)\left|D^2V\right|^4\\&\quad+C_1K^2\left|D^2V\right|^2+C_2K^4\\&\leq\Delta h+16K\left(\varepsilon_1+\varepsilon_2-1\right)K\left|D^3V\right|^2+\left(\frac{1}{\varepsilon_2}+\varepsilon_3-2\right)\left|D^2V\right|^4\\&\quad+\left(\frac{C_1}{4\varepsilon_2}+C_2\right)K^4,\end{aligned} \tag{5.42}$$

where we again applied Young's Inequality with some $\varepsilon_3>0$. Then, since $\left|D^2V\right|^2\leq\frac{h}{8K}$, after an appropriate choice of $\varepsilon_1,\varepsilon_2,\varepsilon_3$, we find that there exists a positive constant C such that

$$\frac{\partial h}{\partial t}\leq\Delta h-\frac{h^2}{CK^2}+CK^4. \tag{5.43}$$

Now considering solutions of the ODE

$$\frac{du}{dt}=-\frac{u^2}{CK^2}+CK^4,$$

we find that

$$u\leq CK^3.$$

Therefore, by the Maximum Principle, we also find that

$$h\leq CK^3,$$

and hence

$$\left|D^2V\right|^2\leq CK^2 \tag{5.44}$$

for some other constant $C > 1$. Note that using (5.44) we can rewrite (5.37) as

$$\frac{\partial\left(\left|D^2V\right|^2\right)}{\partial t} \leq \Delta\left|D^2V\right|^2 - 2\varepsilon_1\left|D^3V\right|^2 + C_3K^3 \tag{5.45}$$

for $\varepsilon_1 < 1$.

Now assuming bounds for the first and second derivative we will obtain a bound for $\left|D^3V\right|^2$. From (5.33) and (3.27), it is not difficult to see that

$$\begin{aligned}\frac{\partial\left(D^3V\right)}{\partial t} &= \Delta_D\left(D^3V\right) + D^3\left(|DV|^2 V\right) + \text{Riem} * D^3V \\ &\quad + R_2 * D^2V + R_2 * DV + R_0\end{aligned} \tag{5.46}$$

where R_i for $i = 0, 1, \ldots, k-1$ are some tensors that combine derivatives of Riem and T. Now, taking the inner product of (5.46) with D^3V and applying (3.29), we obtain

$$\begin{aligned}\frac{\partial\left|D^3V\right|^2}{\partial t} &\leq \Delta\left|D^3V\right|^2 - 2\left|D^4V\right|^2 + \left\langle D^3\left(|DV|^2 V\right), D^3V\right\rangle \\ &\quad + C\left|D^3V\right|^2 + C\left|D^2V\right|\left|D^3V\right| + C\left|DV\right|\left|D^3V\right| + C\left|D^3V\right|.\end{aligned} \tag{5.47}$$

Let us focus on the third term on the right-hand side of (5.47). Schematically, ignoring indices on derivatives, we have

$$\begin{aligned}D^3\left(|DV|^2 V\right) &= \left(\nabla^3|DV|^2\right)V + 3\left(\nabla^2|DV|^2\right)DV + 3\left(\nabla|DV|^2\right)D^2V \\ &\quad + |DV|^2 D^3V\end{aligned} \tag{5.48}$$

and since $D^3\langle V, V\rangle = 0$ we also have

$$\left\langle V, D^3V\right\rangle = -3\left\langle DV, D^2V\right\rangle.$$

Thus,

$$\begin{aligned}\left\langle D^3\left(|DV|^2 V\right), D^3V\right\rangle &= -3\left(\nabla^3|DV|^2\right)\left\langle DV, D^2V\right\rangle \\ &\quad + 3\left(\nabla^2|DV|^2\right)\left\langle DV, D^3V\right\rangle \\ &\quad + 3\left(\nabla|DV|^2\right)\left\langle D^2V, D^3V\right\rangle + |DV|^2\left|D^3V\right|^2,\end{aligned} \tag{5.49}$$

and applying bounds $|DV| \leq CK^{\frac{1}{2}}$, $\left|D^2V\right| \leq CK$, we, get

$$\begin{aligned}\left|\left\langle D^3\left(|DV|^2 V\right), D^3V\right\rangle\right| &\leq CK^2\left|D^4V\right| + CK\left|D^3V\right|^2 + CK^4 \\ &\leq 2\varepsilon\left|D^4V\right|^2 + CK\left|D^3V\right|^2 + CK^4,\end{aligned}$$

where we also used Young's Inequality on the first term. So overall, we have

$$\frac{\partial\left|D^3V\right|^2}{\partial t} \leq \Delta\left|D^3V\right|^2 - 2(1-\varepsilon)\left|D^4V\right|^2 + C_1K\left|D^3V\right|^2 + C_2K^4. \tag{5.50}$$

Similarly as before, let

$$h = \left(8L + \left|D^2V\right|^2\right)\left|D^3V\right|^2, \tag{5.51}$$

where L is a constant such that $K^2 \leq L \leq CK^2$ (which is possible since $C > 1$) and $\left|D^2 V\right|^2 \leq L$. Then, using (5.50) and (5.45), we find that

$$\frac{\partial h}{\partial t} \leq \Delta h - \frac{h^2}{CK^4} + CK^6, \tag{5.52}$$

and this then gives us the bound

$$\left|D^3 V\right|^2 \leq CK^3 \tag{5.53}$$

for some constant C. By induction can similarly obtain bounds (5.32) for higher derivatives. □

Proof of Theorem 5.6 Suppose a solution $V(t)$ to (5.1) exists on the finite maximal time interval $[0, t_{\max})$. We will proceed by contradiction to prove (5.20). Suppose (5.20) does not hold. This implies that there exists a constant K such that

$$\sup_{M\times[0,t_{\max})} \Lambda(x,t) \leq K. \tag{5.54}$$

We then know from (5.32) that for some constant $C_2 > 0$

$$\sup_{M\times[0,t_{\max})} \Lambda^{(2)}(x,t) \leq C_2 K^2.$$

So in particular, $\left|D^2 V\right|$ is bounded, and thus from the flow equation (5.1), there exists a constant $C > 0$ such that

$$\sup_{M\times[0,t_{\max})} \left|\frac{\partial V}{\partial t}\right| \leq C.$$

Then, for any $0 < t_1 < t_2 < t_{\max}$, we have

$$|V(t_2) - V(t_1)| \leq \int_{t_1}^{t_2} \left|\frac{\partial V}{\partial t}\right| dt \leq C(t_2 - t_1). \tag{5.55}$$

Therefore, we see that as $t \longrightarrow t_{\max}$, the octonion sections $V(t)$ converge continuously to a section $V(t_{\max})$. Clearly, this will also have unit norm. Locally, for some $0 < t < t_{\max}$ we can then write

$$V(t_{\max}) = V(t) + \int_t^{t_{\max}} \left(\Delta_D V(s) + |DV(s)|^2 V(s)\right) ds. \tag{5.56}$$

Now, by Theorem 5.10, all derivatives of V are uniformly bounded, hence all derivatives of $V(t_{\max})$ are also bounded. Thus, $V(t_{\max})$ is a smooth section and $V(t)$ converges to it uniformly in any C^m-norm as $t \longrightarrow t_{\max}$. Now we have smoothly extended the flow from $[0, t_{\max})$ to $t_{\max}$. However, using short-time existence and uniqueness of the flow, we can uniquely extend it further starting from $t = t_{\max}$ to $t = t_{\max} + \varepsilon$ for some $\varepsilon > 0$. Therefore, the flow exists on $[0, t_{\max} + \varepsilon)$ and this contradicts the maximality of $t_{\max}$. We then find that (5.54) fails, and can conclude that

$$\lim_{t\longrightarrow t_{\max}} \Lambda(t) = \infty. \tag{5.57}$$

From (5.30), we see that

$$\frac{d(\Lambda(t) + C_0)}{dt} \leq 2(\Lambda(t) + C_0)^2$$

where $C_0 = R_1 + R_2$ is a constant that depends on the curvature and the torsion of the background G_2-structure. Thus,

$$\frac{d\left(\Lambda\left(t\right)+C_0\right)^{-1}}{dt} \geq -2$$

and thus, integrating, and taking the limit as $t \longrightarrow t_{\max}$, we obtain

$$\Lambda\left(t\right) \geq \frac{1}{2\left(t_{\max}-t\right)} - C_0 \tag{5.58}$$

for all $t \in [0, t_{\max})$. Thus, we obtain (5.21). □

6 Monotonicity

In order to be able to get a better control on the flow, it is useful to find quantities that are monotonic, or otherwise well-behaved along the flow. Following Hamilton [27], let k be a positive scalar solution of the backwards heat equation

$$\frac{\partial k}{\partial t} = -\Delta k \tag{6.1}$$

for $0 \leq t \leq t_0$, with some initial condition at $t = t_0$ and evolving towards $t = 0$, such that $\int_M k \,\mathrm{vol} = 1$. Then, consider the quantity

$$Z\left(t\right) = \left(t_0 - t\right) \int_M \left|DV\right|^2 k \,\mathrm{vol}\,. \tag{6.2}$$

Theorem 6.1 *Suppose V is a solution of the flow (5.1) for $0 \leq t < t_0$ with initial energy $\mathscr{E}(0) = \mathscr{E}_0$. Then, there exists a constant $C > 0$, that only depends on the background geometry, such that for any t and τ satisfying $t_0 - 1 \leq \tau \leq t < t_0$, $Z(t)$ satisfies the following relation*

$$Z\left(t\right) \leq CZ\left(\tau\right) + C\left(t-\tau\right)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \tag{6.3}$$

Proof Differentiating $Z(t)$, we find

$$\begin{aligned}\frac{dZ}{dt} = &- \int_M \left|DV\right|^2 k \,\mathrm{vol} + \left(t_0 - t\right)\int_M \frac{\partial}{\partial t}\left(\left|DV\right|^2\right) k \,\mathrm{vol} \\ &- \left(t_0 - t\right)\int_M \left|DV\right|^2 \Delta k \,\mathrm{vol}\,.\end{aligned} \tag{6.4}$$

Consider the second term on the right-hand side of (6.4). We use the evolution equation (5.26) for $|DV|^2$ and then integrate by parts

$$\begin{aligned}\int_M \frac{\partial}{\partial t}\left(\left|DV\right|^2\right) k \,\mathrm{vol} &= 2\int_M \left(\left\langle D_i\left(\Delta_D V\right), D^i V\right\rangle + \left|DV\right|^4\right) k \,\mathrm{vol} \\ &= -2\int_M \left(\left|\Delta_D V\right|^2 - \left|DV\right|^4\right) k \,\mathrm{vol} \\ &\quad - 2\int_M \left\langle \Delta_D V, D^i V\right\rangle \nabla_i k \,\mathrm{vol}\,.\end{aligned} \tag{6.5}$$

Let us now rewrite (6.5) by completing the square $\left|\Delta_D V + |DV|^2 V + \frac{1}{k}\nabla_i k D^i V\right|^2$

$$\int_M \frac{\partial}{\partial t}\left(|DV|^2\right) k \,\mathrm{vol} = -2\int \left|\Delta_D V + |DV|^2 V + \frac{1}{k}\nabla_i k D^i V\right|^2 k \,\mathrm{vol} + 2\int_M \left\langle \Delta_D V, D^i V\right\rangle \nabla_i k \,\mathrm{vol} + 2\int_M \frac{1}{k}\left(\nabla_i k \nabla_j k\right)\left\langle D^i V, D^j V\right\rangle \mathrm{vol} \tag{6.6}$$

and finally, let us integrate the second term by parts, so that overall, we get

$$\begin{aligned}\int_M \frac{\partial}{\partial t}\left(|DV|^2\right) k \,\mathrm{vol} = &-2\int_M \left|\Delta_D V + |DV|^2 V + \frac{1}{k}\nabla_i k D^i V\right|^2 k \,\mathrm{vol} \\ &-2\int_M \left(\left\langle D^j V, D_j D^i V\right\rangle \nabla_i k + \left\langle D^j V, D^i V\right\rangle \nabla_j \nabla_i k\right) \mathrm{vol} \\ &+2\int_M \frac{1}{k}\left(\nabla_i k \nabla_j k\right)\left\langle D^i V, D^j V\right\rangle \mathrm{vol}\,. \end{aligned} \tag{6.7}$$

Now consider the third term of the right-hand side of (6.4). Integrating by parts, we get

$$\begin{aligned}\int_M |DV|^2 \Delta k \,\mathrm{vol} &= -2\int_M \left\langle D_j V, D^i D^j V\right\rangle \nabla_i k \,\mathrm{vol} \\ &= -2\int_M \left\langle D_j V, D^j D^i V + F^{ij}(V)\right\rangle \nabla_i k \,\mathrm{vol} \\ &= -2\int_M \left\langle D_j V, D^j D^i V\right\rangle \nabla_i k \,\mathrm{vol} \\ &\quad +2\int_M \left\langle D^i V, F_{ij}(V)\right\rangle \nabla^j k \,\mathrm{vol}\,. \end{aligned} \tag{6.8}$$

Combining all the terms, (6.4) becomes

$$\begin{aligned}\frac{dZ}{dt} + 2W = &-2(t_0 - t)\int_M \left[\nabla_i \nabla_j k - \frac{1}{k}\left(\nabla_i k \nabla_j k\right) + \frac{k g_{ij}}{2(t_0 - t)}\right]\left\langle D^i V, D^j V\right\rangle \mathrm{vol} \\ &-2(t_0 - t)\int_M \left\langle D^i V, F_{ij}(V)\right\rangle \nabla^j k \,\mathrm{vol}, \end{aligned} \tag{6.9}$$

where we set

$$W = (t_0 - t)\int_M \left|\Delta_D V + |DV|^2 V + \frac{1}{k}\nabla_i k D^i V\right|^2 k \,\mathrm{vol}\,. \tag{6.10}$$

Applying integration by parts to the last term in (6.9), we get

$$\begin{aligned}\frac{dZ}{dt} + 2W = &-2(t_0 - t)\int_M \left[\nabla_i \nabla_j k - \frac{1}{k}\left(\nabla_i k \nabla_j k\right) + \frac{k g_{ij}}{2(t_0 - t)}\right]\left\langle D^i V, D^j V\right\rangle \mathrm{vol} \\ &-2(t_0 - t)\int_M \left\langle D^j V, D^i\left(F_{ij}(V)\right)\right\rangle k \,\mathrm{vol} \\ &-(t_0 - t)\int_M |F(V)|^2 k \,\mathrm{vol}\,. \end{aligned} \tag{6.11}$$

For the first term on the right-hand side of (6.11) we apply Hamilton's matrix Harnack inequality from [26]: there exist constants B and C that depend only on the geometry of M

such that

$$\nabla_i \nabla_j k - \frac{1}{k}\left(\nabla_i k \nabla_j k\right) + \frac{k g_{ij}}{2\left(t_0 - t\right)} \geq -C\left(1 + k \ln\left(\frac{B}{\left(t_0 - t\right)^{\frac{7}{2}}}\right)\right) g_{ij}. \tag{6.12}$$

Note that the trace of this estimate gives the well-known Harnack estimate by Li and Yau [35].

Let us now consider the second term in (6.11). From (3.24), we know that

$$\begin{aligned} D^a\left(F_{ab}\left(V\right)\right) = &\left(\nabla^a \mathrm{Riem}_{ab}\right)\left(V\right) + \left(\mathrm{Riem}_{ab}\nabla^a\right)\left(V\right) - \mathrm{Riem}_{ab}\left(V\right) T^a \\ &- \frac{1}{4}\left(\nabla^a V\right)\left(\pi_7 \mathrm{Riem}\right)_{ab} - \frac{1}{4} V\left(D^a\left(\pi_7 \mathrm{Riem}\right)_{ab}\right) \end{aligned} \tag{6.13}$$

and therefore,

$$\left|\left\langle D^j V, D^i\left(F_{ij}\left(V\right)\right)\right\rangle\right| \leq R_1\left(\left|DV\right|^2 + \left|DV\right|\right), \tag{6.14}$$

where R_1 is a constant that depends on the curvature and torsion. Now using (6.12) and (6.14) in (6.11), and noting that $\int_M k \,\mathrm{vol} = 1$, we have

$$\frac{dZ}{dt} + 2W \leq 2\left(t_0 - t\right) R\left(\mathscr{E}\left(t\right) + 1\right) + 2R\left(1 + \ln\left(\frac{B}{\left(t_0 - t\right)^{\frac{7}{2}}}\right)\right) Z, \tag{6.15}$$

where for convenience we now take R to be the greater of R_1 and C. Let

$$q = \left(t_0 - t\right)\left(\frac{9}{2} + \ln\left(\frac{B}{\left(t_0 - t\right)^{\frac{7}{2}}}\right)\right), \tag{6.16}$$

so that

$$\frac{dq}{dt} = -\left(1 + \ln\left(\frac{B}{\left(t_0 - t\right)^{\frac{7}{2}}}\right)\right) \tag{6.17}$$

and hence,

$$\frac{d}{dt}\left(e^{2Rq} Z\right) + 2e^{2Rq} W \leq 2\left(t_0 - t\right) C e^{2Rq}\left(\mathscr{E}\left(t\right) + \mathscr{E}\left(t\right)^{\frac{1}{2}}\right) \leq 2Ce^{2Rq}\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \tag{6.18}$$

for $t_0 - 1 \leq t < t_0$. We can always take B to be large enough so that $q \geq 0$ and we can also bound e^{2Rq}. Now integrating from τ to t, we find that for any $t_0 - 1 \leq \tau \leq t < t_0$

$$\begin{aligned} Z\left(t\right) &\leq e^{2R\left(q\left(\tau\right) - q\left(t\right)\right)} Z\left(\tau\right) + 2Ce^{-2Rq\left(t\right)}\left(t - \tau\right)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \\ &\leq CZ\left(\tau\right) + C\left(t - \tau\right)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right), \end{aligned} \tag{6.19}$$

thus completing the proof. □

Remark 6.2 In [26], it is shown that in the case when $\nabla\, \mathrm{Ric} = 0$ and the sectional curvature of M is non-negative, the quantity of the left-hand side of (6.12) is actually non-negative, and in [27], this leads to the corresponding quantity Z for the harmonic map flow and the Yang–Mills flow to be monotonically decreasing along the flow. In our case, we have an additional curvature term in (6.11), which doesn't immediately give a non-positive term in this case. On the contrary, in this case, it gives a non-negative $\mathrm{Riem}\left(\nabla v, \nabla v\right)$ term. Therefore, it is not clear if there are some reasonable conditions under which $Z\left(t\right)$ is monotonically decreasing.

7 ε-regularity

In this section we will use the results on the behavior of $Z(t)$ from the previous section as well as the a priori estimates from Sect. 5, to obtain an ε-regularity result and from it, long-time existence for small initial energy density (i.e. small pointwise torsion).

Let $p_{x_0,t_0}(x,t)$ be the backward heat kernel on M, that is, the solution of the backward heat equation (6.1) for $0 \leq t \leq t_0$ that converges to a delta function at $(x,t) = (x_0,t_0)$. Then, given a time-dependent octonion section $V(x,t)$, define the $\mathscr{F}$-functional

$$\mathscr{F}(x_0,t_0,t) = (t_0 - t)\int_M |DV(x,t)|^2\, p_{x_0,t_0}(x,t)\,\mathrm{vol}(x). \tag{7.1}$$

Clearly this is just Z with a particular choice of the backward heat equation solution k. The key result in this section is the following.

Theorem 7.1 *Given $\mathscr{E}_0$, there exist $\varepsilon > 0$ and $\beta > 0$, both depending on M and β also depending on $\mathscr{E}_0$, such that if V is a solution of the flow (5.1) on $M \times [0,t_0)$ with energy bounded by $\mathscr{E}_0$, and if*

$$\mathscr{F}(x_0,t_0,t) \leq \varepsilon \tag{7.2}$$

for $t \in [t_0 - \beta, t_0)$, then V extends smoothly to $U_{x_0} \times [0,t_0]$ for some neighborhood U_{x_0} of x_0 with $|DV|$ bounded uniformly.

Before we go on to prove Theorem 7.1, here is an important corollary.

Corollary 7.2 *There exists an $\varepsilon > 0$ such that if the initial energy density $\Lambda_0 = |DV|^2$ satisfies $\Lambda_0 < \varepsilon$, then a solution V of the flow (5.1) exists for all $t \geq 0$. The limit $V_\infty = \lim_{t\longrightarrow\infty} V(t)$ corresponds to a G_2-structure with divergence-free torsion.*

Proof Suppose V is a solution of the flow (5.1) on a maximal time interval $[0,t_{\max})$ with initial energy $\mathscr{E}_0$. By Theorem 6.1, $\mathscr{F}$ satisfies the following inequality for any t and τ satisfying $t_{\max} - 1 \leq \tau \leq t < t_{\max}$ and any $x_0 \in M$

$$\mathscr{F}(x_0,t_{\max},t) \leq C\mathscr{F}(x_0,t_{\max},\tau) + C(t-\tau)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right). \tag{7.3}$$

Using standard properties of the heat kernel, for some constant C we have

$$\mathscr{F}(x_0,t_{\max},\tau) \leq \frac{C}{(t_{\max}-\tau)^{\frac{5}{2}}}\mathscr{E}(\tau)$$

If $t_{\max} \geq 1$, then set $\tau = t_{\max} - 1$, and then we get a bound on $\mathscr{F}$ in terms of $\mathscr{E}$. Otherwise, set for example $\tau = \frac{t_{\max}}{2}$, and from (5.21) we have $\frac{1}{t_{\max}} \leq 2(\Lambda_0 + C_0)$ for a constant C_0 that only depends on the background geometry, hence in this case,

$$\mathscr{F}(x_0,t_{\max},\tau) \leq C(\Lambda_0 + C_0)^{\frac{5}{2}}\mathscr{E}(\tau).$$

Now, $\mathscr{E}(\tau) \leq \mathscr{E}_0 \leq \Lambda_0 Vol(M)$, where $Vol(M)$ is the total volume of the manifold, so overall from (7.3) we obtain a bound on $\mathscr{F}(x_0,t_{\max},t)$ in terms of Λ_0. Hence, choosing Λ_0 small enough, the conditions of Theorem 7.1 are satisfied, and the solution extends smoothly to $[0,t_{\max}]$. Restarting the flow from $t = t_{\max}$, with initial energy $\mathscr{E}(t_{\max}) \leq \mathscr{E}_0$, by short-time existence we can then extend it to $[0,t_{\max}+\varepsilon)$ for some $\varepsilon > 0$, thus contradicting the maximality of $t_{\max}$.

Now we have a solution that exists for all $t > 0$, with $|DV|$ and, from Theorem 5.10, all higher derivatives bounded uniformly. This means that choosing Λ_0 sufficiently small, we can make sure that $|DV|$ is also sufficiently small, so that it satisfies the conditions of Corollary 5.5 for all time. As in [14], this then implies that $\int_M \left|\operatorname{div} T^{(V)}\right|^2 \mathrm{vol} \longrightarrow 0$ exponentially and hence, $V(t)$ converges in L^1 to a unique limit V_∞. By uniform bounds on the derivatives, the limit is then smooth and has $\operatorname{div} T^{(V)} = 0$. □

To prove Theorem 7.1, similarly as in [19] , we need to carefully understand the local behavior of solutions to the flow (5.1).

Definition 7.3 For any $x_0 \in M$ and $t_0 \in \mathbb{R}$, define a parabolic cylinder $P_r(x_0, t_0) = \bar{B}_r(x_0) \times \left[t_0 - r^2, t_0\right]$, where $\bar{B}_r(x_0)$ is a closed geodesic ball of radius r centered at x_0.

We have the following useful Lemma from [19].

Lemma 7.4 ([19, Lemma 2.1]) *Let M be a compact manifold. There exists a constant $s > 0$, and for every $\gamma < 1$, a constant C_γ, such that if h is a smooth function satisfying*

$$\frac{\partial h}{\partial t} \leq \Delta h - h^2 \tag{7.4}$$

whenever $h \geq 0$ in $P_r(x_0, t_0)$ for some $r \leq s$, then

$$h \leq C_\gamma \left(\frac{1}{r^2} + \frac{1}{t}\right) \tag{7.5}$$

on $P_{\gamma r}(x_0, t_0)$.

Lemma 7.4 can be used to modify the proof of Theorem 5.10 to give a local version on a parabolic cylinder. Define $\Lambda_{B_r(x_0)}(t) = \sup_{x \in B_r(x_0)} \Lambda(x,t)$ and $\Lambda^{(m)}_{B_r(x_0)}(t) = \sup_{x \in B_r(x_0)} \left(|D^m V(x,t)|^2\right)$.

Theorem 7.5 *There exists a constant $s > 0$ and, for any positive integer $m \geq 2$, a constant C_m, that only depend on M and the background G_2-structure, such that, if $V(t)$ is a solution to (5.1) in a parabolic cylinder $P_r(x_0, t_0)$ for $r < \min\{s, 1\}$ and satisfies $\Lambda_{B_r(x_0)} \leq K$ for $K > \frac{1}{r^2}$, then*

$$\Lambda^{(m)}_{B_{r_k}(x_0)}(t) \leq C_m K^m \text{ on } P_{r_k}(x_0, t_0), \tag{7.6}$$

where $r_k = 2^{1-k} r$.

Proof The proof is essentially the same as that of Theorem 5.10. As in [19], the main difference is that when we obtain differential inequalities (5.43) and (5.52) for h and for $\left|D^2 V\right|^2$ and $\left|D^3 V\right|^2$, respectively, we need to make further changes of variables to get these inequalities into the form (7.4). Then, rather than using the Maximum Principle directly, we need to apply Lemma 7.4. In particular, when proving the bound for $\left|D^2 V\right|^2$, we take $h = (8K + \Lambda(x,t)) \left|D^2 V\right|^2$ as in (5.39) and then as in (5.43), we obtain

$$\frac{\partial h}{\partial t} \leq \Delta h - \frac{h^2}{CK^2} + CK^4. \tag{7.7}$$

Now, let

$$\tilde{h} = \frac{h}{CK^2} - K \tag{7.8}$$

for the same constant C. Hence, we have

$$\frac{\partial \tilde{h}}{\partial t} \leq \frac{1}{CK^2}\Delta h - \frac{h^2}{C^2K^4} + K^2$$
$$\leq \Delta\tilde{h} - \tilde{h}^2 - 2K\tilde{h}$$

and therefore, when $\tilde{h} \geq 0$ we have

$$\frac{\partial \tilde{h}}{\partial t} \leq \Delta\tilde{h} - \tilde{h}^2.$$

Therefore, applying Lemma 7.4 with $\gamma = \frac{1}{2}$, we find that for some constant C, on $P_{\frac{r}{2}}(x_0, t_0)$ we have

$$\tilde{h} \leq \frac{C}{r^2} \leq CK \tag{7.9}$$

and thus

$$h \leq CK^3,$$

and for some constant C_2,

$$\Lambda^{(2)}_{B_{r_2}(x_0)}(t) \leq C_2K^2.$$

A similar argument follows for higher derivatives. □

We now need a lemma similar to Lemma 3.1 in [19].

Lemma 7.6 *There exist constants $\delta > 0$ and $\gamma \in (0, 1)$ that depend only on M and the background G_2-structure, such that, if V is a solution of the flow (5.1) in a parabolic cylinder $P_r(x_0, t_0)$ for $r \leq 1$ such that*

$$|DV(x_0, t_0)| = \frac{1}{r}$$

and

$$|DV(x, t)| \leq \frac{2}{r}$$

for all $(x, t) \in P_r(x_0, t_0)$, then for $\theta = t_0 - \gamma r^2$ we have

$$\mathscr{F}(x_0, t_0, \theta) \geq \delta. \tag{7.10}$$

Proof From (5.25), we have

$$\left|\frac{\partial}{\partial t}(DV)\right| \leq |\Delta_D(DV)| + C_1|DV| + C_2$$
$$+2\left|D^2V\right||DV| + |DV|^3. \tag{7.11}$$

□

By hypothesis, $|DV|^2$ is bounded on the parabolic cylinder $P_r(x_0, t_0)$ by $\frac{4}{r^2}$, hence by Theorem 7.5, there exist constants C_2 and C_3 such that $\left|D^2V\right|^2 \leq \frac{C_2}{r^4}$ on $P_{\frac{r}{2}}(x_0, t_0)$ and $\left|D^3V\right|^2 \leq \frac{C_3}{r^6}$ on $P_{\frac{r}{4}}(x_0, t_0)$. Therefore, from (7.11) we find that on $P_{\frac{r}{4}}(x_0, t_0)$,

$$\left|\frac{\partial}{\partial t}(DV)\right| \leq \frac{C}{r^3} \tag{7.12}$$

for some constant $C > 0$. Note that the octonionic derivative D, being metric-compatible, satisfies Kato's Inequality, so in particular we have

$$\left|D^2 V\right| \geq |\nabla |DV||$$

whenever $|DV| \neq 0$. Hence, in some neighborhood around x_0,

$$|\nabla |DV|| \leq \frac{C}{r^2} \tag{7.13}$$

for some constant $C > 0$. Overall, the time-derivative bound (7.12) and space derivative bound (7.13) show that there exists some $\gamma \in (0, 1)$ such that for all $(x, t) \in P_{\gamma r}(x_0, t_0)$,

$$|DV(x, t)| \geq \frac{1}{2r}. \tag{7.14}$$

Now, for $\theta = t_0 - \gamma r^2$, we have

$$\begin{aligned}
\mathscr{F}(x_0, t_0, \theta) &= (t_0 - \theta) \int_M |DV(x, \theta)|^2 \, p_{x_0,t_0}(x, \theta) \operatorname{vol}(x) \\
&\geq \gamma r^2 \int_{B_{\gamma r}(x_0)} |DV(x, \theta)|^2 \, p_{x_0,t_0}(x, \theta) \operatorname{vol}(x) \\
&\geq \frac{1}{4}\gamma \int_{B_{\gamma r}(x_0)} p_{x_0,t_0}(x, \theta) \operatorname{vol}(x).
\end{aligned}$$

However, from Corollary 2.3 of [26], on $P_{\gamma r}(x_0, t_0)$, we have $p_{x_0,t_0}(x, \theta) \geq \frac{c}{r^7}$ for some constant c that depends only on M . Therefore, for some $\delta > 0$, we do obtain (7.10).

Now we can proceed with the proof of Theorem 7.1.

Proof of Theorem 7.1 Suppose first that V is a solution of the flow (5.1) on $M \times [0, t_0]$. From the proof of Theorem 3.2 in [19] and from Theorem 3.1 in [26], we know that for any $\eta > 0$, any constant $C > 1$, and any $x_0 \in M$ and any $\tilde{t}_0 = t_0 - \alpha \in (0, 1]$, there exists a $\rho > 0$ such that for all $(\xi, \tau) \in P_\rho(x_0, t_0)$

$$(\tau - \alpha) \, p_{\xi,\tau}(x, t) \leq C (t_0 - \alpha) \, p_{x_0,t_0}(x, t) + \frac{\eta}{2\mathscr{E}_0}. \tag{7.15}$$

Multiplying (7.15) by $|DV|^2$, and integrating we find

$$\mathscr{F}(\xi, \tau, \alpha) \leq C\mathscr{F}(x_0, t_0, \alpha) + \frac{\eta \mathscr{E}(\alpha)}{2\mathscr{E}_0} \leq \eta \tag{7.16}$$

as long as ε in the hypothesis is chosen such that $\frac{\eta}{2} \geq C\varepsilon$. Then, similarly as in the proof of Theorem 3.2 in [19], define

$$q(x, t) = \min\left\{\rho - d(x_0, x), \sqrt{t - \left(t_0 - \rho^2\right)}\right\}.$$

In some sense this gives the shorter of the distances from (x, t) to the spatial boundary of $P_\rho(x_0, t_0)$ and the lower temporal boundary. Now, the function $q(x, t)\,|DV(x, t)|$ will attain its maximum in $P_\rho(x_0, t_0)$ at some point (ξ, τ) in the interior of $P_\rho(x_0, t_0)$, so that $\sigma = q(\xi, \tau) > 0$. Since $\sigma \leq \rho - d(x_0, \xi)$ and $\sigma^2 \leq \tau - \left(t_0 - r^2\right)$, it is easy to see that $P_\sigma(\xi, \tau) \subset P_\rho(x_0, t_0)$. Moreover, we can also see that $q(x, t) \geq \frac{\sigma}{2}$ on $P_{\frac{\sigma}{2}}(\xi, \tau)$. Now, define r such that $\frac{1}{r} = |DV(\xi, \tau)|$, then for all $(x, t) \in P_\rho(x_0, t_0)$, we have

$$|DV(x, t)| \leq \frac{\sigma}{rq(x, t)}. \tag{7.17}$$

Suppose $r \geq \frac{\sigma}{2}$. Then, since $q(x,t) \geq \frac{\rho}{2}$ on $P_{\frac{\rho}{2}}(x_0, t_0)$, we obtain a bound $|DV(x,t)| \leq \frac{4}{\rho}$ for all $(x,t) \in P_{\frac{\rho}{2}}(x_0, t_0)$. Otherwise, suppose $r \leq \frac{\sigma}{2}$. In that case, from (7.17) we find that for all $(x,t) \in P_r(\xi, \tau)$,

$$|DV(x,t)| \leq \frac{2}{r}. \tag{7.18}$$

We can now apply Lemma 7.6 to obtain a $\delta > 0$ and a $\gamma \in (0,1)$ such that for $\theta = \tau - \gamma r^2$ we have

$$\mathscr{F}(\xi, \tau, \theta) \geq \delta. \tag{7.19}$$

Now, by Theorem 6.1, we find that if $\alpha \leq \theta < \tau$,

$$\begin{aligned} \mathscr{F}(\xi,\tau,\theta) &\leq C\mathscr{F}(\xi,\tau,\alpha) + C(\theta-\alpha)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \\ &\leq C\eta + C(\theta-\alpha)\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) \end{aligned} \tag{7.20}$$

where we have used (7.16). Since $\theta - \alpha \leq t_0 - \alpha$, let us find a β such that $\beta \geq t_0 - \alpha$, and $C\beta\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{1}{2}}\right) < \frac{\delta}{2}$. Choosing $\eta < \frac{\delta}{2C}$ gives us $\mathscr{F}(\xi,\tau,\theta) < \delta$, which contradicts (7.19).

Thus we find that there exist $\varepsilon > 0$ and β (where β depends on $\mathscr{E}_0$), such that for any $(x_0, \alpha) \in M \times [t_0 - \beta, t_0)$ there exists a $\rho > 0$ and a finite B, such that if $\mathscr{F}(x_0, t_0, \alpha) \leq \varepsilon$, then $|DV| \leq B$ is bounded on $P_\rho(x_0, t_0)$. It should be noted that ρ and B only depend on $t_0 - \alpha$, rather than t_0 and α individually. Now suppose the solution V only exists on $M \times [0, t_0)$. Then, by applying the gradient bounds to the translates $\tilde{V}(x,t) = V(x, t-\zeta)$ and taking $\zeta \longrightarrow 0$, we obtain uniform bounds on $|DV|$ for $t < T$. From Theorem 7.5 we then get estimates on higher derivatives, and thus conclude that the solution extends smoothly to $t = t_0$ in some neighborhood of x_0. □

8 Heat flow in the presence of a torsion-free G_2-structure

The flow (5.1) and the octonion covariant derivatives are defined with respect to some fixed background G_2-structure that corresponds to the unit octonion $V = 1$ (or $V = -1$). In fact, due to the covariance of D with respect to change of the background G_2-structure (3.12), this choice is arbitrary. However, if we would like to understand if the flow reaches some particular G_2-structure φ within the given metric class, then we can without loss of generality set the background G_2-structure to be φ, and then all that remains to be checked is whether the flow reaches $V^2 = 1$ within the maximum time interval $[0, t_{\max})$. Equivalently, this corresponds to $v = 0$ where $v = Im\, V$. In this section we will analyze the behavior of the real and imaginary parts of V along the flow, particularly in the case when a torsion-free G_2 -structure exists in the given metric class.

Let $V = f + v$ be the decomposition of the unit octonion V into real and imaginary parts. Then, we also have $f^2 + |v|^2 = 1$. Also, suppose that the initial octonion is given by $V_0 = f_0 + v_0$. The background G_2-structure $\varphi = \sigma_1(\varphi)$ will have torsion T (which we will set to 0 shortly), the initial G_2 -structure $\varphi_0 = \sigma_{V_0}(\varphi)$ will have torsion $T_0 = -(DV_0)V_0^{-1}$, and the G_2-structure $\varphi_V = \sigma_V(\varphi)$ that corresponds to V, will have torsion $T^V = -(DV)V^{-1}$. Here D is with respect to φ.

Lemma 8.1 *The evolution of f and $|v|^2$ along the flow (5.1) is given by*

$$\frac{\partial f}{\partial t} = \Delta f + f\,|\nabla V|^2 + \langle v, \operatorname{div} T\rangle + 2\left\langle \nabla_a V, (1 - fV)\,T^a\right\rangle \tag{8.1a}$$

$$\frac{\partial\,|v|^2}{\partial t} = \Delta\,|v|^2 - 2f^2\,|\nabla V|^2 + 2\,|\nabla f|^2 - 2\,\langle v, f \operatorname{div} T\rangle + 4f\left\langle \nabla_a V, (fV - 1)\,T^a\right\rangle. \tag{8.1b}$$

Proof Taking the inner product of (5.1) with 1, we get

$$\frac{\partial f}{\partial t} = \langle \Delta_D V, 1\rangle + |DV|^2\, f. \tag{8.2}$$

However,

$$\begin{aligned}\Delta f &= \Delta\,\langle V, 1\rangle\\ &= \langle \Delta_D V, 1\rangle + 2\left\langle D_a V, D^a 1\right\rangle + \langle V, \Delta_D 1\rangle\\ &= \langle \Delta_D V, 1\rangle - 2\left\langle \nabla_a V - V T_a, T^a\right\rangle - \left\langle V, \operatorname{div} T + |T|^2\right\rangle\\ &= \langle \Delta_D V, 1\rangle - 2\left\langle \nabla_a V, T^a\right\rangle - \langle v, \operatorname{div} T\rangle + f\,|T|^2\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}|DV|^2 &= |\nabla V - VT|^2\\ &= |\nabla V|^2 - 2\left\langle \nabla_a V, VT^a\right\rangle + |T|^2.\end{aligned} \tag{8.4}$$

Thus, overall, (8.2) becomes

$$\frac{\partial f}{\partial t} = \Delta f + f\,|\nabla V|^2 + \langle v, \operatorname{div} T\rangle + 2\left\langle \nabla_a V, (1 - fV)\,T^a\right\rangle.$$

Multiplying by $2f$, we further obtain

$$\frac{\partial f^2}{\partial t} = \Delta f^2 + 2f^2\,|\nabla V|^2 - 2\,|\nabla f|^2 + 2\,\langle v, f \operatorname{div} T\rangle + 4f\left\langle \nabla_a V, (1 - fV)\,T^a\right\rangle. \tag{8.5}$$

Since $|v|^2 = 1 - f^2$, $\frac{\partial |v|^2}{\partial t} = -\frac{\partial f^2}{\partial t}$, and hence we then get (8.1b). □

Lemma 8.2 *Let $u = f^2 - |v|^2 = 2f^2 - 1$, and suppose $T = 0$. Then, along the flow (5.1), u satisfies the inequality*

$$\frac{\partial u}{\partial t} \geq \Delta u + \left(\frac{u}{1 - u^2}\right)|\nabla u|^2 \tag{8.6}$$

Proof From (8.5), setting $T = 0$, we have

$$\frac{\partial u}{\partial t} = \Delta u + 2\,(u + 1)\,|\nabla v|^2 + 2\,(u - 1)\,|\nabla f|^2. \tag{8.7}$$

Assume first that $u^2 \neq 1$, so that $f \neq 0$ and $v \neq 0$. Since $\nabla u = 4f\nabla f$, we find

$$|\nabla f|^2 = \frac{1}{16f^2}\,|\nabla u|^2 = \frac{1}{8\,(u + 1)}\,|\nabla u|^2. \tag{8.8}$$

With this, (8.7) becomes

$$\frac{\partial u}{\partial t} = \Delta u + 2\,(u+1)\,|\nabla v|^2 + \frac{1}{4}\frac{u-1}{u+1}\,|\nabla u|^2\,. \tag{8.9}$$

From Kato's inequality,

$$|\nabla v|^2 \geq |(\nabla\,|v|)|^2 = \frac{1}{4\,|v|^2}\left|\left(\nabla\left(f^2\right)\right)\right|^2 = \frac{1}{8\,(1-u)}\,|\nabla u|^2\,,$$

using which, (8.9) becomes

$$\frac{\partial u}{\partial t} \geq \Delta u + \left(\frac{u}{1-u^2}\right)|\nabla u|^2\,. \tag{8.10}$$

It should be noted that generally, Kato's inequality holds whenever $v \neq 0$, however in our case, when $v=0$, $\frac{\partial u}{\partial t}=0$ since $u = 1-2\,|v|^2$. However, (8.7) becomes

$$\frac{\partial u}{\partial t} = \Delta u + 4\,|\nabla v|^2 = -4\,|(\nabla\,|v|)|^2 + 4\,|\nabla v|^2 = 0. \tag{8.11}$$

Hence $|\nabla v|^2 = |(\nabla\,|v|)|^2$ and thus the inequality still holds. Now suppose $u=-1$, so that $f=0$ and hence $|v|=1$. Then, (8.7) becomes

$$\frac{\partial u}{\partial t} = \Delta u - 4\,|\nabla f|^2 = 0.$$

On the other hand, in (8.9), as $u \longrightarrow -1$, $\frac{u}{1-u^2}\,|\nabla u|^2 \longrightarrow -4\,|\nabla f|^2$, so

$$0 = \frac{\partial u}{\partial t} \geq -\frac{\varepsilon_2}{2}\,|\nabla u|^2$$

which is of course true. We conclude that (8.10) holds everywhere. □

To be able to apply the Maximum Principle to (8.6) we need to rewrite it in a different form. This will then allow us to obtain lower bounds on u, and hence f.

Lemma 8.3 *Suppose $T=0$, then along the flow (5.1), $f\,(t)^2$ is bounded by*

$$\inf_M\,[f\,(t,x)]^2 \geq \inf_M\,[f\,(0,x)]^2 \tag{8.12}$$

as long as the flow exists.

Proof In (8.6), let $u = \sin\theta$ for some function θ such that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Then,

$$\begin{aligned}\nabla u &= (\cos\theta)\,\nabla\theta\\ \Delta u &= -\,(\sin\theta)\,|\nabla\theta|^2 + (\cos\theta)\,\Delta\theta\end{aligned}$$

and hence we can rewrite (8.6) as

$$(\cos\theta)\,\frac{\partial\theta}{\partial t} \geq (\cos\theta)\,\Delta\theta$$

Overall, for $-1 < u < 1$, and hence $\cos\theta > 0$, the inequality (8.6) becomes

$$\frac{\partial\theta}{\partial t} \geq \Delta\theta. \tag{8.13}$$

Hence, by the Maximum Principle, we conclude that if $\inf_M [f(0)]^2 > 0$, then, as long as the flow exists,

$$\inf_M \theta(t,x) \geq \inf_M \theta(0,x) \tag{8.14}$$

and thus, in any case, $\inf_M [f(t,x)]^2 \geq \inf_M f[(0,x)]^2$. □

Thus, we have shown that in the presence of a torsion-free G_2 -structure, f^2 is bounded below by its initial value and hence, $|v|^2$ is bounded above by its initial value. This shows that pointwise, $V(t)$ never gets further away from the torsion-free G_2-structure than at the initial point. We can do even better though. It turns out that if initially, f is nowhere zero, then the integral of $|f(t)|$ increases monotonically along the flow as long as V is not parallel.

For convenience, we define a new functional

$$\mathscr{G}(t) := \int_M |f(t)| \operatorname{vol} \tag{8.15}$$

which is just the L^1-norm of f at time t. Recall that $\mathscr{E}(t)$ in this case is just the L^2-norm of ∇V.

Lemma 8.4 *Suppose $T = 0$, and $k := \inf_M |f(0,x)| > 0$, then along the flow (5.1),*

$$\frac{\partial \mathscr{G}(t)}{\partial t} \geq k\mathscr{E}(t). \tag{8.16}$$

Proof Recall from (8.1b) that along the flow (5.1), for $T = 0$,

$$\frac{\partial f}{\partial t} = \Delta f + f|\nabla V|^2 \tag{8.17}$$

We know from Lemma 8.3 that in this case, $\inf_M f^2(t,x) \geq \inf_M f^2(0,x) > 0$, so we can rewrite (8.17) as

$$\frac{\partial |f|}{\partial t} = \Delta |f| + |f||\nabla V|^2, \tag{8.18}$$

since f is never zero along the flow. Integrating over M, we get

$$\frac{\partial \mathscr{G}(t)}{\partial t} = \int_M |f||\nabla V|^2 \operatorname{vol} \geq \inf_M |f(t)|\mathscr{E}(t)$$

and hence we get (8.16). □

Remark 8.5 Lemma 8.4 shows that as long as initially $f(t)$ is nowhere zero (and equivalently $\inf_M |f(0,x)| > 0$), its L^1 norm is increasing monotonically as long as $\mathscr{E}(t) \neq 0$. Of course, $|f| \leq 1$, and so $\mathscr{G}(t) \leq Vol(M)$. Recall from Lemma 5.3 that $\mathscr{E}(t)$ is decreasing monotonically, with stationary points corresponding to divergence-free G_2-structures. In particular, if the flow reaches a stationary point with $\operatorname{div} T^{(V)} = 0$, but $\mathscr{E}(t) > 0$, $\mathscr{G}(t)$ will still increase. On the other hand, suppose M has a parallel vector field. An octonion section which has this vector as the imaginary part and has a constant real part will then also define a torsion-free G_2-structure. So if the flow reaches this section, at that point $\mathscr{E}(t)$ will vanish, so it is possible that $|f| = 1$ will never be reached in that case, even though a torsion-free G_2-structure has been reached. If on the other hand, the flow exists for all $t \geq 0$, then we see that it will have to converge to a torsion-free G_2-structure, again not necessarily the one defined by $|f| = 1$.

Combining Corollary 7.2 and Lemma 8.4, we obtain the following theorem. If we assume that a torsion-free G_2-structure is given by $U \in \Gamma(S\mathbb{O}M)$, rather than by the section 1, then the condition $\inf_M |f(0,x)| > 0$ is replaced by the condition that initially $|\langle V(0,x), U\rangle| > 0$.

Theorem 8.6 *Suppose (φ, g) is a G_2 -structure on a compact 7-dimensional manifold M. Suppose there exists a unit octonion section U such that $\sigma_U(\varphi)$ is torsion free. Then there exists $\varepsilon > 0$, such that if the flow (5.1) has initial energy density $\Lambda_0 < \varepsilon$ and the initial octonion section V_0 satisfies $|\langle V_0, U\rangle| > 0$ on M, then a solution exists for all $t \geq 0$, and as $t \longrightarrow \infty$, $V(t) \longrightarrow V_\infty$ where V_∞ defines a torsion-free G_2-structure. If (M, g) admits no parallel vector fields, then $V_\infty = U$.*

Proof From Corollary 7.2 we already know that a solution $V(t)$ will exist for all $t \geq 0$. Recall that we may switch over to the torsion-free G_2-structure $\sigma_U(\varphi)$ as our background G_2-structure. In particular, from (3.10), $\sigma_{V(t)}(\varphi) = \sigma_{V(t)U^{-1}}(\sigma_U(\varphi))$. Hence we can now consider $\tilde{V}(t) = V(t)U^{-1}$. Let us write $\tilde{V}(t) = f(t) + v(t)$. Now the condition $|\langle V_0, U\rangle| > 0$ is equivalent to $|f(0)| > 0$. Thus, from Lemma 8.4 we know that $\mathscr{G}(t)$ is growing monotonically along the flow, however $\mathscr{G}(t)$ is also bounded above by $Vol(M)$, hence it must converge to some $\mathscr{G}_\infty \leq Vol(M)$, and in particular this shows that $\mathscr{E}(t) \longrightarrow 0$. Hence, the limit $V_\infty = \lim_{t\longrightarrow\infty} V(t)$ must define a torsion-free G_2-structure.

If the background G_2-structure is torsion-free, then the torsion of the G_2-structure defined by a unit octonion V will be given by $T^{(V)} = -(\nabla V)V^{-1}$. Hence torsion-free G_2 -structures in the same metric class are given by unit octonion sections V for which $\nabla V = 0$. In particular, the imaginary part of V is then parallel vector field. So any torsion-free G_2-structures apart from the background G_2-structure are defined by parallel vector fields. Hence, if there are no parallel vector fields on M, then the torsion-free G_2 -structure that is compatible with g is unique, and thus $V_\infty = U$. □

9 Concluding remarks

The results in this paper are just the beginning of the study of the heat flow of isometric G_2-structures as well its stationary points: G_2-structures with divergence-free torsion. In the study of the harmonic map heat flow and the Yang–Mills flow, results such as monotonicity formulas and ε-regularity led to a rich study of singularities and solitons of these flows. Clearly, this should also be possible in our setting, with the interesting added challenge of interpreting this in terms of the geometry of G_2-structures. Other related concepts such as entropy, that have been defined in the harmonic map and Yang–Mills cases [4,34] also have an analog and interpretation in our case. Some progress in this direction has been already made in the recent paper [14]. Another possible direction is to consider the flow in some particular simpler settings, such as warped product manifolds with $SU(3)$ -structure that have been considered as models for the Laplacian coflow [22,33], in which case the octonion section should reduce to a unit complex number, or even with $SU(2)$ -structure, in which case the octonion section may reduce to a quaternion section. Understanding the behavior of the flow in such special settings may inform further directions of study.

One property of the flow (5.1) that hasn't been fully used yet is the gauge-invariance, i.e. invariance of the flow under the change of the background G_2-structure, as discussed in Sect. 5. We used this in Sect. 8 to more conveniently describe the behavior of the flow in the presence of a torsion-free G_2-structure. In [14], similar ideas were used to show an

"Uhlenbeck-type trick", using which the evolution of the torsion had a more tractable form. It is however likely that this gauge-invariance can lead to a better understading of the flow.

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