Rapid Communications   Editors' Suggestion

Topology and observables of the non-Hermitian Chern insulator

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Topology now plays a central role in physics, yet its applications have so far been restricted to closed, lossless systems in thermodynamic equilibrium. Given that many physical systems are open and may include gain and loss mechanisms, there is an eminent need to reexamine topology within the context of non-Hermitian theories that describe open, lossy systems. The generalization of the Chern number to non-Hermitian Hamiltonians initiated this reexamination; however, there is no established connection between a non-Hermitian topological invariant and the quantization of an observable. Using field-theoretical techniques, we show that no such relationship exists between the non-Hermitian Chern number and the Hall conductivity, a consequence of the discontinuous nature of Green's functions of non-Hermitian Hamiltonians. Furthermore, we derive an exact formula for the Chern-Simons Hall response of a generic two-level non-Hermitian Hamiltonian and present an illustrative calculation for a non-Hermitian massive Dirac Hamiltonian in $(2+1)$ dimensions. We conclude by clarifying how these results extend to higher-dimensional systems and detailing their implications for recent experiments.

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The topological classification of matter represents a significant enhancement in our understanding of the physical properties of a great variety of systems, both classical [1–3] and quantum mechanical in nature [4,5]. Of central importance within the topological classification of matter is the identification of topological invariants, which are quantities that remain unchanged in the presence of symmetry-allowed perturbations [6–9]. While the topological classification of matter has enjoyed much success, its achievements have to date been limited to idealized closed systems, as described by conventional Hermitian Hamiltonians. Nonetheless, most physical systems are more aptly described as open, defined by a connection to large reservoirs of additional states. Proper theoretical descriptions of open systems must include mechanisms of both loss and gain that account for the flow of energy and particles between the system and additional reservoirs [10–12]. The inclusion of gain and loss mechanisms necessitates a non-Hermitian Hamiltonian, whose complex eigenvalues induce finite quasiparticle lifetimes. Non-Hermitian Hamiltonians permit many topological phenomena that are discordant with their Hermitian counterparts including exceptional points, lines, and surfaces at which two eigenvectors merge into one [13–19], unidirectional optical transport [20,21], bulk Fermi arcs [22], expanded topological classifications [23–26], and a modified bulk-boundary correspondence [15,27–36].

The breakdown of the conventional bulk-boundary correspondence in non-Hermitian topological Hamiltonians calls for the reexamination of other predictions of topology in non-Hermitian systems. One of the sacrosanct tenants of topological physics is the connection between topological invariants and quantized observables. Within the context of gapped Hermitian Hamiltonians, the Chern number of the energy bands is equivalent to the number of chiral edge states, as required by the bulk-boundary correspondence [6,8,9]. The connection between the number of edge states and the Chern number, in turn, leads to a Hall conductivity quantized in units of e^2/h [37]. The Chern number thus provides both a mathematical classification of the Hamiltonian and a physical characterization of the resultant phase.

In this Rapid Communication, we demonstrate that the intimate link between the Hall conductivity and the Chern number no longer holds in a non-Hermitian Chern insulator. Specifically, we show that the Chern-Simons response coefficient in the effective action of a non-Hermitian Chern insulator is not quantized, despite the quantization of the Chern number, as a result of the discontinuous nature of the Green's functions of non-Hermitian Hamiltonians. Importantly, we further show why the nonquantization of the response is not contradictory to the quantization of the Chern number. This is our central result. Additionally, we derive an exact expression for the nonquantized Chern-Simons (CS) Hall response of a generic non-Hermitian two-level system. As a concrete demonstration of the disconnect between topology and observable, we calculate the Hall conductivity of a non-Hermitian massive Dirac Hamiltonian in $(2+1)$ dimensions [$(2+1)$D], which has a nonzero Chern number.

We begin by examining the Hall conductivity of a gapped, translationally invariant Hermitian system in $(2+1)$D. As

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calculated via the Kubo formula, the Hall conductivity is

$$\sigma_{xy} = \frac{ie^2\hbar}{V}\sum_{m,n}(f_m - f_n)\frac{\langle m|\hat{v}_x|n\rangle\langle n|\hat{v}_y|m\rangle}{(\epsilon_m-\epsilon_n)^2}, \tag{1}$$

where V is the volume of the system, $f_i = f(\epsilon_i)$ is the Fermi-Dirac distribution function, $\hat{v}_i = \frac{1}{\hbar}d\hat{H}/dk_i$ are the velocity operators, ϵ_n and $|n\rangle$ are the energies and eigenstates of the Hamiltonian $\hat{H}$, and m, n index the eigenstates of $\hat{H}$. For a gapped Hamiltonian, Eq. (1) may be recast in terms of the Chern number n, an integral of the Berry curvature over the Brillouin zone [37],

$$n = \frac{i}{2\pi}\sum_{q\in\mathrm{occ}}\int_{\mathrm{BZ}}\epsilon_{ij}\langle\partial_i\Psi_q(\mathbf{k})|\partial_j\Psi_q(\mathbf{k})\rangle d^2\mathbf{k}, \tag{2}$$

where q indexes the occupied bands. The Hall conductivity is proportional to the Chern number, which is an integer topological invariant of the bands, and is thus quantized as $\sigma_{xy} = ne^2/h$, where $n \in \mathbb{Z}$.

However, Eqs. (1) and (2) fail for non-Hermitian Hamiltonians as they explicitly rely upon the ability to distinguish occupied and unoccupied eigenstates. The failure is caused by the complex energy eigenvalues possessed by non-Hermitian Hamiltonians, for which the Fermi distribution does not produce occupation probabilities. Although the Kubo approach to the Hall conductivity is inappropriate for non-Hermitian Hamiltonians, we may still construct the effective action of an external $U(1)$ gauge field to obtain the Hall conductivity, an approach that is valid for both free and interacting theories [38,39]. The Hall response of a gapped system is contained in the topological Chern-Simons term of the effective action,

$$S_{\mathrm{CS}}[A] = \frac{C_{\mathrm{CS}}}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{3}$$

where A_μ is the electromagnetic vector potential. The Hall conductivity is proportional to the response coefficient $\sigma_{xy} = C_{\mathrm{CS}}e^2/h$, which is topologically quantized to integer values. Thus the electromagnetic response of the Chern-Simons term is identical to the Kubo formula for the Hall conductivity and we can identify C_{CS} as the Chern number.

Many non-Hermitian Hamiltonians possess a finite spectral density in the gap, which permits the presence of higher-order and nonlocal terms in the effective action which contribute to the Hall conductivity, in addition to the Chern-Simons term. Although progress has been made towards evaluating these contributions [40,41], the quantization of the Hall conductivity fundamentally relies on the quantization of the Chern-Simons coefficient. As in the Hermitian case, the Chern number can be defined as a topological invariant of the bands of non-Hermitian Hamiltonians [13], but the identification of the non-Hermitian Chern number with the Chern-Simons coefficient is not generally possible. By careful evaluation of the Chern-Simons coefficient, we identify the underlying structure of non-Hermitian systems responsible for the disconnect between the Chern number and the Chern-Simons coefficient. This disconnect prevents the quantization of the Hall conductivity arising from the Chern-Simons term, which we refer to as the Chern-Simons Hall response.

In the language of Green's functions, the Chern-Simons Hall response is calculated from the linear, antisymmetric part of the polarization tensor as [42,43]

$$\sigma_{xy} = \frac{e^2}{h}\frac{\epsilon^{\mu\nu\rho}}{24\pi^2}\int d^3p\,\mathrm{Tr}\left[G\frac{\partial G^{-1}}{\partial p_\mu}G\frac{\partial G^{-1}}{\partial p_\nu}G\frac{\partial G^{-1}}{\partial p_\rho}\right], \tag{4}$$

where $p = (\omega, k_x, k_y)$, the frequency ω is integrated along the imaginary axis of the complex plane, and G is the Matsubara Green's function. The Matsubara Green's function in Eq. (4) is defined as

$$G(\omega,\mathbf{k}) = [\omega - H(\mathbf{k}) - \Sigma(\omega,\mathbf{k})]^{-1}, \tag{5}$$

where H is the Hamiltonian and Σ is the self-energy. The self-energy accounts for the presence of energy exchange between the system and reservoirs as well as dissipative interactions, both of which combine to imbue the quasiparticles with a finite lifetime.

To clearly understand the topological quantization of Eq. (4) for Hermitian systems, we recognize that the Green's function represents a homeomorphism, a continuous bijection with a continuous inverse, between $(2+1)$D momentum space and the general linear group $\mathrm{GL}(N,\mathbb{C})$, where N is the number of energy bands. Let us first consider the continuum case, in which momentum space is isomorphic to $\mathbb{R}^3$. Since the Green's function approaches zero in the limits $k\to\infty$ and $\omega\to\infty$, we can compactify momentum space into the three-sphere S^3 by adding a point at infinity. With the point at infinity, the Green's function now defines a three-loop in $\mathrm{GL}(N,\mathbb{C})$ [44]. Therefore, the Green's function is an element of the third homotopy group of the general linear group $\pi_3(\mathrm{GL}(N,\mathbb{C}))$, which is isomorphic to $\mathbb{Z}$. In the lattice case, momentum space can be compactified into a pinched torus, whose third homotopy group is also isomorphic to $\mathbb{Z}$ [42]. Equation (4) identifies to which element of $\mathbb{Z}$ the Green's function corresponds, guaranteeing the integer quantization of the Hall conductivity in the Chern insulator.

In order to evaluate Eq. (4), we must construct the requisite Green's function of the non-Hermitian Hamiltonian, which we refer to as the non-Hermitian Green's function. Consider a general non-Hermitian Hamiltonian, written as

$$H(\mathbf{k}) = H_0(\mathbf{k}) + \Gamma(\mathbf{k}), \tag{6}$$

where the Hamiltonian has been broken up into Hermitian, $H_0 = H_0^\dagger$, and anti-Hermitian, $\Gamma = -\Gamma^\dagger$, components. In this formulation, the anti-Hermitian component is relegated to a self-energy term, giving the Matsubara Green's function

$$G(\omega,\mathbf{k}) = \frac{1}{\omega - H_0(\mathbf{k}) - \Gamma(\mathbf{k})\mathrm{sgn}(\mathrm{Im}\,\omega)}, \tag{7}$$

where $\Sigma(\omega,\mathbf{k}) = \Gamma(\mathbf{k})\mathrm{sgn}(\mathrm{Im}\,\omega)$. In order to preserve causality, we require the eigenvalues of $\Gamma(\mathbf{k})$ to lie on the negative imaginary axis [45].

The salient feature of the non-Hermitian Green's function is the frequency dependence of the self-energy. The self-energy depends on ω only via the signum function because it has been extracted from the Hamiltonian, which has no dependence on ω. The frequency dependence of the self-energy induces a discontinuity in every non-Hermitian Green's function at $\omega = 0$, as demonstrated in the schematic in Fig. 1. This discontinuity is avoided by the self-energy of most common interactions by an additional dependence on ω

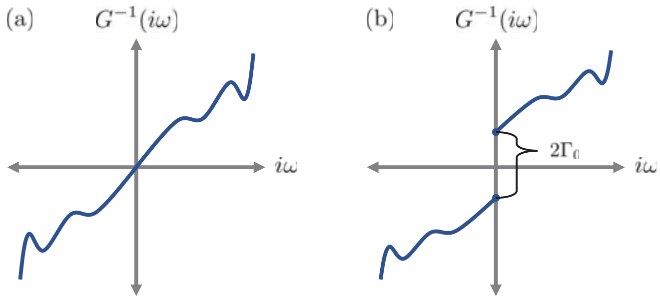

FIG. 1. Schematic representation of the inverse of a (a) conventional Green's function and (b) non-Hermitian Green's function, as a function of $i\omega$. The non-Hermitian self-energy $\Gamma(\mathbf{k})\mathrm{sgn}(\mathrm{Im}\,\omega)$ causes a discontinuity of magnitude $2\Gamma(\mathbf{k}) = 2\Gamma_0$ at $\omega = 0$.

that sets the magnitude of the self-energy to zero at $\omega = 0$. Such a discontinuous Green's function is not a homeomorphism and cannot be identified via Eq. (4) with an element of $\pi_3(\mathrm{GL}(N, \mathbb{C})) \cong \mathbb{Z}$.

The topological invariance of Eq. (4) may be proven by demonstrating that the variation in the CS Hall response induced by a variation of the Green's function is identically zero. Under the general distortion $G \to G + \delta G$, the variation is written as [45]

$$\delta\sigma_{xy} = -\frac{e^2}{h}\frac{\epsilon^{\mu\nu\rho}}{24\pi^2}\int d^3p\,\partial_\mu \mathrm{Tr}[\delta G \partial_\nu G^{-1} G \partial_\rho G^{-1}]. \tag{8}$$

For a smooth, continuous Green's function, this expression can be recast as a surface integral via the divergence theorem. Since the distortion δG must go to zero at the boundary ($\omega \to \pm\infty$), the variation is identically zero and the CS Hall response is a topological invariant. However, the divergence theorem only applies to continuous functions, and thus cannot be used to evaluate the variation of non-Hermitian Green's functions. Since δG is arbitrary, the integral can effectively take any value, thus the variation is finite and the CS Hall response is not a topological invariant. The above discussion is completely general to any non-Hermitian Hamiltonian as we have not *a priori* assumed any particular form of the self-energy.

To illustrate the impact of a discontinuity in the Green's function, we consider a general diagonal self-energy $\Sigma(\omega, \mathbf{k}) = -i\Gamma_0(\omega, \mathbf{k})\mathrm{sgn}(\mathrm{Im}\,\omega)I$, where $\Gamma_0(\omega, \mathbf{k})$ is positive and real. This self-energy can be substituted into the frequency variable in Eq. (8), resulting in a variation in the CS Hall response of the form [45]

$$\delta\sigma_{xy} = \frac{e^2}{h}\frac{\epsilon^{ij}}{24\pi^2} \times \int d^2k\, \mathrm{Tr}\big[\delta G \partial_i G_0^{-1} G_0 \partial_j G_0^{-1}\big]\Big|_{\omega'=-i\Gamma_0(0,\mathbf{k})}^{\omega'=i\Gamma_0(0,\mathbf{k})}, \tag{9}$$

where G_0 is the bare Green's function with no self-energy and the indices i and j span the momenta k_x and k_y. If $\Gamma_0(0, \mathbf{k}) = 0$, this expression is zero and the CS Hall response is a topological invariant. The self-energy arising from any Fermi-liquid interaction, for example, is identically zero at $\omega = 0$, and leaves the Hall conductivity an invariant. However, since $\Sigma(\omega, \mathbf{k}) = \Gamma(\mathbf{k})$ has no frequency dependence for non-Hermitian Green's functions, the terms in this expression do not cancel each other and the result is finite. Since δG is arbitrary we find a finite variation in the CS Hall response, as predicted above.

We can further understand the nonquantization of the CS Hall response by considering a generic, gapped, two-level system described by the Hamiltonian

$$H(\mathbf{k}) = d_0(\mathbf{k})\sigma_0 + \mathbf{d}(\mathbf{k})\cdot\boldsymbol{\sigma}, \tag{10}$$

where $d_0, d_i \in \mathbb{R}$ and $\boldsymbol{\sigma}$ is a vector of the Pauli matrices. The topological quantization of the Hall conductivity is made clear by expressing it as [46]

$$\sigma_{xy} = \frac{e^2}{h}\int \frac{d^2k}{4\pi}\hat{d}\cdot\left(\frac{\partial\hat{d}}{\partial k_x}\times\frac{\partial\hat{d}}{\partial k_y}\right). \tag{11}$$

The integral in this expression measures the solid angle that the vector $\mathbf{d}(\mathbf{k})$ sweeps out on S^2 as the momentum is integrated over the Brillouin zone. This geometric quantity must be an integer, and is formally equivalent to the Chern number.

The non-Hermitian generalization of this Hamiltonian is

$$H(\mathbf{k}) = [d_0(\mathbf{k}) + i\Gamma_0(\mathbf{k})]\sigma_0 + [\mathbf{d}(\mathbf{k}) + i\boldsymbol{\Gamma}(\mathbf{k})]\cdot\boldsymbol{\sigma}, \tag{12}$$

where $\boldsymbol{\Gamma}(\mathbf{k})$ is a vector of real numbers and must satisfy the requirement that the eigenvalues of $H(\mathbf{k})$ have negative imaginary components. In order to make the following calculation more transparent, we suppress any momentum dependence and use the following definitions: $b_0 = d_0 + i\Gamma_0$, $\mathbf{b} = \mathbf{d} + i\boldsymbol{\Gamma}$, and $b = \sqrt{\mathbf{b}\cdot\mathbf{b}}$. Using Eq. (4) [45], we find the CS Hall response of a generic two-level non-Hermitian Hamiltonian to be [47]

$$\sigma_{xy} = -\frac{e^2}{h}\int\frac{d^2k}{2\pi^2}\mathrm{Re}\left\{\hat{b}\cdot\left(\frac{\partial\hat{b}}{\partial k_x}\times\frac{\partial\hat{b}}{\partial k_y}\right) \times\left[\frac{\pi}{2}\mathrm{sgn}(\mathrm{Re}\,b) - \frac{ibb_0}{b^2-b_0^2} - i\,\mathrm{arctanh}\left(\frac{b_0}{b}\right)\right]\right\}. \tag{13}$$

The infinitesimal angle swept out by the vector $\mathbf{b}(\mathbf{k})$ is now multiplied by a function of the momentum, thus the integral does not count the number of times $\mathbf{b}(k)$ covers the sphere. This compact expression for the CS Hall response as an integral over the Brillouin zone makes manifest the absence of a topological interpretation.

To further elucidate the disconnection between Chern number and bulk topological invariant, we now analyze the CS Hall response of a model non-Hermitian Chern insulator in detail. To this end, we utilize an inversion-symmetric massive Dirac Hamiltonian, given by

$$H_0(\mathbf{k}) = -\mu\sigma_0 + v_F\mathbf{k}\cdot\boldsymbol{\sigma} + M\sigma_z, \tag{14}$$

where μ is the chemical potential, v_F is the Fermi velocity, and $M > |\mu|$ is the energy gap. When the chemical potential is within the energy gap, the massive Dirac Hamiltonian has a vanishing longitudinal conductance and a Chern number $C = -\frac{1}{2}$ [48], corresponding to a half-quantized Hall conductance $\sigma_{xy} = -e^2/2h$ [49]. We generalize this model to a non-Hermitian Chern insulator by adding a constant diagonal imaginary term that respects the same symmetry as the

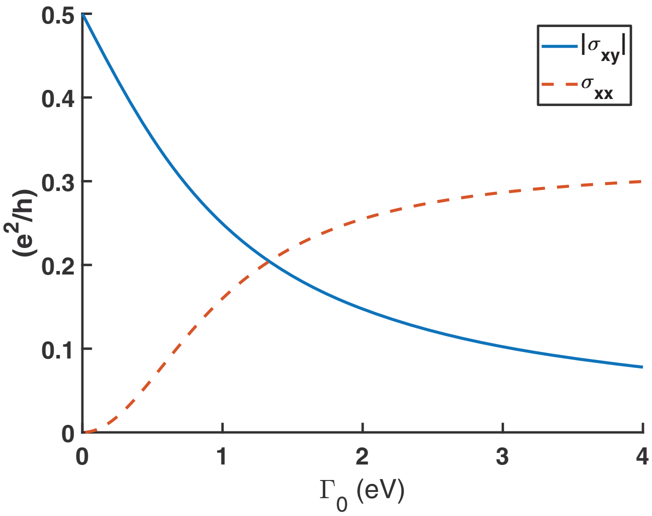

FIG. 2. The longitudinal conductivity and magnitude of the CS Hall response for the non-Hermitian Chern insulator as a function of the broadening Γ_0 with $\mu = 0.1$ eV and $M = 1$ eV. The CS Hall response monotonically decreases from $|\sigma_{xy}| = e^2/2h$ to $\sigma_{xy} = 0$, while the longitudinal conductivity monotonically increases from $\sigma_{xx} = 0$ to $\sigma_{xx} = \frac{1}{\pi}e^2/h$.

Hamiltonian,

$$H(\mathbf{k}) = -(\mu + i\Gamma_0)\sigma_0 + v_F\mathbf{k}\cdot\boldsymbol{\sigma} + M\sigma_z. \tag{15}$$

As the anti-Hermitian component of the Hamiltonian, $\Gamma(\mathbf{k}) = -i\Gamma_0\sigma_0$, is proportional to the identity matrix, the eigenvectors of the Hamiltonian and the Chern number are unchanged from the Hermitian case. Using Eq. (4), we calculate the CS Hall response of this non-Hermitian massive Dirac Hamiltonian to be [45]

$$\sigma_{xy} = \frac{e^2}{h}\frac{M}{2\pi|M|}\left[\arctan\left(\frac{\mu^2+\Gamma_0^2-M^2}{2\Gamma_0|M|}\right) - \frac{\pi}{2}\right], \tag{16}$$

in agreement with previous results on non-Hermitian massive Dirac systems [50,51]. Equation (16) yields the properly quantized value $\sigma_{xy} = -e^2/2h$ in the Hermitian limit $\Gamma_0 \to 0$, as expected. However, for any finite value of broadening, Γ_0, the CS Hall response is reduced from its Hermitian value, as shown in Fig. 2, approaching $\sigma_{xy} = 0$ as $\Gamma_0 \to \infty$. Because the eigenstate topology of this non-Hermitian Hamiltonian is identical to that of the original Hermitian Hamiltonian, the tunable value of the CS Hall response makes clear the disconnect between topology and observable.

With the loss of quantization in the Hall conductivity, one expects an associated response in the longitudinal conductivity [52]. As the broadening increases, a finite spectral density develops in the gap, allowing for conduction through the bulk of the system. We may write the longitudinal conductivity in terms of Green's functions as [53]

$$\sigma_{xx} = -\frac{e^2}{2h}\int\frac{d^2k}{(2\pi)^2}\mathrm{Tr}\left[\mathrm{Im}\,G^A(0,\mathbf{k})\frac{\partial H(\mathbf{k})}{\partial k_x}\right.$$
$$\left.\times\,\mathrm{Im}\,G^A(0,-\mathbf{k})\frac{\partial H(\mathbf{k})}{\partial k_x}\right], \tag{17}$$

where $G^A(\mathbf{k},\omega)$ is the advanced Green's function [45]. Substituting the Green's function of this non-Hermitian Chern insulator into Eq. (17) gives the conductivity

$$\sigma_{xx} = \frac{\mu^2+\Gamma_0^2-M^2}{4\pi\Gamma_0\mu}\left[\frac{2\Gamma_0\mu}{\mu^2+\Gamma_0^2-M^2}\right.$$
$$\left.+\arctan\left(\frac{2\Gamma_0\mu}{M^2+\Gamma_0^2-\mu^2}\right)\right]\frac{e^2}{h}. \tag{18}$$

In examining Eq. (18), we observe that in the Hermitian limit, $\Gamma_0 \to 0$, the longitudinal conductivity goes to zero, as it must for a Hermitian gapped system. In both the massless limit, $M \to 0$, and in the limit of infinite broadening, $\Gamma_0 \to \infty$, the conductivity approaches the theoretical minimum conductivity of a single Dirac cone [53],

$$\lim_{M\to 0}\sigma_{xx} = \lim_{\Gamma_0\to\infty}\sigma_{xx} = \frac{e^2}{\pi h}. \tag{19}$$

Between these two limits, the longitudinal conductivity remains finite.

A natural extension is to consider non-Hermitian systems in dimensions higher than (2 + 1). To this point, we consider the (4 + 1)D quantum Hall insulator, a higher-dimensional analog of the Chern insulator that is described by the Chern-Simons action

$$S_{\mathrm{eff}} = \frac{C_2}{24\pi^2}\int d^4x dt\,\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\partial_\nu A_\rho\partial_\sigma A_\tau, \tag{20}$$

which corresponds a nonlinear Hall response of the form [38,54]

$$j^\mu = \frac{C_2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma\tau}\partial_\nu A_\rho\partial_\sigma A_\tau. \tag{21}$$

Here, the coefficient C_2 is the second Chern number of the non-Abelian Berry phase [38], which may be expressed via Green's functions as

$$C_2 = -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^5p}{(2\pi)^5}\mathrm{Tr}\left[G\frac{\partial G^{-1}}{\partial p_\mu}G\frac{\partial G^{-1}}{\partial p_\nu}\right.$$
$$\left.\times\,G\frac{\partial G^{-1}}{\partial p_\rho}G\frac{\partial G^{-1}}{\partial p_\sigma}G\frac{\partial G^{-1}}{\partial p_\tau}\right]. \tag{22}$$

This integral is a higher-dimensional form of the topological invariant in Eq. (4), as it identifies the Green's function with an element of $\pi_5(\mathrm{GL}(N,\mathbb{C})) = \mathbb{Z}$, resulting in a quantized nonlinear Hall response. The discontinuity in non-Hermitian Green's functions invalidates this topological quantization argument, as it did in the (2 + 1)D case, again leading to a disconnect between a topological invariant and a quantized observable in a higher-dimensional Chern insulator.

The fact that non-Hermiticity results in a nonquantized Hall conductivity despite a quantized Chern number seems to be directly at odds with the clear experimental observations of the quantized Hall conductivity in magnetically doped three-dimensional time-reversal invariant topological insulators [55–59]. Such a mesoscopic system is generally open and disordered, meaning it may be best described by a non-Hermitian Hamiltonian that accounts for finite lifetimes. The reason that the disconnect between topological observable and Chern number is not present in magnetically doped topological insulators is that not all interactions result in non-Hermitian self-energies with finite weight at $\omega = 0$. For example, consider the effect of magnetic impurity scattering on the surface of a topological insulator. The anti-Hermitian component of the self-energy resulting from magnetic impurity scattering is of the form [51]

$$\Sigma = -i\Gamma_0|\omega|\mathrm{sgn}(\mathrm{Im}\,\omega), \tag{23}$$

where Γ_0 quantifies the broadening induced by the magnetic impurity scattering. We immediately notice that the linear dependence of the self-energy on $|\omega|$ circumvents the discontinuity at $\omega = 0$. The vanishing at $\omega = 0$ of self-energies derived from interactions is a common feature and is present, for example, in all Fermi-liquid interactions. The resulting Green's function is continuous for all $\mathbf{k}$ and ω and is a legitimate homeomorphism from momentum space to $\mathrm{GL}(N, \mathbb{C})$. Therefore, Eq. (4) produces a quantized CS Hall response, consistent with experimental results.

In summary, we have studied the connection between observables and topological invariants in non-Hermitian Chern insulators. We have analytically shown via field-theoretical techniques that there exists a disconnect between the Chern number and the CS Hall response in $(2+1)$D non-Hermitian Hamiltonians, the origin of which is the discontinuity present in non-Hermitian Green's functions. This result, which is applicable to all non-Hermitian Hamiltonians with eigenvalues in the lower half of the complex plane, proves that there is no simple relationship between the topology of eigenstates and quantized observables in non-Hermitian systems. We derived an exact formula for the CS Hall response of generic two-level non-Hermitian Hamiltonians that clearly demonstrates the disconnect from the Chern number. For the particular case of a non-Hermitian massive Dirac Hamiltonian, we showed that as broadening is introduced, despite the unchanging eigenstates, the Hall conductivity deviates from its quantized value and the system develops a longitudinal conductivity. We have further shown that the disconnect between topology and observable may be extended to higher-dimensional systems, specifically addressing $(4+1)$D systems characterized by the second Chern number. Importantly, we have illustrated that our results are consistent with the experimental observations of the quantum anomalous Hall effect in magnetically doped topological insulators. Our results demonstrate the necessity for reexamining perceived links between topology and the quantization of observables in non-Hermitian systems.

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