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Resilient Structural Stabilizability of Undirected Networks

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***Abstract*—In this paper, we consider the *structural stabilizability* problem of undirected networks. More specifically, we are tasked to infer the stabilizability of an undirected network from its underlying topology, where the undirected networks are modeled as continuous-time linear time-invariant (LTI) systems involving symmetric state matrices. Firstly, we derive a graph-theoretic necessary and sufficient condition for structural stabilizability of undirected networks. Then, we propose a method to determine the maximum dimension of the stabilizable subspace solely based on the network structure. Based on these results, on one hand, we study the optimal actuator-disabling attack problem, i.e., removing a limited number of actuators to minimize the maximum dimension of the stabilizable subspace. We show this problem is NP-hard. On the other hand, we study the optimal recovery problem with respect to the same kind of attacks, i.e., adding a limited number of new actuators such that the maximum dimension of the stabilizable subspace is maximized. We prove the optimal recovery problem is also NP-hard, and we develop a $(1-1/e)$ approximation algorithm to this problem.**

I. Introduction

In recent years, the control of networked dynamical systems has attracted a great amount of research interest [1–3]. It is of particular interest to study the asymptotic stabilizability of network control systems, i.e., the ability ensuring that all the system states can be steered to the origin by injecting proper controls, such as the undirected consensus network [1], voltage stabilization of grids [2], and formation control with undirected communication links [3].

The existing results on stabilizability analysis highly rely on the assumption that the system parameters can be exactly acquired, which is often violated in practice – see [4–6] and the references therein. It has been shown that the topological structure of a network, which can be obtained accurately, can be exploited to infer the required conditions to ensure the controllability of a network system efficiently [7–9]. This motivates us to investigate the interplay between the network's structure and the stabilizability of a network.

Assessing the stabilizability from the structural information on the system dynamics model has been an active topic of research [10–12]. However, in [10], the authors assumed no control input and proposed conditions on the sparsity pattern of symmetric state matrices such that a specific sparsity pattern sustains a Hurwitz stable state matrix. In addition, the problem considered in [11; 12] is the arbitrary pole placement through output feedback, which is sufficient but not necessary for the stabilizability.

Stabilizability is a crucial concept in network security [13], and there has been a tremendous effort invested in the control of networks under malicious attacks [13–22]. The problems of adding extra actuators/sensors to ensure controllability/observability under attacks are addressed in [14; 15]. The problem of maintaining stabilization under the uncertain feedback-channel failure is considered in [16; 17]. In [18; 19], the problem of optimal attack/recovery on structural controllability is investigated. Although the problems of stabilization under various attacks such as deception attack [13], replay attacks [20], denial-of-service [21] and destabilizing attacks [22], have been widely studied, the crucial problem of optimal attack against stabilizability by manipulating network topological structure (e.g., removing or adding actuators) has not been fully investigated. Moreover, to the best of the authors' knowledge, our paper considers for the first time the problems of optimal attack and recovery on the stabilizable subspace of a network, i.e., the number of stabilizable states or nodes in a network.

Specifically, in this paper, we consider the structural stabilizability problem, and the contributions of this paper are four-fold. First, we derive a graph-theoretic necessary and sufficient condition for structural stabilizability of undirected networks. Second, we propose computationally efficient methods to determine the generic dimension of controllable subspace and the maximum stabilizable subspace of an undirected network system. Third, we formulate the optimal actuator-disabling attack problem, where the attacker disables a limited number of actuators such that the maximum stabilizable subspace is minimized. We prove this problem is NP-hard. Finally, we formulate the optimal recovery problem, where a defender activates a limited number of new actuators such that the dimension of the stabilizable subspace is maximized. We prove this problem is NP-hard, and we propose a $(1-1/e)$ approximation algorithm.

The rest of the paper is organized as follows. In Section II, we formulate the problems considered in this paper. In Section III, we recall several crucial preliminaries. We present the main results in Sections IV and V. In Section VI, we present examples to illustrate our results. Finally, Section VII concludes this paper. Due to the page limitations, all the proofs can be found in the full version of this paper [23].

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II. PROBLEM FORMULATIONS

We consider networks whose interconnection between states are captured by a linear time-invariant (LTI) system, described by

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state vector and input vector, respectively. We refer to matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ as the state matrix and input matrix, respectively. Additionally, we consider the state matrix A to be symmetric, which is motivated by control problems arising in undirected networked dynamical systems [24–26]. Hereafter, we use the pair (A, B) to represent the system (1). To infer the properties of a system modeled by (1) from its structure, we introduce some necessary concepts on structured matrices.

Definition 1 (Structured and Symmetrically Structured Matrices)**.** *A matrix $\bar{M} \in \{0, \star\}^{n\times m}$ is called a* structured matrix*, if $[\bar{M}]_{ij}$, the (i, j)-th entry of $\bar{M}$, is either a fixed zero or an independent free parameter, denoted by $\star$. In particular, a matrix $\bar{M} \in \{0, \star\}^{n\times n}$ is* symmetrically structured*, if the value of the free parameter associated with $[\bar{M}]_{ji}$ is constrained to be the same as the value of the free parameter associated with $[\bar{M}]_{ij}$, for all i and j.*

We refer to $\tilde{M}$ as a *numerical realization* of a (symmetrically) structured matrix $\bar{M}$ if $\tilde{M}$ is matrix obtained by assigning real numbers to $\star$-parameters in $\bar{M}$. Given a pair (A, B), we let the pair $(\bar{A}, \bar{B})$ denote the *structural pattern* of the system (A, B), where $\bar{A} \in \{0, \star\}^{n\times n}$ is a symmetrically structured matrix such that $[\bar{A}]_{ij} = \star$ if $[A]_{ij} \neq 0$ and $[\bar{A}]_{ij} = 0$ otherwise. The structured matrix $\bar{B} \in \{0, \star\}^{n\times m}$ is defined similarly. Recall that a system is stabilizable if and only if the uncontrollable eigenvalues are asymptotically stable [27, Section 2.4]. Hence, to study stabilizability, it is necessary to first investigate controllability. Next, we recall the notion of *structural controllability*.

Definition 2 (Structural Controllability [7])**.** *A structural pair $(\bar{A}, \bar{B})$ is structurally controllable if there exists a numerical realization $(\tilde{A}, \tilde{B})$ such that the controllability matrix $Q(\tilde{A}, \tilde{B}) := [\tilde{B}, \tilde{A}\tilde{B}, \cdots, \tilde{A}^{n-1}\tilde{B}]$ has full row rank.*

Similarly, we define *structural stabilizability* as follows:

Definition 3 (Structural Stabilizability)**.** *A structural pair $(\bar{A}, \bar{B})$ is said to be structurally stabilizable if there exists a stabilizable numerical realization $(\tilde{A}, \tilde{B})$.*

In the next two subsections, we will be focusing on two different main threads: (*i*) analysis, and (*ii*) design.

A. Analysis of Structural Stabilizability

In this subsection, we first formulate the problem of characterizing structural stabilizability using only the structural pattern of a pair, as stated below:

Problem 1. *Given a continuous-time linear time-invariant pair (A, B), we denote by $(\bar{A}, \bar{B})$ the structural pattern of (A, B), where $\bar{A} \in \{0, \star\}^{n\times n}$ is symmetrically structured. Find a necessary and sufficient condition such that $(\bar{A}, \bar{B})$ is structurally stabilizable.*

In addition to the above problem, we also consider how "unstabilizable" a system is, when a system is not stabilizable. To characterize the "unstabilizability", we propose using the dimension of the stabilizable subspace of a system, which can be stated as follows:

Definition 4 (Stabilizable Subspace [28])**.** *Given a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, a subspace $S \subseteq \mathbb{R}^n$ is said to be the stabilizable subspace of (A, B) if for $\forall x(0) \in S$, there exists a control input $u(t) \in \mathbb{R}^m$, for $t \geq 0$, such that $\lim_{t\to\infty} x(t) = \mathbf{0}$.*

As a special case, if a pair (A, B) is stabilizable, then $S = \mathbb{R}^n$. Next, we aim to determine the maximum dimension of the stabilizable subspace, denoted by m-dim$(\bar{A}, \bar{B})$, among all numerical realizations of $(\bar{A}, \bar{B})$, stated formally as follows.

Problem 2. *Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, find* m-dim$(\bar{A}, \bar{B})$.

Upon these problems that concern mainly with the analysis of structural stabilizability, we can now focus on the design aspect of these problems in the following subsection.

B. Optimal Actuator-Attack and Recovery Problems

Stabilizability plays a key role in network security [13]. In this paper, we also consider the network resilient problems. More specifically, we assume that an attacker aims to minimize the maximum dimension of the stabilizable subspace by removing a certain amount of actuation capabilities (i.e., the inputs). We formalize this problem as follows.

Problem 3 (Optimal Actuator-disabling Attack Problem)**.** *Consider a stuctural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n\times n}$ is symmetrically structured, and $\bar{B} \in \{0, \star\}^{n\times m}$ is a structured matrix. Let the set Ω be $\Omega = [m]$, where $[m] := \{1, 2, \cdots, m\}$. Given a budget $k \in \mathbb{N}$, find*

$$\begin{aligned} \mathcal{J}^* = \arg\min_{\mathcal{J}\subseteq\Omega} \ & \text{m–dim}(\bar{A}, \bar{B}(\Omega \setminus \mathcal{J})) \\ \text{s.t. } & |\mathcal{J}| \leq k, \end{aligned} \tag{2}$$

where $\bar{B}(\mathcal{I}) \in \{0, \star\}^{n\times|\mathcal{I}|}$ is a matrix formed by the columns of $\bar{B}$ indexed by $\mathcal{I}$, for some $\mathcal{I} \subseteq \Omega$.

In other words, the Problem 3 concerns about finding an optimal strategy to attack the stabilizability of a network using a fixed budget. Meanwhile, it is also of interest to consider the perspective of a system's designer (or, defender) that is concerned with the resilience of the network, i.e., how to maximize the dimension of the stabilizable subspace by adding actuation capabilities (i.e., the inputs) to the system:

Problem 4 (Optimal Recovery Problem)**.** *Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n\times n}$ is symmetrically structured and $\bar{B} \in \{0, \star\}^{n\times m}$ is structured. Let $\mathcal{U}_{can}$,* *where $|\mathcal{U}_{can}| = m'$, be the set of candidate inputs that can be added to the system, and let $\bar{B}_{\mathcal{U}_{can}} \in \{0,\star\}^{n\times m'}$ be the structured matrix characterizing the interconnection between new inputs and the states in the system. Given a budget $k \in \mathbb{N}$, find*

$$\begin{aligned} \mathcal{J}^* = \arg\max_{\mathcal{J}\subseteq[m']} & \text{ m-dim}(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\mathcal{J})]) \\ \text{s.t. } & |\mathcal{J}| \le k, \end{aligned} \tag{3}$$

where $\bar{B}_{\mathcal{U}_{can}}(\mathcal{J}) \in \{0,\star\}^{n\times|\mathcal{J}|}$ is a structured matrix formed by the columns in $\bar{B}_{\mathcal{U}_{can}}$ indexed by $\mathcal{J}$, and $[\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\mathcal{J})]$ is the concatenation of $\bar{B}$ and $\bar{B}_{\mathcal{U}_{can}}(\mathcal{J})$.

By the duality between stabilizability and detectability [27], all the results obtained on stabilizability in this paper can be readily used to characterize detectability.

III. Preliminaries

To present solutions to Problems 1 – 4, we introduce some relevant notions in structural system theory and graph theory.

A. Structural System Theory

Consider a (symmetrically) structured matrix $\bar{M}$. Let $n_{\bar{M}}$ be the number of its independent $\star$-parameters and associate with $\bar{M}$ a parameter space $\mathbb{R}^{n_{\bar{M}}}$. Let $\mathbf{p}_{\tilde{M}} = (p_1, \dots, p_{n_{\bar{M}}})^\top \in \mathbb{R}^{n_{\bar{M}}}$ to encode the values of the independent $\star$-entries of $\bar{M}$ of a particular numerical realization $\tilde{M}$. In what follows, a set $V \subseteq \mathbb{R}^n$ is called a *variety* if there exist polynomials $\varphi_1, \dots, \varphi_k$, such that $V = \{x \in \mathbb{R}^n : \varphi_i(x) = 0, \forall i \in [k]\}$, and V is *proper* when $V \neq \mathbb{R}^n$. We denote by $V^c = \mathbb{R}^n \setminus V$ its complement.

The *term rank* [29] of a (symmetrically) structured matrix $\bar{M}$, denoted as t–rank$(\bar{M})$, is the largest integer k such that, for some suitably chosen distinct rows $\{i_\ell\}_{\ell=1}^k$ and distinct columns $\{j_\ell\}_{\ell=1}^k$, all of the entries $\{[\bar{M}]_{i_\ell j_\ell}\}_{\ell=1}^k$ are $\star$-entries. Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0,\star\}^{n\times n}$ is symmetrically structured, $(\bar{A}, \bar{B})$ is said to be *irreducible*, if there does not exist a permutation matrix P such that

$$P\bar{A}P^\top = \begin{bmatrix} \bar{A}_{11} & \mathbf{0} \\ \mathbf{0} & \bar{A}_{22} \end{bmatrix}, P\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \end{bmatrix}, \tag{4}$$

where $\bar{A}_{11} \in \{0,\star\}^{p\times p}$, and $\bar{B}_1 \in \{0,\star\}^{p\times m}$.

B. Graph Theory

Given a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, a *path* $\mathcal{P}$ in $\mathcal{D}$ is an ordered sequence of distinct vertices $\mathcal{P} = (v_1, \dots, v_k)$ with $\{v_1, \dots, v_k\} \subseteq \mathcal{V}$ and $(v_i, v_{i+1}) \in \mathcal{E}$ for all $i = 1, \dots, k-1$. A *cycle* is either a path $(v_1, \dots, v_k)$ with the additional edge (v_k, v_1) (denoted as $\mathcal{C} = (v_1, \dots, v_k, v_1)$), or a vertex with an edge to itself (i.e., self-loop, denoted as $\mathcal{C} = (v_1, v_1)$). Given a set $\mathcal{S} \subseteq \mathcal{V}$, we denote the *in-neighbour set* of $\mathcal{S}$ by $\mathcal{N}(\mathcal{S}) = \{v_i \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}, v_j \in \mathcal{S}\}$.

Given a directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ and two sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$, we define the associated *bipartite graph* of $\mathcal{D}$ by $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, whose vertex set is $\mathcal{S}_1 \cup \mathcal{S}_2$ and edge set is $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$. A *matching* $\mathcal{M}$ is a set of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$, $e, e' \in \mathcal{M}$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A matching is said to be *maximum* if it is a matching with the maximum number of edges among all possible matchings. Given a matching $\mathcal{M}$, two vertices s_1 and s_2 are *matched* if $e = (s_1, s_2) \in \mathcal{M}$. The vertex v is said to be *right-unmatched* with respect to a matching $\mathcal{M}$ associated with $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ if $v \in \mathcal{S}_2$ and v does not belong to an edge in the matching $\mathcal{M}$.

Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0,\star\}^{n\times n}$ is symmetrically structured and $\bar{B} \in \{0,\star\}^{n\times m}$ is structured, we associate $(\bar{A}, \bar{B})$ with a directed graph $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, where the vertex sets $\mathcal{X} = \{x_i\}_{i=1}^n$ and $\mathcal{U} = \{u_j\}_{j=1}^m$ are the set of state vertices and input vertices, respectively; and the edge set $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_j, x_i) : [\bar{A}]_{ij} = \star\}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) : [\bar{B}]_{ij} = \star\}$ are the set of edges between state vertices and the set of edges between input vertices and state vertices, respectively. In particular, a state vertex $x_i \in \mathcal{X}$ is said to be (input-)reachable if there exists a path from the input vertex $u_j \in \mathcal{U}$ to it. We also associate $(\bar{A}, \bar{B})$ with a bipartite graph $\mathcal{B}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, which we refer to as the *system bipartite graph*.

IV. Analysis of Structural Stabilizability

In what follows, we provide solutions to Problems 1 and 2. Specifically, in Section IV-A, we obtain Theorem 1 that characterizes the solutions to Problem 1, whereas in Section IV-B, Theorem 2 solves Problem 2 by characterizing the maximum dimension of the stabilizable subspace.

A. Graph-Theoretic Conditions on Structural Stabilizability

Since stabilizability concerns the stability of the uncontrollable part of (A, B), it is necessary to first characterize the controllable and uncontrollable parts from the structural information contained in the pair $(\bar{A}, \bar{B})$. We recall a lemma from [30] that characterizes controllable modes for the numerical realizations of a structural pair.

Lemma 1 ([30]). *Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0,\star\}^{n\times n}$ is symmetrically structured, and t–rank$(\bar{A}) = k$, if $(\bar{A}, \bar{B})$ is irreducible, then there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that for any numerical realization $(\tilde{A}, \tilde{B})$ with $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$, $\tilde{A}$ has k nonzero, simple and controllable modes.*

Lemma 1 shows that the irreducibility of $(\bar{A}, \bar{B})$ guarantees that all the non-zero modes of $(\tilde{A}, \tilde{B})$ are controllable generically. Subsequently, we can claim that given an irreducible pair $(\bar{A}, \bar{B})$, if for any numerical realization $(\tilde{A}, \tilde{B})$ there exists an uncontrollable eigenvalue, then that uncontrollable eigenvalue is 0. This implies that $(\tilde{A}, \tilde{B})$ is not stabilizable. Therefore, if a pair $(\bar{A}, \bar{B})$ is irreducible but not structurally controllable, then $(\bar{A}, \bar{B})$ is not structurally stabilizable. Hence, we have the following lemma.

Lemma 2. *Given an irreducible structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0,\star\}^{n\times n}$ is symmetrically structured, then $(\bar{A}, \bar{B})$ is structurally stabilizable if and only if $(\bar{A}, \bar{B})$ is structurally controllable.*

In addition to Lemma 2, we should also consider the case when $(\bar{A}, \bar{B})$ is reducible. By the definition of reducibility, $(\bar{A}, \bar{B})$ can be permuted to the form of (4). To ensure that $(\bar{A}, \bar{B})$ is structurally stabilizable, there must exist a numerical realization $\tilde{A}_{22}$ whose eigenvalues are all negative. Thus, the existence of a negative definite numerical realization $\tilde{A}_{22}$ determines whether $(\bar{A}, \bar{B})$ is stabilizable.

Lemma 3. *Given a reducible structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n \times n}$ is in the form of (4). Then there exists a numerical realization $\tilde{A}_{22}$ which is negative definite if and only if the diagonal entries of $\bar{A}_{22}$ are all $\star$-entries.*

Combining Lemmas 2 and 3, we have an algebraic condition for structurally stabilizability. In what follows, we present a graph-theoretic interpretation of these conditions.

Theorem 1. *Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured. Let $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ be the digraph associated with $(\bar{A}, \bar{B})$, and $\mathcal{X}_r \subseteq \mathcal{X}$ and $\mathcal{X}_u \subseteq \mathcal{X}$ be the subset of state vertices which are input-reachable and input-unreachable, respectively. The $(\bar{A}, \bar{B})$ is structurally stabilizable if and only if the following two conditions hold simultaneously in $\mathcal{D}(\bar{A}, \bar{B})$:*

1) *the vertex x_i has a self-loop, $\forall x_i \in \mathcal{X}_u$;*
2) *$|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_r$.*

Essentially, to ensure structural stabilizability, two conditions should hold simultaneously: (*i*) every unreachable state vertex should have a self-loop, and (*ii*) the reachable part of the system should be structurally controllable [30]. Next, we utilize Theorem 1 to characterize the maximum dimension of the stabilizable subspace.

B. Maximum Dimension of the Stabilizable Subspace

Similar to the previous subsection, we will first consider the case when $(\bar{A}, \bar{B})$ is irreducible, then extend the solution approach to the general case.

By Lemma 2, when $(\bar{A}, \bar{B})$ is irreducible, the $(\bar{A}, \bar{B})$ is structurally controllable if and only if it is structurally stabilizable. This motivates us to consider the relationship between controllable subspace and stabilizable subspace. Moreover, it is shown in [31] that the maximum dimension of controllable subspace is equal to the generic dimension of controllable subspace of a structural pair without symmetric parameter constraints. We may suspect that equality also holds when symmetric parameter dependency is considered. Motivated by this intuition, we first study the generic dimension of the controllable subspace, and then extend the derived results to obtain a solution of Problem 2.

Given a structured pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, if there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that $\text{rank}(Q(\tilde{A}, \tilde{B})) = k$ when $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$, then we say the *generic dimension* [31] of controllable subspace of $(\bar{A}, \bar{B})$, denoted as d_c, is k. For almost all numerical realizations $(\tilde{A}, \tilde{B})$ with $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$ (except for a proper variety, e.g., $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V$), the dimension of controllable subspace is d_c. We characterize the generic dimension of controllable subspace of $(\bar{A}, \bar{B})$ in the following lemma.

Lemma 4. *Given an irreducible structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n \times n}$ is symmetrically structured and $\bar{B} \in \{0, \star\}^{n \times m}$ is structured, the generic dimension of controllable subspace equals to the term rank of $[\bar{A}, \bar{B}]$, i.e., the concatenation of matrices $\bar{A}$ and $\bar{B}$.*

When $(\bar{A}, \bar{B})$ is reducible, we can permute $(\bar{A}, \bar{B})$ to obtain the form in (4). By Definition 4 and Theorem 1, the maximum dimension of the stabilizable subspace should be the sum of the generic dimension of controllable subspace and the maximum number of negative eigenvalues over all the numerical realizations of the uncontrollable part. This can be formalized in the following result.

Theorem 2. *Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n \times n}$ is symmetrically structured. Then,*

1) *if $(\bar{A}, \bar{B})$ is irreducible, then the maximum dimension of stabilizable subspace of $(\bar{A}, \bar{B})$ equals to the generic dimension of controllable subspace of $(\bar{A}, \bar{B})$;*
2) *if $(\bar{A}, \bar{B})$ is reducible, then we permute the matrix $\bar{A}$ into the form (4). The m-dim$(\bar{A}, \bar{B})$ equals to t–rank$([\bar{A}_{11}, \bar{B}_1]) + k$, where k is the total number of $\star$-entries in the diagonal of $\bar{A}_{22}$.*

Remark 1. *In the form (4), the index of columns of $\bar{A}_{11}$ are corresponding to input-reachable state vertices in $\mathcal{D}(\bar{A}, \bar{B})$, and the index of columns of $\bar{A}_{22}$ are corresponding to the input-unreachable state vertices in $\mathcal{D}(\bar{A}, \bar{B})$. The input-reachable/unreachable vertices can be identified by running a depth-first search [32]. Besides, the term-rank of $([\bar{A}_{11}, \bar{B}_1])$ can be obtained by finding a maximum bipartite matching in $\mathcal{B}(\bar{A}, \bar{B})$ [30]. Thus, the maximum stabilizable subspace can be determined in polynomial time $\mathcal{O}(n^3)$.*

V. Optimal Actuator-Attack and Recovery Problems

In this section, we show that both Problem 3 and Problem 4 are NP-hard in Theorem 3 and Theorem 5, respectively. Then, we introduced a greedy algorithm to solve Problem 4 – see Algorithm 1. Besides, we show that Algorithm 1 achieves a $(1 - 1/e)$ approximation guarantee to an optimal solution of Problem 4 – see Theorem 6.

A. Computational Complexity of Problem 3

Suppose that there is no self-loop in $\mathcal{D}(\bar{A}, \bar{B})$ and the Condition-2) in Theorem 1 is satisfied. Then, we will show that Problem 3 is equivalent to minimizing the number of input-reachable states by removing a limited number of inputs. This leads to the following result.

Theorem 3. *The Optimal Actuator-disabling Attack Problem (Problem 3) is NP-hard.*

Although the problem is NP-hard, that does not imply that all instances of the problem are equally difficult. As a consequence, we now propose to characterize the approximability of Problem 3. We first consider a subclass of instances of Problem 3, which satisfy the following assumption.

Assumption 1. *The symmetrically structured matrix $\bar{A} \in \{0, \star\}^{n \times n}$ is such that for any $\mathcal{S} \subseteq \mathcal{X}$, where $\mathcal{X}$ is the set of state vertices in the state digraph $\mathcal{D}(\bar{A})$, $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$. In addition, there exists no vertex with self-loop in $\mathcal{D}(\bar{A})$.*

Assumption 1 ensures that the Condition-2) in Theorem 1 is always satisfied. In addition, Assumption 1 implies that the diagonal entries of $\bar{A}$ satisfy $[\bar{A}]_{ii} = 0$, for $\forall i \in [n]$. In what follows, we leverage Min-k-Union problem [33] to characterize the approximability of Problem 3.

Theorem 4. *Under Assumption 1, denote by m_1 the total number of sets (i.e., $\{\mathcal{S}_i\}_{i=1}^{m_1}$) in an instance of Min-k-Union problem, and m_2 the total number of candidate inputs in an instance of Problem 3. Additionally, let $\rho : \mathbb{Z} \to \mathbb{R}$. Then, there exists a $\rho(m_1)$-approximation algorithm for Min-k-Union problem if and only if there exists a $\rho(m_2)$-approximation algorithm for Problem 3.*

As a result of Theorem 4, Problem 3 is at least as hard as the Min-k-Union problem.

B. Solution to Problem 4

To investigate the computation complexity of solving Problem 4, we take a similar strategy to that used in the previous section. We show that under Assumption 1, Problem 4 is equivalent to adding a limited number of actuators to maximize the total number of input-reachable state vertices. Subsequently, we have the following theorem.

Theorem 5. *The Optimal Recovery Problem is NP-hard.*

A natural approximation solution to optimal design problems is through greedy algorithms [34]. Although greedy algorithms may not provide an optimal solution, under specific objective functions of the problem, a suboptimal solution with provable approximation guarantees can be provided. Specifically, a particular class of problem with such properties is called submodularity function problems, defined as follows.

Definition 5 (Submodular function [34])**.** *Let Ω be a nonempty finite set. A set function $f\colon 2^{\Omega} \to \mathbb{R}$, where 2^{Ω} denotes the power set of Ω, is a* submodular function *if for every $\mathcal{J}_1, \mathcal{J}_2 \subseteq \Omega$ with $\mathcal{J}_1 \subseteq \mathcal{J}_2$ and every $i \in \Omega \setminus \mathcal{J}_2$, we have $f(\mathcal{J}_2 \cup \{i\}) - f(\mathcal{J}_2) \leq f(\mathcal{J}_1 \cup \{i\}) - f(\mathcal{J}_1)$.*

The greedy algorithm [34] achieves a $(1 - 1/e)$-factor approximation to the optimal solution provided that the objective function is submodular. Hereafter, we show that the objective function in Problem 4 is submodular; hence, the greedy algorithm provides $(1 - 1/e)$-factor approximation.

Theorem 6. *Algorithm 1 returns a $(1-1/e)$-approximation of the optimal solution to Problem 4.*

Remark 2. *In [33], the authors argue that insofar there is no constant factor approximation to the Min-k-Union problem. Thus, together with Theorem 4, we cannot use the greedy algorithm to approximate Problem 3 with guarantee.*

Algorithm 1 $(1-1/e)$ approximation solution to Problem 4

Input: The pair $(\bar{A}, \bar{B})$, $\bar{B}_{\mathcal{U}_{can}} \in \{0, \star\}^{n \times m'}$, and the budget k;
Output: Suboptimal solution $\mathcal{J}$;

Initialize $\mathcal{J} \leftarrow \emptyset$, $\mathcal{L} \leftarrow [m']$;$\triangleright$ $\mathcal{L}$ is the set of indexes of new actuators in $\mathcal{U}_{can}$, the set of new actuators that can be added to the system.
for iteration $i \in [k]$ **do**
  for each $j \in \mathcal{L}$ **do**
    $d_j \leftarrow$ m-dim$(\bar{A}, [\bar{B}, \bar{B}_{can}(\mathcal{J} \cup \{j\})])$;
  end for
  $\mathcal{I} \leftarrow \{i\colon d_i = \max\{d_j\}_{j=1}^{|\mathcal{L}|}\}$;
  Pick a $j \in \mathcal{I}$;
  $\mathcal{J} \leftarrow \mathcal{J} \cup \{j\}$;
  $\mathcal{L} \leftarrow \mathcal{L} \setminus \{j\}$;
end for
return $\mathcal{J}$

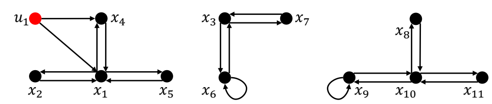

Figure 1: In this figure, we depict the structure of $\mathcal{D}(\bar{A}, \bar{B})$. The red vertex labeled by u_1 and black vertices labeled by $x_1, \ldots, x_{11}$ are the input vertex and state vertices, respectively. The black arrows represent the edges from input vertex to state vertices, as well as edges between state vertices.

VI. Illustrative Examples

In this section, we present examples to illustrate our results on structural stabilizability and Algorithm 1.

A. Maximum Dimension of the Stabilizable Subspace

We consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{11 \times 11}$ is symmetrically structured and $\bar{B} \in \{0, \star\}^{11 \times 1}$ is structured. We depict the digraph representation of the structural pair $(\bar{A}, \bar{B})$, denoted by $\mathcal{D}(\bar{A}, \bar{B})$, in Figure 1. Since x_3 and x_7 are unreachable vertices and they do not have self-loops, the pair $(\bar{A}, \bar{B})$ is not structurally stabilizable due to Theorem 1. Furthermore, the total number of right-matched (with respect to any maximum matching in the associated bipartite graph $\mathcal{B}(\bar{A}, \bar{B})$) reachable vertices is 3, and the total number of unreachable vertices with self-loop is 2. Therefore, by invoking Theorem 2, we conclude that the maximum stabilizable subspace is $3 + 2 = 5$.

B. Optimal Recovery Problem

Now, we present an example to illustrate the use of Algorithm 1. Consider again the structural pair $(\bar{A}, \bar{B})$ specified in the last subsection. The $(\bar{A}, \bar{B})$ is not structurally stabilizable. We let $\mathcal{U}_{can} = \{u_i\}_{i=2}^{7}$ be the set of candidate actuators that can be added into the system and associate it with the structured matrix $\bar{B}_{\mathcal{U}_{can}} \in \{0, \star\}^{11 \times 6}$, of which nonzero entries are captured by the red edges of the digraph $\mathcal{D}(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}])$ depicted in Figure 2.

We have obtained in the last subsection that m-dim$(\bar{A}, \bar{B})$ is 5. Suppose we have a budget $k = 3$, then Problem 4 consists in adding 3 actuators from $\mathcal{U}_{can}$ into the system such that the maximum stabilizable subspace is maximized. In the first iteration of Algorithm 1, u_4 is selected because m-dim$(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\{4\})])$ $-$

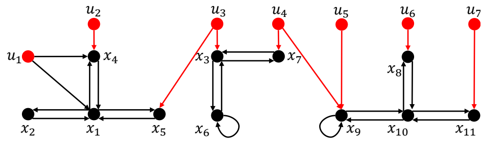

Figure 2: In this figure, we depict the digraph $\mathcal{D}(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}])$. We use red and black vertices to represent input vertices and state vertices, respectively. The black and red arrows represent are the edges in $\mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\{u_1\},\mathcal{X}}$ and edges in $\mathcal{E}_{\mathcal{U}_{can},\mathcal{X}}$, respectively.

m-dim$(\bar{A}, \bar{B}) = 4 \geq$ m-dim$(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\{i\})])$ − m-dim$(\bar{A}, \bar{B}), \forall u_i \in \mathcal{U}_{can}$. Similarly, in the second iteration, u_3 is selected by Algorithm 1. This results that m-dim$(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\{3,4\})]) = 10$. Finally, u_7 is selected and m-dim$(\bar{A}, [\bar{B}, \bar{B}_{\mathcal{U}_{can}}(\{3,4,7\})]) = 11$. Since the maximum possible stabilizable subspace is always less than or equal to the total number of states, in this example, Algorithm 1 returns an optimal solution to Problem 4.

VII. Conclusion

In this paper, we studied the structural stabilizability problem of undirected networked dynamical systems. We proposed a computationally-efficient graph-theoretic method to derive the maximum dimension of the stabilizable subspace of an undirected network. In addition, we formulated the optimal actuator-disabling attack problem and optimal recovery problem. We proved that these two problems are NP-hard, and we developed a $(1 - 1/e)$ approximation algorithm for the optimal recovery problem.

Future work will focus in extending the present framework for arbitrary algebraic constraints on the state space representation.

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