



Explosive predator and mutualistic preys: A comparative study

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ABSTRACT

Mutualism is a powerful strategy by which species can assist each other to survive in various harsh environments. These will often take the form of anti-predator strategies. In the current manuscript we consider two preys that mutually benefit each other in the presence of a sexually reproductive generalist predator. In the absence of the predator, the prey grow logistically, and there is no mutualism between them. However, in the presence of the predator the prey will begin to cooperate. We formulate five different models according to their functional responses which are either prey dependent (Holling type-III, IV and Hassel Varley functional responses) or predator-prey dependent (Beddington-DeAngelis and Crowley-Martin functional responses). We perform dynamical analysis of these five models, and also establish the mathematical restrictions under which the species can blow up at a finite time. Interestingly it is observed that the prey populations will blow-up before the predator can. We use our results to comment on a certain paradox in ecological theory, as well as provide further insight into the nature of predator interference and exploding populations of invasive species.

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1. Introduction

In a mutualistic relationship organisms of different species work together and benefit by an increased fitness. Two prey might have restricted growth rates in the presence of a predator, but might mutually establish a strong anti-predatory strategy. There is a large literature on mutualistic modeling in ecology [1–11]. In the current manuscript “Service-resource relationships” mutualism is focused upon. For example, pollination between angiosperms such as bee, wasp etc. Relationship between ants and aphids, aphids produce honeydew for ants, in return ants protect aphids from predators such as ladybugs. But strict “Service-service” mutualistic interactions are very rare [12]. A rare example of such a relationship is seen between sea anemones and anemone fish in the family Pomacentridae on the reefs of the Atlantic, Indian and Pacific Oceans. The anemones provide the anemone fish with protection from predators which cannot tolerate the stings of the anemone’s tentacles [13,14] and the anemone fish defend the anemones against butterfly fish (family Chaetodontidae), which are predators of anemones [13,14]. A second example is that of the relationship between some ants in the genus *Pseudomyrmex* and trees in the genus *Acacia*, such as the whistling thorn and bullhorn acacia [15,16].

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The ants nest inside the plant's thorns. In exchange for shelter, the ants protect acacias from attacks by herbivores (which they frequently eat, introducing a resource component to this service-service relationship) and competition from other plants, by trimming back vegetation that would shade the acacia. We provide some examples of prey mutualism, in the presence of shared predators. In Indonesian waters, it is often to see that an urchin crab will carry a fire sea urchin on its back, so the crab can hide from predatory birds and even use urchins as weapons. The urchins, meanwhile can use the crabs dispersal to avoid such predatory birds, such as sea gulls, which would otherwise depredate on both the crab and the urchin [17]. Goby fish will play the role of sentries at the entrances of burrows created by host shrimp, and warn them of impending threats by predators, in which case both the goby and the shrimp will retreat into the burrow, thus avoiding the predator [18]. In many such cases, the shared predator can be invasive, an aerial or aquatic predator [19–22].

Depending upon the choice of food, availability of resources and growth structure, food uptake process at higher trophic levels are significantly complicated and can have interesting dynamical impacts on community food chains [22]. When the predator is of generalist type and grows by sexual reproduction, beginning with [23], it has been shown that many models in this class (that is those where the top predator is modeled via the modified Leslie–Gower scheme), have

- A potential to blow-up/explode in finite time, for sufficiently large initial values of top predator density [23].
- The blow-up can occur even for *small* initial values of top predator and middle predator densities [24].

These ideas have also been applied to model invasive populations that seem to be “exploding”, under a variety of ecological scenarios. The findings thus far are that

- Prey refuge for the middle predator (that is prey for the top predator) can prevent blow-up in the top predator [25].
- Interference effects among the top predators increases their propensity to blow-up [26].
- Climate induced mortality in the top predator can prevent blow-up [27].
- Gestation effect in top predator can prevent blow-up [28,29].
- Prey defense at lower trophic levels can prevent blow-up of the top predator at the highest trophic level [30].
- Mixed boundary conditions can prevent blow-up in the predator [31].

In the current manuscript we ask

- What is the effect of mutualism among prey, if the predator is modeled according to the modified Leslie–Gower scheme.
- Could the mutualism work as a strategy to *prevent* blow-up?
- How would the blow-up/blow up-prevention be effected in the case of *various* functional responses/feeding rates of the prey.
- In particular, we ask what would the effects be, if there is predator interference as well.

To answer the above we consider two “Service-service” mutualistic preys (X and Y) with restricted growth rates and one their common predator (Z) where the prey exhibit mutualism in the presence of the predator. Both prey grow logistically in their respective environment. The predator is modeled on the basis of generalist predation, assuming sexual reproduction, that is via the modified Leslie–Gower scheme. We establish and compare different models while varying the functional responses of the predator, some of them are only prey density dependent (Holling type-III, IV and Hassel Varley functional responses) and others are prey–predator density dependent (Beddington–DeAngelis and Crowley–Martin functional responses). We present details of the model formulation next.

1.1. General model formulation

We consider the following system with two prey and a sexually reproductive generalist predator:

$$\begin{aligned}\frac{dX}{dt} &= a_1 X \left(1 - \frac{X}{K_1}\right) + c_1 X (1 - e^{-b_1 Y}) Z - F_1(X(t), Y(t)).Z(t), \\ \frac{dY}{dt} &= a_2 Y \left(1 - \frac{Y}{K_2}\right) + c_2 Y (1 - e^{-b_2 X}) Z - F_2(X(t), Y(t)).Z(t), \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right) Z^2,\end{aligned}$$

where $X(t)$, $Y(t)$ and $Z(t)$ are functions of time representing population densities of the two preys and the sexually reproductive generalist predator respectively and all parameters are positive constants. a_1, a_2 are the intrinsic growth rates of X and Y populations respectively, c_1, c_2 are marginal rate of change of the per capita growth rate of X and Y respectively, b_1, b_2 are the mutualistic impacts of second prey on first and vice-versa, D denotes the growth rate and E denotes the death rate of the generalist predator. α_3 represents the residual loss in Z population due to severe scarcity of its favorite food X and Y . α_3 signifies that Z is the generalist predator and F_1, F_2 represent functional responses of Z population upon X and Y respectively.

The ecological basis in deriving the above model is as follows,

- Both prey (X and Y) grow logistically.
- Each prey has a mutualistic effect on other prey in the presence of the predator (Z).
- In absence of the predator the model predicts pure logistic growth for both prey populations. If we set $Z = 0$ then there are no predation or mutualism terms, in both prey equations. That means, in absence of the predator the prey refrain from mutualism.
- the predator causes a negative feedback on prey density (it is clear from predation terms in both prey equations $-F_1(X(t), Y(t))$ and $-F_2(X(t), Y(t))$). However when there is predator presence, the mutualism takes effect. To describe this, note when $Z > 0$, then $c_1XZ(1 - e^{-\alpha_1 Y}) > 0$, but if you fix X and Z , then see $c_1XZ(1 - e^{-\alpha_1 Y})$ is monotonically increasing as Y increases (similar for the other). That means, in presence of predator Z , prey Y provides a positive feedback on the density of prey X via mutualism, and vice versa [32,33].
- When $Y = 0$, then the mutualism term becomes 0, that means in absence of other prey, there is no mutualism, and there is no effect of predation of predator Z on prey X when $Y = 0$ (vice versa) [34].
- The predator is modeled according to the modified Leslie–Gower scheme. In this formulation the functional response of the predator is not similar to that of its prey. The generalist predator population grows quadratically as Z^2 , signifying population growth is directly proportional to the product of males and females ($Z \times Z = Z^2$). The population decays due to intraspecies competition as $-(\frac{E}{\alpha_3 + X + Y})Z^2$. Thus if the prey population is large, Z has enough prey, and so the competition coefficient is small. On the other hand if the prey population is small, Z has lack of sufficient prey, and so the competition coefficient is large, inducing greater competition amongst the predators Z . The α_3 shows that Z is a true generalist predator, and can switch to alternate prey, in case its favorite prey X or Y goes extinct [29].

2. Specific model formulation

In this section we vary the functional response F_1, F_2 to get five different models in the following forms:

(i) Model 1: Holling III functional response

Here, we consider the predation processes follows Holling III functional response. Therefore, the model system becomes

$$\begin{aligned}\frac{dX}{dt} &= a_1X \left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1X^2Z}{X^2 + h_1}, \\ \frac{dY}{dt} &= a_2Y \left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2Y^2Z}{Y^2 + h_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.\end{aligned}\tag{1}$$

(ii) Model 2: Holling IV functional response

Here, we consider the predation process follows Holling IV functional response. Therefore, the model system becomes

$$\begin{aligned}\frac{dX}{dt} &= a_1X \left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1XZ}{\frac{X^2}{j_1} + X + h'_1}, \\ \frac{dY}{dt} &= a_2Y \left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{\frac{Y^2}{j_2} + Y + h'_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.\end{aligned}\tag{2}$$

(iii) Model 3: Beddington–DeAngelis functional response Here, we consider the predation process follows Beddington–DeAngelis functional response. Therefore, the model system becomes

$$\begin{aligned}\frac{dX}{dt} &= a_1X \left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1XZ}{X + \beta_1Z + \alpha_1}, \\ \frac{dY}{dt} &= a_2Y \left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{Y + \beta_2Z + \alpha_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.\end{aligned}\tag{3}$$

(iv) Model 4: Crowley Martin functional response

Here, we consider the predation process follows Crowley Martin functional response. Therefore, the model system becomes

$$\begin{aligned}\frac{dX}{dt} &= a_1X \left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1XZ}{\delta_1X + \beta'_1Z + \gamma_1XZ + \alpha'_1}, \\ \frac{dY}{dt} &= a_2Y \left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{\delta_2Y + \beta'_2Z + \gamma_2YZ + \alpha'_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.\end{aligned}\quad (4)$$

(v) Model 5: Hassel Varley functional response

Here, we consider the predation process follows Hassel Varley functional response. Therefore, the model system becomes

$$\begin{aligned}\frac{dX}{dt} &= a_1X \left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1XZ}{X^p + v_1}, \\ \frac{dY}{dt} &= a_2Y \left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{Y^q + v_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.\end{aligned}\quad (5)$$

The initial conditions of all the above systems (1)–(5) are given as $X \geq 0, Y \geq 0, Z \geq 0$. By the fundamental theory of functional differential equations [35], we know that there is a unique solution $(X(t), Y(t), Z(t))$ to systems (1)–(5) with above initial conditions. In these models, the mutualism starts between the preys X and Y only when the generalist predator Z enters in the system. In this model, $K_1, K_2, a_1, a_2, b_1, b_2, c_1, c_2, j_1, j_2, D, E, h_1, h_2, h'_1, h'_2, v_1, v_2, \alpha_1, \alpha_2, \alpha_3, \alpha'_1, \alpha'_2, \beta_1, \beta_2, \beta'_1, \beta'_2, \gamma_1, \gamma_2, \delta_1, \delta_2, m_1, m_2, p, q$ are all positive constants. The interaction of preys X, Y and the generalist predator Z is modeled by the modified Leslie–Gower scheme where the loss in predator population is proportional to the reciprocal of per capita availability of its most favorite food. We assume that in the absence of the predator, the prey population density grows according to a logistic law with carrying capacity $K_1(> 0)$ and $K_2(> 0)$ for respective prey X and Y with an intrinsic growth rate constant $a_1(> 0)$ and $a_2(> 0)$ respectively. m_1, m_2, E are the maximum values which per capita reduction rate of X, Y, Z can attain respectively. h_1, h'_1 and h_2, h'_2 are considered as half saturation constant for Holling type III and IV functional response for the prey X and Y respectively in the absence of any inhibitory effect. The parameter j_1 and j_2 are the measure of the predator's immunity from or tolerance of the prey X and Y respectively. α_1, α_2 and α'_1, α'_2 represent the protection provided to the prey by its environment for BD and CM functional responses respectively. β_1, β_2 and β'_1, β'_2 are the intensity of interference between individuals of predator respectively for BD and CM functional responses. δ_1 and δ_2 measure the magnitude of interference among the prey individuals for CM functional response. γ_1 and γ_2 are considered as the inter species interference between prey and predator for CM functional response. D is the growth rate of the generalist predator Z due to sexual reproduction and Z^2 signifies the fact that mating frequency is directly proportional to the number of male as well as female individuals. The mechanism of sexual reproduction of generalist predator is given in [36,37].

Now if we rewrite the systems (1)–(5) in general form, then we get:

$$\begin{aligned}\frac{dX}{dt} &= f_1(X, 0, Z) + \bar{f}_1(X, Y, Z), \\ \frac{dY}{dt} &= f_2(0, Y, Z) + \bar{f}_2(X, Y, Z), \\ \frac{dZ}{dt} &= f_3(X, Y, Z).\end{aligned}\quad (6)$$

All the functions $f_1, f_2, \bar{f}_1, \bar{f}_2, f_3$ are generalized forms of the proposed five models given in Table 1.

3. Stability properties of the systems

Our main aim is to study about the possible solution set of a dynamical model system in a particular environment. Ecological stability possess resilience, persistence, elasticity, amplitude and constancy. The proper definition depends upon in the context of the ecosystem. Dynamical system introduces the idea of neighborhood of stability and the domain of attraction in the ecosystem. Local stability defines that a system is stable over small perturbation and global stability indicates that a system has unique equilibrium point in the entire domain of attraction. We make mathematical model with respect to a given environment and investigate the stability of that model. The simplest way to find the stability of a mathematical model is linearizing the given model. Usually researchers are using Lyapunov stability method for a mathematical model system.

Table 1
Functional form of system (1)–(5).

Model	$f_1(X, 0, Z)$	$\bar{f}_1(X, Y, Z)$	$f_2(0, Y, Z)$	$\bar{f}_2(X, Y, Z)$	$f_3(X, Y, Z)$
(1)	$a_1 X(1 - \frac{X}{K_1}) - \frac{m_1 X^2 Z}{X^2 + h_1}$	$c_1 X(1 - e^{-b_1 Y})Z$	$a_2 Y(1 - \frac{Y}{K_2}) - \frac{m_2 Y^2 Z}{Y^2 + h_2}$	$c_2 Y(1 - e^{-b_1 X})Z$	$(D - \frac{E}{\alpha_3 + X + Y})Z^2$
(2)	$a_1 X(1 - \frac{X}{K_1}) - \frac{m_1 XZ}{\frac{X^2}{j_1} + X + h'_1}$	$c_1 X(1 - e^{-b_1 Y})Z$	$a_2 Y(1 - \frac{Y}{K_2}) - \frac{m_2 YZ}{\frac{Y^2}{j_2} + Y + h'_2}$	$c_2 Y(1 - e^{-b_2 X})Z$	$(D - \frac{E}{\alpha_3 + X + Y})Z^2$
(3)	$a_1 X(1 - \frac{X}{K_1}) - \frac{m_1 XZ}{X + \beta_1 Z + \alpha_1}$	$c_1 X(1 - e^{-b_1 Y})Z$	$a_2 Y(1 - \frac{Y}{K_2}) - \frac{m_2 YZ}{Y + \beta_2 Z + \alpha_2}$	$c_2 Y(1 - e^{-b_2 X})Z$	$(D - \frac{E}{\alpha_3 + X + Y})Z^2$
(4)	$a_1 X(1 - \frac{X}{K_1}) - \frac{m_1 XZ}{\delta_1 X + \beta'_1 Z + \gamma_1 XZ + \alpha'_1}$	$c_1 X(1 - e^{-b_1 Y})Z$	$a_2 Y(1 - \frac{Y}{K_2}) - \frac{m_2 YZ}{\delta_2 Y + \beta'_2 Z + \gamma_2 YZ + \alpha'_2}$	$c_2 Y(1 - e^{-b_2 X})Z$	$(D - \frac{E}{\alpha_3 + X + Y})Z^2$
(5)	$a_1 X(1 - \frac{X}{K_1}) - \frac{m_1 XZ}{X^p + v_1}$	$c_1 X(1 - e^{-b_1 Y})Z$	$a_2 Y(1 - \frac{Y}{K_2}) - \frac{m_2 YZ}{Y^p + v_2}$	$c_2 Y(1 - e^{-b_1 X})Z$	$(D - \frac{E}{\alpha_3 + X + Y})Z^2$

3.1. Stability properties of systems (1)–(5)

Model systems (1)–(5) have eight possible non-negative equilibria, namely $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$, $E_{XZ}(\hat{X}, 0, \hat{Z})$ and $E_*(X^*, Y^*, Z^*)$. Now we show the feasibility and stability properties of the first seven equilibria of systems (1)–(5) in Tables 3–7 along with the corresponding feasibility and stability properties for coexistence equilibrium point E_* .

(i) Stability properties of Model 1

Feasibility and stability conditions of $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ of system (1) are given in Table 3. The coexistence equilibrium point of system (1) is $E_*(X^*, Y^*, Z^*)$, where

$$Y^* = \frac{E}{D} - \alpha_3 - X^*, \quad Z^* = \frac{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}}{\frac{m_2(\frac{E}{D} - \alpha_3 - X^*)}{(\frac{E}{D} - \alpha_3 - X^*)^2 + h_2} - c_2(1 - e^{-b_2 X^*})} = \frac{a_1 - \frac{a_1 X^*}{K_1}}{\frac{m_1 X^*}{X^{*2} + h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}.$$

Let us assume that $0 < X^* < K_1 < \frac{E}{D} - \alpha_3$ and X^* is the positive root of the equation

$$f(X^*) = \frac{\frac{m_1 X^*}{X^{*2} + h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}{\frac{m_2(\frac{E}{D} - \alpha_3 - X^*)}{(\frac{E}{D} - \alpha_3 - X^*)^2 + h_2} - c_2(1 - e^{-b_2 X^*})} - \frac{a_1 - \frac{a_1 X^*}{K_1}}{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}} = 0.$$

We have,

$$f(0) = -\frac{c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3)})}{\frac{m_2(\frac{E}{D} - \alpha_3)}{(\frac{E}{D} - \alpha_3)^2 + h_2}} - \frac{a_1}{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3)}{K_2}} < 0 \text{ and}$$

$$f(K_1) = \frac{\frac{m_1 K_1}{K_1^2 + h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - K_1)})}{\frac{m_2(\frac{E}{D} - \alpha_3 - K_1)}{(\frac{E}{D} - \alpha_3 - K_1)^2 + h_2} - c_2(1 - e^{-b_2 K_1})} > 0,$$

as $\frac{m_1 K_1}{K_1^2 + h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - K_1)}) > 0$ and $\frac{m_2(\frac{E}{D} - \alpha_3 - K_1)}{(\frac{E}{D} - \alpha_3 - K_1)^2 + h_2} - c_2(1 - e^{-b_2 K_1}) > 0$ or $\frac{m_1 K_1}{K_1^2 + h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - K_1)}) < 0$ and $\frac{m_2(\frac{E}{D} - \alpha_3 - K_1)}{(\frac{E}{D} - \alpha_3 - K_1)^2 + h_2} - c_2(1 - e^{-b_2 K_1}) < 0$. Since $f(0)f(K_1) < 0$, there is a positive root of this equation lies in $(0, K_1)$. Therefore, the coexistence equilibrium point $E_*(X^*, Y^*, Z^*)$ exists.

(ii) Stability properties of Model 2

Feasibility and stability conditions of $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ of system (2) are given in Table 4. The coexistence equilibrium of system (2) is $E_*(X^*, Y^*, Z^*)$, where

$$Y^* = \frac{E}{D} - \alpha_3 - X^*, \quad Z^* = \frac{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}}{\frac{m_2}{\frac{(\frac{E}{D} - \alpha_3 - X^*)^2}{j_2} + (\frac{E}{D} - \alpha_3 - X^*) + h'_2} - c_2(1 - e^{-b_2 X^*})} = \frac{a_1 - \frac{a_1 X^*}{K_1}}{\frac{m_1}{\frac{X^{*2}}{j_1} + X^* + h'_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}.$$

Table 2

Entities of Jacobian matrix for system (1)–(5).

Model	Jacobian Matrix entities	Model	Jacobian Matrix entities
(1)	$f_{1X} = a_1(1 - \frac{2X}{K_1}) - \frac{2\alpha_1 m_1 X Z}{(X^2 + h_1)^2}, \bar{f}_{1X} = c_1(1 - e^{-b_1 Y})Z,$ $f_{1Y} = 0, \bar{f}_{1Y} = b_1 c_1 X Z e^{-b_1 Y},$ $f_{1Z} = -\frac{m_1 X^2}{X^2 + h_1}, \bar{f}_{1Z} = c_1(1 - e^{-b_1 Y})X,$ $f_{2X} = 0, \bar{f}_{2X} = b_2 c_2 Y Z e^{-b_2 X},$ $f_{2Y} = a_2(1 - \frac{2Y}{K_2}) - \frac{m_2 h'_2 Z - \frac{m_2 Y^2 Z}{j_2}}{(\frac{Y^2}{j_2} + Y + h'_2)^2}, \bar{f}_{2Y} = c_2(1 - e^{-b_2 X})Z,$ $f_{2Z} = -\frac{m_2 Y^2}{Y^2 + h'_2}, \bar{f}_{2Z} = c_2(1 - e^{-b_2 X})Y,$ $f_{3X} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3X} = 0,$ $f_{3Y} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3Y} = 0,$ $f_{3Z} = 2(D - \frac{E}{\alpha_3 + X + Y})Z, \bar{f}_{3Z} = 0.$	(2)	$f_{1X} = a_1(1 - \frac{2X}{K_1}) - \frac{m_1 h'_1 Z - \frac{m_1 X^2 Z}{j_1}}{(\frac{X^2}{j_1} + X + h'_1)^2}, \bar{f}_{1X} = c_1(1 - e^{-b_1 Y})Z,$ $f_{1Y} = 0, \bar{f}_{1Y} = b_1 c_1 X Z e^{-b_1 Y},$ $f_{1Z} = -\frac{m_1 X}{(\frac{X^2}{j_1} + X + h'_1)}, \bar{f}_{1Z} = c_1(1 - e^{-b_1 Y})X,$ $f_{2X} = 0, \bar{f}_{2X} = b_2 c_2 Y Z e^{-b_2 X},$ $f_{2Y} = a_2(1 - \frac{2Y}{K_2}) - \frac{m_2 h'_2 Z - \frac{m_2 Y^2 Z}{j_2}}{(\frac{Y^2}{j_2} + Y + h'_2)^2}, \bar{f}_{2Y} = c_2(1 - e^{-b_2 X})Z,$ $f_{2Z} = -\frac{m_2 Y}{(\frac{Y^2}{j_2} + Y + h'_2)}, \bar{f}_{2Z} = c_2(1 - e^{-b_2 X})Y,$ $f_{3X} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3X} = 0,$ $f_{3Y} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3Y} = 0,$ $f_{3Z} = 2(D - \frac{E}{\alpha_3 + X + Y})Z, \bar{f}_{3Z} = 0.$
(3)	$f_{1X} = a_1(1 - \frac{2X}{K_1}) - \frac{m_1 \beta_1 Z^2 + m_1 \alpha_1 Z}{(X + \beta_1 Z + \alpha_1)^2}, \bar{f}_{1X} = c_1(1 - e^{-b_1 Y})Z,$ $f_{1Y} = 0, \bar{f}_{1Y} = b_1 c_1 X Z e^{-b_1 Y},$ $f_{1Z} = -\frac{m_1 X^2 + m_1 \alpha_1 X}{(X + \beta_1 Z + \alpha_1)^2}, \bar{f}_{1Z} = c_1(1 - e^{-b_1 Y})X,$ $f_{2X} = 0, \bar{f}_{2X} = b_2 c_2 Y Z e^{-b_2 X},$ $f_{2Y} = a_2(1 - \frac{2Y}{K_2}) - \frac{m_2 \beta_2 Z^2 + m_2 \alpha_2 Z}{(Y + \beta_2 Z + \alpha_2)^2}, \bar{f}_{2Y} = c_2(1 - e^{-b_2 X})Z,$ $f_{2Z} = -\frac{m_2 Y^2 + m_2 Y}{(Y + \beta_2 Z + \alpha_2)^2}, \bar{f}_{2Z} = c_2(1 - e^{-b_2 X})Y,$ $f_{3X} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3X} = 0,$ $f_{3Y} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3Y} = 0,$ $f_{3Z} = 2(D - \frac{E}{\alpha_3 + X + Y})Z, \bar{f}_{3Z} = 0.$	(4)	$f_{1X} = a_1(1 - \frac{2X}{K_1}) - \frac{m_1 \beta'_1 Z^2 + m_1 \alpha'_1 Z}{(\delta_1 X + \beta'_1 Z + \gamma_1 X Z + \alpha'_1)^2}, \bar{f}_{1X} = c_1(1 - e^{-b_1 Y})Z,$ $f_{1Y} = 0, \bar{f}_{1Y} = b_1 c_1 X Z e^{-b_1 Y},$ $f_{1Z} = -\frac{m_1 \delta_1 X^2 + m_1 \alpha'_1 X}{(\delta_1 X + \beta'_1 Z + \gamma_1 X Z + \alpha'_1)^2}, \bar{f}_{1Z} = c_1(1 - e^{-b_1 Y})X,$ $f_{2X} = 0, \bar{f}_{2X} = b_2 c_2 Y Z e^{-b_2 X},$ $f_{2Y} = a_2(1 - \frac{2Y}{K_2}) - \frac{m_2 \beta'_2 Z^2 + m_2 \alpha'_2 Z}{(\delta_2 Y + \beta'_2 Z + \gamma_2 Y Z + \alpha'_2)^2}, \bar{f}_{2Y} = c_2(1 - e^{-b_2 X})Z,$ $f_{2Z} = -\frac{m_2 \delta_2 Y^2 + m_2 \alpha'_2 Y}{(\delta_2 Y + \beta'_2 Z + \gamma_2 Y Z + \alpha'_2)^2}, \bar{f}_{2Z} = c_2(1 - e^{-b_2 X})Y,$ $f_{3X} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3X} = 0,$ $f_{3Y} = \frac{EZ^2}{(\alpha_3 + X + Y)^2}, \bar{f}_{3Y} = 0,$ $f_{3Z} = 2(D - \frac{E}{\alpha_3 + X + Y})Z, \bar{f}_{3Z} = 0.$
(5)	$f_{1X} = a_1(1 - \frac{2X}{K_1}) - \frac{m_1 v_1 Z + m_1 X Z^p - m_1 p Z X^p}{(X^p + v_1)^2},$ $f_{1Y} = 0,$ $f_{1Z} = -\frac{m_1 X}{X^p + v_1},$ $f_{2X} = 0,$ $f_{2Y} = a_2(1 - \frac{2Y}{K_2}) - \frac{m_2 v_2 Z + m_2 Z Y^p - m_2 p Z Y^p}{(Y^p + v_2)^2},$ $f_{2Z} = -\frac{m_2 Y}{Y^p + v_2},$ $f_{3X} = \frac{EZ^2}{(\alpha_3 + X + Y)^2},$ $f_{3Y} = \frac{EZ^2}{(\alpha_3 + X + Y)^2},$ $f_{3Z} = 2(D - \frac{E}{\alpha_3 + X + Y})Z,$		$\bar{f}_{1X} = c_1(1 - e^{-b_1 Y})Z,$ $\bar{f}_{1Y} = b_1 c_1 X Z e^{-b_1 Y},$ $\bar{f}_{1Z} = c_1(1 - e^{-b_1 Y})X,$ $\bar{f}_{2X} = b_2 c_2 Y Z e^{-b_2 X},$ $\bar{f}_{2Y} = c_2(1 - e^{-b_2 X})Z,$ $\bar{f}_{2Z} = c_2(1 - e^{-b_2 X})Y,$ $\bar{f}_{3X} = 0,$ $\bar{f}_{3Y} = 0,$ $\bar{f}_{3Z} = 0.$

Let us assume that $0 < X^* < K_1 < \frac{E}{D} - \alpha_3$ and X^* is the positive root of the equation

$$f(X^*) = \frac{\frac{m_1}{\frac{X^{*2}}{j_1} + X^* + h'_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}{\frac{(\frac{E}{D} - \alpha_3 - X^*)^2}{j_2} + (\frac{E}{D} - \alpha_3 - X^*) + h'_2} - \frac{a_1 - \frac{a_1 X^*}{K_1}}{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}} = 0.$$

We have,

$$f(0) = \frac{\frac{m_1}{h'_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3)})}{\frac{(\frac{E}{D} - \alpha_3)^2}{j_2} + (\frac{E}{D} - \alpha_3) + h'_2} - \frac{a_1}{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3)}{K_2}} < 0$$

Table 3
Stability analysis of boundary equilibria of system (1).

Equilibrium and coordinate	Feasibility condition	Jacobian matrix (J) and eigenvalues ($\lambda_i, i = 1, 2, 3$)	Stability status
(i) $E_0(0, 0, 0)$	always	$J = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(ii) $E_X(K_1, 0, 0)$	always	$J = \begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1^2}{K_1^2 + h_1} \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(iii) $E_Y(0, K_2, 0)$	always	$J = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & -\frac{m_2 K_2^2}{K_2^2 + h_2} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(iv) $E_Z(0, 0, \bar{Z})$ $\bar{Z}$ is any +ve constant	$D = \frac{E}{\alpha_3}$	$J = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ \frac{E\bar{Z}^2}{\alpha_3^2} & \frac{E\bar{Z}^2}{\alpha_3^2} & 2(D - \frac{E}{\alpha_3})\bar{Z} \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 2(D - \frac{E}{\alpha_3})\bar{Z} = 0$	Neutral
(v) $E_{XY}(K_1, K_2, 0)$	always	$J = \begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1^2}{K_1^2 + h_1} + c_1 K_1(1 - e^{-b_1 K_2}) \\ 0 & -a_2 & -\frac{m_2 K_2^2}{K_2^2 + h_2} + c_2 K_2(1 - e^{-b_2 K_1}) \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(vi) $E_{YZ}(0, \bar{Y}, \bar{Z})$ $\bar{Y} = \left(\frac{E}{D} - \alpha_3\right)$ $\bar{Z} = \frac{a_2(1 - \frac{\bar{Y}}{K_2})(\bar{Y}^2 + h_2)}{m_2 \bar{Y}}$	$\alpha_3 < \frac{E}{D} < K_2 + \alpha_3$	$J = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{11}, \lambda_{2,3} = \frac{a_{22} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{22}^2 + 4a_{23}a_{32}$, $a_{11} = a_1 + c_1(1 - e^{-b_1 \bar{Y}})\bar{Z}$, $a_{22} = -\frac{a_2 \bar{Y}}{K_2} - \frac{m_2 \bar{Y} \bar{Z}}{\bar{Y}^2 + h_2} + \frac{2m_2 \bar{Y}^3 \bar{Z}}{(\bar{Y}^2 + h_2)^2}$, $a_{21} = b_2 c_2 \bar{Y} \bar{Z}, a_{23} = -\frac{m_2 \bar{Y}^2}{\bar{Y}^2 + h_2}, a_{31} = \frac{D^2 \bar{Z}^2}{E} = a_{32}$	$a_{11}, a_{22} < 0$: Stable Fig. 1(a) $a_{11}, a_{22} > 0$: Unstable $a_{11} = 0/a_{22} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{11} < 0, a_{22} > 0\} \cup$ $\{\Delta > 0, a_{11} > 0, a_{22} < 0\}$: Saddle
(vii) $E_{XZ}(\hat{X}, 0, \hat{Z})$ $\hat{X} = \left(\frac{E}{D} - \alpha_3\right)$ $\hat{Z} = \frac{a_1(1 - \frac{\hat{X}}{K_1})(\hat{X}^2 + h_1)}{m_1 \hat{X}}$	$\alpha_3 < \frac{E}{D} < K_1 + \alpha_3$	$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{22}, \lambda_{2,3} = \frac{a_{11} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{11}^2 + 4a_{13}a_{31}$, $a_{11} = -\frac{a_1 \hat{X}}{K_1} - \frac{m_1 \hat{X} \hat{Z}}{\hat{X}^2 + h_1} + \frac{2m_1 \hat{X}^3 \hat{Z}}{(\hat{X}^2 + h_1)^2}$, $a_{22} = a_2 + c_2(1 - e^{-b_2 \hat{X}})\hat{Z}$, $a_{12} = b_1 c_1 \hat{X} \hat{Z}, a_{13} = -\frac{m_1 \hat{X}^2}{\hat{X}^2 + h_1}, a_{31} = \frac{D^2 \hat{Z}^2}{E} = a_{32}$	$a_{11}, a_{22} < 0$: Stable Fig. 1(b) $a_{11}, a_{22} > 0$: Unstable $a_{22} = 0/a_{11} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{22} < 0, a_{11} > 0\} \cup$ $\{\Delta > 0, a_{22} > 0, a_{11} < 0\}$: Saddle

as $\frac{m_1}{h_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3)}) < 0$ and

$$f\left(\frac{E}{D} - \alpha_3\right) = \frac{\frac{m_1}{\frac{(\frac{E}{D} - \alpha_3)^2}{J_1} + (\frac{E}{D} - \alpha_3) + h_1'}}{\frac{m_2}{h_2} - c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3)})} - \frac{a_1(1 - \frac{E}{K_1})}{a_2} > 0$$

Table 4

Stability analysis of boundary equilibria of system (2).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(i)	$E_0(0, 0, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(ii)	$E_X(K_1, 0, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{\frac{K_1^2}{J_1} + K_1 + h'_1} \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(iii)	$E_Y(0, K_2, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & -\frac{m_2 K_2}{\frac{K_2^2}{J_2} + K_2 + h'_2} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(iv)	$E_Z(0, 0, \tilde{Z})$ $\tilde{Z}$ is any +ve constant	$D = \frac{E}{\alpha_3}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ \frac{E\tilde{Z}^2}{\alpha_3^2} & \frac{E\tilde{Z}^2}{\alpha_3^2} & 2(D - \frac{E}{\alpha_3})\tilde{Z} \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 2(D - \frac{E}{\alpha_3})\tilde{Z} = 0$	Neutral
(v)	$E_{XY}(K_1, K_2, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{\frac{K_1^2}{J_1} + K_1 + h'_1} + c_1 K_1 (1 - e^{-b_1 K_2}) \\ 0 & -a_2 & -\frac{m_2 K_2}{\frac{K_2^2}{J_2} + K_2 + h'_2} + c_2 K_2 (1 - e^{-b_2 K_1}) \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(vi)	$E_{YZ}(0, \bar{Y}, \bar{Z})$ $\bar{Y} = \left(\frac{E}{D} - \alpha_3\right)$ $\bar{Z} = \frac{a_2 \left(1 - \frac{\bar{Y}}{K_2}\right) \left(\frac{\bar{Y}^2}{J_2} + \bar{Y} + h'_2\right)}{m_2}$	$\alpha_3 < \frac{E}{D} < K_2 + \alpha_3$	$J = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{11}, \lambda_{2,3} = \frac{a_{22} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{22}^2 + 4a_{23}a_{32}$, $a_{11} = a_1 + c_1(1 - e^{-b_1 \bar{Y}})\bar{Z} - \frac{m_1 \bar{Z}}{h'_1},$ $a_{22} = -\frac{a_2 \bar{Y}}{K_2} - \frac{m_2 \bar{Y} \bar{Z}}{\left(\frac{\bar{Y}^2}{J_2} + \bar{Y} + h'_2\right)^2} + \frac{\frac{2m_2 \bar{Y}^2 \bar{Z}}{J_2}}{\left(\frac{\bar{Y}^2}{J_2} + \bar{Y} + h'_2\right)^2},$ $a_{21} = b_2 c_2 \bar{Y} \bar{Z}, a_{23} = -\frac{m_2 \bar{Y}}{\left(\frac{\bar{Y}^2}{J_2} + \bar{Y} + h'_2\right)},$ $a_{31} = \frac{D^2 \bar{Z}^2}{E} = a_{32}$	$a_{11}, a_{22} < 0$: Stable Fig. 1 (c) $a_{11}, a_{22} > 0$: Unstable $a_{11} = 0/a_{22} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{11} < 0, a_{22} > 0\} \cup$ $\{\Delta > 0, a_{11} > 0, a_{22} < 0\}$: Saddle

(continued on next page)

$$\text{as } \frac{m_2}{h'_2} - c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3)}) > 0.$$

Under the above assumptions we have, $f(0)f(\frac{E}{D} - \alpha_3) < 0$. So there is a positive root of this equation lies in $(0, \frac{E}{D} - \alpha_3)$. Therefore, the non-zero equilibrium point $E_*(X^*, Y^*, Z^*)$ exists.

Table 4 (continued).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(vii)	$E_{YZ}(\hat{X}, 0, \hat{Z})$		$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	$a_{11}, a_{22} < 0$: Stable
	$\hat{X} = \left(\frac{E}{D} - \alpha_3 \right)$		$\lambda_1 = a_{22}, \lambda_{2,3} = \frac{a_{11} \pm \sqrt{\Delta}}{2}$	Fig. 1 (d) $a_{11}, a_{22} > 0$: Unstable
	$\hat{Z} = \frac{a_1 \left(1 - \frac{\hat{X}}{K_1} \right) \left(\frac{\hat{X}^2}{J_1} + \hat{X} + h'_1 \right)}{m_1}$	$\alpha_3 < \frac{E}{D} < K_1 + \alpha_3$	where $\Delta = a_{11}^2 + 4a_{13}a_{31}$, $a_{22} = a_2 + c_2(1 - e^{-b_2\hat{X}})\hat{Z} - \frac{m_2\hat{Z}}{h'_2}$, $a_{11} = -\frac{a_1\hat{X}}{K_1} - \frac{m_1\hat{X}\hat{Z}}{\left(\frac{\hat{X}^2}{J_1} + \hat{X} + h'_1 \right)^2} + \frac{2m_1\hat{X}^2\hat{Z}}{\left(\frac{\hat{X}^2}{J_1} + \hat{X} + h'_1 \right)^3}$, $a_{12} = b_1c_1\hat{X}\hat{Z}, a_{13} = -\frac{m_1\hat{X}}{\left(\frac{\hat{X}^2}{J_1} + \hat{X} + h'_1 \right)}$, $a_{31} = \frac{D^2\hat{Z}^2}{E} = a_{32}$	$a_{22} = 0/a_{11} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{22} < 0, a_{11} > 0\} \cup$ $\{\Delta > 0, a_{22} > 0, a_{11} < 0\}$: Saddle

(iii) Stability properties of Model 3

Feasibility and stability conditions of $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ of system (3) are given in Table 5. The coexistence equilibrium of system (3) is $E_*(X^*, Y^*, Z^*)$ and to get the solutions of this model, we have to solve the following system of equations,

$$\begin{aligned} \frac{dX}{dt} &= a_1X\left(1 - \frac{X}{K_1}\right) + c_1X(1 - e^{-b_1Y})Z - \frac{m_1XZ}{X + \beta_1Z + \alpha_1}, \\ \frac{dY}{dt} &= a_2Y\left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{Y + \beta_2Z + \alpha_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2. \end{aligned}$$

From $\frac{dZ^*}{dt} = 0$ we have,

$$X^* = \frac{E}{D} - \alpha_3 - Y^* \text{ or } Y^* = \frac{E}{D} - \alpha_3 - X^*.$$

From $\frac{dY^*}{dt} = 0$ we have,

$$\begin{aligned} c_2\beta_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3 - Y^*)})Z^{*2} + \{c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3 - Y^*)})(Y^* + \alpha_2) + a_2\beta_2\left(1 - \frac{Y^*}{K_2}\right) - m_2\}Z^* \\ + a_2\left(1 - \frac{Y^*}{K_2}\right)(Y^* + \alpha_2) = 0. \end{aligned}$$

From above it is clear that the above equation is function of (Z, Y) variables. Now by numerical method we have to solve the equation to find its roots.

Similarly from $\frac{dX^*}{dt} = 0$ we have,

$$\begin{aligned} c_1\beta_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})Z^{*2} + \{c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})(X^* + \alpha_1) + a_1\beta_1\left(1 - \frac{X^*}{K_1}\right) - m_1\}Z^* \\ + a_1\left(1 - \frac{X^*}{K_1}\right)(X^* + \alpha_1) = 0. \end{aligned}$$

Similarly is a function of (Z, X) variables. So again by numerical method we have to solve the equation to find its roots.

(iv) Stability properties of Model 4

Feasibility and stability conditions of $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ of system (4) are given in Table 5. The coexistence equilibrium of system (4) is $E_*(X^*, Y^*, Z^*)$ and to get

Table 5

Stability analysis of boundary equilibria of system (3).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(i)	$E_0(0, 0, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(ii)	$E_X(K_1, 0, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{K_1 + \alpha_1} \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(iii)	$E_Y(0, K_2, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & -\frac{m_2 K_2}{K_2 + \alpha_2} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(iv)	$E_Z(0, 0, \tilde{Z})$ $\tilde{Z}$ is any +ve constant	$D = \frac{E}{\alpha_3}$	$\begin{pmatrix} a_1 - \frac{m_1 \tilde{Z}}{\beta_1 \tilde{Z} + \alpha_1} & 0 & 0 \\ 0 & a_2 - \frac{m_2 \tilde{Z}}{\beta_2 \tilde{Z} + \alpha_2} & 0 \\ \frac{E \tilde{Z}^2}{\alpha_3^2} & \frac{E \tilde{Z}^2}{\alpha_3^2} & 2(D - \frac{E}{\alpha_3}) \tilde{Z} \end{pmatrix}$ $\lambda_1 = a_1 - \frac{m_1 \tilde{Z}}{(\beta_1 \tilde{Z} + \alpha_1)}, \lambda_2 = a_2 - \frac{m_2 \tilde{Z}}{(\beta_2 \tilde{Z} + \alpha_2)},$ $\lambda_3 = 2(D - \frac{E}{\alpha_3}) \tilde{Z} = 0$	Neutral
(v)	$E_{XY}(K_1, K_2, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{K_1 + \alpha_1} + c_1 K_1 (1 - e^{-b_1 K_2}) \\ 0 & -a_2 & -\frac{m_2 K_2}{K_2 + \alpha_2} + c_2 K_2 (1 - e^{-b_2 K_1}) \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(vi)	$E_{YZ}(0, \bar{Y}, \bar{Z})$ $\bar{Y} = \left(\frac{E}{D} - \alpha_3 \right)$ $\bar{Z} = \frac{\frac{\bar{Y} + \alpha_2}{m_2} - \beta_2}{a_2 \left(1 - \frac{\bar{Y}}{K_2} \right)}$	$\alpha_3 < \frac{E}{D} < K_2 + \alpha_3$	$J = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{11}, \lambda_{2,3} = \frac{a_{22} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{22}^2 + 4a_{23}a_{32}$,	$a_{11}, a_{22} < 0$: Stable Fig. 1 (e) $a_{11}, a_{22} > 0$: Unstable $a_{11} = 0/a_{22} = 0 \cup \Delta < 0$: Neutral $\{ \Delta > 0, a_{11} < 0, a_{22} > 0 \} \cup$ $\{ \Delta > 0, a_{11} > 0, a_{22} < 0 \}$: Saddle

(continued on next page)

the solutions of this model, we have to solve the following system of equations,

$$\frac{dX}{dt} = a_1 X \left(1 - \frac{X}{K_1} \right) + c_1 X (1 - e^{-b_1 Y}) Z - \frac{m_1 X Z}{\delta_1 X + \beta'_1 Z + \gamma_1 X Z + \alpha'_1},$$

Table 5 (continued).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(vii)	$E_{YZ}(\hat{X}, 0, \hat{Z})$		$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	$a_{11}, a_{22} < 0$: Stable
	$\hat{X} = \left(\frac{E}{D} - \alpha_3 \right)$ $\hat{Z} = \frac{\frac{\hat{X} + \alpha_1}{m_1} - \beta_1}{a_1 \left(1 - \frac{\hat{X}}{K_1} \right)}$	$\alpha_3 < \frac{E}{D} < K_1 + \alpha_3$	$\lambda_1 = a_{22}, \lambda_{2,3} = \frac{a_{11} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{11}^2 + 4a_{13}a_{31}$,	Fig. 1 (f) $a_{11}, a_{22} > 0$: Unstable $a_{22} = 0/a_{11} = 0 \cup \Delta < 0$:
			$a_{22} = a_2 + c_2(1 - e^{-b_2\hat{X}})\hat{Z} - \frac{m_2\hat{Z}}{\beta_2\hat{Z} + \alpha_2}$, $a_{11} = -\frac{a_1\hat{X}}{K_1} - \frac{m_1\hat{X}\hat{Z}}{(\hat{X} + \beta_1\hat{Z} + \alpha_1)^2}$, $a_{12} = b_1c_1\hat{X}\hat{Z}, a_{13} = -\frac{m_1\hat{X}}{(\hat{X} + \beta_1\hat{Z} + \alpha_1)} + \frac{m_1\beta_1\hat{X}\hat{Z}}{(\hat{X} + \beta_1\hat{Z} + \alpha_1)^2}$, $a_{31} = \frac{D^2\hat{Z}^2}{E} = a_{32}$	Neutral $\{\Delta > 0, a_{22} < 0, a_{11} > 0\} \cup$ $\{\Delta > 0, a_{22} > 0, a_{11} < 0\}$: Saddle

$$\frac{dY}{dt} = a_2Y\left(1 - \frac{Y}{K_2}\right) + c_2Y(1 - e^{-b_2X})Z - \frac{m_2YZ}{\delta_2Y + \beta_2'Z + \gamma_2YZ + \alpha_2'},$$

$$\frac{dZ}{dt} = \left(D - \frac{E}{\alpha_3 + X + Y}\right)Z^2.$$

From $\frac{dZ^*}{dt} = 0$ we have,

$$X^* = \frac{E}{D} - \alpha_3 - Y^* \text{ or } Y^* = \frac{E}{D} - \alpha_3 - X^*.$$

From $\frac{dY^*}{dt} = 0$ we have,

$$c_2(\beta_2' + \gamma_2Y^*)(1 - e^{-b_2(\frac{E}{D} - \alpha_3 - Y^*)})Z^{*2} + \{c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3 - Y^*)})(\delta_2Y^* + \alpha_2') \\ + a_2(\beta_2' + \gamma_2Y^*)\left(1 - \frac{Y^*}{K_2}\right) - m_2\}Z^* + a_2\left(1 - \frac{Y^*}{K_2}\right)(\delta_2Y^* + \alpha_2') = 0.$$

From above it is clear that the above equation is function of (Z, Y) variables. Now by numerical method we have to solve the equation to find its roots.

Similarly from $\frac{dX^*}{dt} = 0$ we have,

$$c_1(\beta_1' + \gamma_1X^*)(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})Z^{*2} + \{c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})(\delta_1Y^* + \alpha_1') \\ + a_1(\beta_1' + \gamma_1X^*)\left(1 - \frac{X^*}{K_1}\right) - m_1\}Z^* + a_1\left(1 - \frac{X^*}{K_1}\right)(\delta_1X^* + \alpha_1') = 0.$$

Similarly it is a function of (Z, X) variables. So again by numerical method we have to solve the equation to find its roots.

(v) Stability properties of Model 5:

Feasibility and stability conditions of $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \tilde{Z})$, $E_{XY}(K_1, K_2, 0)$, $E_{YZ}(0, \tilde{Y}, \tilde{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ of system (5) are given in Table 7. The coexistence equilibrium point of system (5) is $E_*(X^*, Y^*, Z^*)$, where

$$Y^* = \frac{E}{D} - \alpha_3 - X^*, \quad Z^* = \frac{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}}{\frac{m_2}{(\frac{E}{D} - \alpha_3 - X^*)^p + v_2} - c_2(1 - e^{-b_2X^*})} = \frac{a_1 - \frac{a_1X^*}{K_1}}{\frac{m_1}{X^{*p} + v_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}.$$

Let us assume that $0 < X^* < K_1 < \frac{E}{D} - \alpha_3$ and X^* is the positive root of the equation

$$f(X^*) = \frac{\frac{m_1}{X^{*p} + v_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3 - X^*)})}{\frac{m_2}{(\frac{E}{D} - \alpha_3 - X^*)^p + v_2} - c_2(1 - e^{-b_2X^*})} - \frac{a_1 - \frac{a_1X^*}{K_1}}{a_2 - \frac{a_2(\frac{E}{D} - \alpha_3 - X^*)}{K_2}} = 0.$$

We have,

$$f(0) = \frac{\frac{m_1}{v_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3)})}{\frac{m_2}{(\frac{E}{D} - \alpha_3)^p + v_2}} - \frac{a_1}{a_2(1 - \frac{E}{K_2} - \alpha_3)} < 0,$$

Table 6

Stability analysis of boundary equilibria of system (4).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(i)	$E_0(0, 0, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(ii)	$E_X(K_1, 0, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{(\delta_1 K_1 + \alpha'_1)} \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(iii)	$E_Y(0, K_2, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & -\frac{m_2 K_2}{(\delta_2 K_2 + \alpha'_2)} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(iv)	$E_Z(0, 0, \tilde{Z})$ $\tilde{Z}$ is any +ve constant	$D = \frac{E}{\alpha_3}$	$\begin{pmatrix} a_1 - \frac{m_1 \tilde{Z}}{(\beta'_1 \tilde{Z} + \alpha'_1)} & 0 & 0 \\ 0 & a_2 - \frac{m_2 \tilde{Z}}{(\beta'_2 \tilde{Z} + \alpha'_2)} & 0 \\ \frac{E \tilde{Z}^2}{\alpha_3^2} & \frac{E \tilde{Z}^2}{\alpha_3^2} & 2(D - \frac{E}{\alpha_3}) \tilde{Z} \end{pmatrix}$ $\lambda_1 = a_1 - \frac{m_1 \tilde{Z}}{(\beta'_1 \tilde{Z} + \alpha'_1)}, \lambda_2 = a_2 - \frac{m_2 \tilde{Z}}{(\beta'_2 \tilde{Z} + \alpha'_2)},$ $\lambda_3 = 2(D - \frac{E}{\alpha_3}) \tilde{Z} = 0$	Neutral
(v)	$E_{XY}(K_1, K_2, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{(\delta_1 K_1 + \alpha'_1)} + c_1 K_1 (1 - e^{-b_1 K_2}) \\ 0 & -a_2 & -\frac{m_2 K_2}{(\delta_2 K_2 + \alpha'_2)} + c_2 K_2 (1 - e^{-b_2 K_1}) \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(vi)	$E_{YZ}(0, \tilde{Y}, \tilde{Z})$ $\tilde{Y} = \left(\frac{E}{D} - \alpha_3 \right)$ $\tilde{Z} = \frac{\delta_2 \tilde{Y} + \alpha'_2}{\frac{m_2}{a_2 \left(1 - \frac{\tilde{Y}}{K_2} \right)} - \left(\beta'_2 + \gamma_2 \tilde{Y} \right)}$	$\alpha_3 < \frac{E}{D} < K_2 + \alpha_3$	$J = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{11}, \lambda_{2,3} = \frac{a_{22} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{22}^2 + 4a_{23}a_{32}$, $a_{11} = a_1 + c_1(1 - e^{-b_1 \tilde{Y}}) \tilde{Z} - \frac{m_1 \tilde{Z}}{\beta'_1 \tilde{Z} + \alpha'_1},$ $a_{22} = -\frac{a_2 \tilde{Y}}{K_2} - \frac{m_2 \tilde{Y} \tilde{Z} (\delta_2 + \gamma_2 \tilde{Z})}{(\delta_2 \tilde{Y} + \beta'_2 \tilde{Z} + \gamma_2 \tilde{Y} \tilde{Z} + \alpha'_2)^2},$ $a_{23} = -\frac{m_2 \tilde{Y}}{(\delta_2 \tilde{Y} + \beta'_2 \tilde{Z} + \gamma_2 \tilde{Y} \tilde{Z} + \alpha'_2)} + \frac{m_2 \tilde{Y} \tilde{Z} (\beta'_2 + \gamma_2 \tilde{Y})}{(\delta_2 \tilde{Y} + \beta'_2 \tilde{Z} + \gamma_2 \tilde{Y} \tilde{Z} + \alpha'_2)^2},$ $a_{21} = b_2 c_2 \tilde{Y} \tilde{Z}, a_{31} = \frac{D^2 \tilde{Z}^2}{E} = a_{32}$	$a_{11}, a_{22} < 0$: Stable $a_{11}, a_{22} > 0$: Unstable $a_{11} = 0/a_{22} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{11} < 0, a_{22} > 0\} \cup$ $\{\Delta > 0, a_{11} > 0, a_{22} < 0\}$: Saddle

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Table 6 (continued).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(vii)	$E_{VZ}(\hat{X}, 0, \hat{Z})$		$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	$a_{11}, a_{22} < 0$: Stable
	$\hat{X} = \left(\frac{E}{D} - \alpha_3 \right)$		$\lambda_1 = a_{22}, \lambda_{2,3} = \frac{a_{11} \pm \sqrt{\Delta}}{2}$	Fig. 1 (h) $a_{11}, a_{22} > 0$: Unstable
	$\hat{Z} = \frac{\delta_2 \hat{X} + \alpha'_1}{\frac{m_1}{a_1 \left(1 - \frac{\hat{X}}{K_1} \right)} - \left(\beta'_1 + \gamma_1 \hat{X} \right)}$	$\alpha_3 < \frac{E}{D} < K_1 + \alpha_3$	where $\Delta = a_{11}^2 + 4a_{13}a_{31}$,	$a_{22} = 0/a_{11} = 0 \cup \Delta < 0$: Neutral
			$a_{22} = a_2 + c_2(1 - e^{-b_2 \hat{X}})\hat{Z} - \frac{m_2 \hat{Z}}{\beta'_2 \hat{Z} + \alpha'_2}$, $a_{11} = -\frac{a_1 \hat{X}}{K_1} - \frac{m_1 \hat{X} \hat{Z}(\delta_1 + \gamma_1 \hat{Z})}{(\delta_1 \hat{X} + \beta'_1 \hat{Z} + \gamma_1 \hat{X} \hat{Z} + \alpha'_1)^2}$, $a_{13} = -\frac{m_1 \hat{X}}{(\delta_1 \hat{X} + \beta'_1 \hat{Z} + \gamma_1 \hat{X} \hat{Z} + \alpha'_1)^2} + \frac{m_1 \hat{X} \hat{Z}(\beta'_1 + \gamma_1 \hat{X})}{(\delta_1 \hat{X} + \beta'_1 \hat{Z} + \gamma_1 \hat{X} \hat{Z} + \alpha'_1)^2}$, $a_{12} = b_1 c_1 \hat{X} \hat{Z}, a_{31} = \frac{D^2 \hat{Z}^2}{E} = a_{32}$	$\{\Delta > 0, a_{22} < 0, a_{11} > 0\} \cup \{\Delta > 0, a_{22} > 0, a_{11} < 0\}$: Saddle

$$\text{as } \frac{m_1}{v_1} - c_1(1 - e^{-b_1(\frac{E}{D} - \alpha_3)}) < 0.$$

$$f\left(\frac{E}{D} - \alpha_3\right) = \frac{\frac{m_1}{(\frac{E}{D} - \alpha_3)^p + v_1}}{\frac{m_2}{v_2} - c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3)})} - \frac{a_1(1 - \frac{E}{D} - \frac{\alpha_3}{K_1})}{a_2} > 0$$

$$\text{as } \frac{m_2}{v_2} - c_2(1 - e^{-b_2(\frac{E}{D} - \alpha_3)}) > 0.$$

From the above assumptions we get, $f(0)f(\frac{E}{D} - \alpha_3) < 0$. So there is a positive root of this equation lies in $(0, (\frac{E}{D} - \alpha_3))$. Therefore, the coexistence equilibrium point $E_*(X^*, Y^*, Z^*)$ exists.

From the above feasibility and stability analysis of seven boundary equilibrium points $(E_0(0, 0, 0), E_X(K_1, 0, 0), E_Y(0, K_2, 0), E_Z(0, 0, \bar{Z}), E_{XY}(K_1, K_2, 0), E_{VZ}(0, \bar{Y}, \bar{Z}), E_{XZ}(\hat{X}, 0, \hat{Z}))$ of all the systems (1)–(5), we see that first five equilibrium points are all neutral point where the rest two are showing different dynamics depending upon system parametric conditions. $E_0(0, 0, 0), E_X(K_1, 0, 0), E_Y(0, K_2, 0)$ and $E_{XY}(K_1, K_2, 0)$ are always feasible for all the systems. Also feasibility of $E_Z(0, 0, \bar{Z})$ is same for all the systems. But feasibility of $E_{VZ}(0, \bar{Y}, \bar{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ are depending upon some parametric restrictions. Now we are focusing to determine the stability condition of coexistence equilibrium point $E_*(X^*, Y^*, Z^*)$ for systems (1)–(5), characteristic equation at $E_*(X^*, Y^*, Z^*)$ is given by

$$P(\lambda) = \lambda^3 + \mu_2 \lambda^2 + \mu_1 \lambda + \mu_0 = 0, \quad (7)$$

with coefficients

$$\mu_2 = (J_{11} + J_{22} + J_{33}),$$

$$\mu_1 = J_{11}J_{33} + J_{11}J_{22} + J_{22}J_{33} - J_{32}J_{23} - J_{12}J_{21} - J_{13}J_{31},$$

$$\mu_0 = J_{11}J_{32}J_{23} + J_{22}J_{13}J_{31} + J_{33}J_{12}J_{21} - J_{11}J_{22}J_{33} - J_{31}J_{23}J_{12} - J_{13}J_{21}J_{32},$$

where

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

represents the Jacobian matrix of the model system (1)–(5) calculated at $E_*(X^*, Y^*, Z^*)$. The components of the Jacobian matrix are given below:

$$J_{11} = f_{1X} + \bar{f}_{1X}$$

$$J_{21} = f_{2X} + \bar{f}_{2X}$$

$$J_{31} = f_{3X} + \bar{f}_{3X}$$

$$J_{12} = f_{1Y} + \bar{f}_{1Y}$$

$$J_{22} = f_{2Y} + \bar{f}_{2Y}$$

$$J_{32} = f_{3Y} + \bar{f}_{3Y}$$

$$J_{13} = f_{1Z} + \bar{f}_{1Z}$$

$$J_{23} = f_{2Z} + \bar{f}_{2Z}$$

$$J_{33} = f_{3Z} + \bar{f}_{3Z}$$

The above entities of the Jacobian matrix of the model system (1)–(5) have been listed in the Table 2.

According to Routh–Hurwitz criterion, the coexistence equilibrium point $E_*(X^*, Y^*, Z^*)$ is locally asymptotically stable if $\text{Re}(\lambda) < 0$ for each model system (1)–(5), if and only if these conditions hold:

$$\mu_2 > 0, \mu_1 > 0, \mu_0 > 0 \quad \text{and} \quad [\mu_2 \mu_1 - \mu_0] > 0. \quad (8)$$

Table 7
Stability analysis of boundary equilibria of system (5).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(i)	$E_0(0, 0, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(ii)	$E_X(K_1, 0, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{K_1^q + v_1} \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = a_2, \lambda_3 = 0$	Neutral
(iii)	$E_Y(0, K_2, 0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & -\frac{m_2 K_2}{K_2^q + v_2} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(iv)	$E_Z(0, 0, \tilde{Z})$ $\tilde{Z}$ is any +ve constant	$D = \frac{E}{\alpha_3}$	$\begin{pmatrix} a_1 - \frac{m_1 v_1 \tilde{Z}}{v_1^2} & 0 & 0 \\ 0 & a_2 - \frac{m_2 v_2 \tilde{Z}}{v_2^2} & 0 \\ \frac{E \tilde{Z}^2}{\alpha_3^2} & \frac{E \tilde{Z}^2}{\alpha_3^2} & 2(D - \frac{E}{\alpha_3}) \tilde{Z} \end{pmatrix}$ $\lambda_1 = a_1 - \frac{m_1 v_1 \tilde{Z}}{v_1^2}, \lambda_2 = a_2 - \frac{m_2 v_2 \tilde{Z}}{v_2^2},$ $\lambda_3 = 2(D - \frac{E}{\alpha_3}) \tilde{Z} = 0$	Neutral
(v)	$E_{XY}(K_1, K_2, 0)$	always	$\begin{pmatrix} -a_1 & 0 & -\frac{m_1 K_1}{K_1^q + v_1} + c_1 K_1 (1 - e^{-b_1 K_2}) \\ 0 & -a_2 & -\frac{m_2 K_2}{K_2^q + v_2} + c_2 K_2 (1 - e^{-b_2 K_1}) \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(vi)	$E_{YZ}(0, \bar{Y}, \bar{Z})$ $\bar{Y} = \left(\frac{E}{D} - \alpha_3 \right)$ $\bar{Z} = \frac{a_2 \left(1 - \frac{\bar{Y}}{K_2} \right) \left(\bar{Y}^q + v_2 \right)}{m_2}$	$\alpha_3 < \frac{E}{D} < K_2 + \alpha_3$	$J = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\lambda_1 = a_{11}, \lambda_{2,3} = \frac{a_{22} \pm \sqrt{\Delta}}{2}$ where $\Delta = a_{22}^2 + 4a_{23}a_{32}$, $a_{11} = a_1 + c_1(1 - e^{-b_1 \bar{Y}}) \bar{Z} - \frac{m_1 \bar{Z}}{v_1},$ $a_{22} = -\frac{a_2 \bar{Y}}{K_2} + \frac{q m_2 \bar{Y}^q \bar{Z}}{(\bar{Y}^q + v_2)^2}, a_{21} = b_2 c_2 \bar{Y} \bar{Z},$ $a_{23} = -\frac{m_2 \bar{Y}}{\bar{Y}^q + v_2}, a_{31} = \frac{D^2 \bar{Z}^2}{E} = a_{32}$	$a_{11}, a_{22} < 0$: Stable $a_{11}, a_{22} > 0$: Unstable $a_{11} = 0/a_{22} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{11} < 0, a_{22} > 0\} \cup$ $\{\Delta > 0, a_{11} > 0, a_{22} < 0\}$: Saddle

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To represent any explicit result for each models are difficult due to the highly nonlinear mathematical expressions. So, we show the local stability results of the coexistence equilibrium point $E_*(X^*, Y^*, Z^*)$ for system (1)–(5) numerically by time series Fig. 2, stability domain diagrams w.r.t the phase-portrait in parametric domains Fig. 6 and phase-portrait in initial data domain Fig. 5.

Table 7 (continued).

	Equilibrium and coordinate	Feasibility condition	Jacobian matrix and eigenvalues	Stability status
(vii)	$E_{YZ}(\hat{X}, 0, \hat{Z})$		$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	$a_{11}, a_{22} < 0$: Stable
	$\hat{X} = \left(\frac{E}{D} - \alpha_3 \right)$		$\lambda_1 = a_{22}, \lambda_{2,3} = \frac{a_{11} \pm \sqrt{\Delta}}{2}$	Fig. 1 (j) $a_{11}, a_{22} > 0$: Unstable
	$\hat{Z} = \frac{a_1 \left(1 - \frac{\hat{X}}{K_1} \right) (\hat{X}^p + v_1)}{m_1}$	$\alpha_3 < \frac{E}{D} < K_1 + \alpha_3$	where $\Delta = a_{11}^2 + 4a_{13}a_{31}$, $a_{22} = a_2 + c_2(1 - e^{-b_2\hat{X}})\hat{Z} - \frac{m_2\hat{Z}}{v_2}$, $a_{11} = -\frac{a_1\hat{X}}{K_1} + \frac{pm_1\hat{X}^p\hat{Z}}{(\hat{X}^p + v_1)^2}$, $a_{12} = b_1c_1\hat{X}\hat{Z}$, $a_{13} = -\frac{m_1\hat{X}}{\hat{X}^p + v_1}$, $a_{31} = \frac{D^2\hat{Z}^2}{E} = a_{32}$	$a_{22} = 0/a_{11} = 0 \cup \Delta < 0$: Neutral $\{\Delta > 0, a_{22} < 0, a_{11} > 0\} \cup$ $\{\Delta > 0, a_{22} > 0, a_{11} < 0\}$: Saddle

4. Finite time blow-up

4.1. Motivation and control mechanisms

Invasive predators often wreak havoc on native ecosystems. Data shows that the growth in the population of small Indian mongoose (*Herpestes auropunctatus*) in Fiji, Mauritius and Amami-oshima islands has resulted in a severe decline in the population of native species such as mammals, birds and reptiles [38]. Another example are Zebra mussels (*Dreissena polymorpha*), which are mostly found in Russia, Europe and North America, and are one of the most aggressive freshwater invaders because their population numbers grow so quickly [39]. Several other examples can be found in the literature [20,21,40].

Biological control is an adopted strategy to limit harmful populations [41]. The objective of a biological control is to establish a management strategy that best controls and decreases the harmful population to healthy levels as opposed to high and risky levels. Naturally, how does one define *high* level and further, how well does the biological control actually work, at various high levels? We have recently started investigating this question via the mathematical property of finite time blow-up [25,26,42]. Finite time blow up of model system (1)–(5) has been investigated in this section. We first present our motivation.

Definition 4.1. Given a ODE model of a nonlinear process say,

$$\frac{d\mathbf{u}}{dt} = f(\mathbf{u}), \quad (9)$$

we say finite time blow-up occurs if,

$$\lim_{t \rightarrow T^* < \infty} \|\mathbf{u}\| \rightarrow \infty, \quad (10)$$

where $\|\cdot\|$ is the standard sup norm on $\mathbb{R}^n$, u is the state variable in question that depends on time (t) ($\mathbf{x}$) and T^* is the blow-up time.

In the context of population biology, finite time blow-up has also been well investigated [43–46]. Note, we have now introduced an alternate viewpoint: finite time blow-up, can be viewed as *mimicing* the explosive growth of an invasive species. This is formalized by equating:

$$\text{finite time blow-up} = \text{uncontrollable and unmanageable population level}. \quad (11)$$

Here, the blow-up time T^* is viewed as the *disaster* time, for the ecosystem.

Remark 4.2. Although populations cannot reach infinite values in finite time, they can grow rapidly [47]. For example, experimental evidence suggest that the human population may be growing hyperbolically, rather than logistically [48]. Data on the Burmese python suggests, that its population is growing at least exponentially [19].

Our approach investigates biological control mechanisms, that attempt to lower and control the targeted population *before time* T^* . This approach has distinct advantages:

- (i) There is no ambiguity as to what is a disastrous high level of population.
- (ii) There is a clear demarcation between when or if the disaster occurs.

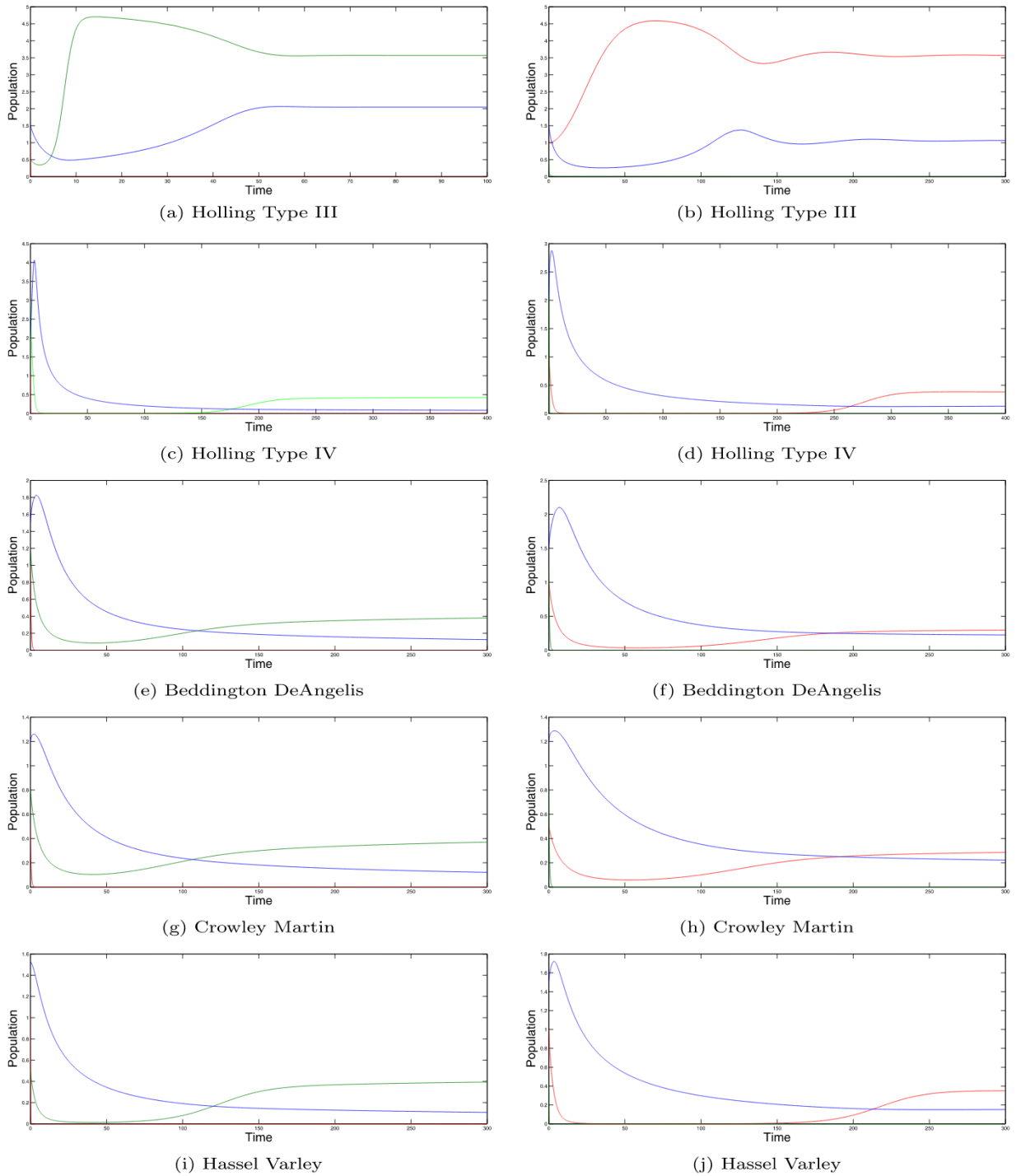


Fig. 1. Time series diagrams for the stability of E_{YZ} and E_{XZ} equilibria of the model systems (1)–(5) are shown in first and second panels respectively. Here red, green and blue colors represent first prey, second prey and generalist predator respectively. Parameter sets are given in Table 9.

(iii) Our controls focus on *avoiding* classical chemical and biological controls.

(iv) This method provides a predictive modeling tool for various ecological settings.

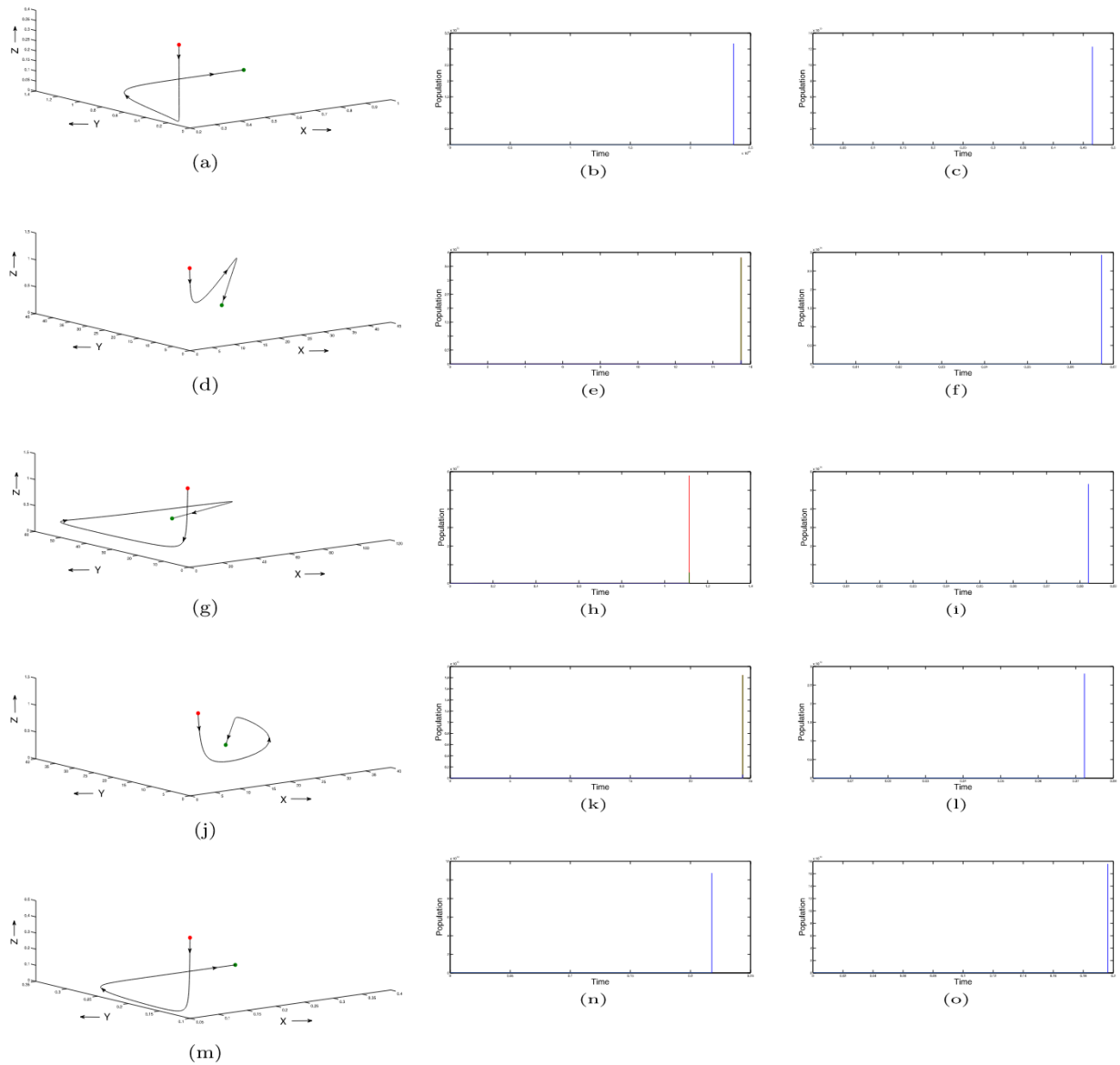


Fig. 2. Phase portrait (first panel: asymptotic stability corresponding to [Tables 10](#) and [11](#)) and time series (second panel corresponding to [Table 10](#) and third panel corresponding to [Table 11](#): blow-up) of system (1)–(5).

4.2. Large data blow-up in the model system

We first prove a classical result on this system

Theorem 4.3. Consider the three species food chain model (1)–(3). $Z(t)$ the solution to (3) blows up in finite time, that is

$$\lim_{t \rightarrow T^* < \infty} \|Z(t)\|_{\infty} \rightarrow \infty, \quad (12)$$

for initial data (X_0, Y_0, Z_0) that is sufficiently large.

Remark 4.4. The proof follows [23].

Proof. Consider (1)–(3), with positive initial conditions.

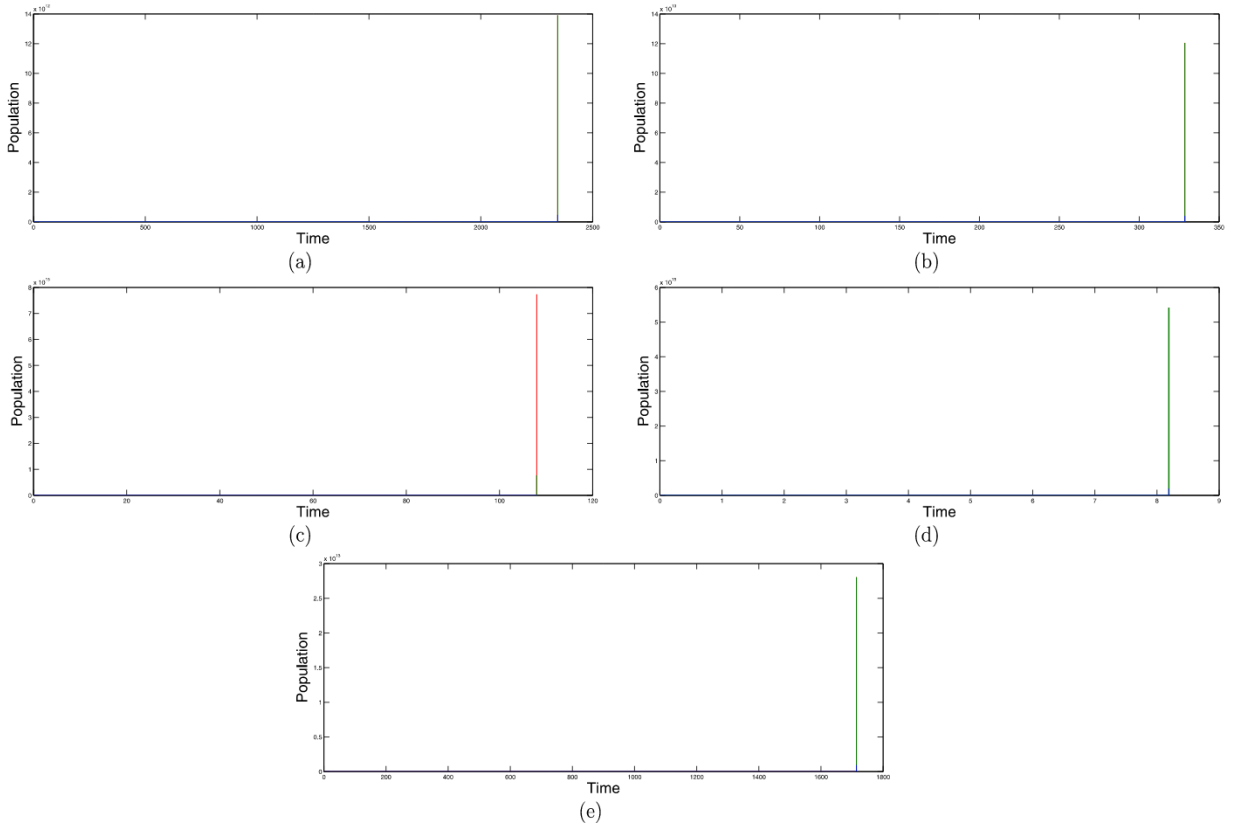


Fig. 3. Prey blows up before than the generalist predator of model system (1)–(5). Here red, green and blue colors represent first prey, second prey and generalist predator respectively. Parameter set is given in Table 12.

By integrating (3) we obtain

$$-\frac{1}{Z} + \frac{1}{Z_0} = -Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3}$$

which gives

$$Z = \frac{1}{\frac{1}{Z_0} - Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3}}.$$

Our goal is to show that the function:

$$\psi(t) = \frac{1}{Z_0} - Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} \rightarrow 0 \text{ as } t \rightarrow T^* < \infty. \quad (13)$$

This will then show that the solution Z will blow-up at the finite time, $t = T^*$. Essentially, for (X_0, Y_0) chosen sufficiently large, there exists a $\delta > 0$ such that

$$\begin{aligned} & \frac{1}{Z_0} - Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} \\ &= \frac{1}{Z_0} + \left[-D + E \frac{1}{t} \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} \right] t \\ &= \frac{1}{Z_0} + \left[-\frac{D}{2} - \frac{D}{2} + E \frac{1}{t} \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} \right] t \\ &< \frac{1}{Z_0} - \frac{D}{2}t, \quad \text{for all } t \in (0, \delta). \end{aligned} \quad (14)$$

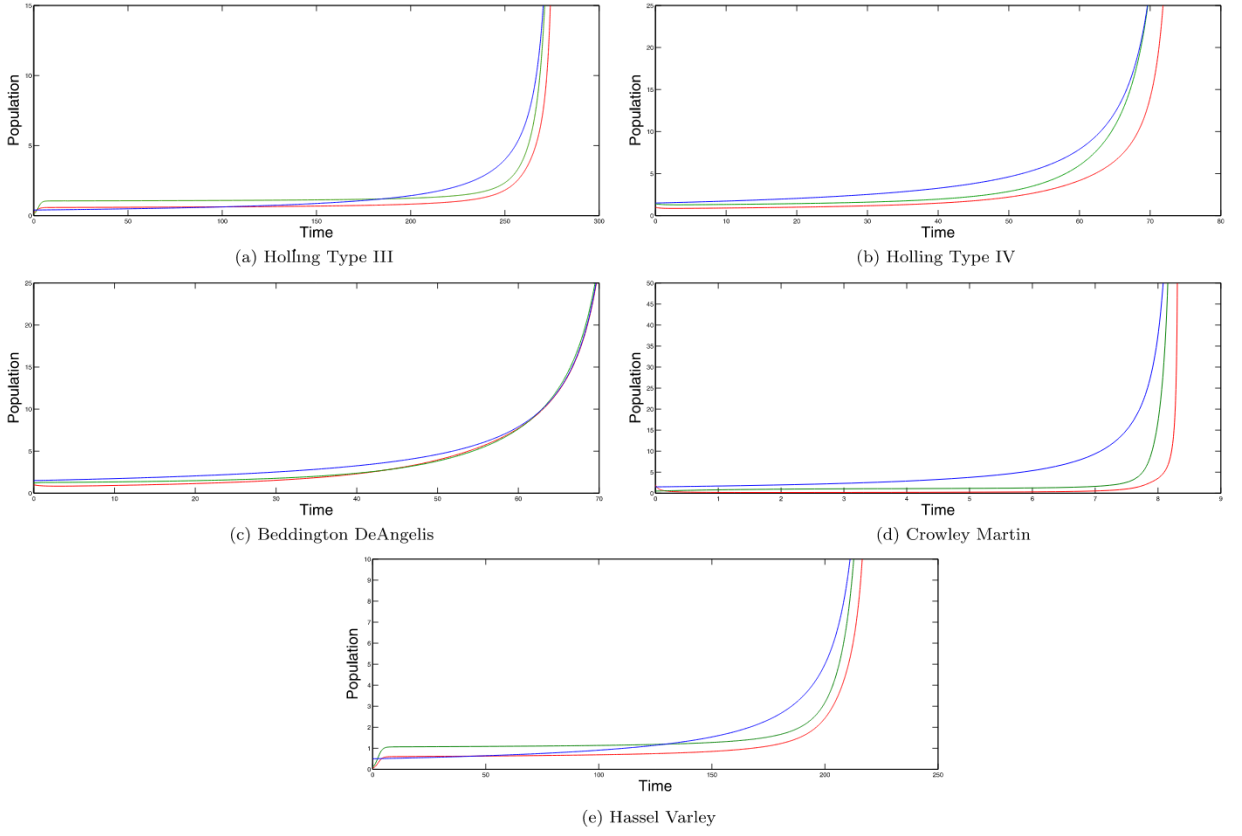


Fig. 4. First prey, second prey and generalist predator blow up together of model system (1)–(5). Here red, green and blue colors represent first prey, second prey and generalist predator respectively. Parameter set is given in Table 12.

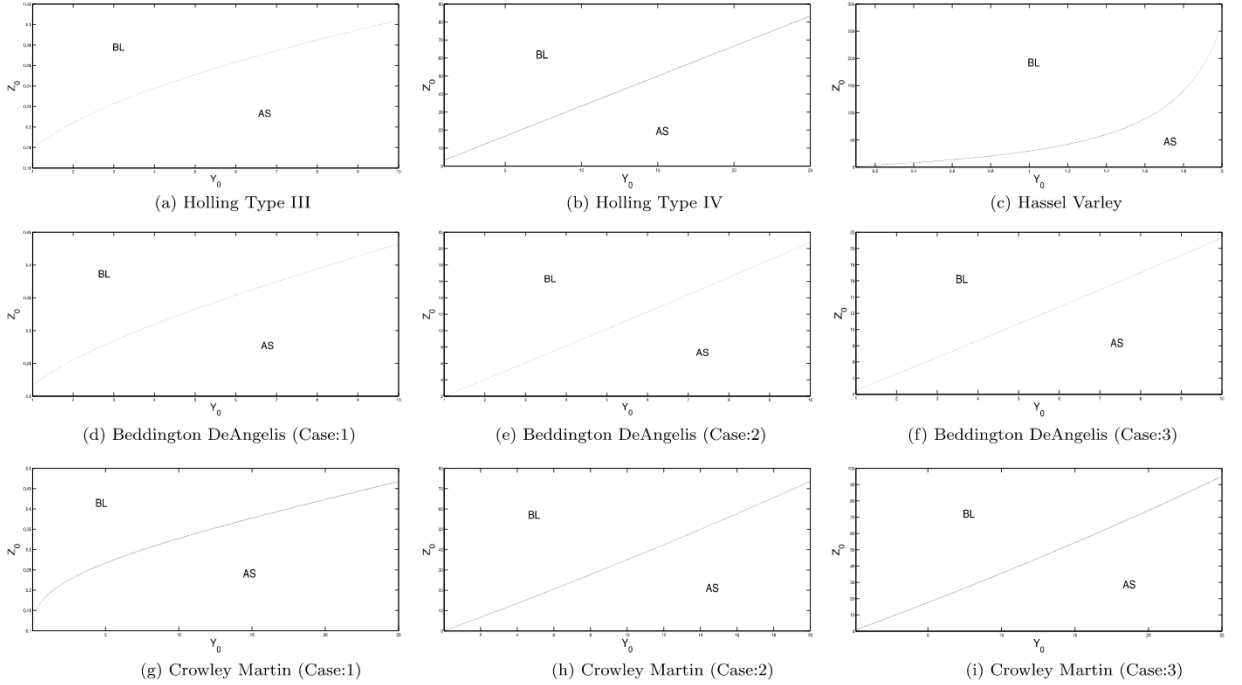


Fig. 5. Stability region with varying initial values Y_0 and Z_0 of model systems (1)–(5). Here AS and BL represent Asymptotically stable and Blow up respectively.

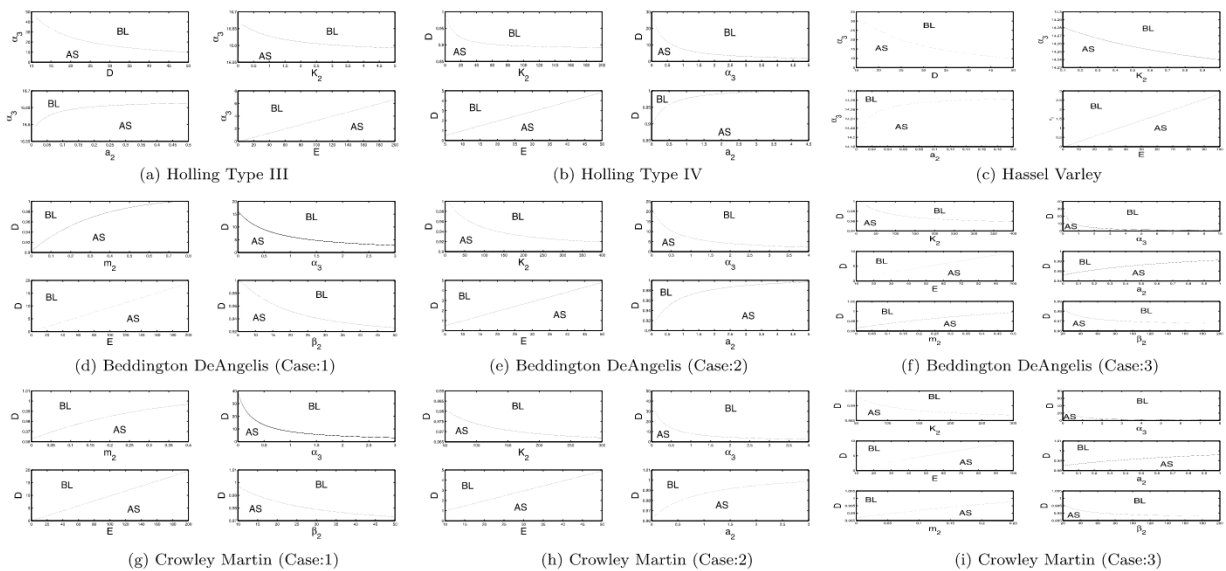


Fig. 6. Stability regions in different parametric phase-planes of model systems (1)–(5) corresponding to the parameters listed in Table 8. Here AS and BL represent Asymptotically stable and Blow up respectively.

This is because by the continuity of the state variables X, Y , we have for δ sufficiently small, and $t \in (0, \delta)$

$$E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} < \frac{Dt}{2} \quad (15)$$

and so

$$E \frac{1}{t} \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} < \frac{D}{2} \quad (16)$$

If now Z_0 is chosen sufficiently large, then we can find $T^{**} \in (0, \delta)$ such that

$$\frac{1}{Z_0} - \frac{D}{2} T^{**} = 0.$$

Now by application of the classical intermediate value theorem on the continuous function ψ , we obtain the existence of some $T^* \in (0, \delta)$, $T^* < T^{**}$, s.t. $\psi(T^*) = 0$. This implies $Z(t)$ the solution of (3), blows-up in finite time, at $t = T^*$, and the theorem is proved. $\square$

4.3. Sufficient conditions for large or small data blow-up in the model system

We now posit sufficient conditions on the initial data so that blow-up occurs, whether the data is large or small. Since all of the presented models have a similar generic structure, we prove the blow-up results for one case. We pick system (4). We will first show that this model blows-up in finite time and prove it in three cases in the following. Consider

$$\begin{aligned} \frac{dX}{dt} &= a_1 X \left(1 - \frac{X}{K_1} \right) + c_1 X (1 - e^{-b_1 Y}) Z - \frac{m_1 X Z}{\delta_1 X + \beta'_1 Z + \gamma_1 X Z + \alpha'_1}, \\ \frac{dY}{dt} &= a_2 Y \left(1 - \frac{Y}{K_2} \right) + c_2 Y (1 - e^{-b_2 X}) Z - \frac{m_2 Y Z}{\delta_2 Y + \beta'_2 Z + \gamma_2 Y Z + \alpha'_2}, \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y} \right) Z^2 \end{aligned} \quad (17)$$

with suitable positive initial condition (X_0, Y_0, Z_0) .

Theorem 4.5. Consider the model system given by (17), for any choice of parameters, and $\delta_1 > 0$, such that $D > \delta_1$, there exists initial data such that if this data meets the largeness condition,

Case 1:

$$|Z_0| \ln \left(\frac{|Y_0|}{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2}, \quad (18)$$

Case 2:

$$|Z_0| \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{|Y_0|} \right) > \frac{a_2}{\delta_1 k_2}, \quad (19)$$

Case 3:

$$|Z_0| \ln \left(\frac{\frac{|Y_0|}{\frac{a_2}{k_2} |Y_0| + \frac{m_2}{\beta_2'}}}{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2'}, \quad (20)$$

then the state variable Z will blow-up in finite time, that is

$$\lim_{t \rightarrow T^* < \infty} \|Z\| \rightarrow \infty. \quad (21)$$

Here the blow-up time $T^* \leq \frac{1}{\delta_1 |Y_0|}$.

Proof.

Case 1 Consider the equation for the top predator

$$\frac{dZ}{dt} = (D - \frac{E}{\alpha_3 + X + Y}) Z^2,$$

blow-up is trivial if $D > \frac{E}{\alpha_3}$ [42]. However if $D < \frac{E}{\alpha_3}$, in order to guarantee blow-up, we must guarantee that

$$Y > \left(\frac{E}{D - \delta_1} - \alpha_3 \right). \quad (22)$$

To get the result, we will solve one of the prey equations

$$\frac{dY}{dt} = a_2 Y \left(1 - \frac{Y}{K_2} \right) + c_2 Y (1 - e^{-b_2 X}) Z - \frac{m_2 Y Z}{\delta_2 Y + \beta_2' Z + \gamma_2 Y Z + \alpha_2'},$$

and we have via positivity

$$\frac{dY}{dt} \geq -\frac{m_2}{\beta_2' Y},$$

now from above we get,

$$|Y| > |Y_0| e^{-\frac{m_2}{\beta_2'} t}.$$

Thus using (22) we have,

$$\frac{\beta_2'}{m_2} \ln \left(\frac{|Y_0|}{\frac{E}{D-\delta_1} - \alpha_3} \right) > t.$$

Note that

$$\frac{dZ}{dt} = \delta_1 Z^2,$$

blows-up at time $T^* = \frac{1}{\delta_1 |Z_0|}$, thus if we choose data such that

$$\frac{\beta_2'}{m_2} \ln \left(\frac{|Y_0|}{\frac{E}{D-\delta_1} - \alpha_3} \right) > t > T^* = \frac{1}{\delta_1 |Z_0|}.$$

Then the above guarantees that Y will remain above the critical level $\frac{E}{D-\delta_1} - \alpha_3$, for sufficiently long enough time, for Z to blow-up. This yields that as long as the following holds

$$|Z_0| \ln \left(\frac{|Y_0|}{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta'_2},$$

Z will blow up in finite time, independent of the choice of the parameters. Thus the theorem is proved.

Case 2

If we will follow similarly as Case1 then we have,

$$\frac{dY}{dt} \geq -\frac{a_2}{K_2} Y^2,$$

and we get,

$$\frac{1}{|Y|} - \frac{1}{|Y_0|} > \frac{a_2}{K_2} t.$$

Now using (22) we have,

$$\frac{K_2}{a_2} \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{|Y_0|} \right) > t.$$

Thus the above guarantees that Y will remain above the critical level $\frac{E}{D-\delta_1} - \alpha_3$, for sufficiently long enough time, for Z to blow-up. This yields that as long as the following holds

$$|Z_0| \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{|Y_0|} \right) > \frac{a_2}{\delta_1 K_2}.$$

Z will blow up in finite time, independent of the choice of the parameters. Thus the theorem is proved.

Case 3

Similarly as previous two cases, we will follow the same steps. In this case, we consider

$$\frac{dY}{dt} \geq -\frac{a_2}{K_2} - \frac{m_2}{\beta'_2} Y,$$

now from above we get,

$$\ln \left(\frac{\frac{|Y_0|}{\frac{a_2}{K_2} |Y_0| + \frac{m_2}{\beta'_2}}}{\frac{|Y|}{\frac{a_2}{K_2} |Y| + \frac{m_2}{\beta'_2}}} \right) > \frac{m_2}{\beta'_2} t.$$

Thus using (22) we have,

$$\frac{\beta'_2}{m_2} \ln \left(\frac{\frac{|Y_0|}{\frac{a_2}{K_2} |Y_0| + \frac{m_2}{\beta'_2}}}{\frac{\frac{E}{D-\delta_1} - \alpha_3}{\frac{a_2}{K_2} (\frac{E}{D-\delta_1} - \alpha_3) + \frac{m_2}{\beta'_2}}} \right) > t.$$

Note that

$$\frac{dZ}{dt} = \delta_1 Z^2,$$

blows-up at time $T^* = \frac{1}{\delta_1 |Z_0|}$, thus if we choose data such that

$$\frac{\beta'_2}{m_2} \ln \left(\frac{\frac{|Y_0|}{\frac{a_2}{K_2} |Y_0| + \frac{m_2}{\beta'_2}}}{\frac{\frac{E}{D-\delta_1} - \alpha_3}{\frac{a_2}{K_2} (\frac{E}{D-\delta_1} - \alpha_3) + \frac{m_2}{\beta'_2}}} \right) > t > T^* = \frac{1}{\delta_1 |Z_0|}.$$

Then the above guarantees that Y will remain above the critical level $\frac{E}{D-\delta_1} - \alpha_3$, for sufficiently long enough time, for Z to blow-up. This yields that as long as the following holds

$$|Z_0| \ln \left(\frac{\frac{|Y_0|}{\frac{a_2}{K_2} |Y_0| + \frac{m_2}{\beta'_2}}}{\frac{\frac{E}{D-\delta_1} - \alpha_3}{\frac{a_2}{K_2} (\frac{E}{D-\delta_1} - \alpha_3) + \frac{m_2}{\beta'_2}}} \right) > \frac{m_2}{\delta_1 \beta'_2},$$

Table 8
Blow-up phenomena of system (1)–(5).

Model System	Blow-up condition X	Blow-up condition Y	Blow-up condition for Z	
			$(D > \frac{E}{\alpha_3})$	$(D < \frac{E}{\alpha_3})$
(1)	Trivial	Trivial	Trivial Fig. 2 (c)	$ Z_0 \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{ Y_0 } \right) > \frac{a_2}{\delta_1 k_2}$. Fig. 2 (b)
(2)	Trivial	Trivial	Trivial Fig. 2 (f)	$ Z_0 \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{ Y_0 } \right) > \frac{a_2}{\delta_1 k_2}$. Fig. 2 (e)
(3)	Trivial	Trivial	Trivial Fig. 2 (i)	Case 1: $ Z_0 \ln \left(\frac{ Y_0 }{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2}$, Case 2: $ Z_0 \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{ Y_0 } \right) > \frac{a_2}{\delta_1 k_2}$, Case 3: $ Z_0 \ln \left(\frac{\frac{ Y_0 }{\left(\frac{a_2}{k_2} Y_0 + \frac{m_2}{\beta_2} \right)}}{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2}$. Fig. 2 (h)
(4)	Trivial	Trivial	Trivial Fig. 2 (l)	Case 1: $ Z_0 \ln \left(\frac{ Y_0 }{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2}$, Case 2: $ Z_0 \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{ Y_0 } \right) > \frac{a_2}{\delta_1 k_2}$, Case 3: $ Z_0 \ln \left(\frac{\frac{ Y_0 }{\left(\frac{a_2}{k_2} Y_0 + \frac{m_2}{\beta_2} \right)}}{\frac{E}{D-\delta_1} - \alpha_3} \right) > \frac{m_2}{\delta_1 \beta_2}$. Fig. 2 (k)
(5)	Trivial	Trivial	Trivial Fig. 2 (o)	$ Z_0 \left(\frac{1}{\frac{E}{D-\delta_1} - \alpha_3} - \frac{1}{ Y_0 } \right) > \frac{a_2}{\delta_1 k_2}$. Fig. 2 (n)

Z will blow up in finite time, independent of the choice of the parameters. Thus the theorem is proved. $\square$

For the results of the rest models, we follow the same process and the results are listed in Table 8.

4.4. Blow-up time comparisons in the model system

Remark 4.6. The three species food chain model (1)–(3), is by far the most interesting food chain model that has appeared in the literature thus far, where the predator Z is modeled via the modified Leslie–Gower scheme. In the other models that have appeared in the literature, only Z has the potential to blow-up in finite time, whereas X, Y do not, and one can actually construct absorbing sets in the phase space for those state variables. However, in (1)–(3), X, Y can both blow up. In fact they blow-up together, and blow-up **before** Z.

Here we posit that if blow-up occurs, it must occur in X, Y before Z. To this end we consider the generic system

$$\begin{aligned} \frac{dX}{dt} &= a_1 X \left(1 - \frac{X}{K_1} \right) + c_1 X (1 - e^{-b_1 Y}) Z - F_1(X(t), Y(t)) \cdot Z(t), \\ \frac{dY}{dt} &= a_2 Y \left(1 - \frac{Y}{K_2} \right) + c_2 Y (1 - e^{-b_2 X}) Z - F_2(X(t), Y(t)) \cdot Z(t), \\ \frac{dZ}{dt} &= \left(D - \frac{E}{\alpha_3 + X + Y} \right) Z^2, \end{aligned}$$

We enforce the following structural restrictions on $F_1(X(t), Y(t))$, $F_2(X(t), Y(t))$

$$\bullet \quad m_1 X > F_1(X(t), Y(t)), \quad m_2 Y > F_1(X(t), Y(t))$$

Remark 4.7. The above structural restrictions are a technical requirement to prove blow-up via [Theorem 4.10](#). Physically, the condition amounts to saying that the linear growth of the prey X, Y , are greater than the depredation/loss caused to their density by the predator Z . If not, the prey could be driven to extinction. Hence this is a persistence condition of sorts. However, we do not investigate this rigorously. It may well be that for super linear F_1, F_2 , blow-up is avoided, and global existence is a possibility.

Lemma 4.8. Consider the three species food chain model given by (1)–(3), given a choice of parameters, there is initial data (X_0, Y_0, Z_0) , and a time $T^* < \infty$, s.t X, Y can blow-up at T^* , that is

$$\lim_{t \rightarrow T^* < \infty} \|X\|_\infty \rightarrow \infty, \quad \lim_{t \rightarrow T^* < \infty} \|Y\|_\infty \rightarrow \infty,$$

whereas Z remains bounded at T^* ,

$$\lim_{t \rightarrow T^* < \infty} \|Z\|_\infty < \infty.$$

Proof. Consider (1)–(3), with positive initial conditions (X_0, Y_0, Z_0) .

By integrating equation (3) in time we obtain

$$Z(t) = \frac{1}{\frac{1}{Z_0} - Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3}}.$$

In the event that we have a globally existing solution, it must be that the continuous function:

$$\psi(t) = \frac{1}{Z_0} - Dt + E \int_0^t \frac{ds}{X(s) + Y(s) + \alpha_3} > 0, \quad \forall t.$$

The first time $t = T^* < \infty$ s.t $\psi(T^*) = 0$, z , blows up in finite time. Note we can have a time $T^{**} < T^*$ s.t

$$\lim_{t \rightarrow T^{**} < \infty} \|X\|_\infty \rightarrow \infty, \quad \lim_{t \rightarrow T^{**} < \infty} \|Y\|_\infty \rightarrow \infty,$$

Whilst $\frac{1}{Z_0} > DT^{**}$, but $\frac{1}{Z_0} = DT^*$. Thus $\psi(T^{**}) > 0$, but $\psi(T^*) = 0$. Thus X, Y could blow up in finite time before Z . $\square$

Remark 4.9. From the form of the solution for Z , it seems the other way is also possible, that is Z might blow-up at a finite time T^* , whereas X, Y remain bounded at T^* . We show that this is actually not possible.

We next show under certain parametric restriction, X, Y will always blow-up before Z .

Theorem 4.10. Consider the three species food chain model given by (1)–(3), for any choice of parameters s.t $c_1 > m_1, c_2 > m_2$, and any choice of initial data (X_0, Y_0, Z_0) , if there exists a time $T^* < \infty$, s.t

$$\lim_{t \rightarrow T^* < \infty} \|Z\|_\infty \rightarrow \infty.$$

Then X, Y must have blown-up at a finite time $T^{**} < T^*$, that is

$$\lim_{t \rightarrow T^{**} < \infty} \|X\|_\infty \rightarrow \infty, \quad \lim_{t \rightarrow T^{**} < \infty} \|Y\|_\infty \rightarrow \infty, \quad \lim_{t \rightarrow T^{**} < \infty} \|Z\|_\infty < \infty.$$

Proof. Note it follows via simple comparison that the X, Y solving (1)–(2) are super solutions to the X, Y solving

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K_1}\right) + c_1 X Z (1 - e^{-\alpha_1 Y}) - m_1 X Z, \quad (23)$$

$$\frac{dY}{dt} = r_2 Y \left(1 - \frac{Y}{K_2}\right) + c_2 Y Z (1 - e^{-\alpha_2 X}) - m_2 Y Z, \quad (24)$$

This follows via positivity of solutions and because of our assumed structural assumptions on F_1, F_2 (similar estimate holds for y).

Now we consider (23)–(24) and (3). Assume Z blows up at a finite time $T^* < \infty$, and so does Y , whilst X remains bounded. Then we have $\forall \epsilon$, and a given IC (X_0, Y_0, Z_0) , $\exists t_\epsilon$ s.t $t_\epsilon \leq T^*$, $e^{-\alpha_1 Y(t_\epsilon)} \leq \epsilon$, and $\lim_{\epsilon \rightarrow 0} t_\epsilon \rightarrow T^*$.

Now (1) can be rewritten,

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K_1}\right) + (c_1 - m_1 - e^{-\alpha_1 Y}) X Z,$$

integrating the above in time from $[0, t_\epsilon]$, we obtain

$$X(t_\epsilon) = X(0) + r_1 \int_0^{t_\epsilon} X(1 - \frac{X}{K_1}) + (c_1 - m_1) \int_0^{t_\epsilon} XZ ds - \int_0^{t_\epsilon} e^{-\alpha_1 Y} XZ ds \quad (25)$$

Note, via the mean value theorem for integrals, that is there exists a C s.t.,

$$\int_0^{t_\epsilon} e^{-\alpha_1 Y} XZ ds = Ce^{-\alpha_1 Y(t_\epsilon)} \int_0^{t_\epsilon} XZ ds < C \int_0^{t_\epsilon} XZ ds. \quad (26)$$

Thus using this in the above equation we obtain

$$X(t_\epsilon) > X(0) + r_1 \int_0^{t_\epsilon} X(1 - \frac{X}{K_1}) + (c_1 - m_1 - C\epsilon) \int_0^{t_\epsilon} XZ ds \quad (27)$$

We now let $\epsilon \rightarrow 0$ to obtain,

$$M \geq X(T^*) > X(0) + r_1 \int_0^{T^*} X(1 - \frac{X}{K_1}) ds + (c_1 - m_1) \int_0^{T^*} XZ ds = +\infty \quad (28)$$

this follows via the blow-up assumption on z , and the boundedness assumption on X , which immediately yields a contradiction. That is if Z blows up at a finite time $T^* < \infty$, and so does Y , then X cannot remain bounded at T^* , and must have blown up before at some $T^{**} < T^*$. The exact same argument is applied integrating (24), and using the above method to show, if z blows up at a finite time $T^* < \infty$, and so does X , then Y cannot remain bounded at T^* , and must have blown up before at some $T^{**} < T^*$. We put these together to obtain the desired result. $\square$

We next state the following corollary

Corollary 4.11. Consider the three species food chain model given by (1)–(3), for any choice of parameters s.t $c_1 > m_1$, $c_2 > m_2$, and large enough initial data (X_0, Y_0, Z_0) , there exists a time $T^* < \infty$, s.t X, Y must have blown-up by T^* , whereas z will blow-up at some $T^{**} \geq T^*$.

Proof. The proof is a simple application of Lemma 1 in conjunction with Theorem 1. $\square$

Remark 4.12. Corollary 4.11 tells us that X, Y, Z could all possibly blow-up at the same finite time. This is also seen numerically.

5. Numerical simulation

In this section, we validate the analytical results of our model systems (1)–(5) which are obtained by the numerical simulation. First we consider the parameter sets twice for each model system as shown as in the Table 9 and observe the stability of $E_{YZ}(0, \bar{Y}, \bar{Z})$ and $E_{XZ}(\bar{X}, 0, \bar{Z})$ equilibrium points respectively as shown in the first and second panels respectively of Fig. 1.

Now we consider the five different parameter sets of systems (1)–(5) with two different initial values for each model as shown in Table 10 and investigate that each of the model system (1)–(5) is asymptotically stable at coexistence equilibrium point (E_*) w.r.t. the first initial condition (first panel of Fig. 2), here we are using the red and green bullets to identify the initial and coexistence equilibrium point (E_*) respectively. But blows up happen for the second initial condition as shown in the second panel of the Fig. 2. Now we choose two different parameter sets and a common initial condition for each model systems (1)–(5) as shown in Table 11, where we observe that each model system is asymptotically stable at coexistence equilibrium point (E_*) (first column of Fig. 2) for first parameter set and blows up (third column of Fig. 2) for second parameter set.

In Table 8, we illustrate the finite time blow up properties of model system (1)–(5). We elucidate the blow up property of the generalist predator in two parts for each of the model system (1)–(5). The first part is $D > \frac{E}{\alpha_3}$ and the second part is $D < \frac{E}{\alpha_3}$. From the differential equations for the prey species of the model system (1)–(5), we notice that if the generalist predator blows up then the prey species blow up obviously. So from Table 8, one can clearly understand that the blow up conditions for prey species are trivial for the condition $D > \frac{E}{\alpha_3}$. Now we concentrate our focus on the blow up property of the generalist predator (Z). For the first part ($D > \frac{E}{\alpha_3}$), the blow up condition for the generalist predator is trivial and for the second part ($D < \frac{E}{\alpha_3}$), we expatiate all the possibilities of the blow up condition for the generalist predator.

In third panel of Fig. 2, we depict the trivial case of finite time blow up for the generalist predator and for the other cases we follow the second panel of the Fig. 2. Fig. 3 shows in certain parameter set shown in Table 12, both the preys blow up before the generalist predator for the model system (1)–(5). Fig. 4 illustrates the phenomena of together blow up along the parameter set listed in the Table 12 of the model system (1)–(5).

In Fig. 5, we observe the dynamical changes of each system (1)–(5) with respect to the varying initial conditions Y_0 and Z_0 along parameter set as in Table 8 where stable and blow up regions are stated as AS (Asymptotic Stable) and BL

Table 9Parameter sets for the stability of E_{YZ} and E_{XZ} equilibria of model system (1)–(5) in first and second panels respectively.

Model	Parameter set for stability of E_{YZ}	Figure	Parameter set for stability of E_{XZ}	Figure
(1)	$a_1 = 0.01, K_1 = 5, c_1 = 0.1$ $b_1 = 0.1, m_1 = 8.5, h_1 = 0.01$ $a_2 = 1, K_2 = 1.2, c_2 = 0.1$ $b_2 = 1, m_2 = 0.5, h_2 = 0.05$ $D = 0.7, E = 9.5, \alpha_3 = 10$ IC: (1, 0.5, 1.5)	1 (a)	$a_1 = 0.1, K_1 = 1.1, c_1 = 0.1$ $b_1 = 1, m_1 = 0.1, h_1 = 60$ $a_2 = 0.01, K_2 = 5, c_2 = 0.01$ $b_2 = 0.01, m_2 = 2, h_2 = 60$ $D = 0.7, E = 9.5, \alpha_3 = 10$ IC: (1, 0.5, 1.5)	1 (b)
(2)	$a_1 = 0.1, K_1 = 0.3, c_1 = 0.1$ $b_1 = 0.25, m_1 = 5.5, h'_1 = 0.6$ $a_2 = 0.1, K_2 = 0.5, c_2 = 1$ $b_2 = 0.15, m_2 = 0.5, h'_2 = 2.06$ $D = 0.9, E = 9.5, \alpha_3 = 10$ $j_1 = 0.525, j_2 = 0.625$ IC: (1, 3, 2)	1 (c)	$a_1 = 0.1, K_1 = 0.5, c_1 = 0.1$ $b_1 = 0.15, m_1 = 0.5, h'_1 = 2.06$ $a_2 = 0.1, K_2 = 0.3, c_2 = 0.1$ $b_2 = 0.25, m_2 = 5.5, h'_2 = 0.6$ $D = 0.92, E = 9.5, \alpha_3 = 10$ $j_1 = 0.625, j_2 = 0.525$ IC: (1, 3, 2)	1 (d)
(3)	$a_1 = 0.1, K_1 = 0.3, c_1 = 0.1$ $b_1 = 0.25, m_1 = 5.5, \alpha_1 = 0.6$ $a_2 = 0.1, K_2 = 0.5, c_2 = 1$ $b_2 = 0.15, m_2 = 0.5, \alpha_2 = 2.06$ $D = 0.9, E = 9.5, \alpha_3 = 10$ $\beta_1 = 2, \beta_2 = 2$ IC: (1, 1.2, 1.5)	1 (e)	$a_1 = 0.1, K_1 = 0.5, c_1 = 0.1$ $b_1 = 0.15, m_1 = 0.5, \alpha_1 = 2.06$ $a_2 = 0.1, K_2 = 0.3, c_2 = 0.1$ $b_2 = 0.25, m_2 = 5.5, \alpha_2 = 0.6$ $D = 0.92, E = 9.5, \alpha_3 = 10$ $\beta_1 = 2, \beta_2 = 2$ IC: (1, 1.2, 1.5)	1 (f)
(4)	$a_1 = 0.1, K_1 = 0.3, c_1 = 0.1$ $b_1 = 0.25, m_1 = 5.5, \alpha'_1 = 0.6$ $a_2 = 0.1, K_2 = 0.5, c_2 = 1$ $b_2 = 0.15, m_2 = 0.5, \alpha'_2 = 2.06$ $D = 0.9, E = 9.5, \alpha_3 = 10$ $\beta'_1 = 2, \beta'_2 = 2$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$ IC: (0.5, 0.8, 1.2)	1 (g)	$a_1 = 0.1, K_1 = 0.5, c_1 = 0.1$ $b_1 = 0.15, m_1 = 0.5, \alpha'_1 = 2.06$ $a_2 = 0.1, K_2 = 0.3, c_2 = 0.1$ $b_2 = 0.25, m_2 = 5.5, \alpha'_2 = 0.6$ $D = 0.92, E = 9.5, \alpha_3 = 10$ $\beta'_1 = 2, \beta'_2 = 2$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$ IC: (0.5, 0.8, 1.2)	1 (h)
(5)	$a_1 = 0.1, K_1 = 0.3, c_1 = 0.1$ $b_1 = 0.25, m_1 = 5.5, v_1 = 0.6$ $a_2 = 0.1, K_2 = 0.5, c_2 = 1$ $b_2 = 0.15, m_2 = 0.5, v_2 = 2.06$ $D = 0.9, E = 9.5, \alpha_3 = 10$ $p = 0.7, q = 0.5$ IC: (1, 0.5, 1.5)	1 (i)	$a_1 = 0.1, K_1 = 0.5, c_1 = 0.1$ $b_1 = 0.15, m_1 = 0.5, v_1 = 2.06$ $a_2 = 0.1, K_2 = 0.3, c_2 = 0.1$ $b_2 = 0.25, m_2 = 5.5, v_2 = 0.6$ $D = 0.92, E = 9.5, \alpha_3 = 10$ $p = 0.5, q = 0.7$ IC: (1, 0.5, 1.5)	1 (j)

(Blow up) respectively. We distinguish the parametric domains of stability and blow up properties of the model system (1)–(5) in Fig. 6 in different parametric planes. Similarly stable and blow up regions are stated as AS (Asymptotic Stable) and BL (Blow up) respectively. In both of the figures we define the conditional blow up phenomenon which is shown in Table 8.

6. Conclusions and discussions

In this paper, we elaborately discuss about the behavior of the dynamical system consisting two mutualistic prey and a sexually reproductive generalist predator. We formulate five general structure of dynamical models with five different prey dependent and predator dependent functional responses. The stability properties of each of model system has been studied into deep. We observe the stability analysis of boundary equilibrium points and as well as we investigate the stability properties of the coexistence equilibrium point for each of the model system (1)–(5). In this paper we have eight possible non-negative equilibria points from each of the model system of (1)–(5). $E_0(0, 0, 0)$, $E_X(K_1, 0, 0)$, $E_Y(0, K_2, 0)$, $E_Z(0, 0, \bar{Z})$, $E_{XY}(K_1, K_2, 0)$ are the equilibrium points of the model systems and the stability region becomes neutral for each of the equilibrium point of the corresponding model system. For the equilibrium points $E_{YZ}(0, \bar{Y}, \bar{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$, we observe that stability of the model systems are varying according to their eigenvalues are changed. We show the stability time series diagram of $E_{YZ}(0, \bar{Y}, \bar{Z})$ and $E_{XZ}(\hat{X}, 0, \hat{Z})$ in Fig. 1 and notice that under some parametric restriction,

Table 10

Stability and Blow-up phenomena w.r.t different initial data with fixed parametric values of system (1)–(5).

Model	Parameter set	Initial Condition (IC)	Stability status	Blow-Up	Initial Condition (IC)	Blow-Up
(1)	$a_1 = 0.01, K_1 = 1.1, c_1 = 1, b_1 = 0.50, m_1 = 0.75, h_1 = 60$ $a_2 = 0.05, K_2 = 1.2, c_2 = 1, b_2 = 0.25, m_2 = 0.50, h_2 = 60$ $D = 30, E = 500, \alpha_3 = 12$	(0.2, 0.1, 0.4)	Asymptotically stable (Figure) 2 (a)	No	(10, 15, 25)	Yes (Figure) 2 (b)
(2)	$a_1 = 1, K_1 = 30, c_1 = 1, b_1 = 0.50, m_1 = 0.75, h'_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1, b_2 = 0.25, m_2 = 0.50, h'_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 10, j_1 = 0.625, j_2 = 0.525$	(1, 1.4, 1.5)	Asymptotically stable (Figure) 2 (d)	No	(30, 25, 20)	Yes (Figure) 2 (e)
(3)	$a_1 = 0.1, K_1 = 50, c_1 = 1, b_1 = 0.50, m_1 = 0.25, \alpha_1 = 50$ $a_2 = 1, K_2 = 50, c_2 = 1, b_2 = 0.25, m_2 = 0.20, \alpha_2 = 50$ $D = 0.1, E = 25, \alpha_3 = 10, \beta_1 = 20, \beta_2 = 20$	(1, 1.2, 1.5)	Asymptotically stable (Figure) 2 (g)	No	(10, 15, 20)	Yes (Figure) 2 (h)
(4)	$a_1 = 1, K_1 = 30, c_1 = 1, b_1 = 0.5, m_1 = 0.25, \alpha'_1 = 50$ $a_2 = 1, K_2 = 30, c_2 = 1, b_2 = 0.25, m_2 = 0.20, \alpha'_2 = 50$ $D = 0.01, E = 10, \alpha_3 = 10, \beta'_1 = 20, \beta'_2 = 20$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$	(2, 0.5, 1.5)	Asymptotically stable (Figure) 2 (j)	No	(15, 20, 20)	Yes (Figure) 2 (k)
(5)	$a_1 = 0.01, K_1 = 0.5, c_1 = 0.01, b_1 = 1.50, m_1 = 0.75, v_1 = 60$ $a_2 = 0.05, K_2 = 0.3, c_2 = 0.01, b_2 = 2.25, m_2 = 0.50, v_2 = 60$ $D = 35, E = 500, \alpha_3 = 12, p = 0.7, q = 0.5$	(0.05, 0.1, 0.5)	Asymptotically stable (Figure) 2 (m)	No	(1, 2, 3)	Yes (Figure) 2 (n)

Table 11

Stability and Blow-up phenomena w.r.t different parametric values with same initial data of system (1)–(5).

Model	Initial Condition (IC)	Parameter set	Stability status	Blow-Up	Parameter set	Blow-Up
(1)	(0.2, 0.1, 0.4)	$a_1 = 0.01, K_1 = 1.1, c_1 = 1$ $b_1 = 0.50, m_1 = 0.75, h_1 = 60$ $a_2 = 0.05, K_2 = 1.2, c_2 = 1$ $b_2 = 0.25, m_2 = 0.50, h_2 = 60$ $D = 30, E = 500, \alpha_3 = 12$	Asymptotically stable (Figure) 2 (a)	No	$a_1 = 0.01, K_1 = 1.1, c_1 = 1$ $b_1 = 0.50, m_1 = 0.75, h_1 = 60$ $a_2 = 0.05, K_2 = 1.2, c_2 = 1$ $b_2 = 0.25, m_2 = 0.50, h_2 = 60$ $D = 30, E = 500, \alpha_3 = 20$	Yes (Figure) 2 (c)
(2)	(1, 1.4, 1.5)	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.25, m_1 = 0.50, h'_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.50, h'_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 10$ $j_1 = 0.625, j_2 = 0.525$	Asymptotically stable (Figure) 2 (d)	No	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.25, m_1 = 0.50, h'_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.50, h'_2 = 60$ $D = 10.01, E = 10, \alpha_3 = 10$ $j_1 = 0.625, j_2 = 0.525$	Yes (Figure) 2 (f)
(3)	(1, 1.2, 1.5)	$a_1 = 0.1, K_1 = 50, c_1 = 1$ $b_1 = 0.50, m_1 = 0.25, \alpha_1 = 50$ $a_2 = 1, K_2 = 50, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha_2 = 50$ $D = 0.1, E = 25, \alpha_3 = 10$ $\beta_1 = 20, \beta_2 = 20$	Asymptotically stable (Figure) 2 (g)	No	$a_1 = 0.1, K_1 = 50, c_1 = 1$ $b_1 = 0.50, m_1 = 0.25, \alpha_1 = 50$ $a_2 = 1, K_2 = 50, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha_2 = 50$ $D = 10.1, E = 25, \alpha_3 = 10$ $\beta_1 = 20, \beta_2 = 20$	Yes (Figure) 2 (i)
(4)	(2, 0.5, 1.5)	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.25, \alpha'_1 = 50$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha'_2 = 50$ $D = 0.01, E = 10, \alpha_3 = 10$ $\beta'_1 = 20, \beta'_2 = 20$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$	Asymptotically stable (Figure) 2 (j)	No	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.25, \alpha'_1 = 50$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha'_2 = 50$ $D = 10.01, E = 10, \alpha_3 = 10$ $\beta'_1 = 20, \beta'_2 = 20$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$	Yes (Figure) 2 (l)
(5)	(0.05, 0.1, 0.5)	$a_1 = 0.01, K_1 = 0.5, c_1 = 0.01$ $b_1 = 1.50, m_1 = 0.75, v_1 = 60$ $a_2 = 0.05, K_2 = 0.3, c_2 = 0.01$ $b_2 = 2.25, m_2 = 0.50, v_2 = 60$ $D = 35, E = 500, \alpha_3 = 12$ $p = 0.7, q = 0.5$	Asymptotically stable (Figure) 2 (m)	No	$a_1 = 0.01, K_1 = 0.5, c_1 = 0.01$ $b_1 = 1.50, m_1 = 0.75, v_1 = 60$ $a_2 = 0.05, K_2 = 0.3, c_2 = 0.01$ $b_2 = 2.25, m_2 = 0.50, v_2 = 60$ $D = 35, E = 500, \alpha_3 = 20$ $p = 0.7, q = 0.5$	Yes (Figure) 2 (o)

stability region of $E_{YZ}(0, \bar{Y}, \bar{Z})$ switches into the stability region of $E_{XZ}(\hat{X}, 0, \hat{Z})$. We investigate asymptotic stable region for the coexistence equilibrium points for each of the model systems under the restriction of the parameter set.

From Table 8 we determine that for each model system (1)–(5), the condition for the trivial case of blow up of the generalist predator is $D > \frac{E}{\alpha_3}$. Otherwise (in the case of $D < \frac{E}{\alpha_3}$), we have seen a conditional blow up phenomena which depends on initial values of the model system (1)–(5). In Table 8, we have shown that the conditional blow up

Table 12

Parameter sets for prey blows up before predator and parameter sets for all three species blow up together of model system (1)–(5) in first and second panels respectively.

Model	Parameter set for any prey blow up	Figure	Parameter set for all species blow up	Figure
(1)	$a_1 = 0.1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.75, h_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.5, h_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 1000$ IC: (0.2, 0.1, 0.4)	3 (a)	$a_1 = 1, K_1 = 0.5, c_1 = 1$ $b_1 = 0.5, m_1 = 0.75, h_1 = 60$ $a_2 = 1, K_2 = 1, c_2 = 1$ $b_2 = 0.25, m_2 = 0.5, h_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 10000$ IC: (0.2, 0.1, 0.4)	4 (a)
(2)	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.75, h'_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.5, h'_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 1000$ $j_1 = 0.625, j_2 = 0.525$ IC: (1, 1.4, 1.5)	3 (b)	$a_1 = 1, K_1 = 0.5, c_1 = 1$ $b_1 = 0.5, m_1 = 0.75, h'_1 = 60$ $a_2 = 1, K_2 = 1, c_2 = 0.1$ $b_2 = 0.25, m_2 = 5.5, h'_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 10000$ $j_1 = 0.625, j_2 = 0.525$ IC: (1, 1.4, 1.5)	4 (b)
(3)	$a_1 = 0.1, K_1 = 50, c_1 = 1$ $b_1 = 0.5, m_1 = 0.25, \alpha_1 = 50$ $a_2 = 1, K_2 = 50, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha_2 = 50$ $D = 0.01, E = 10, \alpha_3 = 1000$ $\beta_1 = 0.25, \beta_2 = 0.20$ IC: (1, 1.2, 1.5)	3 (c)	$a_1 = 0.1, K_1 = 0.1, c_1 = 1$ $b_1 = 0.5, m_1 = 0.25, \alpha_1 = 50$ $a_2 = 1, K_2 = 1, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha_2 = 50$ $D = 0.01, E = 10, \alpha_3 = 10$ $\beta_1 = 0.25, \beta_2 = 0.20$ IC: (1, 1.2, 1.5)	4 (c)
(4)	$a_1 = 1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.25, \alpha'_1 = 50$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha'_2 = 50$ $D = 0.1, E = 100, \alpha_3 = 5000$ $\beta'_1 = 0.25, \beta'_2 = 0.20$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$ IC: (2, 0.5, 1.5)	3 (d)	$a_1 = 1, K_1 = 0.1, c_1 = 1$ $b_1 = 0.50, m_1 = 0.5, \alpha'_1 = 50$ $a_2 = 1, K_2 = 1, c_2 = 1$ $b_2 = 0.25, m_2 = 0.20, \alpha'_2 = 50$ $D = 0.1, E = 100, \alpha_3 = 5000$ $\beta'_1 = 0.25, \beta'_2 = 0.20$ $\gamma_1 = 0.03, \gamma_2 = 0.05, \delta_1 = 0.4, \delta_2 = 0.2$ IC: (2, 0.5, 1.5)	4 (d)
(5)	$a_1 = 0.1, K_1 = 30, c_1 = 1$ $b_1 = 0.5, m_1 = 0.75, v_1 = 60$ $a_2 = 1, K_2 = 30, c_2 = 1$ $b_2 = 0.25, m_2 = 0.5, v_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 1000$ $p = 0.7, q = 0.5$ IC: (0.05, 0.1, 0.5)	3 (e)	$a_1 = 1, K_1 = 0.5, c_1 = 1$ $b_1 = 0.50, m_1 = 0.75, v_1 = 60$ $a_2 = 1, K_2 = 1, c_2 = 1$ $b_2 = 0.25, m_2 = 0.5, v_2 = 60$ $D = 0.01, E = 10, \alpha_3 = 10000$ $p = 0.7, q = 0.5$ IC: (0.05, 0.1, 0.5)	4 (e)

phenomena depends on the initial value Y_0 (here for analytic purpose we consider the differential equation $\frac{dY}{dt}$ of the second prey species for each of the model system (1)–(5)). However, it also depends on the initial value X_0 . For example, if we take the differential equation $\frac{dX}{dt}$ of the first prey species of the model system (1) for analytic purpose in place of $\frac{dY}{dt}$, we get the blow up condition of Z when $D < \frac{E}{\alpha_3}$ is $|Z_0| \left(\frac{1}{\frac{E}{\delta_1} - \alpha_3} - \frac{1}{|X_0|} \right) > \frac{a_1}{\delta_1 k_1}$. In the same fashion we can observe the blow up phenomena of Z in the case $D < \frac{E}{\alpha_3}$ for rest of systems depend on initial value X_0 and Z_0 as well. Therefore we conclude that the initial values of each of the model system (1)–(5) has a significant impact in their blow up phenomena. Note the structural assumptions on F_1, F_2 are a persistence condition of sorts. However, we do not investigate this rigorously. It may be that for F_1, F_2 growing super linearly, blow-up is avoided, and global existence is a possibility. This may be true for any initial condition, and so a global in time existence result, may be posited for global data. The effects of such growth conditions on the damping of blow-up solutions in the current model would make for interesting future investigations.

Invasive species pose direct (or indirect) threats to the ecosystems and also decrease biodiversity. In the event that a predator invades an ecosystem, whose population has a possibility of exploding, our key finding is that mutualism amongst its target prey can actually cause their own population to explode. This will occur most times before the actual explosion of the predator population (and at a few times together). Thus in this sense mutualism amongst prey benefits

them, as they continue to survive. In the absence of mutualism $c_1 = c_2 = 0$, blow up in the predator Z is trivial, and would then lead to it wiping out its prey in most realistic scenarios.

Also note, theory claims that generalist predators are poor agents for biological control – but this is not observed in field studies [49]. Herein, clearly prey mutualism could further reaffirm this theory, because if the prey were cooperating, in the presence of an introduced predator – it would not be as effective as when they were not. All in all we establish that in the current modeling framework, mutualism is a key survival strategy for prey, in the presence of an explosive generalist predator.

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