Minimal Edge Addition for Network Controllability

Ximing Chen, Sérgio Pequito, *Member, IEEE*, George J. Pappas, *Fellow, IEEE*, and Victor M. Preciado

***Abstract*—We address the problem of optimally modifying the topology of a directed dynamical network to ensure structural controllability. More precisely, given the structure of a directed dynamical network (i.e., an existing networked infrastructure), we propose a framework to find the minimum number of directed edges that need to be added to the network topology in order to render a structurally controllable system. Our main contribution is twofold: first, we provide a full characterization of all optimal network modifications, and second, we propose an algorithm able to find an optimal solution in polynomial time. We illustrate the validity of our algorithm via numerical simulations in random networked systems.**

***Index Terms*—Algorithm design and analysis, controllability, network topology.**

I. Introduction

NETWORK control theory provides a plethora of tools to analyze the behavior of dynamical processes taking place in complex networked systems, such as epidemic outbreaks in human contact networks [1], information spreading in social networks [2], or synchronization in power systems [3]. The analysis and design of complex networks using tools from graph theory have gained a growing interest in recent years [4]; in particular, the classical control problem of steering the state of a dynamical network toward a desired state [5], [6]. However, in many practical scenarios, an exact quantitative description of the edges in the network may not be available due to measurement errors and/or modeling uncertainties [7]. In this scenario, it is still possible to analyze network control problems resorting to tools developed in the context of structural systems theory [8]–[11].

Structural controllability extends the classical controllability concept to the case of networks with uncertain edges. Loosely speaking, a network is structurally controllable if it is controllable for almost all realizations of edge weights (see Section II for a formal description of this concept). In this context, given a structurally uncontrollable system, one may be interested in enforcing structural controllability by either 1) adding actuation capabilities to the networked system, or 2) modifying the topology of the dynamical network by, for example, adding new edges to the network topology. The former case is explored in [12]–[19]. Briefly, in [12], [18], and [19], the authors proposed graph-theoretical algorithms to find the minimum number of driving nodes to ensure structural controllability in complex networks. In [13] and [14], the authors complement this work to obtain the minimum number of driven nodes in polynomial time. Subsequently, the minimum number of driven nodes required while accounting for actuation costs was addressed in [15] and [16]. Alternatively, if one seeks the minimum collection of inputs from an *a priori* defined collection of actuation capabilities, then the problem is NP-hard [17]. Notwithstanding, there are several cases when adding actuation capabilities to the network is either too expensive or not feasible. Therefore, whenever possible or cost-efficient, one can opt to modify the topology of the dynamical network. This case is the focus of the this paper, where we propose a polynomial-time algorithm to determine the minimum number of extra connections that must be added to a given structural system in order to ensure structural controllability.

Wang *et al.* [20] proposed an approach to perturb the structure of an *undirected* network to ensure structurally controllability when only one driving node was considered. Ding *et al.* [21] studied a similar problem for directed networks. However, they assumed that all the nodes are already reachable from the driving nodes. Although they solved the problem using a constrained integer program which is, in general, NP-hard [22], they did not discuss the complexity of their algorithm. In contrast with previous works, we address the case of arbitrary directed network topologies with any number of driving nodes and show that the problem can be solved in polynomial time without any assumption on reachability. The following are the contributions of this paper: first, we characterize all possible solutions to the problem of determining the minimum number of additional edges required to ensure structural controllability, and second, we provide a polynomial-time algorithm to find a solution suitable for large complex networks.

The rest of the paper is organized as follows. A formal description of the problem under consideration are introduced in Section II. Preliminaries on graph theory and structural system theory are introduced in Section III. The main results are provided in Section IV. In Section V, we illustrate our results in several complex network topologies. Finally, conclusions and discussion of future research are presented in Section VI.

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II. Problem Statement

The dynamics of a linear networked dynamical system can be described as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n\times n}$ is the state transition matrix, and $B \in \mathbb{R}^{n\times m}$ is the input matrix. In the following, we refer to system (1) by the matrix pair (A, B), and if the system is controllable, we say that the pair (A, B) is controllable. Furthermore, we define $\bar{A} \in \{0,1\}^{n\times n}$ to be the structural pattern of A, i.e., $\bar{A}_{ij} = 0$ if $[A]_{ij} = 0$, and $\bar{A}_{ij} = 1$ otherwise. Similarly, $\bar{B} \in \{0,1\}^{n\times m}$ encodes the sparsity pattern of B where $\bar{B}_{ij} = 0$ of $[B]_{ij} = 0$, and $\bar{B}_{ij} = 1$ otherwise. We say that the structural pattern $(\bar{A}, \bar{B})$ is *structually controllable* if there exists a pair $(\hat{A}, \hat{B})$ with the same structural pattern as $(\bar{A}, \bar{B})$ that is controllable [23]. Furthermore, if such pair $(\hat{A}, \hat{B})$ exists, then almost all possible matrix pairs with the same structural pattern as $(\bar{A}, \bar{B})$ are controllable [23].

In this paper, given a structurally uncontrollable pair $(\bar{A}, \bar{B})$, we are interested in the problem of adding a minimum number of entries in $\bar{A}$ to obtain a structurally controllable system. Intuitively, if we add sufficient edges in the network such that the resulting network is a complete graph, then the resulting system is structurally controllable, provided that at least one node is actuated, i.e., $\bar{B} \neq 0$. Nonetheless, adding new edges corresponds, in practice, to building new infrastructure. Therefore, from a design and implementation perspective, one seeks to add the minimum number of edges to attain the design objective, which, in our case, consists in ensuring structural controllability. Formally, the problem is described as follows:

Problem 1: Given the pair $(\bar{A}, \bar{B})$ with $\bar{B} \neq 0$, find

$$\tilde{A}^* = \arg \min_{\tilde{A}\in\{0,1\}^{n\times n}} \|\tilde{A}\|_0 \tag{2}$$

$$\text{s.t. } (\bar{A} + \tilde{A}, \bar{B}) \text{ is structurally controllable}$$

where $\|\tilde{A}\|_0$ denotes the number of nonzero entries in a matrix $\tilde{A}$, and the operator $+ : \{0,1\}^{n\times n} \times \{0,1\}^{n\times n} \to \{0,1\}^{n\times n}$ is the element-wise exclusive-or for binary matrices.

If $(\bar{A} + \tilde{A}, \bar{B})$ is structurally controllable, we refer to matrix $\tilde{A}$ as a *feasible edge-addition matrix*, and to $\tilde{A}^*$ in (2) as the *optimal edge-addition matrix*. As part of the solution proposed in this paper, we provide a characterization of all possible optimal edge-addition matrices by resorting to graph-theoretical tools. Furthermore, we provide a polynomial-time algorithm to obtain one such solution.

III. Notation and Preliminaries

In the rest of the paper, $|\mathcal{S}|$ denotes the cardinality of a set $\mathcal{S}$. Let $G = (\mathcal{V}, \mathcal{E})$ denote a directed graph with vertex-set $\mathcal{V} = \{1, \ldots, n\}$, and edge-set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Given an edge $(i, j) \in \mathcal{E}$, we say that the "tail" vertex i is pointing toward the "head" vertex j, which we denote by $i \to j$. A *path* of length K in G is defined as an ordered sequence of distinct vertices $(v_0, v_1, \ldots, v_K)$ with $v_k \in \mathcal{V}$ and $(v_k, v_{k+1}) \in \mathcal{E}$ for all $k = 0, \ldots, K-1$. A *cycle* is either a path $(v_0, v_1, \ldots, v_K)$ with an additional edge (v_K, v_0), or a vertex with an edge to itself (i.e., self-loop). A vertex $v_2 \in \mathcal{V}$ is *reachable* from $v_1 \in \mathcal{V}$ if there exists a path in G from v_1 to v_2. A directed graph $G_s = (\mathcal{V}_s, \mathcal{E}_s)$ is a *subgraph* of G if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E}$. In particular, if $\mathcal{V}_s = \mathcal{V}$, then G_s is said to *span* G. Given a vertex set $\mathcal{S} \subseteq \mathcal{V}$, we define the *$\mathcal{S}$-induced subgraph* of G by $\mathcal{S}$ as $G_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}})$, where $\mathcal{E}_{\mathcal{S}} = \mathcal{E} \cap (\mathcal{S} \times \mathcal{S})$.

A graph is said to be *strongly connected* if there exists a path between any two vertices in the graph. A *strongly connected component* (SCC) is a maximal subgraph G_s that is strongly connected. A *condensation* of G is a *directed acyclic graph* (DAG) generated by representing each SCC in G as a virtual vertex in the condensation and a directed edge between two virtual vertices in the condensation exists, if and only if, there exists a directed edge connecting the corresponding SCCs in G [24]. An SCC is said to be *linked* if it has at least one incoming/outgoing edge from another SCC. In particular, a *source SCC* has no incoming edges from another SCC and a *sink SCC* has no outgoing edges to another SCC.

Given a directed graph $G = (\mathcal{V}, \mathcal{E})$ and two vertex sets $S_1, S_2 \subseteq \mathcal{V}$, we define the (undirected) *bipartite graph* $\mathcal{B}(S_1, S_2, \mathcal{E}_{S_1,S_2})$ as a graph, whose vertex set is $S_1 \cup S_2$ and edge set[1] $\mathcal{E}_{S_1,S_2} = \{\{s_1, s_2\} \in \mathcal{E} : s_1 \in S_1, s_2 \in S_2\}$. Given $\mathcal{B}(S_1, S_2, \mathcal{E}_{S_1,S_2})$, a *matching* M is a set of edges in $\mathcal{E}_{S_1,S_2}$ that do not share vertices, i.e., given edges $e = \{s_1, s_2\}$ and $e' = \{s_1', s_2'\}$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. The vertex v is said to be *right-unmatched* (resp., *left-unmatched*) with respect to a matching M associated with $\mathcal{B}(S_1, S_2, \mathcal{E}_{S_1,S_2})$ if $v \in S_2$ (resp., S_1), and v does not belong to an edge in the matching M. A matching is said to be maximum if it is a matching with the maximum number of edges among all possible matchings. Additionally, a matching is called a *perfect matching* if it does not contain right-unmatched vertices. Given a bipartite graph $\mathcal{B}(S_1, S_2, \mathcal{E}_{S_1,S_2})$, the maximum matching problem can be solved efficiently in $\mathcal{O}(\sqrt{|S_1 \cup S_2|}|\mathcal{E}_{S_1,S_2}|)$ time [24].

Given a structural pair $(\bar{A}, \bar{B})$, we associate a directed graph $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, which we refer to as the *system digraph*, where $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{U} = \{u_1, \ldots, u_m\}$ denote the set of state vertices and input vertices, and $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : [\bar{A}]_{ji} \neq 0\}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) : [\bar{B}]_{ij} \neq 0\}$ denote its edge sets. In the remaining of the paper, unless otherwise specified, a state vertex being reachable means that it is reachable from some input vertex. Similarly, a vertex set is reachable if every vertex in the set is reachable. Also, due to the graph representation of the pair $(\bar{A}, \bar{B})$, when $(\bar{A}, \bar{B})$ is structural controllable, we interchangeably say that $G(\bar{A}, \bar{B})$ is structurally controllable. In addition, we can associate an undirected bipartite graph with $G(\bar{A}, \bar{B})$, called the *system bipartite graph* and denoted by $\mathcal{B}(\bar{A}, \bar{B}) = \mathcal{B}(\mathcal{X}^+ \cup \mathcal{U}^+, \mathcal{X}^-, \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-})$, in which $\{x_i^+, x_j^-\} \in \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-}$ if $(x_i, x_j) \in \mathcal{E}_{\mathcal{X},\mathcal{X}}$, and $\{u_i^+, x_j^-\} \in \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-}$ if $(u_i, x_j) \in \mathcal{E}_{\mathcal{U},\mathcal{X}}$. Subsequently, for ease of notation, we use a *signal-notation mapping* $s : \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \to \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-}$ to map edges from the system digraph into edges of the

[1] We denote undirected edges using curly brackets $\{v_i, v_j\}$, in contrast with directed edges, for which we use parenthesis.

system bipartite graph, as follows: $s((u_i, x_j)) = \{u_i^+, x_j^-\}$ and $s((x_i, x_j)) = \{x_i^+, x_j^-\}$. In addition, due to the bijectivity of the signal-notation mapping, we have that $s^{-1}(\{u_i^+, x_j^-\}) = (u_i, x_j)$ and $s^{-1}(\{x_i^+, x_j^-\}) = (x_i, x_j)$.

The concepts introduced in this section can be used to determine if a structural system is structurally controllable, as follows.

Theorem 1 ([11], [13]): The pair $(\bar{A}, \bar{B})$ is structurally controllable if and only if the following two conditions hold.

1) Every state vertex $x \in \mathcal{X}$ in the system digraph $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ is reachable (from some input vertex $u \in \mathcal{U}$).
2) Any maximum matching M of the system bipartite graph $\mathcal{B}(\bar{A}, \bar{B}) = \mathcal{B}(\mathcal{X}^+ \cup \mathcal{U}^+, \mathcal{X}^-, \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-})$ has no right-unmatched vertices. $\diamond$

Notice that both conditions in Theorem 1 can be verified in polynomial time [11]. Hence, one could naively try to ensure both conditions by adding edges iteratively, but such an approach is, in general, nonoptimal and does not provide optimality guarantees.

IV. Minimum Topological Changes to Ensure Structural Controllability

In this section, we provide the main results of the paper. First, in Section IV-A, we reformulate Problem 1 as a graph-theoretical problem. Next, in Section IV-B, we sharpen our intuition by exploring two particular network topologies. In Section IV-C, we show that iterative solutions are suboptimal. Next, using graph-theoretical tools, we characterize the set of feasible solutions to Problem 1 (see Theorem 2). Subsequently, we obtain a feasible solution containing the minimum number of additional edges to ensure structural controllability (see Theorem 3). Finally, we provide a polynomial-time algorithm (see Algorithm 3) to obtain an optimal solution to Problem 1, whose correctness and computational complexity are proved in Theorem 4.

A. Graph-Theoretical Optimization Problem

At a first glance, Problem 1 may seem a purely combinatorial problem. Naively, one may find a solution by exhaustively exploring the set of $n \times n$ binary matrices. However, Theorem 1 can be leveraged to shrink the search domain of (2). This motivates us to recast (2) as the following graph-theoretical problem.

Recall that the system digraph is given by $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$. Therefore, given a feasible edge-addition matrix $\tilde{A}$, we can associate a digraph with the perturbed structural system $(\bar{A} + \tilde{A}, \bar{B})$, which we denote by $G(\bar{A} + \tilde{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \tilde{\mathcal{E}})$, where the edge set $\tilde{\mathcal{E}} \subseteq \mathcal{X} \times \mathcal{X}$ is such that $(x_i, x_j) \in \tilde{\mathcal{E}}$ if and only if $\tilde{A}_{ji} = 1$. Subsequently, since there is an one-to-one correspondence between $\tilde{\mathcal{E}}$ and the structural matrix $\tilde{A}$, we can provide the following equivalent formulation of Problem 1.

Problem 2: Given the system digraph $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, find

$$\begin{aligned} \tilde{\mathcal{E}}^* = \arg \min_{\tilde{\mathcal{E}} \subseteq \mathcal{X} \times \mathcal{X}} \quad & |\tilde{\mathcal{E}}| \\ \text{s.t. } & G(\bar{A} + \tilde{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \tilde{\mathcal{E}}) \\ & \text{is structurally controllable.} \end{aligned}$$

Additionally, we define a *feasible edge-addition configuration* as a set of edges that is a feasible solution of Problem 2. Also, an *optimal edge-addition configuration* is defined as an optimal solution of Problem 2.

B. Special Cases

Next, before showing that iterative strategies can be suboptimal, we discuss two special cases to sharpen our intuition. First, recall that according to Theorem 1, the pair $(\bar{A}, \bar{B})$ is structurally controllable, if and only if, two conditions are satisfied. Therefore, we explore two special cases, where in each case only one of the conditions in Theorem 1 is satisfied; hence, only the remaining condition needs to be ensured to attain feasibility.

Case I: Consider a structured system $(\bar{A}, \bar{B})$ such that only Condition 1) in Theorem 1 holds, while Condition 2) is not satisfied. In other words, all state vertices are reachable while there exists a maximum matching of the system bipartite graph with right-unmatched vertices. As a result, the cardinality of a maximum matching M with respect to $\mathcal{B}(\bar{A}, \bar{B})$ is strictly less than n. Subsequently, let us denote by $U_L = \{v_i^l : i \in \{1, \ldots, n_l\}\}$ and $U_R = \{v_i^r : i \in \{1, \ldots, n_r\}\}$ the left- and right-unmatched vertices associated with a maximum matching M, respectively. In particular, notice that $n_l \geq n_r$ since $|\mathcal{X}^+ \cup \mathcal{U}^+| \geq |\mathcal{X}^-|$, and $|M| = n - n_r$. Therefore, to ensure that $G(\bar{A} + \tilde{A}, \bar{B})$ is structurally controllable, it is sufficient to add edges between U_L and U_R without common end points and such that all right-unmatched vertices belong to one of such edges. However, such approach is not necessarily a solution to Problem 2 since some of the newly considered edges may correspond to edges between input and state vertices, while we are only allowed to connect pairs of state vertices. Consequently, let (without loss of generality) $U_L^{\mathcal{X}} = \{v_i^l : i \in \{1, \ldots, n_r\}\} \subseteq U_L$ be the set of n_r left-unmatched state vertices. Therefore, an optimal edge-addition configuration can be obtained as $\mathcal{E}^* = \{(v_i^l, v_i^r) : v_i^l \in U_L^{\mathcal{X}}, v_i^r \in U_R, i \in \{1, \ldots, n_r\}\}$. In other words, $M \cup \mathcal{E}^*$ is a maximum matching with respect to the bipartite graph $\mathcal{B}(\bar{A} + \tilde{A}, \bar{B})$ without right-unmatched vertices, which implies that Condition 2) in Theorem holds. Thus, $\tilde{\mathcal{E}}^* = \{s^{-1}(\{v_i^l, v_i^r\})) : \{v_i^l, v_i^r\} \in \mathcal{E}^*\}$ is an optimal solution to Problem 2. Since there may exist multiple maximum matchings of the system bipartite graph, the optimal edge-addition configuration constructed using the above procedure may not be unique. However, the number of right-unmatched vertices are the same for all maximum matchings due to maximality. As a result, in this case, all optimal edge-addition configurations contain n_r edges.

Remark 1: Under the assumption that all state vertices in the system digraph are reachable, Problem 1 can also be solved via an integer program, as proposed in [21].

Case II: Suppose that a network $(\bar{A}, \bar{B})$ is such that Condition 2) in Theorem 1 holds, while Condition 1) does not, i.e., some state vertex might be unreachable in $G(\bar{A}, \bar{B})$. Since at least one state vertex is assumed to be actuated (i.e., $\bar{B} \neq 0$), the set of reachable state vertices is nonempty. Therefore, we propose to partition the state vertices of the system digraph into two disjoint sets according to their reachability. Let $\mathcal{R}_1$ and $\mathcal{N}$ be the sets containing all the reachable and unreachable state vertices, respectively. Then, we define G_r (respectively, G_u) as the $\mathcal{R}_1$-induced (respectively, $\mathcal{N}$-induced) subgraph.

Now, notice that if an edge is added to ensure the reachability of any vertex v in some source SCC in G_u (i.e., the tail of the edge is a reachable state vertex), then all state vertices reachable from this particular source SCC become reachable as well. Consequently, to ensure reachability of all state vertices, it is sufficient to add edges to ensure reachability of one vertex per each unreachable source SCCs. Additionally, it is also necessary to have an edge pointing toward each source SCC in G_u, since otherwise the vertices belonging to it remain unreachable. Therefore, we first need to identify the source SCCs in the DAG associated with the unreachable subgraph G_u (these source SCCs can be efficiently found using, for example, [24]). Also, without loss of generality, assume there are r of these source SCCs, whose vertex sets are denoted by $\mathcal{S}_j \subseteq \mathcal{N}$, $j = 1, \ldots, r$. Subsequently, to ensure the reachability of all state vertices in $\mathcal{N}$, we need to add r edges whose tails are in a reachable vertex and each head points toward one of the vertices in one of the r source SCCs. Thus, the set $\tilde{\mathcal{E}}^* = \{(v_r, v_j) : v_r \in \mathcal{R}_1, v_j \in \mathcal{S}_j, j \in \{1, \ldots, r\}\}$ is an optimal edge-addition configuration. Notwithstanding, notice that $\tilde{\mathcal{E}}^*$ does not characterize all possible optimal edge-addition configurations, since when an edge is added from a reachable vertex toward an unreachable source SCC, all state vertices reachable from this particular source SCC become reachable; thus, the tail of an edge in an optimal edge-addition configuration should be in $\mathcal{R}_1$ and its head should be in an unreachable source SCC.

From Case I, we notice that selecting new edges for the edge-addition configuration do not increase the number of right-unmatched vertex associated with the system bipartite graph. Similarly, adding more edges never decreases the number of reachable state vertices in the system digraph. As a consequence, one may select edges to ensure both conditions in Theorem 1 are satisfied iteratively. Nonetheless, such a selection scheme often leads to suboptimal solutions, as we show next.

C. Iterative Solutions Are Suboptimal

In order to motivate the need for an algorithm that solves a general instance of the problem proposed in Problem 2, we describe below a naive iterative approach leading to suboptimal solutions. The steps in this iterative algorithm are based on the cases described in Section IV-B. Specifically, each iteration consists of a two-stage process. In the first stage, we find the minimum number of edges required to satisfy Condition 2) in Theorem 1 using the methodology described in Case I. The second stage in each iteration is described in Case II, whose aim is to satisfy Condition 1) in Theorem 1.

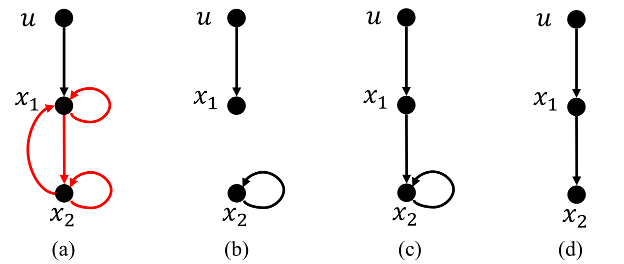

Fig. 1. In (a), we illustrate a system digraph $G(\{u, x_1, x_2\}, \{(u, x_1)\})$ with three vertices and one edge depicted in black. The goal is to find the smallest subset of state edges (depicted by red edges) to ensure structural controllability. Let us consider the iterative strategy described in Section IV-C. In (b), we depict a possible solution to the first step described in Case I, i.e., the edge (x_2, x_2) suffices to satisfy Condition 2) in Theorem 1. In (c), we depict a possible solution to the second step described in Case II when the system digraph considered is the one depicted in (b). In (d), in contrast, the edge (x_1, x_2) suffices to satisfy Condition 1) in Theorem 1, resulting in the system digraph.

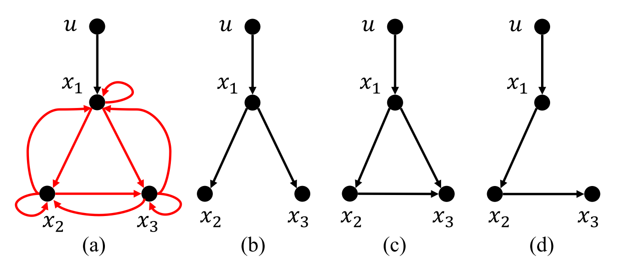

Fig. 2. In (a), we illustrate a system digraph $G(\{u, x_1, x_2, x_3\}, \{(u, x_1)\})$ in black. The goal is to find the smallest subset of state edges (depicted by red) to ensure structural controllability. Let us consider the iterative strategy described in Section IV-C. In (b), we depict a possible solution to the first step described in Case II. In (c), we depict a possible solution to the second step, which was computed by performing the solution described in Case I when the system digraph considered is the one depicted in (b). In (d) , in contrast, the edge (x_2, x_3) suffices to satisfy Condition 2) in Theorem 1, resulting in the system digraph.

To show how this iterative approach can lead to suboptimal solutions, we show in Fig. 1 an instance where we initially use the method proposed in Case I to ensure that Condition 2) in Theorem 1 holds, followed by the method proposed in Case II is applied to ensure Condition 1) in Theorem 1. As we explain in the caption of Fig. 1, the naive strategy requires two edges, whereas the digraph depicted in Fig. 1(d) is also feasible and requires only one edge. Alternatively, in Fig. 2, we provide an instance where the strategy adopted aims first to ensure Condition 1) in Theorem 1, followed by Condition 2) in Theorem 1, using the solutions in Case II and Case I, respectively. Again, in this case, the naive strategy requires three edges, whereas the digraph depicted in Fig. 2(d) is also feasible and requires only two edges. In summary, naive strategies are (in general) suboptimal.

D. General Case

Hereafter, we characterize the solutions to Problem 2 when no assumptions are made on the topology of the network. First,

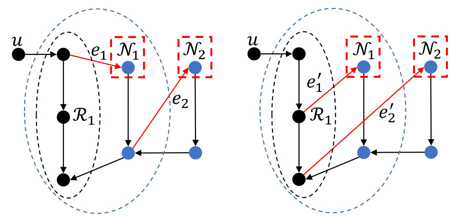

Fig. 3. Illustration of Algorithm 1. All vertices (blue or black), together with all black edges, form the initial system digraph $G(\bar{A}, \bar{B})$. The black vertices, except the input vertex u, constitute the set of reachable state vertices $\mathcal{R}_1$ (enclosed by the black dashed ellipsoid). Blue vertices constitute the set of unreachable state vertices $\mathcal{N}$. The unreachable state source SCCs, $\mathcal{N}_1$ and $\mathcal{N}_2$, are contained in red dashed squares. In (a), we depict one possible result for Algorithm 1. In the initialization step, our algorithm initializes S_B as the set containing edge e_1 only. Subsequently, after e_1 is added to S_B, all the states reachable from $\mathcal{N}_1$ become reachable [we encircle these reachable states by a blue dashed ellipsoid in (a)]. Afterward, in the FOR loop, edge e_2 in (a) is added to S_B (in Step 5 of Algorithm 1), resulting in a digraph in which all vertices are reachable from the input node. (b) Alternative output of Algorithm 1. Notice that both in (a) and (b), all vertices are reachable after adding two red edges. Therefore, $S_B = \{e_1, e_2\}$ and $S'_B = \{e'_1, e'_2\}$ are two possible sets of bridging edges.

we introduce a definition required to characterize the smallest collection of edges needed to attain reachability, i.e., satisfy Condition 1) in Theorem 1. In order to introduce this definition, we need to define the following notation. Let $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ be the system digraph, and partition the set of state vertices $\mathcal{X}$ into two sets based on their reachability (from an input), namely, $\mathcal{X} = \mathcal{R}_1 \cup \mathcal{N}$, where $\mathcal{R}_1$ is the set of reachable vertices and $\mathcal{N}$ is the set of unreachable vertices. Additionally, without loss of generality, let us assume there are r source SCCs that are unreachable, which vertex sets are denoted by $\mathcal{N}_1, \ldots, \mathcal{N}_r \subseteq \mathcal{N}$. Also, let $\Delta(\mathcal{N}_h)$ denote the set of vertices that are reachable in $G(\bar{A}, \bar{B})$ from the vertices in $\mathcal{N}_h$, for $h = 1, \ldots, r$.

Definition 1: A set S_B is called a set of bridging edges if it can be generated by the following recursive algorithm.

Algorithm 1: Set of Bridging Edges.

Input: Sets $\mathcal{R}_1$ and $\mathcal{N}_1, \ldots, \mathcal{N}_r$;
1: Initialize $\mathcal{K} = \{1, \ldots, r\}$, t_1 as any value in $\mathcal{K}$, and the set S_B to contain a single edge (i, j) where i is any vertex in $\mathcal{R}_1$ and j is any vertex in $\mathcal{N}_{t_1}$;
2: **for** $k = 2 : r$ **do**
3: $\mathcal{R}_k \leftarrow \mathcal{R}_{k-1} \cup \Delta(\mathcal{N}_{t_{k-1}})$;
4: Assign t_k to any value in $\mathcal{K} \setminus \bigcup_{h=1}^{k-1}\{t_h\}$;
5: $S_B \leftarrow S_B \cup \{(i, j)\}$ for any $i \in \mathcal{R}_k$ and any $j \in \mathcal{N}_{t_k}$;
6: **end for**

Algorithm 1 is illustrated in Fig. 3. In particular, notice that at the end of this algorithm $\mathcal{N} = \bigcup_{h=1}^{r} \Delta(\mathcal{N}_{t_k})$, which implies that all unreachable states become reachable. Furthermore, notice that the set of bridging edges contains the minimum number edges required to ensure that all state vertices are reachable. In fact, it readily follows that the solutions to Case II in Section IV-B can be characterized by the possible sets of bridging edges. Furthermore, the set of bridging edges only ensure Condition 1) in Theorem 1, which is not sufficient to ensure structural controllability in general. More specifically, to ensure structural controllability and, subsequently, to obtain a feasible edge-addition configuration, two types of edges are required: 1) a set of bridging edges, and 2) edges that connect left-unmatched state vertices to right-unmatched vertices in some maximum matching associated with the system bipartite graph (recall Case I in Section IV-B). In what follows, we state necessary and sufficient conditions to obtain a feasible edge-addition configuration.

Theorem 2: Let $G(\bar{A}, \bar{B})$ be a system digraph and $\mathcal{B}(\bar{A}, \bar{B})$ be its bipartite representation. Furthermore, let M be a maximum matching associated with $\mathcal{B}(\bar{A}, \bar{B})$ and $U_L(M) = \{v_i^l : i \in \{1, \ldots, n_l\}\}$ and $U_R(M) = \{v_i^r : i \in \{1, \ldots, n_r\}\}$ be the left- and right-unmatched vertices of M. Without loss of generality, let $U_L^{\mathcal{X}}(M) = \{v_i^l : i \in \{1, \ldots, n_r\}\}$ denotes the set of n_r left-unmatched state vertices of M. A set $\tilde{\mathcal{E}}$ is a feasible edge-addition configuration if and only if it contains the union of the following two sets.

a) S_B is the set of bridging edges.
b) $S_M = \{s^{-1}(\{v_i^l, v_i^r\}) : v_i^l \in U_L^{\mathcal{X}}(M), v_i^r \in U_R(M)$, and $i = \{1, \ldots, n_r\}\}$, for some maximum matching M associated with the system bipartite graph.

◇

From Theorem 2, we can readily obtain a lower bound on the number of edges in a feasible edge-addition configuration.

Corollary 1: The cardinality of an optimal edge-addition configuration $\tilde{\mathcal{E}}^*$ satisfies $|\tilde{\mathcal{E}}^*| \geq \max\{n_r, r\}$, where n_r is the number of right-unmatched vertices of any given maximum matching M associated with the system bipartite graph $\mathcal{B}(\bar{A}, \bar{B})$, and r is the number of unreachable state source SCCs in the DAG associated with the system digraph $G(\bar{A}, \bar{B})$.

In particular, it is easy to verify that the equality in Corollary 1 is ensured when both special cases addressed in Section IV-B are considered.

Although Theorem 2 characterizes feasible edge-addition configurations, we seek to find a feasible edge-addition configuration of minimum cardinality. To achieve this goal, we notice that it is preferable to obtain a maximum matching whose set of right-unmatched vertices are spread across different unreachable source SCCs. This is because the edges connecting left- to right-unmatched vertices in this particular maximum matching are useful to simultaneously satisfy both Conditions (a) and (b) in Theorem 2. To formalize this reasoning, we introduce the following concept.

Definition 2: Let $G(\bar{A}, \bar{B})$ be the system digraph and M be a maximum matching associated with its bipartite representation $\mathcal{B}(\bar{A}, \bar{B})$. Furthermore, denote by $U_R(M)$ the set of right-unmatched vertices of M. An unreachable state source SCC of the DAG associated with the system digraph $G(\bar{A}, \bar{B})$ is said to be unreachable assignable if it contains at least one right-unmatched vertex in $U_R(M)$.

Whether an unreachable state source SCC $\mathcal{S}$ is unreachable assignable depends on the specific maximum matching M. In other words, given two sets $U_R(M_1)$ and $U_R(M_2)$ of

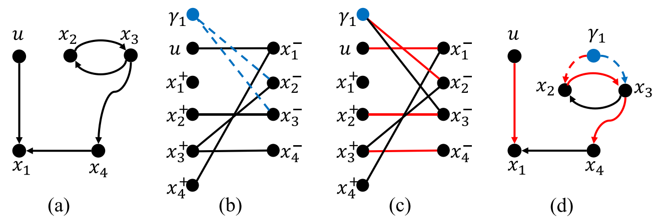

Fig. 4. Example illustrating Algorithm 2. The black vertices and edges in (a) form the initial system digraph $G(\bar{A}, \bar{B})$. In this case, $\mathcal{N} = \{x_2, x_3, x_4\}$ is the set of unreachable state vertices. Moreover, there is only one unreachable source SCC, whose vertex set is $\mathcal{N}_1 = \{x_2, x_3\}$. The black vertices and edges in (b) constitute the original system bipartite graph $\mathcal{B}(\bar{A}, \bar{B})$, while the blue vertex γ_1 represents a slack variable associated with $\mathcal{N}_1$. In addition, the blue dashed edges $\{\gamma_1, x_2\}$ and $\{\gamma_1, x_3\}$ together constitute $\mathcal{E}_\mathcal{I}$. The minimum-weighted maximum matching M' of $\mathcal{B}_w$ is depicted using red edges in (c). By removing $\{\gamma_1, x_2^-\} \in \mathcal{E}_\mathcal{I}$, we have that $M = \{\{u, x_1^-\}, \{x_2^+, x_3^-\}, \{x_3^+, x_4^-\}\}$ is a maximum matching of $\mathcal{B}(\bar{A}, \bar{B})$. In (d), we depict in red the edges from the system digraph $G(\bar{A}, \bar{B})$ associated with those in the maximum matching M. Notice that x_2 is a right-unmatched vertex of M and it is in $\mathcal{N}_1$; hence, M is a maximum matching attaining the USAN of $G(\bar{A}, \bar{B})$.

right-unmatched vertices associated with two different maximum matchings M_1 and M_2, it is possible that $U_R(M_1)$ contains a vertex from $\mathcal{S}$ while $U_R(M_2)$ does not. We introduce the following definition to characterize the maximum number of possible unreachable-assignable state source SCCs.

Definition 3: The *unreachable source assignability number* (USAN) of the system digraph $G(\bar{A}, \bar{B})$ is defined as the maximum number of unreachable-assignable state source SCCs among all the maximum matchings associated with the system bipartite graph $\mathcal{B}(\bar{A}, \bar{B})$.

Remark 2: According to Definition 3, for every system digraph $G(\bar{A}, \bar{B})$, the USAN must be less or equal to the number of right-unmatched vertices associated with any maximum matching of the $\mathcal{B}(\bar{A}, \bar{B})$ and the total number of unreachable state source SCCs in $G(\bar{A}, \bar{B})$.

To find a maximum matching associated with the system bipartite graph that attains the USAN, one can naively enumerate all possible maximum matchings associated with $\mathcal{B}(\bar{A}, \bar{B})$, but this approach incurs into a problem that is computationally $\sharp P$-complete[2] [25]. Instead of using an exhaustive search, it is possible to determine in *polynomial time* a maximum matching attaining the USAN using the following algorithm.

Remark 3: The proof of correctness of the algorithm described above is very similar to the proof of [13, Th. 11 in Sec. VI].

Essentially, in order to find a maximum matching attaining the USAN, we associate a slack vertex γ_i with each unreachable source SCC $\mathcal{N}_i$. We create additional edges from each slack vertex to every state vertex of its corresponding SCC. In other words, we let $\mathcal{E}_\mathcal{I} = \bigcup_{i=1}^{r}\{\{\gamma_i, x_j^-\} : x_j \in \mathcal{N}_i\}$. Next, we set the weights of edges $\mathcal{E}_\mathcal{I}$ higher than the weights of edges in $\mathcal{B}(\bar{A}, \bar{B})$. With this particular selection of weights, the minimum-weighted maximum matching M' prefers selecting edges in $\mathcal{B}(\bar{A}, \bar{B})$ to edges in $\mathcal{E}_\mathcal{I}$. In particular, edges are selected from $\mathcal{E}_\mathcal{I}$ if it

[2]The class of $\sharp P$-complete problems is a class of computationally equivalent counting problems that are at least as difficult as the *NP-complete* problems.

Algorithm 2: Maximum Matching Attaining the USAN.

Input: A system digraph $G(\bar{A}, \bar{B})$;
Output: A maximum matching M attaining the USAN;
1: Partition the set of state vertices in the system digraph $G(\bar{A}, \bar{B})$ based on their reachability. Obtain the set containing all the unreachable vertices of $G(\bar{A}, \bar{B})$, denoted as $\mathcal{N}$, and its $\mathcal{N}$-induced subgraph, denoted as G_u.
2: Obtain the source SCCs of G_u and denote their vertex sets as $\mathcal{N}_1, \ldots, \mathcal{N}_r$, where r is the total number of source SCCs in G_u;
3: Define a vertex set $\mathcal{I} = \{\gamma_1, \ldots, \gamma_r\}$ comprising r slack vertices. Construct a weighted bipartite graph $\mathcal{B}_w = \mathcal{B}(\mathcal{X}^+ \cup \mathcal{U}^+ \cup \mathcal{I}, \mathcal{X}^-, \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-} \cup \mathcal{E}_\mathcal{I})$, where $\mathcal{E}_\mathcal{I} = \bigcup_{i=1}^{r}\{\{\gamma_i, x_j^-\} : x_j \in \mathcal{N}_i\}$. The weights in $\mathcal{B}_w$ are as follows: every edge in $\mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-}$ is assigned to have unit weight, whereas every edge in $\mathcal{E}_\mathcal{I}$ has weight two;
4: Let M' be the minimum-weighted maximum matching of $\mathcal{B}_w$;
5: Return $M = M' \setminus \mathcal{E}_\mathcal{I}$.

helps to increase the matching. As a consequence, the vertices that are matched using edges in $\mathcal{E}_\mathcal{I}$ must correspond to right-unmatched vertices in the matching $M' \setminus \mathcal{E}_\mathcal{I}$. Furthermore, these right-unmatched vertices are spread across different unreachable source SCCs. Finally, due to maximality of matching, we can ensure that M achieves the USAN. To further illustrate the algorithm, we present an example in Fig. 4.

Remark 4: Due to maximality, the USAN is unique for every system digraph $G(\bar{A}, \bar{B})$. Nonetheless, there may exist multiple maximum matchings that attains this value. Algorithm 2 obtains one particular solution.

Although the maximum matching that achieves the USAN can be efficiently obtained as described in Algorithm 2, this is not sufficient to obtain an optimal feasible edge-addition configuration. To illustrate this claim, let us consider the example depicted in Fig. 5. In this case, the optimal feasible edge-addition configuration depends on the maximum matching achieving the USAN. Specifically, if all the left-unmatched vertices are unreachable state vertices, then after fulfilling Condition (b) in Theorem 2, we should add extra edges to form a set of bridging edges to ensure Condition (a) in Theorem 2. This would result in a suboptimal solution.

Since $\|B\|_0 \neq 0$, one can find a path rooted at an input vertex $u \in \mathcal{U}$ whose end vertex is some state vertex $x \in \mathcal{X}$. Thus, x^- is a left-unmatched vertex in the maximum matching containing the path. Consequently, it is always possible to obtain a maximum matching associated with $\mathcal{B}(\bar{A}, \bar{B})$ with at least one reachable left-unmatched state vertex—See Proof of Theorem 3 in Appendix A for more details. Moreover, when an edge is added from the reachable left-unmatched vertex to a right-unmatched state vertex in an unreachable source SCC, the set of reachable state vertices can be extended. We will use this fact to circumvent the suboptimality issue mentioned above. In our

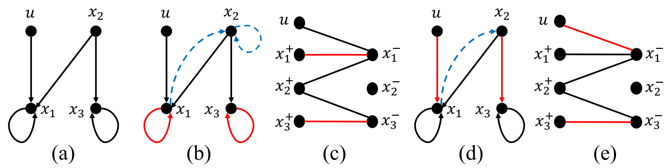

Fig. 5. This figure presents two examples where different maximum matchings lead to sets of feasible edge-addition configurations with different cardinalities. The black vertices and edges in (a) form the initial system digraph $G(\bar{A}, \bar{B})$. The red edges in (c) and (e) constitute two different maximum matchings associated with $\mathcal{B}(\bar{A}, \bar{B})$. The red edges in (b) and (d) are direct graph representations of the edges determined by the maximum matchings in (b) and (d), respectively. The edge-set $\tilde{\mathcal{E}}_2 = \{(x_1, x_2)\}$ [depicted by blue dashed arrows in (d)] is a feasible edge-addition configuration, since the addition of (x_1, x_2) ensures both conditions in Theorem 1. In contrast, in (b) we also need to add edge (x_2, x_2) [in addition to (x_1, x_2)] to ensure that Theorem 2-(*b*) holds, which leads to a feasible edge-addition configuration given by $\tilde{\mathcal{E}}_1 = \{(x_1, x_2), (x_2, x_2)\}$. Thus, $\tilde{\mathcal{E}}_2$ is an optimal edge-addition configuration with cardinality 1 while $\tilde{\mathcal{E}}_1$ is not.

next result, we characterize the relationship between the USAN and the optimal value to Problem 2.

Theorem 3: Given the system digraph $G(\bar{A}, \bar{B})$ and its bipartite representation $\mathcal{B}(\bar{A}, \bar{B})$, if $\|\bar{B}\|_0 > 0$, then the cardinality of an optimal edge-addition configuration $p^* = |\tilde{\mathcal{E}}^*|$ satisfies

$$p^* = n_r + r - q \tag{3}$$

where n_r is the number of right-unmatched vertices in any maximum matching associated with $\mathcal{B}(\bar{A}, \bar{B})$, r is the number of unreachable source state SCCs in the DAG associated with $G(\bar{A}, \bar{B})$, and q is the USAN.

In fact, based on the constructive proof of Theorem 3 in Appendix A, we propose a procedure (described in Algorithm 3) to find an optimal edge-addition configuration in polynomial time. Briefly, Algorithm 3 consists of the following four main steps: *Step 1*—Decompose the system digraph based on the reachability of state vertices. *Step 2*—Determine a maximum matching that achieves the USAN; if the obtained maximum matching admits no reachable left-unmatched vertex, then we alter the matching by finding a path rooted at certain input vertex. *Step 3*—Based on the obtained maximum matching, in order to ensure both conditions in Theorem 2, select the edges from reachable left-unmatched vertices to right-unmatched vertices in unreachable source SCCs iteratively. (Step 4) If the system is still not structurally controllable, then add the smallest collection of edges ensuring that both conditions in Theorem 2 hold independently. The correctness and computational complexity of this procedure are described in the following result.

Theorem 4: Given the system digraph $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, Algorithm 3 provides an optimal solution to Problem 2. Furthermore, the computational complexity of Algorithm 3 is $\mathcal{O}(|\mathcal{X} \cup \mathcal{U}|^3)$.

Remark 5: The computational complexity incurred by Algorithm 3 is comparable to that incurred by the algorithms required to solve the special cases described in Section IV-B. Specifically, the solution to Case I can be determined through the computation of a maximum matching, whose computational complexity is given by $\mathcal{O}(\sqrt{|\mathcal{X} \cup \mathcal{U}|}|\mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-}|)$ [24].

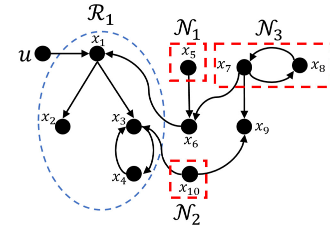

Fig. 6. System digraph $G(\bar{A}, \bar{B})$ containing a single input vertex u and ten state vertices $\{x_1, \ldots, x_{10}\}$ (depicted in black dots). Black arrows correspond to the edges of $G(\bar{A}, \bar{B})$. The dashed blue ellipsoid contains all the reachable state vertices, i.e., $\mathcal{R}_1 = \{x_1, \ldots, x_4\}$, whereas each red dashed square contains an unreachable source SCC, whose vertex sets are $\mathcal{N}_1 = \{x_5\}$, $\mathcal{N}_2 = \{x_{10}\}$, and $\mathcal{N}_3 = \{x_7, x_8\}$, respectively.

Alternatively, the solution to Case II can be obtained by determining the SCCs of the system digraph, which can be obtained by running a depth-first search algorithm twice [24] and incurring in $\mathcal{O}(|\mathcal{X} \cup \mathcal{U}|^2)$ computational complexity. A MATLAB implementation of Algorithm 3 can be found in [26].

V. Simulations

In this section, we illustrate the use of the main results of this paper. In particular, given a structurally uncontrollable system, we determine the minimum number of additional edges required for ensuring structural controllability in a some artificial network models. First, in Section V-A, we provide a pedagogical example capturing the outcome of the different steps of Algorithm 1. In Section V-B, we evaluate the minimum number of edges required in the context of large-scale randomly generated networks.

A. Illustrative Example

Consider the pair $(\bar{A}, \bar{B})$, whose system digraph is depicted in Fig. 6. Notice that the system is not structurally controllable since both conditions in Theorem 1 fail to hold. Therefore, additional edges are required to ensure structural controllability. Toward this goal, we invoke Algorithm 3 to obtain an optimal edge-addition configuration that solves Problem 1 given $(\bar{A}, \bar{B})$. In this algorithm, we need to decompose the system digraph $G(\bar{A}, \bar{B})$ according to the reachability of its state vertices. In particular, the set of reachable state vertices is given by $\mathcal{R}_1 = \{x_1, \ldots, x_4\}$, while the set of unreachable state vertices is $\mathcal{N} = \{x_5, \ldots, x_{10}\}$. Subsequently, we find the unreachable source SCCs, whose vertex sets are denoted by $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ in Fig. 6; hence, the set of states in unreachable source SCCs is $\{x_5, x_7, x_8, x_{10}\}$. Step 2 of Algorithm 3 computes a maximum matching $\bar{M}$ using Algorithm 2. In Fig. 7(a), we present in red such maximum matching, whose set of left-unmatched state vertices and right-unmatched vertices are $U_L^{\mathcal{X}}(\bar{M}) = \{x_2, x_9\}$ and $U_R(\bar{M}) = \{x_5, x_{10}\}$, respectively. Notice that x_5 and x_{10} belong to two different unreachable source SCCs; hence, the USAN equals two, i.e., $q = 2$. As a result, by invoking Theorem 3, it follows that an optimal edge-addition configuration consists of $p^* = 3$ edges.

Now, notice that x_2 is a reachable left-unmatched vertex, i.e., $x_2 \in U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{R}_1$. Thus, Step 2 of Algorithm 3 sets

Algorithm 3: Computing an Optimal Edge-Addition Configuration $\tilde{\mathcal{E}}^*$ to Problem 2.

Input: The system digraph $G(\bar{A}, \bar{B})$;
Output: An optimal edge-addition configuration $\tilde{\mathcal{E}}^*$;

Step 1: System digraph decomposition

Obtain the set of all reachable (resp. unreachable) state vertices $\mathcal{R}_1$ (resp. $\mathcal{N}$ in $G(\bar{A}, \bar{B})$.

Step 2: Maximum matching attaining the USAN

Obtain a maximum matching $\bar{M}$ associated with $\mathcal{B}(\bar{A}, \bar{B})$ attaining the USAN q using Algorithm 2;
if $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{R}_1 = \emptyset$ **then**
  Find v such that $v \in \mathcal{R}_1$ and $(u, v) \in \mathcal{E}_{\mathcal{U},\mathcal{X}} \setminus \bar{M}$;
  Find $\hat{v}$ such that $\{\hat{v}^+, v^-\} \in \bar{M}$;
  $M \leftarrow \left(\bar{M} \setminus \{\{\hat{v}^+, v^-\}\}\right) \cup \{\{u^+, v^-\}\}$;
else
  Set M equal to $\bar{M}$;
end if

Step 3: Add edges to satisfy (*a*) and (*b*) in Theorem 2

Obtain the unique set of disjoint paths $\mathcal{P} = \bigcup_{i=1}^{q} \mathcal{P}_i$ in the matching M, where the starting vertex of each $\mathcal{P}_i$ is in some unreachable source SCC and the end vertex is a left-unmatched state vertex;
$\triangleright$ We remark that the uniqueness of $\mathcal{P}$ is a direct consequence of M being a matching.
Construct two sets of vertices $\mathcal{S} = \{s_1, \ldots, s_q\}$ and $\mathcal{T} = \{t_1, \ldots, t_q\}$ such that s_i and t_i are the starting and ending vertices of each path $\mathcal{P}_i$, respectively;
Let $\tilde{\mathcal{E}}^* \leftarrow \emptyset$ and $k \leftarrow 1$;
if $\mathcal{T} \cap \mathcal{R}_1 = \emptyset$ **then**
  Select a t_0 such that $t_0^+ \in U_L^{\mathcal{X}}(M)$ and $t_0 \in \mathcal{R}_1$;
  for $k \le q$ **do**
    $\tilde{\mathcal{E}}^* \leftarrow \tilde{\mathcal{E}}^* \cup \{(t_{k-1}, s_k)\}$; $k \leftarrow k + 1$;
  end for
  $U_L^{\mathcal{X}}(M) \leftarrow U_L^{\mathcal{X}}(M) \setminus \{t_0^+, \ldots, t_{q-1}^+\}$;
else
  Find and apply a permutation of the i indexes associated to the paths $\mathcal{P}_i$ such that $t_1 \in \mathcal{R}_1$ (accordingly, permute the elements in $\mathcal{S}$ and $\mathcal{T}$);
  for $k < q$ **do**
    $\tilde{\mathcal{E}}^* \leftarrow \tilde{\mathcal{E}}^* \cup \{(t_k, s_{k+1})\}$; $k \leftarrow k + 1$;
  end for
  $\tilde{\mathcal{E}}^* \leftarrow \tilde{\mathcal{E}}^* \cup \{(t_q, s_1)\}$; $U_L^{\mathcal{X}}(M) \leftarrow U_L^{\mathcal{X}}(M) \setminus \mathcal{T}$;
end if
$U_R(M) \leftarrow U_R(M) \setminus \mathcal{S}$;

Step 4: Add extra edges to satisfy Theorem 2

for $v_l^+ \in U_L^{\mathcal{X}}(M)$ **do**
% to satisfy Theorem 2-(*b*)
  if $U_R(M) \neq \emptyset$ **then**
    $\tilde{\mathcal{E}}^* \leftarrow \tilde{\mathcal{E}}^* \cup \{(v_l, v_r)\}$, for some $v_r^- \in U_R(M)$;
    $U_L^{\mathcal{X}}(M) \leftarrow U_L^{\mathcal{X}}(M) \setminus v_l^+$;
    $U_R(M) \leftarrow U_R(M) \setminus v_r^-$;
  end if
end for

Algorithm 3: Continued.

Construct a graph $G_{\text{aug}} = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \tilde{\mathcal{E}}^*)$. Let $C_i, i = 1, \ldots, \beta$, be the vertex-sets of β unreachable source SCCs in the DAG of G_{aug}. Additionally, let $\mathcal{R}_{\text{aug}}$ be the set of all reachable vertices in G_{aug};
for $i = 1 : \beta$ **do** % to satisfy Theorem 2-(*a*)
  $\tilde{\mathcal{E}}^* \leftarrow \tilde{\mathcal{E}}^* \cup \{(v_i, z_i)\}$, for some $v_i \in \mathcal{R}_{\text{aug}}, z_i \in C_i$.
end for

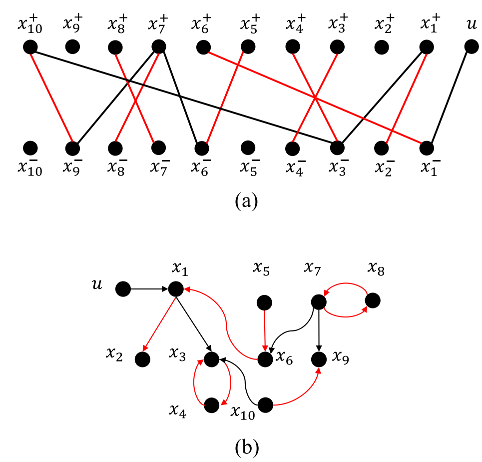

Fig. 7. Maximum matching $\bar{M}$ obtained using Step 2 in Algorithm 3. In (a), we depict the system bipartite graph associated with the pair $(\bar{A}, \bar{B})$, whose edges are depicted in black and red (edges in red are those in the maximum matching $\bar{M}$). In (b), we depict in red the edges from the system digraph $G(\bar{A}, \bar{B})$ associated with those in the maximum matching $\bar{M}$.

M equal to $\bar{M}$. To obtain an optimal edge-addition configuration $\tilde{\mathcal{E}}^*$, we should add an edge with tail in x_2 and head in some right-unmatched unreachable state vertex. According to M, we obtain $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1 = \{x_5, x_6, x_1, x_2\}$ and $\mathcal{P}_2 = \{x_{10}, x_9\}$. From $\mathcal{P}$, the set $\mathcal{S} = \{s_1 = x_5, s_2 = x_{10}\}$ and $\mathcal{T} = \{t_1 = x_2, t_2 = x_9\}$ are constructed accordingly. As a result, Step 3 in Algorithm 3 adds the edge (x_2, x_{10}) to the edge-addition configuration $\tilde{\mathcal{E}}^*$. By selecting this edge, all vertices reachable from x_{10} become reachable. Subsequently, the algorithm adds (x_9, x_5) to $\tilde{\mathcal{E}}^*$, after which Condition (*b*) in Theorem 2 is satisfied, since $M \cup \tilde{\mathcal{E}}_{\mathcal{B}}^*$ is a maximum matching of $G(\bar{A} + \tilde{A}, \bar{B})$ without right-unmatched vertices, where $\tilde{\mathcal{E}}_{\mathcal{B}}^* = \{(x_2^-, x_{10}^+), (x_9^-, x_5^+)\}$ represents the bipartite representation of the edges in $\tilde{\mathcal{E}}^*$ in $G(\bar{A} + \tilde{A}, \bar{B})$.

Finally, it remains to ensure that every state vertex is reachable, i.e., that Condition (*a*) in Theorem 2 is satisfied by $G(\bar{A} + \tilde{A}, \bar{B})$. Toward this end, notice that the only remaining unreachable state source SCC is given by $\mathcal{N}_3 = \{x_7, x_8\}$. Consequently, it suffices to add (x_1, x_7) into $\tilde{\mathcal{E}}^*$ to ensure their reachability. However, there are multiple choices of edges to ensure the reachability of $\mathcal{N}_3$. More specifically, instead of adding (x_1, x_7) into $\tilde{\mathcal{E}}^*$, one can add any edge (x_i, x_j) with $i \in \{1, \ldots, 6, 10\}$ and $j \in \{7, 8\}$ as an alternative. In summary, an optimal edge-addition configuration, i.e., a solution to

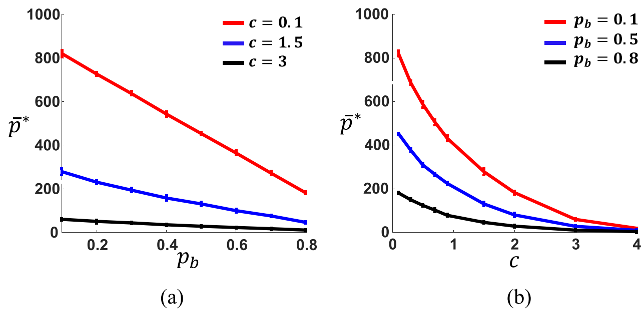

Fig. 8. Evolution of the average value of p^* as c and p_b vary. In (a), we fix the value of c and show the evolution of p^* versus p_b, when p_b ranges from 0.1 to 0.8 with step size 0.1. The red, blue, and black lines correspond to $c = 0.1$, $c = 1.5$, and $c = 3$, respectively. In (b), we plot the evolution of $\bar{p}^*$ when c varies in the interval $c \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1.5, 2, 3, 4\}$, while fixing p_b. The red, blue, and black lines show the value of $\bar{p}^*$ when $p_b = 0.1$, $p_b = 0.5$, and $p_b = 0.8$, respectively. In both figures, the error bars represent the standard deviation of p^*.

Problem 2, is given by $\tilde{\mathcal{E}}^* = \{(x_2, x_{10}), (x_9, x_5), (x_1, x_7)\}$, which contains $p^* = 3$ edges, as prescribed by Theorem 3.

B. Random Networks

In this section, we explore the minimum number of edges p^* contained in an optimal edge-addition configuration $\tilde{\mathcal{E}}^*$ required to ensure structural controllability of random networks. We assume that the structure of $\bar{A}$ is generated using an Erdős–Renyi model, i.e., $[\bar{A}]_{ij} = 1$ with probability $0 < p_a < 1$ for all i, j; 0 otherwise. In our simulations, the size of $\bar{A}$ is assumed to be $n = 1000$. We let $c \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1.5, 2, 3, 4\}$ and define $p_a = \frac{c}{n}$ for every c accordingly. Thus, c represents the average sum of in-degree and out-degree of each vertex in the graph represented by A. Moreover, we assume $\bar{B}$ to be a random diagonal matrix with $p_b n$ entries equal to 1, and 0 otherwise, where $p_b \in (0, 1)$ represents the fraction of vertices to be set equal to 1. With this particular setup, we examine the value of p^* as we vary c and p_b, independently.

In Fig. 8, we plot the empirical average of p^* (over ten random realizations). Notice that p^* decreases as c or p_b increase. Intuitively, a larger value of c results in a denser state digraph. Thus, both conditions in Theorem 1 are more likely to be satisfied. In other words, the number of right-unmatched vertices associated with the maximum matching of the system bipartite graph and the number of unreachable state vertices are smaller as c increases. Furthermore, when p_b becomes close to one, almost every state vertex is actuated by an individual input. Thus, (a) in Theorem 1 holds with high probability. Since $p^* = n_r + r - q$, it follows that p^* decreases as c or p_b increase.

To emphasize the effect of varying p_b (respectively, c) on the minimum number of additional edges to ensure structural controllability, we plot in Fig. 8(a) [respectively, Fig. 8(b)] the evolution of p^* when c is fixed (respectively, p_b is fixed). In Fig. 8(a), we observe that for a reasonably small value of c (e.g., $c = 3$), the impact of p_b in the size of the optimal edge-addition configuration is almost negligible. Intuitively, as c increases toward $\log(n)$, the number of isolated vertices in the random subgraph induced by state vertices decreases. In particular, if $c \approx \log(n)$, then the state digraph presents a unique giant SCC [27]. Subsequently, p^* is small even when there is only one state being actuated by an input. Indeed, in our experiment, $\bar{p}^* = 1.1$ when $c = 7$ and $p_b = 0.001$. In Fig. 8(b), we observe an almost exponential decrease of p^* with respect to c.

VI. Conclusion

We have addressed the problem of designing the topology of a networked dynamical system in order to achieve structural controllability. In particular, given a system digraph, we have developed an efficient methodology to find the minimum number of edges that must be added to the digraph to render a structurally controllable system. As part of our analysis, we have characterized the set of all possible solutions to this problem, and provided a polynomial-time algorithm to obtain an optimal solution. Additionally, we have presented scalable algorithms to solve our problem under additional assumptions that are commonly found in engineering applications. Finally, we have numerically illustrated our results in the context of random networked systems. In future research, we will extend these results to the case when the cost of adding a particular edge is not a fixed value. Furthermore, since structural controllability can be achieved by either (1) adding edges to the system or (2) actuating more state vertices, we will explore the tradeoffs between these two alternative strategies. In certain scenarios, it is of definite practical and theoretical interest to find efficient suboptimal algorithms with quality guarantees. Finally, it would be interesting to solve the optimal design problem under consideration when only a subset of variables is required to be under control.

Appendix A

Proof of Theorem 2: First, we show that if the set of edges $\tilde{\mathcal{E}}$ contains S_M and S_B as subsets, then it must be a feasible edge-addition configuration. We notice that, given the system digraph $G(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, it suffices to show that $S_M \cup S_B$ satisfies both conditions in Theorem 1 when the graph $G_{\text{aug}} \equiv (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup S_M \cup S_B)$ is considered. Hereafter, we denote the bipartite representation of G_{aug} by $\mathcal{B}_{\text{aug}} \equiv \mathcal{B}(\mathcal{X}^+ \cup \mathcal{U}^+, \mathcal{X}^-, \mathcal{E}_{\mathcal{X}^+,\mathcal{X}^-} \cup \mathcal{E}_{\mathcal{U}^+,\mathcal{X}^-} \cup S_M^{\pm} \cup S_B^{\pm})$, where $S_M^{\pm} = \{s(e) : e \in S_M\}$ and $S_B^{\pm} = \{s(e) : e \in S_B\}$.

To verify Condition 1) of Theorem 1, we decompose the set of state vertices $\mathcal{X}$, into $\mathcal{R}_1$ and $\mathcal{N}$ based on their reachability as in Definition 1. Specifically, $\mathcal{R}_1$ contains all the reachable state vertices and $\mathcal{N}$ contains all the unreachable state vertices. Since $\mathcal{N} = \bigcup_{h=1}^{r} \Delta(\mathcal{N}_{t_h})$, every state vertex $v \in \mathcal{N}$ must be contained in some $\Delta(\mathcal{N}_{t_h})$ for some iteration step h. By the recursive construction of the bridging set S_B as described in Definition 1, $\mathcal{N}_{t_h}$ is reachable provided that $\mathcal{N}_{t_{h-1}}$ is also reachable. Thus, we conclude that all $v \in \mathcal{N}$ become reachable in G_{aug}.

To verify Condition 2) of Theorem 1, let M be a maximum matching associated with the system bipartite graph. Next, we propose to consider a bipartite graph $\mathcal{B}_M \equiv \mathcal{B}(\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup$

$\mathcal{E}_{\mathcal{U},\mathcal{X}} \cup S_M^{\pm}$), which is a subgraph of the bipartite graph $\mathcal{B}_{\text{aug}}$. By the construction of S_M, $M \cup S_M$ is a matching in $\mathcal{B}_M$. Furthermore, it is a maximum matching since it has no right-unmatched vertices in $\mathcal{B}_M$. Since $\mathcal{B}_{\text{aug}}$ has the same set of vertices as $\mathcal{B}_M$, it follows that $M \cup S_M$ is also a maximum matching associated with $\mathcal{B}_{\text{aug}}$. Subsequently, $M \cup S_M$ satisfies Condition (*b*) in Theorem 1 for the system bipartite graph $\mathcal{B}_{\text{aug}}$.

Therefore, if $S_M \cup S_B$ is added to the system digraph, the resulting system is structurally controllable, which implies that $S_M \cup S_B$ is a feasible edge-addition configuration.

Next, we show that if $\tilde{\mathcal{E}}$ is a feasible edge-addition configuration, then it must contain the union of the two sets as described in the theorem. Assume, by contradiction, that there is no such S_B in $\tilde{\mathcal{E}}$, then there is a source SCC containing only state vertices that is unreachable. This implies that none of its states are reachable, which precludes the Condition 1) in Theorem 1 to hold; hence, a contradiction is attained. On the other hand, assume that for any maximum matchings M associated with $\mathcal{B}(\bar{A}, \bar{B})$, we have $S_M \setminus \tilde{\mathcal{E}} \neq \emptyset$, then there exists at least one right-unmatched vertex corresponding to the head of an edge in $S_M^{\pm} \setminus M$, which precludes Condition 2) in Theorem 1 to hold; hence, a contradiction is attained. Thus, a set $\tilde{\mathcal{E}}$ is a feasible edge-addition configuration if and only if it contains S_M and S_B as subsets. ∎

Proof of Corollary 1: From Theorem 2, any feasible edge-addition configuration contains S_M, for some maximum matching M associated with the system bipartite graph, and S_B, the bridging edges as subsets, i.e., $\tilde{\mathcal{E}} \supseteq S_M \cup S_B$. Consequently, an optimal edge-addition configuration should satisfy $|\tilde{\mathcal{E}}^*| \geq |S_M| = n_r$ and $|\tilde{\mathcal{E}}^*| \geq |S_B| = r$. ∎

Proof of Theorem 3: Briefly, the proof requires the following steps. First, we show that an optimal edge-addition configuration $\tilde{\mathcal{E}}^*$ must satisfy $|\tilde{\mathcal{E}}^*| \geq n_r + r - q$. Then, we construct a feasible edge-addition configuration such that its cardinality achieves $n_r + r - q$.

From Theorem 2, a feasible edge-addition configuration must satisfy $\tilde{\mathcal{E}} \supseteq S_M \cup S_B$. As a result, the cardinality of a feasible edge-addition configuration should satisfy $|\tilde{\mathcal{E}}| \geq |S_M \cup S_B|$, which implies that $|\tilde{\mathcal{E}}| \geq |S_M| + |S_B| - |S_M \cap S_B|$. Notice that, $S_M = n_r$ and $|S_B| = r$, then $|\tilde{\mathcal{E}}| \geq n_r + r - |S_M \cap S_B|$. Thus, an optimal edge-addition configuration, which we denote as $\tilde{\mathcal{E}}^*$, must satisfy $|\tilde{\mathcal{E}}^*| \geq n_r + r - \max_{M,S_B} |S_M \cap S_B|$, where the maximum is taken over all possible maximum matchings M of the system bipartite graph and possible bridging sets S_B for the system digraph. To obtain the value of $\max_{M,S_B} |S_M \cap S_B|$, we recall that maximizing the intersection between S_M and S_B gives the maximum number of right-unmatched vertices across all possible maximum matchings associated with $\mathcal{B}(\bar{A}, \bar{B})$ in the unreachable source SCCs, i.e., the USAN q, from Definition 3. Therefore, we have that $\max_{M,S_B} |S_M \cap S_B| = q$, which implies that $|\tilde{\mathcal{E}}^*| \geq n_r + r - q$. Next, we show that there exists a feasible edge-addition configuration that achieves $p^* = n_r + r - q$, which we approach by construction.

Given the system digraph $G(\bar{A}, \bar{B})$, we partition its state vertices based on reachability. Specifically, we denote $\mathcal{R}_1$ as the set of all reachable state vertices and $\mathcal{N}$ as the set of all unreachable state vertices. Moreover, we use $\mathcal{N}_1, \ldots, \mathcal{N}_r \subseteq \mathcal{N}$ to denote the vertex sets of r source SCCs that are unreachable, as in Definition 1. Furthermore, let G_r be the $\mathcal{R}_1$-induced subgraph of $G(\bar{A}, \bar{B})$.

Next, we obtain a maximum matching $\bar{M}$ that attains the USAN using Algorithm 2. Without loss of generality, we assume there are q unreachable-assignable source SCCs whose vertex sets are denoted as $\mathcal{N}_1, \ldots, \mathcal{N}_q$ with $q \leq r$. Let $U_L^{\mathcal{X}}(\bar{M})$ and $U_R(\bar{M})$ be the set of left-unmatched and right-unmatched state vertices associated with $\bar{M}$, respectively. We can obtain a digraph $G(\mathcal{V}(s^{-1}(\bar{M})), \mathcal{E}(s^{-1}(\bar{M})))$ from $\bar{M}$, where $\mathcal{E}(s^{-1}(\bar{M}))) = \{s^{-1}(e) : e \in \bar{M}\}$ and $\mathcal{V}(s^{-1}(\bar{M}))$, the vertices used by the edges belonging to $\mathcal{E}(s^{-1}(\bar{M}))$. In particular, the set of edges $\mathcal{E}(s^{-1}(\bar{M}))$ is spanned by a disjoint union of paths $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ and cycles $\{\mathcal{C}_j\}_{j\in\mathcal{J}}$, where $\mathcal{I}$ and $\mathcal{J}$ denote their indices. Furthermore, to construct an optimal edge-addition configuration, we define the following sets according to the correspondence between the maximum matching attaining the USAN q and the path and cycle decomposition captured by $G(\mathcal{V}(s^{-1}(\bar{M})), \mathcal{E}(s^{-1}(\bar{M})))$. Let $\mathcal{V}_L$ be the set of ending vertices of paths in $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ whose starting vertex is in $\mathcal{U}$. Let $\mathcal{S}$ be the set containing q starting vertices corresponding to disjoint paths in $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ and belonging to different unreachable source SCCs. Finally, let $\mathcal{S}^{\pm} = \{x_i^+ : x_i \in \mathcal{V}_L\}$, which by construction is a subset of left-unmatched vertices associated with $\bar{M}$. Thus, either $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} \neq \emptyset$ or $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$ holds.

We now begin to construct a feasible edge-addition configuration that achieves p^* under the assumption that $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} \neq \emptyset$ holds. We first initialize $\tilde{\mathcal{E}}^*$ to be an empty set. Then, at the initialization ($k = 1$), we add an edge (v_1, z_1) into $\tilde{\mathcal{E}}^*$, where $v_1^+ \in U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm}$ and z_1^- is a right-unmatched vertex associated with $\bar{M}$ in some unreachable source SCCs, i.e., $z_1 \in \mathcal{N}_l$ for some $l \in \{1, \ldots, q\}$. Since $v_1^+ \in \mathcal{S}^{\pm}$, it follows that $v_1 \in \mathcal{R}_1$. Consequently, if we add the edge (v_1, z_1) to the system digraph, then the vertex z_1 becomes reachable, which implies that all the state vertices in $\Delta(\mathcal{N}_l)$ become reachable as well. On the other hand, if $z_1^- \in U_R(\bar{M})$, then there must exist a path in $G(\mathcal{V}(s^{-1}(\bar{M})), \mathcal{E}(s^{-1}(\bar{M})))$ departing from z_1. In addition, the end of this path is a left-unmatched state vertex $v_2^+ \in U_L^{\mathcal{X}}(\bar{M})$ with $v_2^+ \neq v_1^+$. In particular, $v_2 \in \Delta(\mathcal{N}_l)$ since it is reachable from z_1. Then, we can add another edge departing from v_2^+ to another right-unmatched vertex z_2^- in a different unreachable source SCC, i.e., to add the edge (v_2, z_2) to $\tilde{\mathcal{E}}$. We iterate this procedure for another $q - 1$ steps, i.e., $k = 2, \ldots, q$, until all q unreachable-assignable SCCs become reachable by adding edges into $\tilde{\mathcal{E}}^*$.

Now, without loss of generality, let $\tilde{\mathcal{E}}^* = \{(v_k, z_k) : k = 1, \ldots, q\}$, where $v_k^+ \in U_L^{\mathcal{X}}(\bar{M})$ and $z_k^- \in U_R(\bar{M})$ for all $k = 1, \ldots, q$, respectively. Nonetheless, there are $r - q$ remaining unreachable source SCCs, i.e., $\mathcal{N}_{q+1}, \ldots, \mathcal{N}_r$. To ensure reachability of all state vertices, it suffices to add edges from the set of reachable state vertices to each one of the remaining unreachable source SCCs. Consequently, the complementary set of edges to account in $\tilde{\mathcal{E}}^*$ is a set of bridging edges containing r edges by Definition 1. However, as implied by Theorem 2, to construct a feasible edge-addition configuration, we still need to include S_M as a subset. Toward this service, we notice that q right-unmatched vertices, i.e., those in the unreachable-assignable SCCs, have been matched during the iterative procedure. Consequently, it suffices to add $n_r - q$ edges to ensure that all the remaining right-unmatched state vertices are matched, i.e., those in $U_R(\bar{M}) \setminus \{z_1^-, \ldots, z_q^-\}$. As such, we have constructed a set of edges considered to be added, i.e., $\tilde{\mathcal{E}}^*$, that contains a set of bridging edges and $S_{\bar{M}}$ for the maximum matching $\bar{M}$. As a result, $\tilde{\mathcal{E}}^*$ is a feasible edge-addition configuration by Theorem 2. In addition, it contains $n_r + r - q$ edges, which implies that it is an optimal edge-addition configuration—the construction considered in this paragraph leads to Step 4 of Algorithm 3.

Next, we discuss the case when $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$. First, we define $\mathcal{G}_r^{\pm} = \{x_i^- : x_i \in \mathcal{R}_1\}$ as the set of left-unmatched state vertices in G_r. As a consequence, two particular cases may happen: either $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} = \emptyset$ or $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} \neq \emptyset$ holds. Consider the first case, where $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} = \emptyset$, since $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$, then the subgraph of $G(\mathcal{V}(s^{-1}(\bar{M})), \mathcal{E}(s^{-1}(\bar{M})))$ constrained to the vertices in G_r consists only of cycles. Therefore, and without loss of generality, we let c_r be the number of those cycles, whose set of vertices are denoted as $\mathcal{C}_i, i = 1, \ldots, c_r$. According to the assumption $\|\bar{B}\|_0 \neq 0$, there exists an edge $(u, v) \in \mathcal{E}_{\mathcal{U},\mathcal{X}}$, with $u \in \mathcal{U}$ and $v \in \mathcal{V}$. Additionally, v belongs to the vertex set of some cycle, i.e., $v \in \mathcal{C}_j$ for some $j \leq c_r$, which we represent by the ordered sequence $(v, v_1, \ldots, v_k, v)$. If we replace the cycle $(v, v_1, \ldots, v_k, v)$ by the path $(u, v, v_1, \ldots, v_k)$, then the new digraph will correspond to another maximum matching $\hat{M}$ associated with $\mathcal{B}(\bar{A}, \bar{B})$ with a reachable left-unmatched state vertex v_k. Additionally, $U_L^{\mathcal{X}}(\hat{M}) \cap \mathcal{S}^{\pm} \neq \emptyset$, and, as a result, we may reduce the case with assumptions $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$ and $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} = \emptyset$ to the case previously discussed by constructing a new maximum matching $\hat{M}$—this procedure corresponds to steps 3–9 in Algorithm 3.

Now, we suppose that $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$ and $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} \neq \emptyset$ hold simultaneously. Then, there exists $v_1 \in U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm}$ and $v_r \in U_R(\bar{M})$ such that $(v_r, \ldots, v_1)$ is a path whose edges are associated with those in $\bar{M}$ through a signal-notation mapping. In particular, $v_r \notin \mathcal{U}$. If v_r is not a vertex in some unreachable source SCCs, then we may apply the procedure introduced in the case when $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} \neq \emptyset$ to construct a feasible edge-addition configuration containing p^* edges. Nonetheless, if v_r is a vertex in some unreachable source SCCs, then a modification of the iterative construction must be adopted. Specifically, recall that previously, at the basis step of iteration, we add (v_1, z_1) into $\tilde{\mathcal{E}}^*$, in which $z_1 \in \mathcal{N}_l$ is arbitrarily chosen. Now, if z_1 is chosen to be equal to v_r, then (v_1, v_r) is added into $\tilde{\mathcal{E}}^*$ and follow-up iteration steps cannot be performed since the end of the path starting at z_1 is v_1. Consequently, we must adopt the following modification: if $q = 1$, then we must add an edge (v_1, v_r) into $\tilde{\mathcal{E}}^*$; otherwise, we add an edge (v_1, z_1) into $\tilde{\mathcal{E}}^*$ with $z_1^- \in U_R(\bar{M})$ being a vertex in some unreachable source SCCs and $z_1 \neq v_r$ at the basis step. In other words, when constructing the first q steps of a feasible edge-addition configuration, we force $z_i^- \in U_R(\bar{M})$, $z_i \in \mathcal{N}_l$ and $z_i \neq v_r$ for all $i = 1, \ldots, q-1$ and $z_q = v_r$, whereas the rest of the construction readily follows as previously discussed. As such, we can obtain a feasible edge-addition configuration achieving p^* if $U_L^{\mathcal{X}}(\bar{M}) \cap \mathcal{S}^{\pm} = \emptyset$ and $U_L^{\mathcal{X}}(\bar{M}) \cap G_r^{\pm} \neq \emptyset$ simultaneously—This construction procedure is summarized in steps 20–32 in Algorithm 3.

Therefore, we conclude that if $\|\bar{B}\|_0 > 0$, we can construct a feasible edge-addition configuration achieving $p^* = n_r + r - q$.

Proof of Theorem 4: The correctness of the algorithm follows from the proof of Theorem 3. To determine the computational complexity of the algorithm, we consider the computational complexity incurred by each one of the major steps in the algorithm. Specifically, Step 1 requires the computation of SCCs, which can be achieved by applying the depth-first search algorithm twice with complexity $\mathcal{O}(|\mathcal{X} \cup \mathcal{U}| + |\mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}}|)$ [24]. Finding a minimum-weighted maximum matching in Step 2 incurs in $\mathcal{O}(|\mathcal{X} \cup \mathcal{U}|^3)$, and can be achieved as described in Algorithm 2, and we can guarantee that exists at least one left-unmatched vertex of $\bar{M}$ that is reachable in $\mathcal{O}(|\mathcal{X}|)$. In Step 3, we iteratively construct an optimal edge-addition configuration as described in the proof of Theorem 3, which can be attained in $\mathcal{O}(|\mathcal{X}| + |\mathcal{U}|)$, since it searches over the computed maximum matching and the source SCCs in the system digraph. Finally, in Step 4, we add the remaining edges to ensure conditions in Theorem 2, which incurs in $\mathcal{O}(|\mathcal{X}|)$. In summary, the computational complexity of Algorithm 3 is dominated by the second step, which implies an overall computational complexity in $\mathcal{O}(|\mathcal{X} \cup \mathcal{U}|^3)$. ∎

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