Dynamics of a parabolic-ODE competition system in heterogeneous environments

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Abstract

This work is concerned with the large time behavior of the solutions of a parabolic-ODE hybrid system, modeling the competition of two populations which are identical except their movement behaviors: one species moves by random dispersal while the other does not diffuse. We show that the non-diffusing population will always drive the diffusing one to extinction in environments with sinks. In contract, the non-diffusing and diffusing populations can coexist in environments without sinks.

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1 Introduction

In this work we investigate the asymptotic behaviors of classical solutions of the following parabolic-ODE competition system:

$$\begin{cases} u_t = d\Delta u + u(a(x) - u - v) & x \in \Omega,\ t > 0, \\ v_t = v(a(x) - u - v) & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $d > 0$ is a constant, $a(\cdot) \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$. The functions $u(x,t)$ and $v(x,t)$ denote the density functions of two populations residing in the same habitat Ω and competing for a common limited resource. Hence, the initial conditions $u_0(x)$ and $v_0(x)$ are assumed to

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be non-negative, not identically zero and continuous functions on $\overline{\Omega}$. The function $a(x)$ represents their common intrinsic growth rate, and throughout this paper it is assumed to be non-constant and positive somewhere in Ω, reflecting that the environment is heterogeneous in space. The region $\{x \in \bar{\Omega} : a(x) < 0\}$ is referred to as the sink (low quality habitat), where the growth rate of the population is negative. n denotes the outward unit normal vector on $\partial\Omega$, and the boundary condition for u means that no individuals cross the boundary.
We note that system (1.1) is a Lotka-Volterra competition model, in which the species $v(x,t)$ has zero diffusion rate, while $u(x,t)$ has a positive diffusion rate. In the recent years there has been increasing interest in the dynamics of two-species Lotka–Volterra competition models in heterogeneous environments; see [2, 3, 4, 7, 8, 9, 10, 11, 12, 13] and the references therein. To motivate our work, consider the following fully-parabolic Lotka-Volterra competition system:

$$\begin{cases} u_t = d\Delta u + u(a(x) - u - v) & x \in \Omega,\ t > 0, \\ v_t = \varepsilon\Delta v + v(a(x) - u - v) & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.2}$$

where $\varepsilon, d > 0$ are constant positive numbers. When $\varepsilon = 0$, system (1.2) reduces to system (1.1). Hence the system studied in this work can be regarded as the limiting case of (1.2) by formally letting $\varepsilon \to 0$. Concerning the large time behavior of the solutions of (1.2), it is well known that the relationship between d and ε plays an important role. For our interest, we suppose that $0 < \varepsilon < d$. In [5], Dockery et. al. showed that the species with the smaller diffusion rate is always favored by the competition, provided that $a(x)$ is non-constant. More precisely, if we assume that $a(x)$ is non-constant and that in the absence of competition, there exist two semi-trivial steady states of (1.2), denoted as $(u_d^*(x), 0)$ and $(0, v_\varepsilon^*(x))$, with $0 < \min\{u_d^*(x), v_\varepsilon^*(x)\} < \max\{u_d^*(x), v_\varepsilon^*(x)\} < \infty$, then $(0, v_\varepsilon^*(x))$ is globally asymptotically stable. This conclusion is often referred to as the evolution of slow dispersal [6]. It is thus natural to inquire about the dynamics of (1.2) when $\varepsilon = 0$, in particular, whether the non-diffusing population is still able to drive the diffusing one to extinction, as in the case of $\epsilon \in (0, d)$.
To state our main result on system (1.1), we first introduce a few notations. Let $C(\overline{\Omega})$ denote the Banach space of uniformly continuous functions on Ω endowed with the usual sup-norm, and $[C(\overline{\Omega})]^+$ denotes the closed subspace of $C(\overline{\Omega})$ consisting of non-negative functions.

Definition 1.1. *For given $u_0, v_0 \in C(\overline{\Omega})$ with $u_0(x) \geq 0$ and $v_0(x) \geq 0$ and $T \in (0, \infty]$, we say that $(u(x,t;u_0,v_0), v(x,t;u_0,v_0))$ is a classical solution of (1.1) on $[0,T) \times \Omega$ with $(u(x,0;u_0,v_0), v(x,0;u_0,v_0)) = (u_0(x), v_0(x))$ if the followings hold:*

1) *for every $p > N$,*

$$u \in C([0,T) : [C(\overline{\Omega})]^+) \cap C^1((0,T) : C(\overline{\Omega})) \cap C((0,T) : W^{2,p}(\Omega)), \tag{1.3}$$

2)

$$v \in C([0,\infty) : [C(\overline{\Omega})]^+) \cap C^1((0,\infty) : C(\overline{\Omega})), \tag{1.4}$$

3)
$$\lim_{t\to 0}[\|u(\cdot,t;u_0,v_0)-u_0\|_\infty+\|v(\cdot,t;u_0,v_0)-v_0\|_\infty]=0, \tag{1.5}$$

4) $(v(x,t;u_0,v_0))$ *satisfies the second equation of* (1.1) *in classical sense,*

5) $u(x,t;u,v_0)$ *is a solution of the first equation of* (1.1) *as an abstract evolution equation in* $C(\overline{\Omega})$.

By a global classical solution of (1.1), *we mean a classical solution on* $[0,\infty)\times\Omega$.

We first note from the continuous embedding results of $W^{2,p}(\Omega)$ in $C^{1+\beta}(\Omega)$ for $0<\beta\ll 1$ and $p>N$, that for any solution $(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$ of (1.1) in the sense of Definition 1.1 that both $\partial_t u(x,t;u_0,v_0)$ and $\partial_x u(x,t;u_0,v_0)$ exist in classical sense and are continuous. When $\|u_0\|_\infty>0$, it follows from the comparison principle for parabolic equations that $u(x,t;u_0,v_0)>0$ for all $t>0$ and $x\in\Omega$ and that $v(x,t;u_0,v_0)>0$ for every $t>0$ and $x\notin\{y\in\Omega:\ v_0(y)=0\}$. The following result addresses the existence of global classical solution of (1.1):

Proposition 1.2. *Given any* $u_0,v_0\in[C(\overline{\Omega})]^+$, *system* (1.1) *has a unique global classical solution on* $[0,\infty)\times\Omega$. *Moreover, it holds that*
$$\limsup_{t\to\infty}\max\{\|u(\cdot,t;u_0,v_0)\|_\infty,\|v(\cdot,t;u_0,v_0)\|_\infty\}\le\|a\|_\infty. \tag{1.6}$$

When $a(x)$ changes sign in Ω, the following result provides a rather complete feature of the behavior of solutions for large time:

Theorem 1.3. *Suppose that*
$$\{x\in\overline{\Omega}\ :\ a(x)\le 0\}\ne\emptyset. \tag{1.7}$$
For every non-negative initial $u_0,v_0\in C(\overline{\Omega})$ *satisfying*
$$\{x\in\overline{\Omega}\ :\ v_0(x)=0\}\subset\{x\in\overline{\Omega}\ :\ a(x)\le 0\}, \tag{1.8}$$
we have that
$$\lim_{t\to\infty}(u(x,t;u_0,v_0),v(x,t;u_0,v_0))=(0,a_+(x)),\quad\forall x\in\Omega. \tag{1.9}$$
Moreover the convergence $u(x,t;u_0,v_0)\to 0$ *as* $t\to\infty$ *is uniform in* $x\in\Omega$.

Biologically, Theorem 1.3 implies that the non-diffusing population will always drive the diffusing one to extinction in environments with sink. When (1.7) fails to hold, i.e. if $a(x)$ is strictly positive, Theorems 5.1, 5.3, and 5.5 provide some partial answers on the large time behaviors of solutions, which illustrate that the non-diffusing and diffusing populations can coexist in environments without sinks. We also note that Theorem 1.3 can not hold without the condition (1.8).

2 Preliminaries

In this section we recall a few results from the literature on the single species model that will be needed for our discussion and prove Theorem 2.2 (see below). It turns out that Theorem 2.2 will be essential for our proof of Theorem 1.3.
We first consider the single species equation

$$\begin{cases} u_t = d\Delta u + u(a(x) - u) & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{2.1}$$

We denote by $u(x,t;u_0)$ the classical solution of (2.1). Next, we let $u^*(x)$ denotes the unique non-negative steady solution of (2.1) attracting all positive solutions of (2.1). The existence of $u^*(x)$ is well known; see [2]. Throughout this section we shall suppose that $u^*(x) > 0$. Hence, $u^*(x)$ is the only solution of the elliptic equation

$$\begin{cases} 0 = d\Delta u^* + u^*(a(x) - u^*) & x \in \Omega, \\ u^*(x) > 0 & x \in \Omega, \\ \frac{\partial u^*}{\partial n} = 0 & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Hence, for every non-negative and not identically zero initial function $u_0 \in C(\overline{\Omega})$, it holds that

$$\lim_{t\to\infty} \|u(t,\cdot;u_0) - u^*(\cdot)\|_\infty = 0. \tag{2.3}$$

Note that a sufficient condition to ensure that $u^*(x) > 0$ is to require that $\int_\Omega a(x)dx > 0$. Note also that $u(x,t;u^*) = u^*(x)$ for every $x \in \Omega$, $t \geq 0$. It is important to note that $u^*(x)$ is not a constant function. Indeed, otherwise, by (2.2), we would have that $a(x) = u^*(x)$ for all $x \in \Omega$. So, $a(x)$ must also be a constant function, which contradicts our standing assumption on $a(x)$. We note that the constant solution $u(x,t) := \|a\|_\infty$, is a super-solution of (2.1), hence by (2.3), the fact that $u^*(x)$ is not a constant function, and the comparison principle for scalar parabolic equations, we must have that

$$u^*(x) = \inf_{t>0} u(t,x;\|a\|_\infty) < \|a\|_\infty, \quad x \in \Omega.$$

Thus, since $u(x) := \|a\|_\infty$ is a super-solution of (2.2), it follows by Hopf boundary lemma that

$$\max_{x\in\Omega} u^*(x) < \|a\|_\infty. \tag{2.4}$$

The following result holds.

Lemma 2.1. *There holds that*

$$\int_\Omega u^*(x)(a(x) - u^*(x))dx = 0, \tag{2.5}$$

and

$$\Omega^* := \{x \in \Omega \ : \ a(x) - u^*(x) > 0\} \neq \emptyset. \tag{2.6}$$

Proof. Integrating the first equation in (2.2), then use integration by part formula yields (2.5). Observe that (2.6) easily follows from (2.4). ☐

Next, we consider the sequence $\{u_k^*\}_{k\geq 0}$ with $u_0^*(x) = u^*(x)$ defined as follows. Suppose that $u_k^*(x)$ is being defined, we let $u_{k+1}^*(x)$ denotes the unique non-negative attracting set of solutions of the PDE

$$\begin{cases} u_t = d\Delta u + u(a(x) - [a(x) - u_k^*(x)]_+ - u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.7}$$

where we adopt the conventional notation $[a(x) - u_k^*(x)]_+ = \max\{0, a(x) - u_k^*(x)\}$. Note that $u_{k+1}^*(x)$ is a non-negative steady state solution of (2.7) for every $k \geq 0$. Let $u_{k+1}(t, x; w)$ denotes the solution of (2.7) with $u_{k+1}(0, x) = w(x)$. We prove the following result.

Theorem 2.2. *Let $\{u_k^*\}_{k\geq 0}$ be defined as above.*

(i) $u_{k+1}^*(x) \leq u_k^*(x)$ *for every* $x \in \Omega$ *and* $k \geq 0$.

(ii) *It holds that*

$$\lim_{k\to\infty} \|u_k^* - [a_{\min}]_+\|_\infty = 0, \tag{2.8}$$

where $a_{min} := \min_{x\in\overline{\Omega}} a(x)$ *and* $[a_{\min}]_+ = \max\{0, a_{min}\}$.

Proof. (i) We prove this by induction. We first note $u_0^*(x) = u^*(x)$ is super-solution of (2.7) with $k = 0$ and initial condition $u^*(x)$. Hence, by the comparison principle for parabolic equations, we obtain that

$$u_1^*(x) = \lim_{t\to\infty} u_1(t, x; u_0^*) \leq u_1(1, x; u_0^*) \leq u_0^*(x), \quad \forall\ x \in \Omega.$$

Suppose by induction hypothesis that (i) holds for $k = 1, \cdots, m$, with $m \geq 1$. As in the previous case we note that $u_m^*(x)$ is also a super-solution of (2.7) with $k = m - 1$ and initial condition $u_m^*(x)$ because $a(x) - (a(x) - u_m^*(x))_+ \leq a(x) - (a(x) - u_{m-1}^*(x))_+$ by induction hypothesis. Therefore similar arguments yields that the result also holds for $k = m + 1$.

(ii) By (i), we have that $u_k^*(x) \to U^*(x)$ as $k \to \infty$ for some $U^* \in L^\infty(\Omega)$. Moreover, since $\sup_{k\geq 0} \|u_k^*\| \leq \|a\|_\infty$, by estimates for elliptic equations, we have that $u_k^*(x) \to U^*(x) \in W^{2,p}(\Omega)$ for every $p > N$, and $U^*(x)$ satisfies

$$\begin{cases} 0 = d\Delta U^* + U^*(a(x) - [a(x) - U^*(x)]_+ - U^*) & x \in \Omega, \\ \frac{\partial U^*}{\partial n} = 0 & x \in \partial\Omega. \end{cases} \tag{2.9}$$

Observe that $a(x) - [a(x) - U^*(x)]_+ - U^* = -[a(x) - U^*(x)]_-$ and integrating the first equation of (2.9) yields that

$$\int_\Omega U^* [a(x) - U^*(x)]_- \, dx = 0,$$

which implies that

$$U^*(x)\,[a(x)-U^*(x)]_- = 0, \tag{2.10}$$

since $x \mapsto U^*(x)\,[a(x)-U^*(x)]_-$ is continuous. Thus, since $U^*(x) \geq 0$, we conclude that

$$0 \leq U^*(x) \leq \max\{0, a(x)\} \quad \forall\, x \in \Omega.$$

Hence, by (2.9), we conclude that $U^*(x) = c^*$ for some non-negative constant c^*. So, by (2.10), we obtain that $c^* \leq [a_{\min}]_+$. Hence if $[a_{\min}]_+ = 0$, the result follows. So, it remains to consider the case $[a_{\min}]_+ > 0$. That is, $[a_{\min}]_+ = a_{\min} > 0$. In this case, it is clear that $u(t,x) = a_{\min}$ is a sub-solution of (2.7) with initial condition a_{min}. Hence, we have

$$a_{\min} \leq \lim_{t\to\infty} u_k(x,t;a_{\min}) = u_k^+(x) \quad \forall x \in \Omega,\ k \geq 0.$$

As a result, we have that $a_{\min} \leq U^*(x) = c^* \leq a_{\min}$. This completes the proof. □

3 Proof of Proposition 1.2

In this section we present the proof of Proposition 1.2. We start with the local existence of classical solutions.

Lemma 3.1. *For every $u_0, v_0 \in [C(\overline{\Omega})]^+$, there is a unique $T_{max} > 0$ such that* (1.1) *has a unique classical solution $(u(x,t;u_0,v_0), v(x,t;u_0,v_0))$ satisfying $(u(x,0;u_0,v_0), v(x,0;u_0,v_0)) = (u_0(x), v_0(x))$ on $[0,T_{\max})\times\Omega$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t\to T_{\max}^-} \|u(\cdot,t;u_0,v_0)\|_\infty = +\infty. \tag{3.1}$$

Proof. We first note that if (1.1) has a local classical solution on some $[0,T)\times\Omega$, the uniqueness and extension criterion (3.1) follows from classical extension argument in the literature. Therefore, we will only show that (1.1) has a local classical solution. We used fixed point arguments to prove the result. Let $R > \|u_0\|_\infty$ and $T > 0$ be given, and define

$$\mathcal{S}_{R,T} := \{u \in C([0,T] : C(\overline{\Omega}))\ :\ u(x,0) = u_0(x)\,\text{and}\,\|u\|_\infty \leq R\}$$

endowed with sup-norm. Next, for every $\lambda > \|a\|_\infty + R$ and $u \in \mathcal{S}_{R,T}$, define the integral operator

$$\mathcal{T}(u)(t) = T_{\lambda,d}(t)[u_0] + \int_0^t T_{\lambda,d}(t-s)[(\lambda + a - u(s) - \mathcal{V}(u)(s))u(s)]ds \tag{3.2}$$

where

$$\mathcal{V}(u)(s) = \frac{v_0(x)e^{\int_0^t (a(x)-u(x,s))ds}}{1 + v_0(x)\int_0^t \left[e^{\int_0^s (a(x)-u(x,\tau))d\tau}\right]ds} \tag{3.3}$$

and $T_{\lambda,d}(t)$ denotes the analytic c_0-semigroup generated by $A_p u = d\Delta u - \lambda u$ on $L^p(\Omega)$ with $\mathrm{Dom}(A_p) = \{w \in W^{2,p}(\Omega) : \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega\}$. It is clear from (3.3) that

$$\mathcal{V}(u) \in C([0,\infty) : [C(\overline{\Omega})]^+) \cap C^1((0,\infty) : C(\overline{\Omega})) \quad \forall\, u \in \mathcal{S}_{R,T}, \tag{3.4}$$

with

$$\partial_t \mathcal{V}(u) = (a - u - \mathcal{V}(u))\mathcal{V}(u), \tag{3.5}$$

and

$$\lim_{t\to\infty} \|\mathcal{V}(u)(t) - v_0\|_{C(\overline{\Omega})} = 0. \tag{3.6}$$

We note that by considering

$$\mathrm{Dom}(A) := \{u \in W^{2,p}(\Omega),\ \ p > N,\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ Au \in C(\overline{\Omega})\}$$

where $Au = d\Delta u - \lambda u$, it is well known (see [14, Theorem 2]) that A generates an analytic semigroup on $C(\overline{\Omega})$, given by $T_{\lambda,p}(t)$. Thus it follows from (3.2) that $\mathcal{T}(u) \in C([0,T] : C(\overline{\Omega}))$ for every $u \in \mathcal{S}_{R,T}$. Note from the choice of λ that $u(x,t) \geq 0$. Hence, the maximum principle implies that $\mathcal{T}(u) \in C([0,T] : [C(\overline{\Omega})]^+)$ for every $u \in \mathcal{S}_{R,T}$.
Next, we show that for $0 < T \ll 1$, the map $\mathcal{T}$ maps $\mathcal{S}_{R,T}$ into itself. Indeed, this easily follows from the continuity of the maps $\mathcal{T}$ at $t = 0$, since $\lim_{t\to 0} \mathcal{T}(u)(t) = u_0$ uniformly in $u \in \mathcal{S}_{R,T}$ and $\|u_0\|_{C(\overline{\Omega})} < R$.
Finally, we claim that the map $\mathcal{T}$ is a contraction for $0 < T \ll 1$. To this end, observe that it is enough to show that the map $t \mapsto \mathcal{V}(u)(t)$ is Lipschitz continuous. This in turn follows from the fact that

$$\begin{aligned}
&\left| e^{\int_0^t (a(x)-u_1(x,s))ds} - e^{\int_0^t (a(x)-u_2(x,s))ds} \right| \\
\leq& \left| \int_0^t (u_1(x,s) - u_2(x,s))ds \right| \sup_{\theta\in[0,1]} e^{\int_0^t [\theta u_1(x,s)+(1-\theta)u_2(x,s)]ds} \\
\leq& Te^{RT} \|u_1 - u_2\|_{\mathcal{S}_{R,T}},
\end{aligned} \tag{3.7}$$

for every $u_1, u_2 \in \mathcal{S}_{R,T}$, where we have used the mean-value theorem. Therefore, for $0 < T \ll 1$, it follows from the contraction mapping theorem that the map $\mathcal{S}_{R,T} \ni u \mapsto \mathcal{T}(u) \in \mathcal{S}_{R,T}$ has a unique fixed point.
To complete the proof of the regularity of the function $[0,T] \ni t \mapsto u(\cdot,t) \in C([0,T] : C(\overline{\Omega}))$ we set

$$f(t) = (\lambda + a - u(t) - v(t))u(t)$$

where $v(t) = \mathcal{V}(u)(t)$ and show that the function $(0,T) \ni t \mapsto f(t) \in C(\overline{\Omega})$ is locally Hölder continuous. Hence, the regularity follows by [1, Theorem 1.2.1, Page 43]. Again by the regularity of $v(t)$, to show that $f(t)$ is locally Hölder continuous, it is enough to show that the function

$$t \mapsto u(t) \in C(\overline{\Omega}) \tag{3.8}$$

is locally Hölder continuous. Let $0 < \alpha < 1$ and let X^α denotes the fractional power space of $-A_p$. By $L_p - L_q$ estimates, there exist constants $c_\alpha > 0$ and $\omega > 0$ such that

$$\|(T_{\lambda,p}(h) - I)T_{\lambda,p}(t)w\|_{C(\overline{\Omega})} \leq c_\alpha h^\alpha t^{-\alpha} e^{-\omega t} \|w\|_{C(\overline{\Omega})} \quad \forall w \in C(\overline{\Omega}).$$

Therefore, for every $0 < t < t + h < T$, we have

$$\begin{aligned}
&\|u(t+h) - u(t)\|_{C(\overline{\Omega})} \\
\leq &\|(T_{\lambda,d}(h) - I)T_{\lambda,d}(t)u_0\|_{C(\overline{\Omega})} + \int_0^t \|(T(h) - I)T(t-s)f(s)\|_{C(\overline{\Omega})} ds \\
&+ \int_t^{t+h} \|T(t+h-s)f(s)\|_{C(\overline{\Omega})} ds \\
\leq &M\left(h^\alpha + h^\alpha + h\right),
\end{aligned}$$

where M is a constant depending on R, α, p and T. As a result, we have the desired result. □

Next, we complete the proof of Proposition 1.2.

Proof of Proposition 1.2. Let $u_0, v_0 \in [C(\overline{\Omega})]^+$ be given and let $(u(x,t;u_0,v_0), v(x,t;u_0,v_0))$ denotes the classical solution given by Lemma 3.1. Since $v(x,t;u_0,v_0) \geq 0$, we have that

$$u_t \leq d\Delta u + u(a(x) - u), \quad \forall\, x \in \Omega, t \in (0, T_{\max}). \tag{3.9}$$

Hence, the comparison principle implies that $u(x,t;u_0,v_0) \leq \max\{\|u_0\|_\infty, \|a\|_\infty\}$ for all $x \in \Omega$ and $t \in (0, T_{\max})$. So, by (3.1), we conclude that $T_{\max} = +\infty$. Again by (3.9) and the comparison principle for parabolic equations, we have that $\limsup_{t\to\infty} \|u(\cdot,t;u_0,v_0)\|_\infty \leq \|a\|_\infty$. Similar arguments yield that $\limsup_{t\to\infty} \|v(\cdot,t;u_0,v_0)\|_\infty \leq \|a\|_\infty$. □

4 Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3. Throughout this section we suppose $u_0(x)$ and $v_0(x)$ are chosen fixed and satisfy the assumption of Theorem 1.3. We first prove a few preliminary results.

Lemma 4.1. *It holds that*

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \geq [a(x) - u^*(x)]_+, \quad \forall\, x \in \Omega, \tag{4.1}$$

where $u^*(x)$ is given by (2.2). *In particular, the first species can not drive the second species to extinction.*

Proof. Since $v(x,t;u_0,v_0) \geq 0$ for every $x \in \Omega$ and $t \geq 0$, it follows that $u(x,t;u_0,v_0) \leq u(x,t;u_0)$ for all $x \in \Omega$ and $t \geq 0$, where $u(x,t;u_0)$ denotes the unique classical solution of (2.1). Thus, by (2.3), it holds that

$$\limsup_{t\to\infty} \sup_{x\in\Omega} [u(x,t;u_0,v_0) - u^*(x)] \leq \limsup_{t\to\infty} \sup_{x\in\Omega} [u(x,t;u_0) - u^*(x)] = 0.$$

So, for every $\varepsilon > 0$ there is $t_\varepsilon \gg 1$ such that $u(x,t;u_0,v_0) < u^*(x) + \varepsilon$ for all $x \in \Omega$ and $t \geq t_\varepsilon$. Whence, from the second equation in (1.1), we deduce that

$$v_t \geq v(a(x) - u^*(x) - \varepsilon - v), \quad \forall t \geq t_\varepsilon, x \in \Omega. \tag{4.2}$$

The comparison principle for ODEs thus implies that

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \geq [a(x) - u^*(x) - \varepsilon]_+, \quad x \in \Omega.$$

Letting $\varepsilon \to 0^+$ in the last inequality leads to (4.1). The last statement of Lemma 4.1 follows from Lemma 2.1 and inequality (4.1). □

Next, we improve the previous Lemma to

Lemma 4.2. *Suppose that $u^*(x) > 0$. For every $x_0 \in \Omega^*$, it holds that*

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) > a(x_0) - u^*(x_0) > 0.$$

Furthermore, there exist $0 < \varepsilon_0 \ll 1$ and $T_0 \gg 1$ such that

$$u(x,t;u_0,v_0) \leq (1-\varepsilon_0)u^*(x), \quad \forall t \geq T_0,\ x \in \Omega. \tag{4.3}$$

Proof. Let $x_0 \in \Omega^*$ and set

$$\underline{v}(x_0) := \liminf_{t\to\infty} v(x_0,t;u_0,v_0).$$

By Lemma 4.1 and the comparison principle for ODEs, it holds that

$$a(x_0) \geq \underline{v}(x_0) \geq a(x_0) - u^*(x_0) > 0. \tag{4.4}$$

Let $\mu > 0$ so that

$$m_\mu := \min\{a(x) - u^*(x),\ |x - x_0| \leq \mu\} > 0.$$

By (4.2), for every $\beta > 1$, there is $t_\beta \gg 1$ so that

$$v(x,t;u_0,v_0) \geq \frac{1}{\beta} m_\mu, \quad \forall\ t \geq t_\beta, \quad |x - x_0| \leq \mu.$$

So we can perturb $a(x)$ around x_0 and get a function $a_\beta(x)$ satisfying

$$\begin{cases} a(x) - v(x,t;u_0,v_0) \le a_\beta(x), \quad \forall x \in \Omega, t \ge t_\beta; \\ a_\beta(x) \le a(x), \quad \forall x \in \Omega; \\ a_\beta(x) = a(x), \quad |x - x_0| \ge \mu; \\ a_\beta(x_0) < a(x_0). \end{cases}$$

Let u_β^* denote the unique positive solution of (2.2) with $a(x)$ being replaced by $a_\beta(x)$. Hence, it holds that

$$u(x,t;u_0,v_0) \le u_\beta(x,t;u(\cdot,t_\beta,u_0,v_0)), \quad \forall\ t \ge t_\beta, \tag{4.5}$$

where $u_\beta(x,t;u(\cdot,t_\beta,u_0,v_0))$ denotes the classical solution of (2.1), with $a(x)$ being replaced by a_β, satisfying $u_\beta(x,t_\beta;u(\cdot,t_\beta,u_0,v_0)) = u(x,t_\beta,u_0,v_0)$. Similarly, as in the proof of Lemma 4.1, we have that

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \ge [a(x) - u_\beta^*(x)]_+, \quad \forall\ x \in \Omega.$$

But since $u_\beta^*(x)$ is a sub-solution of (2.2), by the comparison principle, it holds that

$$u_\beta^*(x) < u^*(x), \quad \forall\ x \in \overline{\Omega}. \tag{4.6}$$

Hence

$$\underline{v}(x_0) \ge a(x_0) - u_\beta^*(x_0) > a(x_0) - u^*(x_0).$$

Now, observe from (4.5) that

$$\limsup_{t\to\infty} \sup_{x\in\Omega} [u(x,t;u_0,v_0) - u_\beta^*(x)] \le 0.$$

This together with (4.6) yield the last assertion of the Lemma. □

Theorem 4.3. *Suppose that $u^*(x) > 0$. Let $\varepsilon_0 > 0$ be given by Lemma 4.2. It holds that*

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \ge [a(x) - (1-\varepsilon)u^*(x)]_+, \quad \textit{uniformly in} \ \ x \in \Omega \tag{4.7}$$

for every $0 \le \varepsilon < \varepsilon_0$.

Proof. Let $0 \le \varepsilon < \varepsilon_0$ be fixed. Note that it is enough to show that (4.7) holds on the set $\Omega_\varepsilon := \{x \in \Omega \ : \ a(x) \ge (1-\varepsilon)u^*(x)\}$. For, by Lemma 4.2, we have that

$$v_t \ge v(a(x) - (1-\varepsilon_0)u^* - v), \quad \forall\ t \ge T_0, x \in \Omega.$$

Hence, by the comparison principle for ODEs, we conclude that

$$v(x,t;u_0,v_0) \geq \frac{1}{e^{-(a(x)-(1-\varepsilon_0)u^*(x))(t-T_0)}\left[\frac{1}{v(x,T_0;u_0,v_0)} - \frac{1}{(a(x)-(1-\varepsilon_0)u^*(x))}\right] + \frac{1}{a(x)-(1-\varepsilon_0)u^*(x)}} \tag{4.8}$$

for every $x \in \Omega_\varepsilon$ and $t \geq T_0$. Observe that

$$\sup_{x\in\Omega_\varepsilon}\left|\frac{1}{v(x,T_0;u_0,v_0)} - \frac{1}{(a(x)-(1-\varepsilon_0)u^*(x))}\right| < \infty$$

and

$$\begin{aligned}\sup_{x\in\Omega_\varepsilon} e^{-(a(x)-(1-\varepsilon_0)u^*(x))(t-T_0)} \leq & \sup_{x\in\Omega_\varepsilon} e^{-(\varepsilon_0-\varepsilon)u^*(x)(t-T_0)} \\ \leq & e^{-(\varepsilon_0-\varepsilon)u^*_{\inf}(t-T_0)} \to 0 \quad \text{as } t\to\infty.\end{aligned}$$

Hence, we conclude that the expression at the right hand side of the inequality (4.8) converges to $a(x) - (1-\varepsilon_0)u^*(x)$ uniformly on Ω_ε, which combined with inequality (4.8) and the fact that $\varepsilon_0 > \varepsilon$ lead to (4.7). □

Lemma 4.4. *Let $\{u^*_k\}_{k\geq 0}$ be the sequence of Theorem 2.2. Then for every $k \geq 0$ such that $u^*_k(x) > 0$ there is $\varepsilon_k > 0$ such that $\liminf_{t\to\infty} v(t,x;u_0,v_0) \geq [a(x) - (1-\varepsilon_k)u^*_k(x)]_+$ uniformly in $x \in \Omega$.*

Proof. Let $k \geq 0$ such that $u^*_k(x) > 0$. If $k = 0$, the result follows from Theorem 4.3. So, we may suppose that $k \geq 1$. Let

$$K = \max\{j : 0 \leq j \leq k \text{ such that the lemma holds}\}.$$

We will show that $K = k$. Suppose not. Hence $0 \leq K \leq k-1$. Since $k > K$, by Theorem 2.2(i) we note that $\min_{x\in\overline{\Omega}} u^*_K \geq \min_{x\in\overline{\Omega}} u^*_k > 0$. Now, by induction hypothesis, for every $0 \leq \varepsilon < \varepsilon_K$, there is $T_\varepsilon \gg 1$ such that

$$u_t \leq d\Delta u + u(a - [a(x) - (1-\varepsilon)u^*_K(x)]_+ - u), \quad \forall\ t \geq T_\varepsilon.$$

Let $u^*_{K+1,\varepsilon}(x)$ denotes the unique non-negative attracting solution of

$$\begin{cases} u_t = d\Delta u + u(a(x) - [a(x) - (1-\varepsilon)u^*_K(x)]_+ - u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.9}$$

Note that $u^*_{K+1,\varepsilon}(x)$ is a non-negative steady state solution of (4.9). Since $u(x,t;u_0,v_0)$ is a sub-solution of (4.9) for $t \geq T_\varepsilon$, we then conclude that

$$\limsup_{t\to\infty}\sup_{x\in\Omega}[u(x,t;u_0,v_0) - u^*_{K+1,\varepsilon}(x)] \leq 0. \tag{4.10}$$

It is clear that $u^*_{K+1,\varepsilon}(x) < u^*_{K+1}(x)$ for every $x \in \overline{\Omega}$. Hence, by (4.10), there exist $\tilde{\varepsilon}_{K+1} > 0$ and $T_{K+1} \gg 1$ such that

$$u(x,t;u_0,v_0) \leq (1-\tilde{\varepsilon}_{K+1})u^*_{K+1}(x), \quad \forall t \geq T_{K+1},\ x \in \Omega. \tag{4.11}$$

Observe that inequality (4.11) is equivalent to inequality (4.3). Therefore, by the arguments of the proof of Theorem 4.3, we can show that there is $0 < \varepsilon_{K+1} \ll 1$ such that

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \geq [a(x) - (1-\varepsilon)u^*_{K+1}(x)]_+$$

uniformly in $x \in \Omega$ for all $0 \leq \varepsilon < \varepsilon_{K+1}$. Hence we must have that $K \geq K+1$, which is absurd. Therefore $K = k$. □

Now, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. We suppose $a(x)$ satisfies hypothesis (1.7). We first note that when $u^* \equiv 0$, then the result easily follows from Lemma 4.1. So, we may suppose that $u^* > 0$. Hence by Theorem 2.2 (ii), we have that $\|u^*_k\|_\infty \to 0$ as $k \to \infty$. Thus, by Lemma 4.4, we conclude that $\liminf_{t\to\infty} v(x,t;u_0,v_0) \geq a_+(x)$ for every $x \in \Omega$. On the other hand, since $u(x,t;u_0,v_0) \geq 0$ for every $x \in \Omega$ and $t \geq 0$, it follows from the comparison principle for ODEs that $\limsup_{t\to\infty} v(x,t;u_0,v_0) \leq a_+(x)$ for every $x \in \Omega$. Hence, $\lim_{t\to\infty} v(x,t;u_0,v_0) = a_+(x)$ uniformly in x and accordingly, $\lim_{t\to\infty} \|u(\cdot,t;u_0,v_0)\|_\infty = 0$ uniformly in x. □

5 Dynamics of solutions of (1.1) when $a(x)$ is strictly positive

This section is devoted to the study of dynamics of system (1.1) when $a(x)$ is strictly positive, i.e. $a_{\min} > 0$. Thanks to Theorem 2.2 and Lemma 4.4, we have the following *a priori* estimate on the solutions of (1.1):

Theorem 5.1. *Suppose that $a_{\min} > 0$. Then for every non-negative and not identically zero initial condition $u_0(x), v_0(x) \in C(\overline{\Omega})$ satisfying*

$$\{x \in \overline{\Omega}\ :\ v_0(x) = 0\} \subset \{x \in \overline{\Omega}\ :\ a(x) = a_{\min}\}, \tag{5.1}$$

we have that

$$\limsup_{t\to\infty} \sup_{x\in\Omega} u(x,t;u_0,v_0) \leq a_{\min}, \tag{5.2}$$

and

$$\liminf_{t\to\infty} v(x,t;u_0,v_0) \geq [a(x) - a_{\min}]_+ \quad \textit{uniformly in} \ \ x \in \Omega. \tag{5.3}$$

Note that (5.1) automatically holds if $v_0(x) > 0$ for all $x \in \bar{\Omega}$. In particular, Theorem 5.1 applies when u_0 is non-negative and $v_0 > 0$ in $\bar{\Omega}$.
Next, we find a sufficient condition on (u_0, v_0) to ensure that the equality holds for (5.2). We start with the following lemma.

Lemma 5.2. *Suppose that* $0 < \min\{u_{0\min}, v_{0\min}\}$. *Let* $(u,v)(x,t;u_0,v_0)$ *be the classical solution of* (1.1) *and define*

$$\mathcal{M}(t) := \int_\Omega \ln(\frac{v}{u})dx. \tag{5.4}$$

Then

$$\frac{d}{dt}\mathcal{M}(t) = -d\|\|\nabla \ln u(\cdot,t;u_0,v_0)|\|^2_{L^2(\Omega)}. \tag{5.5}$$

Hence,

$$\mathcal{M}(t) = \mathcal{M}(0) - d\int_0^t \|\|\nabla \ln u(\cdot,s;u_0,v_0)|\|^2_{L^2(\Omega)}ds, \quad \forall t \geq 0. \tag{5.6}$$

Proof. Notice form (1.1) that

$$\partial_t \ln(\frac{v}{u}) = \frac{v_t}{v} - \frac{u_t}{u} = -d\frac{\Delta u}{u}$$

Hence, integrating with respect to the space variable yields

$$\frac{d}{dt}\int_\Omega \ln(\frac{v}{u})dx = -d\int_\Omega \frac{\Delta u}{u}dx = d\int_\Omega < \nabla(\frac{1}{u}), \nabla u > dx = -d\int_\Omega \frac{|\nabla u|^2}{u^2}dx.$$

Hence the lemma holds. ☐

We introduce the following definition:

$$\mathcal{M}_\# = \Big\{ \int_\Omega \ln(\tfrac{a(x)-a_{\min}}{a_{\min}})dx \quad \text{if } \ln(\tfrac{a(x)-a_{\min}}{a_{\min}})dx \in L^1(\Omega; cr - \infty \quad \text{otherwise}. \tag{5.7}$$

Theorem 5.3. *Suppose that* $0 < \min\{u_{0\min}, v_{0\min}\}$. *If*

$$\mathcal{M}(0) = \int_\Omega \ln(\frac{v_0(x)}{u_0(x)})dx \leq \mathcal{M}_\#, \tag{5.8}$$

then

$$\limsup_{t\to\infty} \sup_{x\in\Omega} u(x,t;u_0,v_0) = a_{\min}. \tag{5.9}$$

Proof. Suppose to the contrary that (5.9) is false. Then by Theorem 5.1, there exist $\epsilon \in (0, a_{\min})$ and $T_\varepsilon \gg 1$ such that

$$u(x,t;u_0,v_0) < a_{\min} - \varepsilon \text{ and } v(x,t;u_0,v_0) \geq a(x) - a_{\min} + \varepsilon \quad \forall\, x \in \Omega,\, t \geq T_\varepsilon.$$

Hence

$$\int_\Omega \ln(\frac{a(x) - a_{\min} + \varepsilon}{a_{\min} - \varepsilon})dx \leq \mathcal{M}(t), \quad \forall\, t \geq T_\varepsilon,$$

which combined with (5.6) yield that, for $t \geq T_\varepsilon$,

$$\int_\Omega \ln(\frac{a(x) - a_{\min} + \varepsilon}{a_{\min} - \varepsilon})dx \leq \mathcal{M}(t) = \mathcal{M}(0) - d\int_0^t \|\|\nabla \ln u(\cdot, s; u_0, v_0)\|\|^2_{L^2(\Omega)} ds. \quad (5.10)$$

Observe that

$$\frac{a(x) - a_{\min} + \varepsilon}{a_{\min} - \varepsilon} < \frac{a(x) - a_{\min} + \tilde{\varepsilon}}{a_{\min} - \tilde{\varepsilon}}, \quad \forall x \in \Omega, \ \varepsilon < \tilde{\varepsilon} < a_{\min}.$$

Hence, it follows from (5.10) that $\mathcal{M}_\# < \mathcal{M}(0)$, contradicting (5.8). Thus we must have that (5.9) holds. □

Remark 5.4. *We note the collection of functions* $(U_c(x), V_c(x)) = (c, a(x) - c)$ *with* $0 < c < a_{\min}$*, forms a continuum of positive steady states of* (1.1)*. In particular, for* $(u_0(x), v_0(x)) = (a_{\min}, a(x) - a_{\min})$*, we have that* $\mathcal{M}(0) = \mathcal{M}_\#$ *and* $u(x, t; u_0, v_0) = a_{\min}$ *for all* $t \geq 0$ *and* $x \in \Omega$*.*

For $(x_i, y_i) \in \mathbb{R}^2$ we define the partial order

$$(u_1, v_1) \precsim (u_2, v_2) \Leftrightarrow u_1 \leq u_2 \text{ and } v_1 \geq v_2.$$

As a consequence of Remark 5.4 and Theorem 5.1, we obtain the following result.

Theorem 5.5. *Suppose that* (u_0, v_0) *satisfy* (5.1)*. If in addition,* $(u_0(x), v_0(x))$ *satisfy*

$$(a_{\min}, a(x) - a_{\min}) \precsim (u_0(x), v_0(x)), \quad \forall\, x \in \Omega, \quad (5.11)$$

then

$$\lim_{t\to\infty} (u(x, t; u_0, v_0), v(x, t; u_0, v_0)) = (a_{\min}, a(x) - a_{\min}), \quad \forall x \in \Omega. \quad (5.12)$$

Proof. By the comparison principle for competitive systems, it holds that

$$(a_{\min}, a(x) - a_{\min}) \precsim (u(x, t; u_0, v_0), v(x, t; u_0, v_0)), \quad \forall\, t \geq 0, x \in \Omega.$$

Thus, the result follows from Theorem 5.1. □

We conclude with some comments on Theorems 5.3 and 5.5. We first note that the hypotheses of Theorem 5.5 implies that $\{x \in \overline{\Omega} \,:\, v_0(x) = 0\} = \{x \in \overline{\Omega} \,:\, a(x) = a_{\min}\}$. In Theorem 5.3, the hypothesis that $0 < \min\{u_{0\min}, v_{0\min}\}$ was essential to justify that $\mathcal{M}(t)$ is well defined by noticing that $0 < \min\{u_{\min}(t), v_{\min}(t)\}$ for every $t \geq 0$. Note that the right hand side of equation (5.6) clearly suggests that $\mathcal{M}(t)$ might be well defined under a more general weaker assumption. Now, if $\mathcal{M}(0)$ is well defined, then (5.11) implies that $\mathcal{M}(0) \leq \mathcal{M}_\#$. Thus, we clearly see that these two theorems only complement each other and one does not imply the other. Moreover, they both provide sufficient conditions for the first species $u(x, t)$ to reach its possible maximum state at infinity when (5.1) holds.

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