

THE GENERALISED PRINCIPAL EIGENVALUE OF TIME-PERIODIC NONLOCAL DISPERSAL OPERATORS AND APPLICATIONS

YUAN-HANG SU¹, WAN-TONG LI^{1,*}, YUAN LOU² AND FEI-YING YANG¹

ABSTRACT. This paper is mainly concerned with the generalised principal eigenvalue for time-periodic nonlocal dispersal operators. We first establish the equivalence between two different characterisations of the generalised principal eigenvalue. We further investigate the dependence of the generalised principal eigenvalue on the frequency, the dispersal rate and the dispersal spread. Finally, these qualitative results for time-periodic linear operators are applied to time-periodic nonlinear KPP equations with nonlocal dispersal, focusing on the effects of the frequency, the dispersal rate and the dispersal spread on the existence and stability of positive time-periodic solutions to nonlinear equations.

Key words: Time-periodic nonlocal dispersal operator; Generalised principal eigenvalue; KPP equation; Positive time-periodic solution; Asymptotic behavior

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¹ School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, PRC

² Department of Mathematics, The Ohio State University, Columbus, Ohio 43210, USA

*Corresponding author (wtli@lzu.edu.cn).

1. INTRODUCTION

In recent years nonlocal dispersal evolution equations have been widely used to model non-adjacent diffusive phenomena which exhibit long range internal interactions; See [5, 14, 17, 22, 25] and references therein. The principal eigenvalues of nonlocal dispersal operators serve as a basic tool for the investigation of nonlocal dispersal equations. Many studies have been devoted to the understanding of the principal eigenvalues for elliptic-type nonlocal dispersal operators and their qualitative properties; See [3, 4, 9–11, 15, 17, 20, 23, 32, 34, 38, 39, 41, 42] and references therein. As far as time-periodic nonlocal dispersal operators are concerned, however, there is less understanding for the associated principal eigenvalue, especially the dependence of the principal eigenvalue with respect to the underlying parameters. Similar to the time-periodic random dispersal operators [16, 26, 27], the principal eigenvalues for time-periodic nonlocal dispersal operators are relevant when a time-periodic environment is involved.

In this paper, we are interested in the following time-periodic nonlocal dispersal operators:

$$(1.1) \quad L_{\Omega}^{\tau, \mu, \sigma, m}[v](x, t) := -\tau v_t(x, t) + \frac{\mu}{\sigma^m} \left(\int_{\Omega} J_{\sigma}(x-y)v(y, t)dy - h^{\sigma}(x)v(x, t) \right) + a(x, t)v(x, t),$$

where $(x, t) \in \bar{\Omega} \times \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\tau > 0$ is the frequency, $\mu > 0$ is the dispersal rate, $\sigma > 0$ is the dispersal spread which characterises the dispersal range, $m \geq 0$ is the cost parameter, $J_{\sigma}(\cdot) = \frac{1}{\sigma^N} J(\frac{\cdot}{\sigma})$ is the scaled dispersal kernel. Throughout the paper, we will make the following assumptions on the dispersal kernel J , a family of functions $\{h^{\sigma}\}_{\sigma>0}$ and function a :

- (J) $J \in C(\mathbb{R}^N)$ is nonnegative symmetric with compact support on the unit ball $B_1(0)$, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(z)dz = 1$;
- (H) $h^{\sigma} \in C(\bar{\Omega})$ and there exists a constant $M > 0$ such that $\|h^{\sigma}\|_{C(\bar{\Omega})} \leq M$ for all $\sigma > 0$;
- (A) $a \in C_1(\bar{\Omega} \times \mathbb{R}) := \{v \in C(\bar{\Omega} \times \mathbb{R}) \mid v(x, t+1) = v(x, t), (x, t) \in \bar{\Omega} \times \mathbb{R}\}$.

Define the spaces $\chi_{\Omega}, \chi_{\Omega}^+, \chi_{\Omega}^{++}$ as follows:

$$\begin{aligned} \chi_{\Omega} &= \{v \in C^{0,1}(\bar{\Omega} \times \mathbb{R}) \mid v(x, t+1) = v(x, t), (x, t) \in \bar{\Omega} \times \mathbb{R}\}, \\ \chi_{\Omega}^+ &= \{v \in \chi_{\Omega} \mid v(x, t) \geq 0, (x, t) \in \bar{\Omega} \times \mathbb{R}\}, \\ \chi_{\Omega}^{++} &= \{v \in \chi_{\Omega} \mid v(x, t) > 0, (x, t) \in \bar{\Omega} \times \mathbb{R}\}, \end{aligned}$$

where $C^{0,1}(\bar{\Omega} \times \mathbb{R})$ denotes the class of functions that are continuous in x and C^1 in t . The operator $L_{\Omega}^{\tau, \mu, \sigma, m}$ is then considered as an unbounded linear operator on the space $C_1(\bar{\Omega} \times \mathbb{R})$ with domain χ_{Ω} , namely,

$$L_{\Omega}^{\tau, \mu, \sigma, m} : \chi_{\Omega} \subset C_1(\bar{\Omega} \times \mathbb{R}) \rightarrow C_1(\bar{\Omega} \times \mathbb{R}).$$

It may be worthwhile to point out that the time-periodic nonlocal dispersal operators of the form (1.1) include several kinds of boundary conditions, such as Dirichlet, Neumann and mixed type; See [7, 28, 29].

The principal eigenvalues for time-periodic nonlocal dispersal operators have been studied in [2, 19, 29–31, 33, 36, 40]. In this paper, we adopt the approaches as in Berestycki, Nirenberg and Varadhan [6] and Berestycki, Coville and Vo [4] for the definition of the generalised principal eigenvalue $\lambda_p(L_{\Omega}^{\tau, \mu, \sigma, m})$:

$$\lambda_p(L_{\Omega}^{\tau, \mu, \sigma, m}) := \sup\{\lambda \in \mathbb{R} \mid \exists v \in \chi_{\Omega}^{++} \text{ s.t. } (L_{\Omega}^{\tau, \mu, \sigma, m} + \lambda)[v] \leq 0 \text{ in } \bar{\Omega} \times \mathbb{R}\}.$$

Another definition for the generalised principal eigenvalue of $L_\Omega^{\tau,\mu,\sigma,m}$ is given by

$$\lambda_p'(L_\Omega^{\tau,\mu,\sigma,m}) := \inf\{\lambda \in \mathbb{R} \mid \exists v \in \chi_\Omega^{++} \text{ s.t. } (L_\Omega^{\tau,\mu,\sigma,m} + \lambda)[v] \geq 0 \text{ in } \bar{\Omega} \times \mathbb{R}\},$$

motivated by the works of Donsker and Varadhan in [13] and Berestycki, Coville and Vo in [4].

Shen and Vo proved in [36] that $\lambda_p = \lambda_p'$ when λ_p is the principal eigenvalue and $h^\sigma(x) \equiv 1$ for all $\sigma > 0$. Our first main result proves that for general h^σ , $\lambda_p = \lambda_p'$ always holds, and λ_p can be characterised as the infimum of the spectrum of $-L_\Omega^{\tau,\mu,\sigma,m}$, whether λ_p is an eigenvalue or not.

Theorem 1.1. *Assume that (J), (H) and (A) hold. Then*

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = \lambda_p'(L_\Omega^{\tau,\mu,\sigma,m}) = \lambda_1,$$

where $\lambda_1 = \inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(-L_\Omega^{\tau,\mu,\sigma,m})\}$ and $\sigma(-L_\Omega^{\tau,\mu,\sigma,m})$ is the spectrum of $-L_\Omega^{\tau,\mu,\sigma,m}$.

Next, we turn to study the influences of the frequency τ , the dispersal rate μ and the dispersal spread σ on the generalised principal eigenvalue $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$. The following result establishes the monotonicity and asymptotic behaviors of the generalised principal eigenvalue $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ with respect to the frequency τ :

Theorem 1.2. *Assume that (J), (H) and (A) hold. Then the following conclusions hold:*

- (i) *The function $\tau \mapsto \lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ is non-decreasing and continuous on $(0, \infty)$. Moreover, if $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ is a principal eigenvalue, then the following assertions hold:*
 - (a) *If $a(x, t) = \hat{a}(x) + g(t)$, then $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ is constant for $\tau > 0$.*
 - (b) *Otherwise $\frac{\partial \lambda_p}{\partial \tau}(L_\Omega^{\tau,\mu,\sigma,m}) > 0$ for every $\tau > 0$;*
- (ii) *If $a \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, then there holds*

$$\lim_{\tau \rightarrow 0^+} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = \int_0^1 \lambda_p(N_\Omega^t) dt.$$

Here, for each fixed $t \in [0, 1]$, $\lambda_p(N_\Omega^t)$ is the generalised principal eigenvalue of the operator N_Ω^t

$$N_\Omega^t[v](x) := \frac{\mu}{\sigma^m} \left(\int_\Omega J_\sigma(x - y)v(y)dy - h^\sigma(x)v(x) \right) + a(x, t)v(x);$$

- (iii) *If $a \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, then there holds*

$$\lim_{\tau \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = \lambda_p(N_\Omega),$$

where $\lambda_p(N_\Omega)$ is the generalised principal eigenvalue of the operator N_Ω

$$N_\Omega[v](x) := \frac{\mu}{\sigma^m} \left(\int_\Omega J_\sigma(x - y)v(y)dy - h^\sigma(x)v(x) \right) + \hat{a}(x)v(x)$$

with

$$\hat{a}(x) := \int_0^1 a(x, t)dt, \quad x \in \bar{\Omega}.$$

Theorem 1.2 is motivated by the recent work of Liu et al. [24] for time-periodic parabolic operators; See also [18, 26]. Biologically, this reflects that in a spatio-temporal heterogeneous environment, when the temporal variability increases, it becomes harder for a single species to persist. Part (i) also implies that if a is a time-periodic function with period T , then the generalised principal eigenvalue is a non-increasing function of T .

Now, we investigate the effects of the dispersal on the generalised principal eigenvalue. On one hand, we study the dependence of the generalised principal eigenvalue λ_p on the dispersal rate μ . For this purpose, we consider the non-scaled operators $L_\Omega^{\tau,\mu,1,0}$. On the other hand, we also intend to understand the effects of the dispersal spread and the dispersal budget on the generalised principal eigenvalue λ_p . The concept of the dispersal budget was introduced by Hutson et al. [17]. They showed that the dispersal rate is characterised by $\frac{\mu}{\sigma^m}$ under proper conditions. From the biological point of view, the species can “choose” to disperse a few offspring over a long distance or many offspring over a short distance or some other combinations.

Theorem 1.3. *Assume that (J), (H) and (A) hold. Then the following conclusions hold:*

- (i) *The function $\mu \mapsto \lambda_p(L_\Omega^{\tau,\mu,1,0})$ is continuous on $(0, \infty)$ and there holds*

$$\lim_{\mu \rightarrow 0^+} \lambda_p(L_\Omega^{\tau,\mu,1,0}) = -\max_{\Omega} \hat{a};$$

- (ii) *The function $\sigma \mapsto \lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ is continuous on $(0, \infty)$ and*

- (a) *If $m > 0$, then there holds*

$$\lim_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = -\max_{\Omega} \hat{a};$$

- (b) *If $m = 0$ and $\lim_{\sigma \rightarrow \infty} h^\sigma(x) = c$, then there holds*

$$\lim_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = \mu c - \max_{\Omega} \hat{a}.$$

Remark 1.4. For the case $m = 0$, when the nonlocal dispersal operators take Dirichlet boundary conditions, $h^\sigma(x) \equiv 1$; When the nonlocal dispersal operators take Neumann boundary conditions, i.e., $h^\sigma(x) = \int_{\Omega} J_\sigma(x - y)dy$, we have $\lim_{\sigma \rightarrow \infty} h^\sigma(x) = 0$. This implies that the boundary conditions play an important role in the persistence of the populations, i.e. the large spread strategy with Neumann boundary conditions may be more advantageous for species to persist, in comparison to Dirichlet boundary conditions [36].

For later applications to time-periodic nonlinear KPP equations with nonlocal dispersal, we also investigate the time-periodic nonlocal dispersal operators with Neumann boundary conditions. More precisely, we have

Theorem 1.5. *Assume that (J) and (A) hold. If $h^\sigma(x) = \int_{\Omega} J_\sigma(x - y)dy$, then the following conclusions hold:*

- (i) *There exists $\mu_1 > 0$ such that $\lambda_p(L_\Omega^{\tau,\mu,1,0})$ is the principal eigenvalue of $L_\Omega^{\tau,\mu,1,0}$ for all $\mu \geq \mu_1$. Moreover, there holds*

$$\lim_{\mu \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,1,0}) = -\bar{\hat{a}},$$

where $\bar{\hat{a}} = \frac{1}{|\Omega|} \int_{\Omega} \hat{a}(x)dx$;

- (ii) *There exists $\sigma_0 > 0$ such that $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$ is the principal eigenvalue of $L_\Omega^{\tau,\mu,\sigma,m}$ for all $0 < \sigma \leq \sigma_0$. Moreover, if J is symmetric with respect to each component and $0 \leq m < 2$, then there holds*

$$\lim_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = -\max_{\Omega} \hat{a}.$$

For the case $m = 2$, it is shown in [31] that $\lim_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau,\mu,\sigma,2}) = \lambda_r$, where λ_r is the principal eigenvalue of the corresponding time-periodic random dispersal eigenvalue problem. For the

case $m > 2$, we conjecture that $\lim_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) = -\bar{a}$, which has been proved by [38, Theorem 1.3 (iii)] for the time-independent operators.

In the second part of this paper, we consider the applications of previous results for the generalised principal eigenvalue to the nonlocal dispersal equation in spatio-temporally heterogeneous environments

$$(1.2) \quad \begin{cases} \tau u_t(x, t) = \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y)(u(y, t) - u(x, t)) dy + f(x, t, u(x, t)), & (x, t) \in \bar{\Omega} \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \bar{\Omega}, \end{cases}$$

and the time-periodic nonlocal dispersal KPP equation with Neumann boundary conditions

$$(1.3) \quad \begin{cases} \tau u_t(x, t) = \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y)(u(y, t) - u(x, t)) dy + f(x, t, u(x, t)), & (x, t) \in \bar{\Omega} \times \mathbb{R}, \\ u(x, t + 1) = u(x, t), & (x, t) \in \bar{\Omega} \times \mathbb{R}, \end{cases}$$

where $u(x, t)$ represents the population density at location x and time t . Since we only integrate over Ω , we assume that diffusion takes place only in Ω . The individuals may not enter or leave the domain, which is called **nonlocal Neumann boundary condition**; See [1, 8]. The nonlinearity $f(x, t, u)$ satisfies the following assumptions:

(F) $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is of KPP type and satisfies:

- (1) $f(\cdot, t, u) \in C^1(\bar{\Omega})$, $f(x, \cdot, u) \in C(\mathbb{R})$ and $f(x, t, \cdot) \in C^1(\mathbb{R})$;
- (2) $f(x, t, 0) = 0$ for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$ and

$$f(x, t + 1, u) = f(x, t, u), \quad \forall (x, t, u) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R};$$

- (3) For all $(x, t) \in \bar{\Omega} \times \mathbb{R}$, the function $u \mapsto f(x, t, u)/u$ is decreasing on $(0, \infty)$;
- (4) There exists $M > 0$ such that

$$f(x, t, u) \leq 0, \quad \forall (x, t, u) \in \bar{\Omega} \times \mathbb{R} \times [M, \infty).$$

From now on, we set

$$a(x, t) = f_u(x, t, 0), \quad (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

Then, $L_\Omega^{\tau, \mu, \sigma, m}$, defined in (1.1), is the linear operator associated to the linearization of (1.3) at $u \equiv 0$.

Nonlocal dispersal evolution equations of the form (1.2) have attracted a lot of attentions in recent years; See [29, 31, 33, 36, 40] and references therein. The case $f(x, t, u) = f(x, u)$ in equations (1.2) has been well studied; See [3, 5, 7, 9, 12, 20, 34, 35, 37–39, 42]. We first recall the following results of the existence and non-existence of positive time-periodic solutions to (1.3) by Rawal and Shen [29] and Shen and Vo [36]:

Lemma 1.6. *Assume that (J) and (F) hold. Let $u(x, t; u_0)$ be a solution of (1.2) with initial data $u_0 \in C(\bar{\Omega})$, which is non-negative and not identically zero. The following statements hold:*

- (i) *If $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) < 0$, then (1.3) admits a unique solution u^* in χ_Ω^{++} and there holds*

$$\|u(\cdot, t; u_0) - u^*(\cdot, t)\|_\infty \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $\|\cdot\|_\infty$ is the sup norm on $C(\bar{\Omega})$;

- (ii) *If $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) > 0$, then (1.3) admits no solution in $\chi_\Omega^+ \setminus \{0\}$ and there holds*

$$\|u(\cdot, t; u_0)\|_\infty \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Now, we discuss the effects of the frequency on the persistence of populations. The following conclusion is a direct corollary of Theorem 1.2 and Lemma 1.6.

Corollary 1.7. *Assume that (J) and (F) hold. Then the following statements hold:*

- (i) *If $\int_0^1 \lambda_p(N_\Omega^t) dt > 0$, then (1.3) admits no solution in $\chi_\Omega^+ \setminus \{0\}$ and zero solution is globally asymptotically stable for all $\tau \in (0, \infty)$;*
- (ii) *If $\int_0^1 \lambda_p(N_\Omega^t) dt < 0$, $\lambda_p(N_\Omega) > 0$ and $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ is a principal eigenvalue of the operator $L_\Omega^{\tau, \mu, \sigma, m}$, then there is a constant $\tau^* > 0$ such that*
 - (a) *If $\tau < \tau^*$, then (1.3) admits a unique solution $u_\tau^* \in \chi_\Omega^{++}$ that is globally asymptotically stable.*
 - (b) *If $\tau > \tau^*$, then (1.3) admits no solution in $\chi_\Omega^+ \setminus \{0\}$ and zero solution is globally asymptotically stable;*
- (iii) *If $\lambda_p(N_\Omega) < 0$, then (1.3) admits a unique solution $u_\tau^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\tau \in (0, \infty)$.*

In the spatially and temporally varying environment, Corollary 1.7 (ii) suggests that increasing the frequency of oscillations in the resources may be disadvantageous to the persistence of populations. It should be pointed out that the condition of Corollary 1.7 (i)-(iii) may be satisfied respectively; See Theorem 3.1 for more details.

We turn to study the effects of the dispersal rate μ on the persistence of populations. The existence and asymptotic behaviors of positive time-periodic solutions associated to (1.3) in the non-scaled case with $m = 0$ and $\sigma = 1$ are obtained as μ tends to zero or infinity.

Theorem 1.8. *Assume that (J) and (F) hold. Then the following statements hold:*

- (i) *If $\max_{\bar{\Omega}} \hat{a} > 0$, then there exists $\mu_0 > 0$ such that (1.3) admits a unique solution $u_\mu^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\mu \in (0, \mu_0)$. Moreover, if $\min_{\bar{\Omega}} \hat{a} > 0$, then*

$$\lim_{\mu \rightarrow 0^+} u_\mu^*(x, t) = v^*(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^(x, t)$ is the unique positive 1-periodic solution of the equation $\tau v_t = f(x, t, v)$ for every $x \in \bar{\Omega}$.*

- (ii) *If $\bar{\hat{a}} > 0$, then there exists $\mu_1 > 0$ such that (1.3) admits a unique solution $u_\mu^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\mu \in (\mu_1, \infty)$. Moreover,*

$$\lim_{\mu \rightarrow \infty} u_\mu^*(x, t) = v^*(t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^(t)$ is the unique positive 1-periodic solution of the equation*

$$(1.4) \quad \tau v_t(t) = \frac{1}{|\Omega|} \int_{\Omega} f(x, t, v(t)) dx.$$

We see from Theorem 1.8 that the populations with small dispersal rate can persist while the populations with large dispersal rate die out, provided that $\bar{\hat{a}} < 0 < \max_{\bar{\Omega}} \hat{a}$. This shows that the small dispersal rates are better dispersal strategies than the larger ones in proper situations.

Now, we are interested in the effects of the dispersal spread and the dispersal budget on the persistence of populations. We establish the existence, uniqueness and stability of positive time-periodic solutions to (1.3) when σ is sufficiently small or large. Furthermore, we analyse the asymptotic limits of the positive time-periodic solutions as σ tends to zero or infinity. As in [17, 36], these asymptotics for $\sigma \ll 1$ or $\sigma \gg 1$ represent two completely different dispersal strategies: The limit $\sigma \rightarrow 0^+$ can be associated to a strategy of dispersing many offspring on a

short range, while the limit $\sigma \rightarrow +\infty$ corresponds to a strategy that disperses a few offspring over a long distance. More precisely, we obtain

Theorem 1.9. *Assume that (J) and (F) hold. Then the following statements hold:*

- (i) *Let $m \geq 0$. If $\max_{\bar{\Omega}} \hat{a} > 0$, then there exists $\sigma_1 > 0$ such that (1.3) admits a unique solution $u_\sigma^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\sigma \in (\sigma_1, \infty)$. Moreover, if $\min_{\bar{\Omega}} \hat{a} > 0$, then*

$$\lim_{\sigma \rightarrow \infty} u_\sigma^*(x, t) = v^*(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^(x, t)$ is the unique positive 1-periodic solution of the equation $\tau v_t = f(x, t, v)$ for every $x \in \bar{\Omega}$;*

- (ii) *Let $0 \leq m < 2$. If J is symmetric with respect to each component and $\max_{\bar{\Omega}} \hat{a} > 0$, then there exists $\sigma_0 > 0$ such that (1.3) admits a unique solution $u_\sigma^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\sigma \in (0, \sigma_0)$. Moreover, if $\min_{\bar{\Omega}} \hat{a} > 0$, then*

$$\lim_{\sigma \rightarrow 0^+} u_\sigma^*(x, t) = v^*(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^(x, t)$ is the same as in (i).*

In addition, Shen and Xie proved in [31] that for the case $m = 2$ and $\lambda_r < 0$, there exists $\sigma_0 > 0$ such that (1.3) admits a unique solution $u_\sigma^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\sigma \in (0, \sigma_0)$ and

$$\lim_{\sigma \rightarrow 0^+} u_\sigma^*(x, t) = v(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where v is the positive 1-periodic solution of the corresponding reaction diffusion equation. For the case $m > 2$, it seems reasonable to conjecture that when $\bar{a} > 0$, there exists $\sigma_0 > 0$ such that (1.3) admits a unique solution $u_\sigma^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\sigma \in (0, \sigma_0)$ and there holds

$$\lim_{\sigma \rightarrow 0^+} u_\sigma^*(x, t) = v^*(t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^*(t)$ is the unique positive 1-periodic solution of (1.4). We refer interested readers to [38, Theorem 1.8 (iii)] for the time-independent case.

The rest of the paper is organised as follows. In Section 2, we first establish the equivalence of different definitions of the generalised principal eigenvalue and a characterisation of the generalised principal eigenvalue by the infimum of the spectrum. Then we study the influences of the frequency, the dispersal rate and the dispersal spread on the generalised principal eigenvalue. Section 3 is devoted to investigating the effects of the frequency, the dispersal rate and the dispersal spread on persistence criteria of populations.

2. TIME-PERIODIC NONLOCAL DISPERSAL OPERATORS

In this section we consider the eigenvalue problem

$$(2.1) \quad \begin{cases} L_\Omega^{\tau, \mu, \sigma, m}[v](x, t) + \lambda v(x, t) = 0, & (x, t) \in \bar{\Omega} \times \mathbb{R}, \\ v(x, t+1) = v(x, t), & (x, t) \in \bar{\Omega} \times \mathbb{R}. \end{cases}$$

As shown in [9, 29, 34, 36], the operator $L_\Omega^{\tau, \mu, \sigma, m}$ may not have any principal eigenvalue. However, the generalised principal eigenvalue $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ can become the surrogate of the principal eigenvalue. Here, we establish the equivalent definitions of $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ and study the dependence of $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ on the frequency, the dispersal rate and the dispersal spread.

2.1. The equivalence of the generalised principal eigenvalue.

We consider the following general form of nonlocal dispersal operators

$$M_\Omega(b)[v](x, t) := -\tau v_t(x, t) + \mu \int_\Omega J(x - y)v(y, t)dy + b(x, t)v(x, t), \quad (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $b \in C_1(\bar{\Omega} \times \mathbb{R})$. We define

$$\lambda_1 = \inf\{Re\lambda \mid \lambda \in \sigma(-M_\Omega(b))\}.$$

Firstly, we recall two lemmas in [36, Theorem 3.3 and Proposition 6.1 (iii)].

Lemma 2.1. *Assume that (J) holds and $b \in C_1(\bar{\Omega} \times \mathbb{R})$. For any $\epsilon > 0$, there exists $b_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that the following conclusions hold:*

- (i) *There holds $\max_{\bar{\Omega} \times \mathbb{R}} |b - b_\epsilon| < \epsilon$;*
- (ii) *λ_1^ϵ is the principal eigenvalue of $M_\Omega(b_\epsilon)$, where $\lambda_1^\epsilon = \inf\{Re\lambda \mid \lambda \in \sigma(-M_\Omega(b_\epsilon))\}$;*
- (iii) *There holds $|\lambda_1^\epsilon - \lambda_1| < \epsilon$.*

Lemma 2.2. *Assume that (J) holds and $b \in C_1(\bar{\Omega} \times \mathbb{R})$. Then $\lambda_p(M_\Omega(b))$ is a Lipschitz continuous function with respect to b . More precisely, for every $b_1, b_2 \in C_1(\bar{\Omega} \times \mathbb{R})$, we have*

$$|\lambda_p(M_\Omega(b_1)) - \lambda_p(M_\Omega(b_2))| \leq \sup_{t \in [0, 1]} \|b_1(\cdot, t) - b_2(\cdot, t)\|_\infty.$$

Next, we prove the following two results, from which Theorem 1.1 follows as a consequence.

Theorem 2.3. *Assume that (J) holds and $b \in C_1(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$\lambda_p(M_\Omega(b)) = \lambda_1.$$

Proof. The proof is divided into two cases.

Case 1. We prove the result under the additional assumption that λ_1 is the principal eigenvalue. By the definition of the principal eigenvalue, there is $\varphi_1 \in \chi_\Omega^{++}$ such that

$$M_\Omega(b)[\varphi_1] + \lambda_1 \varphi_1 = 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

Thanks to the definition of $\lambda_p(M_\Omega(b))$, we have $\lambda_1 \leq \lambda_p(M_\Omega(b))$. It remains to establish the inequality $\lambda_p(M_\Omega(b)) \leq \lambda_1$, which is similar to the proof of [36, Theorem 2.3]. Here, we omit it. Thus, we get $\lambda_p(M_\Omega(b)) = \lambda_1$.

Case 2. If λ_1 is not the principal eigenvalue, we can use an approximation argument. More precisely, applying Lemma 2.1, we find that for each $\epsilon > 0$, there exists $b_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that

$$(2.2) \quad \max_{\bar{\Omega} \times \mathbb{R}} |b_\epsilon - b| < \epsilon, \quad |\lambda_1 - \lambda_1^\epsilon| < \epsilon$$

and λ_1^ϵ is the principal eigenvalue of $M_\Omega(b_\epsilon)$. Then, we apply **Case 1** to conclude

$$(2.3) \quad \lambda_p(M_\Omega(b_\epsilon)) = \lambda_1^\epsilon.$$

Since $\lambda_p(M_\Omega(b))$ is Lipschitz continuous with respect to b in Lemma 2.2 and the inequalities (2.2), setting $\epsilon \rightarrow 0$ in (2.3) yields $\lambda_p(M_\Omega(b)) = \lambda_1$. $\square$

Theorem 2.4. *Assume that (J) holds and $b \in C_1(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$\lambda_p(M_\Omega(b)) = \lambda_p'(M_\Omega(b)).$$

Proof. We first show that $\lambda_p(M_\Omega(b)) \leq \lambda'_p(M_\Omega(b))$. Let us assume by contradiction that

$$\lambda'_p(M_\Omega(b)) < \lambda_p(M_\Omega(b)).$$

Pick now $\lambda \in (\lambda'_p(M_\Omega(b)), \lambda_p(M_\Omega(b)))$, then, by the definition of $\lambda_p(M_\Omega(b))$ and $\lambda'_p(M_\Omega(b))$, there exist $\varphi \in \chi_\Omega^{++}$ and $\psi \in \chi_\Omega^{++}$ such that

$$(2.4) \quad M_\Omega(b)[\varphi](x, t) + \lambda\varphi(x, t) \leq 0 \quad \text{in } \bar{\Omega} \times \mathbb{R},$$

$$(2.5) \quad M_\Omega(b)[\psi](x, t) + \lambda\psi(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

By taking λ bigger if necessary, we assume that ψ satisfies

$$(2.6) \quad M_\Omega(b)[\psi](x, t) + \lambda\psi(x, t) > 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

Set $w := \frac{\psi}{\varphi} \in \chi_\Omega^{++}$. Using (2.4), a direct computation yields

$$\begin{aligned} M_\Omega(b)[\psi] &= M_\Omega(b)[w\varphi] \\ &= -\tau(w\varphi)_t + \mu \int_\Omega J(x-y)w(y, t)\varphi(y, t)dy + b(x, t)w\varphi \\ &= -\tau w_t\varphi + \mu \int_\Omega J(x-y)\varphi(y, t)(w(y, t) - w(x, t))dy - \lambda w\varphi \\ &\quad + w(-\tau\varphi_t + \mu \int_\Omega J(x-y)\varphi(y, t)dy + b(x, t)\varphi + \lambda\varphi) \\ &\leq -\tau w_t\varphi + \mu \int_\Omega J(x-y)\varphi(y, t)(w(y, t) - w(x, t))dy - \lambda\psi. \end{aligned}$$

By (2.6), we find

$$(2.7) \quad 0 < -\tau w_t\varphi + \mu \int_\Omega J(x-y)\varphi(y, t)(w(y, t) - w(x, t))dy \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

Since $w \in \chi_\Omega^{++}$, there exists $(x_0, t_0) \in \bar{\Omega} \times [0, 1]$ such that

$$w(x_0, t_0) = \max_{\bar{\Omega} \times \mathbb{R}} w, \quad w_t(x_0, t_0) = 0.$$

Hence, setting $(x, t) = (x_0, t_0)$ in (2.7) yields

$$0 < -\tau w_t(x_0, t_0)\varphi(x_0, t_0) + \mu \int_\Omega J(x_0-y)\varphi(y, t_0)(w(y, t_0) - w(x_0, t_0))dy \leq 0,$$

which is a contradiction. Therefore, $\lambda_p(M_\Omega(b)) \leq \lambda'_p(M_\Omega(b))$.

To complete the proof, it suffices to establish

$$(2.8) \quad \lambda'_p(M_\Omega(b)) \leq \lambda_p(M_\Omega(b)) + 2\delta \quad \text{for all } \delta > 0.$$

We claim that for any $\delta > 0$, there exists $\varphi_\delta \in \chi_\Omega^{++}$ such that

$$M_\Omega(b)[\varphi_\delta] + (\lambda_p(M_\Omega(b)) + 2\delta)\varphi_\delta \geq 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

Indeed, thanks to Lemma 2.1 and Theorem 2.3, there is $b_\delta \in C_1(\bar{\Omega} \times \mathbb{R})$ such that

$$(2.9) \quad \max_{\bar{\Omega} \times \mathbb{R}} |b_\delta - b| < \delta, \quad |\lambda_p(M_\Omega(b)) - \lambda_p(M_\Omega(b_\delta))| < \delta$$

and $\lambda_p(M_\Omega(b_\delta))$ is the principal eigenvalue of $M_\Omega(b_\delta)$. Thus, there exists $\varphi_\delta \in \chi_\Omega^{++}$ such that

$$(2.10) \quad M_\Omega(b_\delta)[\varphi_\delta] + \lambda_p(M_\Omega(b_\delta))\varphi_\delta = 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

Owing to (2.9) and (2.10), we get

$$M_\Omega(b)[\varphi_\delta] + (\lambda_p(M_\Omega(b)) + 2\delta)\varphi_\delta$$

$$\begin{aligned}
&= M_\Omega(b_\delta)[\varphi_\delta] + (b(x, t) - b_\delta(x, t))\varphi_\delta + (\lambda_p(M_\Omega(b)) + 2\delta)\varphi_\delta \\
&= (\lambda_p(M_\Omega(b)) - \lambda_p(M_\Omega(b_\delta)))\varphi_\delta + (b(x, t) - b_\delta(x, t))\varphi_\delta + 2\delta\varphi_\delta \\
&\geq -\delta\varphi_\delta - \delta\varphi_\delta + 2\delta\varphi_\delta = 0
\end{aligned}$$

The proof of the claim is complete.

Moreover, it follows from this claim and the definition of $\lambda'_p(M_\Omega(b))$ that (2.8) holds. In conclusion, we obtain $\lambda_p(M_\Omega(b)) = \lambda'_p(M_\Omega(b))$. $\square$

2.2. Influences of the frequency.

This subsection concerns the dependence of $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ on τ . Consider

$$M_\tau[v](x, t) := -\tau v_t(x, t) + \mu \int_\Omega J(x - y)v(y, t)dy + b(x, t)v(x, t), \quad (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $b \in C_1(\bar{\Omega} \times \mathbb{R})$. Our goal is to prove the following two results, from which the conclusions of Theorem 1.2 follow:

Theorem 2.5. *Assume that (J) holds and $b \in C_1(\bar{\Omega} \times \mathbb{R})$. Then the function $\tau \mapsto \lambda_p(M_\tau)$ is continuous non-decreasing on $(0, \infty)$. Moreover, if $\lambda_p(M_\tau)$ is a principal eigenvalue, then the following assertions hold:*

- (i) *If $b(x, t) = \hat{b}(x) + g(t)$, then $\lambda_p(M_\tau)$ is constant for $\tau > 0$;*
- (ii) *Otherwise $\frac{\partial \lambda_p}{\partial \tau}(M_\tau) > 0$ for every $\tau > 0$.*

Proof. The proof is divided into two cases.

Case 1. We prove the result under the additional assumption that $\lambda_p(M_\tau)$ is a principal eigenvalue for all $\tau > 0$. By the definition of the principal eigenvalue, there exists $\varphi_\tau \in \chi_\Omega^{++}$ s.t.

$$\begin{aligned}
(2.11) \quad &\begin{cases} M_\tau[\varphi_\tau](x, t) = -\tau \partial_t \varphi_\tau + \mu \int_\Omega J(x - y)\varphi_\tau(y, t)dy + b(x, t)\varphi_\tau = -\lambda_p(M_\tau)\varphi_\tau & \text{in } \bar{\Omega} \times [0, 1], \\ \varphi_\tau(x, 1) = \varphi_\tau(x, 0) & \text{in } \bar{\Omega}. \end{cases}
\end{aligned}$$

Note that there is $\psi_\tau \in \chi_\Omega^{++}$ such that ψ_τ satisfies the adjoint problem of (2.11)

$$\begin{aligned}
(2.12) \quad &\begin{cases} M_\tau^*[\psi_\tau](x, t) := \tau \partial_t \psi_\tau + \mu \int_\Omega J(x - y)\psi_\tau(y, t)dy + b(x, t)\psi_\tau = -\lambda_p(M_\tau)\psi_\tau & \text{in } \bar{\Omega} \times [0, 1], \\ \psi_\tau(x, 1) = \psi_\tau(x, 0) & \text{in } \bar{\Omega}. \end{cases}
\end{aligned}$$

For convenience, we denote $\mathcal{C} := \Omega \times (0, 1)$. We normalize φ_τ and ψ_τ such that $\int_{\mathcal{C}} \varphi_\tau^2 = \int_{\mathcal{C}} \varphi_\tau \psi_\tau = 1$ for any $\tau > 0$.

A family of closed operators $\{M_\tau\}_{\tau>0}$ is a holomorphic family by [21, Chapter 7, Section 2.1]. As $\lambda_p(M_\tau)$ is an isolated eigenvalue, the continuous differentiability of $\tau \mapsto (\lambda_p(M_\tau), \varphi_\tau)$ follows from the classical perturbation theory in [21, Chapter 7, Section 6.2]. We can differentiate the equation (2.11) with respect to τ to find

$$\begin{cases} -\partial_t \varphi_\tau + M_\tau[\varphi'_\tau] = -\frac{\partial \lambda_p(M_\tau)}{\partial \tau} \varphi_\tau - \lambda_p(M_\tau) \varphi'_\tau & \text{in } \bar{\Omega} \times [0, 1], \\ \varphi'_\tau(x, 1) = \varphi'_\tau(x, 0) & \text{in } \bar{\Omega}. \end{cases}$$

Multiplying the above equation by ψ_τ and integrating the resulting equation over $\mathcal{C}$, we obtain

$$-\int_{\mathcal{C}} \psi_\tau \partial_t \varphi_\tau + \int_{\mathcal{C}} M_\tau[\varphi'_\tau] \psi_\tau = -\frac{\partial \lambda_p(M_\tau)}{\partial \tau} \int_{\mathcal{C}} \varphi_\tau \psi_\tau - \lambda_p(M_\tau) \int_{\mathcal{C}} \varphi'_\tau \psi_\tau.$$

By the adjoint problem (2.12) and the normalization $\int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} = 1$, we find that

$$\frac{\partial \lambda_p(M_{\tau})}{\partial \tau} = \int_{\mathcal{C}} \psi_{\tau} \partial_t \varphi_{\tau}.$$

Due to the definition of M_{τ} and M_{τ}^* , we derive

$$\begin{aligned} \int_{\mathcal{C}} \psi_{\tau} \partial_t \varphi_{\tau} &= \frac{1}{2\tau} \int_{\mathcal{C}} \psi_{\tau} (M_{\tau}^*[\varphi_{\tau}] - M_{\tau}[\varphi_{\tau}]) \\ &= \frac{1}{2\tau} \int_{\mathcal{C}} (\varphi_{\tau} M_{\tau}[\psi_{\tau}] - \psi_{\tau} M_{\tau}[\varphi_{\tau}]) \\ &= \frac{1}{2\tau} \left(K_{\tau}(\psi_{\tau}) - K_{\tau}(\varphi_{\tau}) \right), \end{aligned}$$

where functional K_{τ} is defined by

$$K_{\tau}(\zeta) := \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \left(\frac{M_{\tau}[\zeta]}{\zeta} \right), \quad \zeta \in \chi_{\Omega}^{++}.$$

We claim that

Claim 2.6. *For any $\zeta \in \chi_{\Omega}^{++}$, we have*

$$K_{\tau}(\zeta) - K_{\tau}(\varphi_{\tau}) \geq 0.$$

Assume for the moment that the claim holds true, then it implies that

$$(2.13) \quad \frac{\partial \lambda_p(M_{\tau})}{\partial \tau} = \frac{1}{2\tau} \left(K_{\tau}(\psi_{\tau}) - K_{\tau}(\varphi_{\tau}) \right) \geq 0 \quad \text{for all } \tau > 0.$$

It remains to prove parts (i) and (ii). When $b(x, t) = \hat{b}(x) + g(t)$ for some 1-periodic function $g(t)$, we set $\phi_{\tau}(x, t) := e^{-\frac{1}{\tau} \int_0^t g(s) ds} \varphi_{\tau}(x, t)$, which satisfies

$$\begin{cases} -\tau \partial_t \phi_{\tau} + \mu \int_{\Omega} J(x-y) \phi_{\tau}(y, t) dy + \hat{b}(x) \phi_{\tau} = -\lambda_p(M_{\tau}) \varphi_{\tau} & \text{in } \bar{\Omega} \times [0, 1], \\ \phi_{\tau}(x, 1) = \phi_{\tau}(x, 0) & \text{in } \bar{\Omega}. \end{cases}$$

It is clear that $\lambda_p(M_{\tau})$ is constant for $\tau > 0$. This proves part (i).

Finally, we show that $\frac{\partial \lambda_p(M_{\tau})}{\partial \tau} > 0$ for every $\tau > 0$ if $b(x, t)$ does not take the form of $b(x, t) = \hat{b}(x) + g(t)$. Suppose that there is some $\tau_0 > 0$ such that $\frac{\partial \lambda_p(M_{\tau_0})}{\partial \tau} = 0$. According to the formula (2.18) and $J(0) > 0$, we obtain

$$\frac{\varphi_{\tau_0}(x, t)}{\psi_{\tau_0}(x, t)} \cdot \frac{\psi_{\tau_0}(y, t)}{\varphi_{\tau_0}(y, t)} \equiv 1 \quad \text{for each } x, y \in \bar{\Omega}, t \in [0, 1].$$

Thus, we have $\varphi_{\tau_0} = c(t) \psi_{\tau_0}$ for some 1-periodic function $c(t) > 0$. Substituting $\varphi_{\tau_0} = c(t) \psi_{\tau_0}$ into $M_{\tau_0}[\varphi_{\tau_0}] = -\lambda_p(M_{\tau_0}) \varphi_{\tau_0}$ and using $M_{\tau_0}^*[\psi_{\tau_0}] = -\lambda_p(M_{\tau_0}) \psi_{\tau_0}$, we deduce that

$$c'(t) \psi_{\tau_0} + 2c(t) \partial_t \psi_{\tau_0} = 0.$$

It then follows that $\partial_t \ln \psi_{\tau_0} = -\frac{c'(t)}{2c(t)}$ in $\mathcal{C}$, which depends only on t . Hence, ψ_{τ_0} is of the form $\psi_{\tau_0} = X_{\tau_0}(x) T_{\tau_0}(t)$ with some 1-periodic function $T_{\tau_0}(t) > 0$ in $[0, 1]$ and function $X_{\tau_0}(x) > 0$ in $\bar{\Omega}$. By $M_{\tau_0}^*[\psi_{\tau_0}] = -\lambda_p(M_{\tau_0}) \psi_{\tau_0}$, we have

$$\tau_0 \frac{T'_{\tau_0}(t)}{T_{\tau_0}(t)} + \frac{\mu \int_{\Omega} J(x-y) X_{\tau_0}(y) dy}{X_{\tau_0}(x)} + b(x, t) = -\lambda_p(M_{\tau_0}).$$

Thus, it is necessary that b has the form of $b(x, t) = \hat{b}(x) + g(t)$, which contradicts the previous assumption. This completes the proof of part (ii).

Case 2. If $\lambda_p(M_\tau)$ is not the principal eigenvalue for some $\tau > 0$, then we can use an approximation argument. More precisely, applying Lemma 2.1 and Theorem 2.3, we find that for each $\epsilon > 0$, there exists $b_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that

$$\max_{\bar{\Omega} \times \mathbb{R}} |b_\epsilon - b| < \epsilon, \quad |\lambda_p(M_\tau(b_\epsilon)) - \lambda_p(M_\tau(b))| < \epsilon$$

and $\lambda_p(M_\tau(b_\epsilon))$ is the principal eigenvalue for all $\tau > 0$, where $M_\tau(b_\epsilon)$ is $M_\tau(b)$ with b being replaced by b_ϵ . We then apply **Case 1** to conclude that for each $\epsilon > 0$, the function $\tau \mapsto \lambda_p(M_\tau(b_\epsilon))$ is continuous non-decreasing on $(0, \infty)$, i.e., for every $\tau_0 > 0$, there exists $\delta_0 > 0$ such that for all $|\tau - \tau_0| < \delta_0$, we have

$$|\lambda_p(M_\tau(b_\epsilon)) - \lambda_p(M_{\tau_0}(b_\epsilon))| < \epsilon.$$

By Lemma 2.2, $\lambda_p(M_\tau(b))$ is Lipschitz continuous with respect to b , i.e.,

$$|\lambda_p(M_\tau(b)) - \lambda_p(M_\tau(b_\epsilon))| \leq \sup_{t \in [0,1]} \|b(\cdot, t) - b_\epsilon(\cdot, t)\|_\infty < \epsilon.$$

Hence, for every given constant $\epsilon > 0$, there exist $\delta_0 > 0$ and $b_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that for all $|\tau - \tau_0| < \delta_0$, we have

$$\begin{aligned} & |\lambda_p(M_{\tau_0}(b)) - \lambda_p(M_\tau(b))| \\ & \leq |\lambda_p(M_{\tau_0}(b)) - \lambda_p(M_{\tau_0}(b_\epsilon))| + |\lambda_p(M_{\tau_0}(b_\epsilon)) - \lambda_p(M_\tau(b_\epsilon))| + |\lambda_p(M_\tau(b_\epsilon)) - \lambda_p(M_\tau(b))| \\ & < \epsilon + \epsilon + \epsilon = 3\epsilon, \end{aligned}$$

which implies that $\lambda_p(M_\tau(b))$ is continuous with respect to τ . Thus, the function $\tau \mapsto \lambda_p(M_\tau(b))$ is continuous non-decreasing on $(0, \infty)$. The proof is complete. $\square$

Proof of Claim 2.6. First, we claim that φ_τ is a critical point of K_τ in the sense that

$$(2.14) \quad \mathbf{D}K_\tau(\varphi_\tau)\eta = 0 \quad \text{for all } \eta \in \chi_\Omega,$$

where $\mathbf{D}K_\tau(\varphi_\tau)$ is the Fréchet derivative of K_τ at the point $\varphi_\tau \in \chi_\Omega^{++}$.

For any $\eta \in \chi_\Omega$, we have

$$\mathbf{D}K_\tau(\varphi_\tau)\eta = \int_{\mathcal{C}} \varphi_\tau \psi_\tau \left(\frac{M_\tau[\eta]}{\varphi_\tau} - \frac{M_\tau[\varphi_\tau]\eta}{\varphi_\tau^2} \right).$$

On one hand, it follows from $M_\tau[\varphi_\tau] = -\lambda_p \varphi_\tau$ and $M_\tau^*[\psi_\tau] = -\lambda_p \psi_\tau$ that

$$\begin{aligned} (2.15) \quad \mathbf{D}K_\tau(\varphi_\tau)\eta &= \int_{\mathcal{C}} \left(\psi_\tau M_\tau[\eta] - \frac{M_\tau[\varphi_\tau]\psi_\tau \eta}{\varphi_\tau} \right) \\ &= \int_{\mathcal{C}} \left(M_\tau^*[\psi_\tau]\eta - \frac{M_\tau[\varphi_\tau]\psi_\tau \eta}{\varphi_\tau} \right) \\ &= 0. \end{aligned}$$

On the other hand, a simple calculation yields

$$\begin{aligned} (2.16) \quad & \mathbf{D}K_\tau(\varphi_\tau)\eta \\ &= \int_{\mathcal{C}} \varphi_\tau \psi_\tau \left(\frac{M_\tau[\eta]}{\varphi_\tau} - \frac{M_\tau[\varphi_\tau]\eta}{\varphi_\tau^2} \right) \\ &= \int_{\mathcal{C}} \varphi_\tau \psi_\tau \frac{-\tau(\eta_t \varphi_\tau - (\varphi_\tau)_t \eta) + \mu \int_{\Omega} J(x-y) [\eta(y, t) \varphi_\tau(x, t) - \varphi_\tau(y, t) \eta(x, t)] dy}{\varphi_\tau^2}. \end{aligned}$$

A direct calculation shows that

$$K_\tau(\zeta) - K_\tau(\varphi_\tau)$$

$$\begin{aligned}
&= \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \left(\frac{M_{\tau}[\zeta]}{\zeta} - \frac{M_{\tau}[\varphi_{\tau}]}{\varphi_{\tau}} \right) \\
&= \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \left(\frac{-\tau \zeta_t}{\zeta} + \frac{\tau(\varphi_{\tau})_t}{\varphi_{\tau}} \right) + \mu \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \int_{\Omega} J(x-y) \left(\frac{\zeta(y,t)}{\zeta(x,t)} - \frac{\varphi_{\tau}(y,t)}{\varphi_{\tau}(x,t)} \right) dy.
\end{aligned}$$

Taking $\eta = \varphi_{\tau} \ln\left(\frac{\zeta}{\varphi_{\tau}}\right)$ in (2.16), we obtain

$$\begin{aligned}
(2.17) \quad &\mathbf{D}K_{\tau}(\varphi_{\tau})\eta \\
&= \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \left(\frac{-\tau \zeta_t}{\zeta} + \frac{\tau(\varphi_{\tau})_t}{\varphi_{\tau}} \right) + \mu \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \int_{\Omega} J(x-y) \frac{\varphi_{\tau}(y,t)}{\varphi_{\tau}(x,t)} \ln\left(\frac{\zeta(y,t)\varphi_{\tau}(x,t)}{\zeta(x,t)\varphi_{\tau}(y,t)}\right) dy.
\end{aligned}$$

By formulas (2.15) and (2.17), we have

$$\begin{aligned}
(2.18) \quad &K_{\tau}(\zeta) - K_{\tau}(\varphi_{\tau}) \\
&= \mathbf{D}K_{\tau}(\varphi_{\tau})\eta + \mu \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \int_{\Omega} J(x-y) \frac{\varphi_{\tau}(y,t)}{\varphi_{\tau}(x,t)} \left[\frac{\zeta(y,t)\varphi_{\tau}(x,t)}{\zeta(x,t)\varphi_{\tau}(y,t)} - 1 - \ln\left(\frac{\zeta(y,t)\varphi_{\tau}(x,t)}{\zeta(x,t)\varphi_{\tau}(y,t)}\right) \right] dy \\
&= \mu \int_{\mathcal{C}} \varphi_{\tau} \psi_{\tau} \int_{\Omega} J(x-y) \frac{\varphi_{\tau}(y,t)}{\varphi_{\tau}(x,t)} \left[\frac{\zeta(y,t)\varphi_{\tau}(x,t)}{\zeta(x,t)\varphi_{\tau}(y,t)} - 1 - \ln\left(\frac{\zeta(y,t)\varphi_{\tau}(x,t)}{\zeta(x,t)\varphi_{\tau}(y,t)}\right) \right] dy.
\end{aligned}$$

Define

$$f(z) = z - 1 - \ln z, \quad z \in (0, \infty).$$

As $f(z) \geq 0$ and $f(z) = 0$ if and only if $z = 1$, thus we obtain $K_{\tau}(\zeta) - K_{\tau}(\varphi_{\tau}) \geq 0$. $\square$

Theorem 2.7. Assume that (J) holds and $b \in C_1^{0,1}(\bar{\Omega} \times \mathbb{R})$. Then the followings hold:

(i) There holds

$$\lim_{\tau \rightarrow 0^+} \lambda_p(M_{\tau}) = \int_0^1 \lambda_p(N_{\Omega}^t) dt.$$

Here, for each fixed $t \in [0, 1]$, $\lambda_p(N_{\Omega}^t)$ is the generalised principal eigenvalue of the operator N_{Ω}^t

$$N_{\Omega}^t[v](x) := \mu \int_{\Omega} J_{\sigma}(x-y)v(y)dy + b(x,t)v(x);$$

(ii) There holds

$$\lim_{\tau \rightarrow \infty} \lambda_p(M_{\tau}) = \lambda_p(N_{\Omega}),$$

where $\lambda_p(N_{\Omega})$ is the generalised principal eigenvalue of the operator N_{Ω}

$$N_{\Omega}[v](x) := \mu \int_{\Omega} J_{\sigma}(x-y)v(y)dy + \hat{b}(x)v(x).$$

Proof. (i) The proof is divided into two cases.

Case 1. We prove the result under the additional assumption that $\lambda_p(N_{\Omega}^t)$ is a principal eigenvalue for all $t \in [0, 1]$. For fixed $t \in [0, 1]$, there is $v(\cdot, t) \in C(\bar{\Omega})$ and $v(\cdot, t) > 0$ in $\bar{\Omega}$ s.t.

$$(2.19) \quad N_{\Omega}^t[v](x, t) + \lambda_p(N_{\Omega}^t)v(x, t) = 0 \quad \text{in } \bar{\Omega}.$$

It follows from the perturbation theory [21, Chapter 7, Section 6.2] that $v \in C^1([0, 1]; C(\bar{\Omega}))$ and $v(x, t+1) = v(x, t)$.

Define $\varphi(x, t) = \rho(t)v(x, t)$ for 1-periodic function

$$\rho(t) = e^{\frac{1}{\tau}} \left[t \int_0^1 \lambda_p(N_{\Omega}^s) ds - \int_0^t \lambda_p(N_{\Omega}^s) ds \right].$$

Given arbitrary $\epsilon_0 > 0$, there is sufficiently small $\tau_0 > 0$ such that $\tau|\partial_t v| \leq \epsilon_0 v$ for all $\tau \leq \tau_0$. Moreover, a direct calculation yields

$$M_\tau[\varphi] + \left(\int_0^1 \lambda_p(N_\Omega^t) dt - \epsilon_0 \right) \varphi \leq -\tau \varphi_t + N_\Omega^t[\varphi] + \left(\int_0^1 \lambda_p(N_\Omega^t) dt - \epsilon_0 \right) \varphi \leq 0$$

By the definition of $\lambda_p(M_\tau)$, we know that

$$(2.20) \quad \int_0^1 \lambda_p(N_\Omega^t) dt - \epsilon_0 \leq \lambda_p(M_\tau) \quad \text{for all } \tau \leq \tau_0.$$

In a similar manner, we obtain

$$M_\tau[\varphi] + \left(\int_0^1 \lambda_p(N_\Omega^t) dt + \epsilon_0 \right) \varphi \geq -\tau \varphi_t + N_\Omega^t[\varphi] + \left(\int_0^1 \lambda_p(N_\Omega^t) dt + \epsilon_0 \right) \varphi \geq 0.$$

By the definition of $\lambda'_p(M_\tau)$, we know that

$$(2.21) \quad \lambda'_p(M_\tau) \leq \int_0^1 \lambda_p(N_\Omega^t) dt + \epsilon_0 \quad \text{for all } \tau \leq \tau_0.$$

Combining Theorem 2.4 and inequalities (2.20), (2.21), we obtain

$$\lim_{\tau \rightarrow 0^+} \lambda_p(M_\tau) = \int_0^1 \lambda_p(N_\Omega^t) dt.$$

Case 2. If $\lambda_p(N_\Omega^t)$ is not a principal eigenvalue for some $t \in [0, 1]$, then we can use similar approximation argument as in **Case 2** in the proof of Theorem 2.5 to deduce the result.

(ii) The proof is also divided into two cases.

Case 1. We prove the result under the additional assumption that $\lambda_p(M_\tau)$ is a principal eigenvalue for all $\tau > 0$. Choose a sequence of $\{\tau_n\}_{n=1}^\infty$ such that $\tau_n \rightarrow +\infty$ and let the eigenpairs $(\lambda_p(M_{\tau_n}), \varphi_{\tau_n})$ be defined by

$$(2.22) \quad \begin{cases} -\tau_n \partial_t \varphi_{\tau_n} + \mu \int_\Omega J(x-y) \varphi_{\tau_n}(y, t) dy + b(x, t) \varphi_{\tau_n} + \lambda_p(M_{\tau_n}) \varphi_{\tau_n} = 0 & \text{in } \bar{\Omega} \times [0, 1], \\ \varphi_{\tau_n} \in \chi_\Omega^{++}, \quad \|\varphi_{\tau_n}\|_{L^2(\Omega \times (0, 1))} = 1. \end{cases}$$

Multiplying equation (2.22) by φ_{τ_n} and integrating over $\Omega \times (0, 1)$, we get

$$\begin{aligned} \lambda_p(M_{\tau_n}) &= \mu \int_0^1 \int_\Omega \int_\Omega J(x-y) \varphi_{\tau_n}(y, t) \varphi_{\tau_n}(x, t) dy dx dt + \int_0^1 \int_\Omega b(x, t) \varphi_{\tau_n}^2(x, t) dx dt \\ &\leq \mu |\Omega| \max_{\mathbb{R}^N} J + \max_{\bar{\Omega} \times [0, 1]} |b|. \end{aligned}$$

Owing to the monotone non-decreasing of $\lambda_p(M_\tau)$ on $\tau > 0$, one gets

$$\lim_{n \rightarrow \infty} \lambda_p(M_{\tau_n}) = \lambda_p^\infty.$$

Multiplying equation (2.22) by $\partial_t \varphi_{\tau_n}$ and integrating over $\Omega \times (0, 1)$ yield

$$\begin{aligned} \tau_n \int_0^1 \int_\Omega |\partial_t \varphi_{\tau_n}|^2 dx dt &= \mu \int_0^1 \int_\Omega \int_\Omega J(x-y) \varphi_{\tau_n}(y, t) \partial_t \varphi_{\tau_n}(x, t) dy dx dt \\ &\quad + \int_0^1 \int_\Omega [b(x, t) + \lambda_p(M_{\tau_n})] \varphi_{\tau_n}(x, t) \partial_t \varphi_{\tau_n}(x, t) dx dt \\ &= \frac{1}{2} \int_0^1 \int_\Omega \partial_t b(x, t) \varphi_{\tau_n}^2(x, t) dx dt \\ &\leq \frac{1}{2} \max_{\bar{\Omega} \times [0, 1]} |\partial_t b|, \end{aligned}$$

which implies that

$$\|\partial_t \varphi_{\tau_n}\|_{L^2(\Omega \times (0,1))} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Due to the above result and $\|\varphi_{\tau_n}\|_{L^2} = 1$, up to extraction, there exists $w \in W^{1,2}((0,1); L^2(\Omega))$ such that

$$\varphi_{\tau_n} \rightharpoonup w \quad \text{and} \quad \partial_t \varphi_{\tau_n} \rightharpoonup \partial_t w.$$

Moreover, we have $\|\partial_t w\|_{L^2(\Omega \times (0,1))} \leq \liminf_{\tau \rightarrow 0^+} \|\partial_t \varphi_{\tau_n}\|_{L^2(\Omega \times (0,1))} = 0$ and thus w does not depend on t .

Passing to the limit $n \rightarrow \infty$ in (2.22), we find that w is a weak solution of the equation

$$\mu \int_{\Omega} J(x-y)w(y)dy + b(x,t)w(x) + \lambda_p^\infty w(x) = 0 \quad \text{in } \bar{\Omega} \times [0,1].$$

Integrating the above equation over $(0,1)$ yields

$$\mu \int_{\Omega} J(x-y)w(y)dy + \hat{b}(x)w(x) + \lambda_p^\infty w(x) = 0 \quad \text{in } \bar{\Omega}.$$

So $w \in C(\bar{\Omega})$ and $w > 0$ in $\bar{\Omega}$, which implies that λ_p^∞ is the principal eigenvalue of the operator N_Ω . It is easy to know that $\lambda_p^\infty = \lambda_p(N_\Omega)$. Thus, we have

$$\lim_{\tau \rightarrow \infty} \lambda_p(M_\tau) = \lambda_p(N_\Omega).$$

Case 2. If $\lambda_p(M_\tau)$ is not a principal eigenvalue for some $\tau > 0$, then we can use the approximation argument as in **Case 2** in the proof of Theorem 2.5 to deduce the result. $\square$

2.3. Influences of the dispersal rate and the dispersal spread.

In this subsection, we investigate the influences of the dispersal rate μ and the dispersal spread σ on the generalised principal eigenvalue $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$. Firstly, we establish the upper bound of the generalised principal eigenvalue $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$.

Lemma 2.8. *Assume that (J), (H) and (A) hold. Then*

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \leq \min_{x \in \bar{\Omega}} \left\{ \frac{\mu}{\sigma^m} h^\sigma(x) - \hat{a}(x) \right\}.$$

Proof. Fix $\lambda < \lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$. By the definition of $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$, there exists $\varphi \in \chi_\Omega^{++}$ such that

$$L_\Omega^{\tau,\mu,\sigma,m}[\varphi](x,t) + \lambda\varphi(x,t) \leq 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

It is easy to check that

$$H_\Omega^{\tau,\mu,\sigma,m}[\varphi](x,t) + \lambda\varphi(x,t) \leq 0 \quad \text{in } \bar{\Omega} \times \mathbb{R},$$

where $H_\Omega^{\tau,\mu,\sigma,m}[\varphi] = -\tau\varphi_t - \frac{\mu}{\sigma^m}h^\sigma(x)\varphi + a(x,t)\varphi$. This implies that $\lambda \leq \lambda_p(H_\Omega^{\tau,\mu,\sigma,m})$. Thus

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \leq \lambda_p(H_\Omega^{\tau,\mu,\sigma,m}).$$

It follows from [29, Propositions 3.4-3.5] that

$$\lambda_p(H_\Omega^{\tau,\mu,\sigma,m}) = \min_{x \in \bar{\Omega}} \left\{ \frac{\mu}{\sigma^m} h^\sigma(x) - \hat{a}(x) \right\}.$$

This completes the proof. $\square$

Now, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. (i) For any $\mu_0 \in (0, \infty)$, applying Lemma 2.1, we find that for each $\epsilon > 0$, there exists $b_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that $\max_{\bar{\Omega} \times \mathbb{R}} |b_\epsilon - b| < \epsilon$, $|\lambda_1 - \lambda_1^\epsilon| < \epsilon$ and λ_1^ϵ is an isolated principal eigenvalue with finite multiplicity of $L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon)$, where $b(x, t) = a(x, t) - \mu_0 h^1(x)$, $a_\epsilon(x, t) = b_\epsilon(x, t) + \mu_0 h^1(x)$. In fact, we rewrite $L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon)$ as

$$L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon) = L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon) + U_{\mu, \mu_0},$$

where

$$U_{\mu, \mu_0}[\varphi](x) = (\mu - \mu_0) \left(\int_{\Omega} J(x - y) \varphi(y) dy - h^1(x) \varphi(x) \right).$$

Note that U_{μ, μ_0} is a linear bounded operator and $U_{\mu, \mu_0} \rightarrow 0$ in norm as $\mu \rightarrow \mu_0$. It follows from the classical perturbation theory of isolated eigenvalues [21, Chapter 4, Section 3.5], there exists $\delta_0 > 0$ such that for all $|\mu - \mu_0| < \delta_0$, we have

$$|\lambda_1(L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon)) - \lambda_1(L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon))| < \epsilon.$$

Thanks to Theorem 1.1, we obtain

$$|\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon)) - \lambda_p(L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon))| < \epsilon.$$

By Lemma 2.2, $\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a))$ is Lipschitz continuous with respect to a , i.e.,

$$|\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon)) - \lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a))| \leq \|a - a_\epsilon\|_{L^\infty(\Omega)} < \epsilon.$$

In a word, for every given constant $\epsilon > 0$, there exist $\delta_0 > 0$ and $a_\epsilon \in C_1(\bar{\Omega} \times \mathbb{R})$ such that for all $|\mu - \mu_0| < \delta_0$, we have

$$\begin{aligned} & |\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a)) - \lambda_p(L_\Omega^{\tau, \mu_0, 1, 0}(a))| \\ & \leq |\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a)) - \lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon))| + |\lambda_p(L_\Omega^{\tau, \mu, 1, 0}(a_\epsilon)) - \lambda_p(L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon))| \\ & \quad + |\lambda_p(L_\Omega^{\tau, \mu_0, 1, 0}(a_\epsilon)) - \lambda_p(L_\Omega^{\tau, \mu_0, 1, 0}(a))| \\ & < \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

So $\lambda_p(L_\Omega^{\tau, \mu, 1, 0})$ is continuous with respect to μ .

Now, we prove the asymptotic behavior of $\lambda_p(L_\Omega^{\tau, \mu, 1, 0})$ as $\mu \rightarrow 0^+$. For simplicity, we write $\lambda_p^\mu := \lambda_p(L_\Omega^{\tau, \mu, 1, 0})$. We first claim that for each $\epsilon > 0$, there is $\mu_\epsilon > 0$ such that

$$(2.23) \quad \lambda_\epsilon^{max} \leq \lambda_p^\mu \leq \lambda_\epsilon^{min} \quad \text{for all } \mu \in (0, \mu_\epsilon),$$

where $\lambda_\epsilon^{max} = -\max_{\bar{\Omega}} \hat{a} - \epsilon$, $\lambda_\epsilon^{min} = -\min_{\bar{\Omega}} \hat{a} + \epsilon$. In fact, it is easy to check that the function $\phi(x, t) := e^{\int_0^t [a(x, s) - \hat{a}(x)] ds}$ is a positive 1-periodic solution of $\phi_t = a(x, t)\phi - \hat{a}(x)\phi$ for $(x, t) \in \bar{\Omega} \times \mathbb{R}$. In particular, $\phi \in \chi_{\bar{\Omega}}^{++}$. A simple computation yields

$$\begin{aligned} (L_\Omega^{\tau, \mu, 1, 0} + \lambda_\epsilon^{max})[\phi](x, t) &= \mu \left(\int_{\Omega} J(x - y) \phi(y, t) dy - h^1(x) \phi(x, t) \right) \\ & \quad + \left(\hat{a}(x) - \max_{\bar{\Omega}} \hat{a} - \epsilon \right) \phi(x, t), \\ (L_\Omega^{\tau, \mu, 1, 0} + \lambda_\epsilon^{min})[\phi](x, t) &= \mu \left(\int_{\Omega} J(x - y) \phi(y, t) dy - h^1(x) \phi(x, t) \right) \\ & \quad + \left(\hat{a}(x) - \min_{\bar{\Omega}} \hat{a} + \epsilon \right) \phi(x, t). \end{aligned}$$

Thus, there exists $\mu_\epsilon > 0$ such that

$$(L_\Omega^{\tau, \mu, 1, 0} + \lambda_\epsilon^{max})[\phi] \leq 0 \quad \text{and} \quad (L_\Omega^{\tau, \mu, 1, 0} + \lambda_\epsilon^{min})[\phi] \geq 0 \quad \text{for all } \mu \in (0, \mu_\epsilon).$$

Moreover, by Theorem 1.1 and the definitions of $\lambda_p(L_\Omega^{\tau,\mu,1,0})$ and $\lambda_p'(L_\Omega^{\tau,\mu,1,0})$, there holds (2.23) for all $\mu \in (0, \mu_\epsilon)$.

Next, thanks to Lemma 2.8 and the inequality (2.23), for each $\epsilon > 0$ there is $\mu_\epsilon > 0$ such that

$$-\max_{\Omega} \hat{a} - \epsilon \leq \lambda_p^\mu \leq \min_{x \in \Omega} \left[\mu h^1(x) - \hat{a}(x) \right] \quad \text{for all } \mu \in (0, \mu_\epsilon).$$

Passing $\mu \rightarrow 0^+$, we find

$$-\max_{\Omega} \hat{a} - \epsilon \leq \liminf_{\mu \rightarrow 0^+} \lambda_p^\mu \leq \limsup_{\mu \rightarrow 0^+} \lambda_p^\mu \leq -\max_{\Omega} \hat{a} \quad \forall \epsilon > 0,$$

which leads to

$$\lambda_p^\mu \rightarrow -\max_{\Omega} \hat{a} \quad \text{as } \mu \rightarrow 0^+.$$

(ii) The proof of continuity is similar to (i). Hence, we omit it. It remains to prove parts (a) and (b).

(a) By Lemma 2.8, we have

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \leq \min_{x \in \Omega} \left\{ \frac{\mu}{\sigma^m} h^\sigma(x) - \hat{a}(x) \right\}.$$

As $m > 0$ and $\sigma \rightarrow \infty$, there holds

$$\frac{\mu}{\sigma^m} h^\sigma(x) \rightarrow 0 \quad \text{for all } x \in \bar{\Omega},$$

which implies that

$$\limsup_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \leq -\max_{\Omega} \hat{a}.$$

To complete our proof, it remains to show

$$-\max_{\Omega} \hat{a} \leq \liminf_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}).$$

For fixed constant $\phi_0 > 0$, it is easy to check that for every $x \in \bar{\Omega}$, the function

$$(2.24) \quad \phi(x, t) = e^{\int_0^t (a(x,s) - \hat{a}(x)) ds} \phi_0, \quad t \in \mathbb{R},$$

is a positive 1-periodic solution of the ordinary differential equation $v_t = a(x, t)v - \hat{a}(x)v$ with the initial condition $v(x, 0) = \phi_0$. In particular, $\phi \in \chi_\Omega^{++}$ and we can choose ϕ_0 such that $\sup_{\bar{\Omega} \times \mathbb{R}} \phi = 1$. For every $\epsilon > 0$, we have

$$\begin{aligned} & (L_\Omega^{\tau,\mu,\sigma,m} - \max_{\Omega} \hat{a} - \epsilon)[\phi](x, t) \\ &= -\tau \phi_t(x, t) + \left(a(x, t) - \max_{\Omega} \hat{a} - \epsilon \right) \phi(x, t) \\ & \quad + \frac{\mu}{\sigma^m} \left(\int_{\Omega} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) \right) \\ & \leq \frac{\mu}{\sigma^m} \left(\int_{\Omega} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) \right) - \epsilon \phi(x, t). \end{aligned}$$

Using $\sup_{\bar{\Omega} \times \mathbb{R}} \phi = 1$, there holds

$$\begin{aligned} & \left\| \frac{\mu}{\sigma^m} \left(\int_{\Omega} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) \right) \right\|_\infty \\ & \leq \left\| \frac{\mu}{\sigma^m} \int_{\Omega} J_\sigma(x - y) dy \right\|_\infty + \frac{\mu M}{\sigma^m} \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty, \end{aligned}$$

which implies that there is $\sigma_\epsilon > 0$ such that

$$(L_\Omega^{\tau,\mu,\sigma,m} - \max_{\bar{\Omega}} \hat{a} - \epsilon)[\phi] \leq 0 \quad \text{for all } \sigma \geq \sigma_\epsilon.$$

Thanks to the definition of $\lambda_p(L_\Omega^{\tau,\mu,\sigma,m})$, there holds

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \geq -\max_{\bar{\Omega}} \hat{a} - \epsilon \quad \text{for all } \sigma \geq \sigma_\epsilon.$$

Since ϵ is an arbitrary constant, we have

$$\liminf_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) \geq -\max_{\bar{\Omega}} \hat{a}.$$

Thus, we get

$$\lim_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,m}) = -\max_{\bar{\Omega}} \hat{a}.$$

(b) Again, following from Lemma 2.8, we have

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,0}) \leq \min_{x \in \bar{\Omega}} \left\{ \mu h^\sigma(x) - \hat{a}(x) \right\}.$$

Owing to $\lim_{\sigma \rightarrow \infty} h^\sigma(x) = c$, this implies that

$$\limsup_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,0}) \leq \mu c - \max_{\bar{\Omega}} \hat{a}.$$

To complete our proof, it remains to obtain

$$\mu c - \max_{\bar{\Omega}} \hat{a} \leq \liminf_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau,\mu,\sigma,0}).$$

For fixed constant $\phi_0 > 0$, it is easy to check that for every $x \in \bar{\Omega}$, the function

$$(2.25) \quad \phi(x, t) = e^{\int_0^t (a(x, s) - \hat{a}(x)) ds} \phi_0, \quad t \in \mathbb{R},$$

is a positive 1-periodic solution of the ordinary differential equation $v_t = a(x, t)v - \hat{a}(x)v$ with the initial condition $v(x, 0) = \phi_0$. In particular, $\phi \in \chi_\Omega^{++}$ and we can choose ϕ_0 such that $\sup_{\bar{\Omega} \times \mathbb{R}} \phi = 1$. For every $\epsilon > 0$, we have

$$\begin{aligned} & (L_\Omega^{\tau,\mu,\sigma,0} - \max_{\bar{\Omega}} \hat{a} + \mu c - \epsilon)[\phi](x, t) \\ &= -\tau \phi_t(x, t) + \left(a(x, t) - \max_{\bar{\Omega}} \hat{a} - \epsilon \right) \phi(x, t) \\ & \quad + \mu \left(\int_{\bar{\Omega}} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) + c \phi(x, t) \right) \\ & \leq \mu \left(\int_{\bar{\Omega}} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) + c \phi(x, t) \right) - \epsilon \phi(x, t). \end{aligned}$$

Using $\sup_{\bar{\Omega} \times \mathbb{R}} \phi = 1$, there holds

$$\begin{aligned} & \left\| \mu \left(\int_{\bar{\Omega}} J_\sigma(x - y) \phi(y, t) dy - h^\sigma(x) \phi(x, t) + c \phi(x, t) \right) \right\|_\infty \\ & \leq \left\| \mu \int_{\bar{\Omega}} J_\sigma(x - y) dy \right\|_\infty + \|c - h^\sigma\|_\infty \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty, \end{aligned}$$

which implies that there is $\sigma_\epsilon > 0$ such that

$$(L_\Omega^{\tau,\mu,\sigma,0} - \max_{\bar{\Omega}} \hat{a} + \mu c - \epsilon)[\phi] \leq 0 \quad \text{for all } \sigma \geq \sigma_\epsilon.$$

Thanks to the definition of $\lambda_p(L_\Omega^{\tau,\mu,\sigma,0})$, there holds

$$\lambda_p(L_\Omega^{\tau,\mu,\sigma,0}) \geq \mu c - \max_{\bar{\Omega}} \hat{a} - \epsilon \quad \text{for all } \sigma \geq \sigma_\epsilon.$$

Since ϵ is an arbitrary constant, we have

$$\liminf_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau, \mu, \sigma, 0}) \geq \mu c - \max_{\Omega} \hat{a}.$$

Thus, we get

$$\lim_{\sigma \rightarrow \infty} \lambda_p(L_\Omega^{\tau, \mu, \sigma, 0}) = \mu c - \max_{\Omega} \hat{a}.$$

This completes the proof of Theorem 1.3. $\square$

Next, we recall the following lemma in [29, Corollary D].

Lemma 2.9. *Assume that (J) and (A) hold. Let $N_\Omega^{\mu, \sigma, m}[\varphi](x) := \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y)(\varphi(y) - \varphi(x))dy + \hat{a}(x)\varphi(x)$. If $\lambda_p(N_\Omega^{\mu, \sigma, m})$ is the principal eigenvalue of $N_\Omega^{\mu, \sigma, m}$ and $h^\sigma(x) = \int_\Omega J_\sigma(x - y)dy$, then $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ is the principal eigenvalue of $L_\Omega^{\tau, \mu, \sigma, m}$.*

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. (i) By [32, Theorem 2.2 (3)], there exists $\mu_1 > 0$ such that $\lambda_p(N_\Omega^{\mu, 1, 0})$ is the principal eigenvalue of $N_\Omega^{\mu, 1, 0}$ for all $\mu \geq \mu_1$. It follows from Lemma 2.9 that $\lambda_p(L_\Omega^{\tau, \mu, 1, 0})$ is the principal eigenvalue of $L_\Omega^{\tau, \mu, 1, 0}$ for all $\mu \geq \mu_1$.

Since λ_p^μ is the principal eigenvalue of $L_\Omega^{\tau, \mu, 1, 0}$ for all $\mu \geq \mu_1$, there exists $\varphi \in \chi_\Omega^{++}$ with $\int_0^1 \int_\Omega \varphi^2(x, t) dx dt = 1$ such that

$$(2.26) \quad -\tau \varphi_t(x, t) + \mu \int_\Omega J(x - y)(\varphi(y, t) - \varphi(x, t))dy + (a(x, t) + \lambda_p^\mu) \varphi(x, t) = 0 \quad \text{in } \bar{\Omega} \times \mathbb{R}.$$

On one hand, divide (2.26) by φ and integrate over $\Omega \times (0, 1)$ to obtain

$$\begin{aligned} \int_0^1 \int_\Omega \lambda_p^\mu dx dt &= \tau \int_0^1 \int_\Omega \frac{\varphi_t(x, t)}{\varphi(x, t)} dx dt - \int_0^1 \int_\Omega a(x, t) dx dt \\ &\quad - \mu \int_0^1 \int_\Omega \int_\Omega J(x - y) \frac{\varphi(y, t) - \varphi(x, t)}{\varphi(x, t)} dy dx dt. \end{aligned}$$

Owing to

$$\int_0^1 \int_\Omega \frac{\varphi_t(x, t)}{\varphi(x, t)} dx dt = \int_\Omega \int_0^1 \frac{\varphi_t(x, t)}{\varphi(x, t)} dt dx = 0$$

and

$$\begin{aligned} &\mu \int_0^1 \int_\Omega \int_\Omega J(x - y) \frac{\varphi(y, t) - \varphi(x, t)}{\varphi(x, t)} dy dx dt \\ &= \frac{\mu}{2} \int_0^1 \int_\Omega \int_\Omega J(x - y) \frac{(\varphi(y, t) - \varphi(x, t))^2}{\varphi(x, t)\varphi(y, t)} dy dx dt, \end{aligned}$$

we get

$$\begin{aligned} &\int_0^1 \int_\Omega \lambda_p^\mu dx dt \\ &= - \int_0^1 \int_\Omega a(x, t) dx dt - \frac{\mu}{2} \int_0^1 \int_\Omega \int_\Omega J(x - y) \frac{(\varphi(y, t) - \varphi(x, t))^2}{\varphi(x, t)\varphi(y, t)} dy dx dt. \end{aligned}$$

This implies that

$$(2.27) \quad \lambda_p^\mu \leq -\bar{a}.$$

On the other hand, multiplying (2.26) by φ and integrating over $\Omega \times (0, 1)$ yield

$$\int_0^1 \int_\Omega \lambda_p^\mu \varphi^2(x, t) dx dt = \tau \int_0^1 \int_\Omega \varphi_t(x, t) \varphi(x, t) dx dt - \int_0^1 \int_\Omega a(x, t) \varphi^2(x, t) dx dt$$

$$- \mu \int_0^1 \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(y,t) - \varphi(x,t))\varphi(x,t)dydxdt.$$

In view of $\varphi \in \chi_{\Omega}^{++}$ and the symmetry of J , we have

$$(2.28) \quad \lambda_p^{\mu} = \frac{\mu}{2} \int_0^1 \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(y,t) - \varphi(x,t))^2 dydxdt - \int_0^1 \int_{\Omega} a(x,t)\varphi^2(x,t)dxdt,$$

which implies that

$$(2.29) \quad \lambda_p^{\mu} \geq - \int_0^1 \int_{\Omega} a(x,t)\varphi^2(x,t)dxdt \geq - \max_{\Omega \times [0,1]} a.$$

By combining (2.27) and (2.29), we obtain

$$(2.30) \quad - \max_{\Omega \times [0,1]} a \leq \lambda_p^{\mu} \leq -\bar{a}.$$

Rewriting (2.28) as

$$\begin{aligned} & \int_0^1 \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(y,t) - \varphi(x,t))^2 dydxdt \\ &= \frac{2}{\mu} \int_0^1 \int_{\Omega} (\lambda_p^{\mu} + a(x,t))\varphi^2(x,t)dxdt, \end{aligned}$$

it follows from (2.30) that

$$(2.31) \quad \begin{aligned} & \int_0^1 \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(y,t) - \varphi(x,t))^2 dydxdt \\ & \leq \frac{2}{\mu} \left(\max_{\Omega \times [0,1]} a - \bar{a} \right). \end{aligned}$$

Let $\psi(x,t) := \varphi(x,t) - \bar{\varphi}(t)$, where $\bar{\varphi}(t) = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x,t)dx$. Then we have $\int_{\Omega} \psi(x,t)dx = 0$. Observe that

$$(2.32) \quad \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(y,t) - \varphi(x,t))^2 dydx = \int_{\Omega} \int_{\Omega} J(x-y)(\psi(y,t) - \psi(x,t))^2 dydx.$$

By [32, Page 1688, Formula (5.6)], there exists $C > 0$ such that

$$(2.33) \quad \int_{\Omega} \psi^2(x,t)dx \leq \frac{1}{2C} \int_{\Omega} \int_{\Omega} J(x-y)(\psi(y,t) - \psi(x,t))^2 dydx \quad \text{for all } \mu \gg 1.$$

It deduces from (2.31), (2.32) and (2.33) that

$$(2.34) \quad \begin{aligned} \int_0^1 \int_{\Omega} \psi^2(x,t)dxdt & \leq \frac{1}{2C} \int_0^1 \int_{\Omega} \int_{\Omega} J(x-y)(\psi(y,t) - \psi(x,t))^2 dydxdt \\ & \leq \frac{1}{C\mu} \left(\max_{\Omega \times [0,1]} a - \bar{a} \right). \end{aligned}$$

Now, integrating (2.26) over Ω and substituting $\psi = \varphi - \bar{\varphi}$ yield

$$(2.35) \quad \tau \bar{\varphi}_t = \frac{1}{|\Omega|} \int_{\Omega} (\lambda_p^{\mu} + a(x,t))dx \bar{\varphi} + \frac{1}{|\Omega|} \int_{\Omega} (\lambda_p^{\mu} + a(x,t))\psi(x,t)dx.$$

In view of (2.34), we have

$$\int_0^1 \int_{\Omega} (\lambda_p^{\mu} + a(x,t))\psi(x,t)dxdt = O(\mu^{-\frac{1}{2}}) \quad \text{as } \mu \rightarrow \infty.$$

Using the integrating form in this first order differential equation, we find that

$$(2.36) \quad \bar{\varphi}(t) = \bar{\varphi}(0)e^{\frac{1}{\tau|\Omega|} \int_0^t \int_{\Omega} (\lambda_p^{\mu} + a(x,t))dxdt} + O(\mu^{-\frac{1}{2}}) \quad \text{as } \mu \rightarrow \infty.$$

Since $\bar{\varphi}(1) = \bar{\varphi}(0)$, we get

$$\int_0^1 \int_{\Omega} (\lambda_p^\mu + a(x, t)) dx dt \rightarrow 0 \quad \text{or} \quad \bar{\varphi}(0) \rightarrow 0 \quad \text{as } \mu \rightarrow \infty.$$

If $\bar{\varphi}(0) \rightarrow 0$, then, by (2.36), also $\bar{\varphi}(t) \rightarrow 0$ uniformly in $t \in [0, 1]$ as $\mu \rightarrow \infty$. Thanks to (2.31) and the symmetry of J , we get

$$\begin{aligned} & \int_0^1 \int_{\Omega} \varphi^2(x, t) dx dt \\ & \leq C_0 \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) \varphi^2(x, t) dy dx dt \\ & = C_0 \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) (\varphi^2(x, t) - \varphi(y, t) \varphi(x, t)) dy dx dt \\ & \quad + C_0 \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) \varphi(y, t) \varphi(x, t) dy dx dt \\ & \leq \frac{C_0}{2} \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) (\varphi(y, t) - \varphi(x, t))^2 dy dx dt \\ & \quad + |\Omega|^2 C_0 M \int_0^1 \bar{\varphi}^2(t) dt \\ & \leq \frac{C_0}{\mu} \left(\max_{\bar{\Omega} \times [0, 1]} a - \bar{a} \right) + |\Omega|^2 C_0 M \int_0^1 \bar{\varphi}^2(t) dt, \end{aligned}$$

where $C_0 = \left(\min_{x \in \bar{\Omega}} \int_{\Omega} J(x - y) dy \right)^{-1}$ and $M = \max_{(x, y) \in \bar{\Omega} \times \bar{\Omega}} J(x - y)$. This implies that

$$\int_0^1 \int_{\Omega} \varphi^2(x, t) dx dt \rightarrow 0 \quad \text{as } \mu \rightarrow \infty,$$

in contradiction to the normalization of φ . Thus, we have

$$\int_0^1 \int_{\Omega} (\lambda_p^\mu + a(x, t)) dx dt \rightarrow 0 \quad \text{as } \mu \rightarrow \infty,$$

that is,

$$\lambda_p^\mu \rightarrow -\bar{a} \quad \text{as } \mu \rightarrow \infty.$$

(ii) By [38, Theorem 1.2], there exists $\sigma_0 > 0$ such that $\lambda_p(N_{\Omega}^{\mu, \sigma, m})$ is the principal eigenvalue of $N_{\Omega}^{\mu, \sigma, m}$ for all $\sigma \leq \sigma_0$. It follows from Lemma 2.9 that $\lambda_p(L_{\Omega}^{\tau, \mu, \sigma, m})$ is the principal eigenvalue of $L_{\Omega}^{\tau, \mu, \sigma, m}$ for all $\sigma \leq \sigma_0$.

Let ϕ be as in (2.24). Without loss of generality, we assume that $a \in C_1^{2,1}(\bar{\Omega} \times \mathbb{R})$. Then there holds $\phi \in C_1^{2,1}(\bar{\Omega} \times \mathbb{R}) \cap \chi_{\Omega}^{++}$. For every $\epsilon > 0$, we have

$$\begin{aligned} & (L_{\Omega}^{\tau, \mu, \sigma, m} - \max_{\bar{\Omega}} \hat{a} - \epsilon)[\phi](x, t) \\ (2.37) \quad & \leq \frac{\mu}{\sigma^m} \int_{\Omega} J_{\sigma}(x - y) (\phi(y, t) - \phi(x, t)) dy - \epsilon \phi(x, t) \\ & \leq \frac{\mu}{\sigma^m} \int_{\frac{\Omega - x}{\sigma}} J(z) (\phi(x + \sigma z, t) - \phi(x, t)) dz - \epsilon \phi(x, t). \end{aligned}$$

For σ small enough, say $\sigma \leq \sigma_1$, we obtain $B(0, 1) \subset \frac{\Omega-x}{\sigma}$ for all $x \in \Omega$. Thus, by Taylor's expansion and the symmetric of J , there holds

$$\begin{aligned}
 & \frac{\mu}{\sigma^m} \int_{\frac{\Omega-x}{\sigma}} J(z)(\phi(x + \sigma z, t) - \phi(x, t))dz \\
 &= \frac{\mu}{\sigma^m} \int_{\mathbb{R}^N} J(z)(\phi(x + \sigma z, t) - \phi(x, t))dz \\
 (2.38) \quad &= \frac{\mu}{\sigma^m} \int_{\mathbb{R}^N} J(z) \left(D\phi(x, t)(\sigma z) + \frac{1}{2}(\sigma z)^T D^2\phi(x, t)(\sigma z) + o(\sigma^2) \right) dz \\
 &= \frac{\mu\sigma^{2-m}}{2} \int_{\mathbb{R}^N} J(z) z^T D^2\phi(x, t) z dz + o(\sigma^{2-m}).
 \end{aligned}$$

By combining (2.37) with (2.38), there exists $0 < \sigma_\epsilon < \sigma_1$ such that

$$(L_\Omega^{\tau, \mu, \sigma, m} - \max_{\Omega} \hat{a} - \epsilon)[\phi] \leq 0 \quad \text{for all } \sigma \leq \sigma_\epsilon.$$

Using the definition of $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$, there holds

$$\lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) \geq -\max_{\Omega} \hat{a} - \epsilon \quad \text{for all } \sigma \leq \sigma_\epsilon.$$

Therefore, we obtain

$$\liminf_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) \geq -\max_{\Omega} \hat{a}.$$

It remains to show that

$$\limsup_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) \leq -\max_{\Omega} \hat{a}.$$

By Theorem 1.1 and [29, Theorem C], it follows that

$$(2.39) \quad \lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) \leq \lambda_p(N_\Omega^{\mu, \sigma, m}).$$

Using [38, Theorem 1.2], there holds

$$(2.40) \quad \lim_{\sigma \rightarrow 0^+} \lambda_p(N_\Omega^{\mu, \sigma, m}) = -\max_{\Omega} \hat{a}.$$

In view of (2.39) and (2.40), we have

$$\limsup_{\sigma \rightarrow 0^+} \lambda_p(L_\Omega^{\tau, \mu, \sigma, m}) \leq -\max_{\Omega} \hat{a}.$$

The proof is complete. $\square$

3. TIME-PERIODIC NONLOCAL DISPERSAL KPP EQUATIONS

In this section we apply the results for the generalised principal eigenvalues to the time-periodic nonlocal dispersal KPP equation with Neumann boundary conditions. Firstly, we study the effects of the frequency on the persistence of populations. Next, we discuss the effects of the dispersal rate and the dispersal spread on the positive time-periodic solutions to the equation. More precisely, we obtain the existence, uniqueness and stability of positive time-periodic solutions when μ or σ is sufficiently small or large. Furthermore, we analyse the asymptotic limits of the positive time-periodic solutions as μ or σ tends to zero or infinity.

3.1. Effects of the frequency.

The following result provides some sufficient conditions such that Corollary 1.7 holds.

Theorem 3.1. *Assume that (J) and (F) hold. Then the following statements hold:*

- (i) *If $\int_0^1 \max_{x \in \Omega} \{a(x, t)\} dt < 0$, then (1.3) admits no solution in $\chi_\Omega^+ \setminus \{0\}$ and zero solution is globally asymptotically stable for all $\tau \in (0, \infty)$;*
- (ii) *If $\int_0^1 \max_{x \in \Omega} \{a(x, t)\} dt - \frac{\mu}{\sigma^m} > 0$, $\max_{x \in \Omega} \{\hat{a}(x)\} < 0$ and $\lambda_p(L_\Omega^{\tau, \mu, \sigma, m})$ is a principal eigenvalue of the operator $L_\Omega^{\tau, \mu, \sigma, m}$, then there is some constant $\tau^* > 0$ such that*
 - (a) *If $\tau < \tau^*$, then (1.3) admits a unique solution $u_\tau^* \in \chi_\Omega^{++}$ that is globally asymptotically stable.*
 - (b) *If $\tau > \tau^*$, then (1.3) admits no solution in $\chi_\Omega^+ \setminus \{0\}$ and zero solution is globally asymptotically stable;*
- (iii) *If $\max_{x \in \Omega} \{\hat{a}(x)\} - \frac{\mu}{\sigma^m} > 0$, then (1.3) admits a unique solution $u_\tau^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\tau \in (0, \infty)$.*

Proof. (i) By the definition of $\lambda_p(N_\Omega^t)$, it is easy to deduce

$$(3.1) \quad -\max_{x \in \Omega} \{a(x, t)\} \leq \lambda_p(N_\Omega^t) \leq -\max_{x \in \Omega} \left\{ a(x, t) - \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y) dy \right\}.$$

Thus, we have

$$\int_0^1 \lambda_p(N_\Omega^t) dt \geq \int_0^1 -\max_{x \in \Omega} \{a(x, t)\} dt > 0.$$

By Corollary 1.7, we deduce the conclusion of part (i).

(ii) Owing to the inequality (3.1), we have

$$\int_0^1 \lambda_p(N_\Omega^t) dt \leq \int_0^1 -\max_{x \in \Omega} \left\{ a(x, t) - \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y) dy \right\} dt \leq -\int_0^1 \max_{x \in \Omega} \{a(x, t)\} dt + \frac{\mu}{\sigma^m} < 0.$$

By the definition of $\lambda_p(N_\Omega)$, we obtain

$$(3.2) \quad -\max_{x \in \Omega} \{\hat{a}(x)\} \leq \lambda_p(N_\Omega) \leq -\max_{x \in \Omega} \left\{ \hat{a}(x) - \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y) dy \right\},$$

which implies that

$$\lambda_p(N_\Omega) \geq -\max_{x \in \Omega} \{\hat{a}(x)\} > 0.$$

The conclusion of part (ii) thus follows from Corollary 1.7.

(iii) By the inequality (3.2), it implies that

$$\lambda_p(N_\Omega) \leq -\max_{x \in \Omega} \left\{ \hat{a}(x) - \frac{\mu}{\sigma^m} \int_\Omega J_\sigma(x - y) dy \right\} \leq -\max_{x \in \Omega} \{\hat{a}(x)\} + \frac{\mu}{\sigma^m} < 0.$$

The conclusion of part (iii) thus follows from Corollary 1.7. $\square$

3.2. Effects of the dispersal rate.

This subsection is devoted to the proof of Theorem 1.8. We recall the following result in [36]:

Lemma 3.2. *Assume that f satisfies (F). If $\min_{\bar{\Omega}} \hat{a} > 0$, then for each $x \in \bar{\Omega}$, the equation*

$$\tau v_t = f(x, t, v)$$

has a unique positive 1-periodic solution, denoted by $v^(x, t)$, which is continuous in x .*

Although the proof of Theorem 1.8 (i) is similar to [36, Theorem C], we still present the proof for the convenience of the reader.

Proof of Theorem 1.8. (i) By Theorem 1.3 (ii), there exists $\mu_0 > 0$ such that $\lambda_p(L_\Omega^{\tau, \mu, 1, 0}) \leq -\max_{\bar{\Omega}} \hat{a}/2 < 0$ for all $\mu \in (0, \mu_0)$. Thus, it follows from Lemma 1.6 that (1.3) admits a unique solution $u_\mu^* \in \chi_\Omega^{++}$ which is globally asymptotically stable for all $\mu \in (0, \mu_0)$.

We claim that for each $0 < \epsilon \ll 1$, there exists $\mu_\epsilon > 0$ such that for each $\mu \in (0, \mu_\epsilon)$,

$$v^*(x, t) - \epsilon \leq u_\mu^*(x, t) \leq v^*(x, t) + \epsilon, \quad (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

Let us prove the lower bound only as the upper bound follows from similar arguments. Let $0 < \epsilon \ll 1$. By $\min_{\bar{\Omega} \times \mathbb{R}} v^* > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$v(x, t) := (1 - \delta)v^*(x, t) \geq v^*(x, t) - \epsilon > 0, \quad (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

Note that for each $(x, t) \in \bar{\Omega} \times \mathbb{R}$,

$$\begin{aligned} & -\tau v_t(x, t) + \mu \int_{\Omega} J(x - y)(v(y, t) - v(x, t))dy + f(x, t, v(x, t)) \\ &= -(1 - \delta)\tau v_t^*(x, t) + (1 - \delta)\mu \int_{\Omega} J(x - y)(v^*(y, t) - v^*(x, t))dy + f(x, t, v(x, t)) \\ &= (1 - \delta)\mu \int_{\Omega} J(x - y)(v^*(y, t) - v^*(x, t))dy + f(x, t, v(x, t)) - (1 - \delta)f(x, t, v^*(x, t)). \end{aligned}$$

We see that as $\mu \rightarrow 0^+$,

$$(3.3) \quad (1 - \delta)\mu \int_{\Omega} J(x - y)(v^*(y, t) - v^*(x, t))dy \rightarrow 0 \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

By (F)-(3), there holds for each $(x, t) \in \bar{\Omega} \times \mathbb{R}$,

$$f(x, t, v(x, t)) - (1 - \delta)f(x, t, v^*(x, t)) = v(x, t) \left[\frac{f(x, t, v(x, t))}{v(x, t)} - \frac{f(x, t, v^*(x, t))}{v^*(x, t)} \right] > 0.$$

Thus, there exists $\mu_\epsilon > 0$ such that for each $\mu \in (0, \mu_\epsilon)$,

$$(3.4) \quad \tau v_t(x, t) \leq \mu \int_{\Omega} J(x - y)(v(y, t) - v(x, t))dy + f(x, t, v(x, t)) \quad \text{for all } (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

It remains to show that for each $\mu \in (0, \mu_\epsilon)$, there holds $v(x, t) \leq u_\mu^*(x, t)$ for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$. To do so, let us fix any $\mu \in (0, \mu_\epsilon)$ and define

$$\alpha_* = \inf\{\alpha > 0 \mid v(x, t) \leq \alpha u_\mu^*(x, t) \text{ for all } (x, t) \in \bar{\Omega} \times \mathbb{R}\}.$$

Since $u_\mu^* \in \chi_\Omega^{++}$ and v is bounded, α_* is well-defined and positive. Thanks to the continuity of u_μ^* and v , there holds $v(x, t) \leq \alpha_* u_\mu^*(x, t)$ and there exists $(x_0, t_0) \in \bar{\Omega} \times \mathbb{R}$ such that $v(x_0, t_0) = \alpha_* u_\mu^*(x_0, t_0)$.

If $\alpha_* \leq 1$, then we are done. Therefore, let us assume $\alpha_* > 1$. By the inequality (3.4) and (F)-(3), we see that $w(x, t) := v(x, t) - \alpha_* u_\mu^*(x, t)$ satisfies

$$\begin{aligned} (3.5) \quad & \tau w_t(x, t) \leq \mu \int_{\Omega} J(x - y)(w(y, t) - w(x, t))dy + f(x, t, v(x, t)) - \alpha_* f(x, t, u_\mu^*(x, t)) \\ & < \mu \int_{\Omega} J(x - y)(w(y, t) - w(x, t))dy + f(x, t, v(x, t)) - f(x, t, \alpha_* u_\mu^*(x, t)). \end{aligned}$$

Hence, setting $(x, t) = (x_0, t_0)$ in (3.5) yields

$$0 = \tau w_t(x_0, t_0) < \mu \int_{\Omega} J(x_0 - y)(w(y, t_0) - w(x_0, t_0))dy$$

$$+ f(x_0, t_0, v(x_0, t_0)) - f(x_0, t_0, \alpha_* u_\mu^*(x_0, t_0)) \leq 0,$$

which is a contradiction. Hence, $\alpha_* \leq 1$ and the proof is complete.

(ii) By Theorem 1.5(i), there exists $\mu_1 > 0$ such that $\lambda_p(L_\Omega^{\tau, \mu, 1, 0}) \leq -\bar{a}/2 < 0$ for all $\mu \in (\mu_1, \infty)$. Thus, it follows from Lemma 1.6 that (1.3) admits a unique solution $u_\mu^* \in \chi_\Omega^{++}$ that is globally asymptotically stable for all $\mu \in (\mu_1, \infty)$. Thus, we have

$$(3.6) \quad \tau(u_\mu^*)_t(x, t) = \mu \int_\Omega J(x-y)(u_\mu^*(y, t) - u_\mu^*(x, t))dy + f(x, t, u_\mu^*(x, t)), \quad (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

It is easy to check that there holds

$$\sup_{t \in [0, 1]} \|f(\cdot, t, u_\mu^*(\cdot, t))\|_{L^\infty(\Omega)} \leq \sup_{t \in [0, 1]} \|a(\cdot, t)u_\mu^*(\cdot, t)\|_{L^\infty(\Omega)} \leq M \max_{(x, t) \in \Omega \times [0, 1]} |a(x, t)| := MA.$$

Multiplying (3.6) by u_μ^* and integrating over $\Omega \times (0, 1)$ yield

$$\begin{aligned} \tau \int_0^1 \int_\Omega (u_\mu^*)_t u_\mu^* dx dt &= \int_0^1 \int_\Omega f(x, t, u_\mu^*(x, t)) u_\mu^*(x, t) dx dt \\ &\quad + \mu \int_0^1 \int_\Omega \int_\Omega J(x-y)(u_\mu^*(y, t) - u_\mu^*(x, t)) u_\mu^*(x, t) dy dx dt. \end{aligned}$$

In view of $u_\mu^* \in \chi_\Omega^{++}$ and the symmetry of J , we have

$$\frac{\mu}{2} \int_0^1 \int_\Omega \int_\Omega J(x-y)(u_\mu^*(y, t) - u_\mu^*(x, t))^2 dy dx dt = \int_0^1 \int_\Omega f(x, t, u_\mu^*(x, t)) u_\mu^*(x, t) dx dt,$$

which implies that

$$\begin{aligned} (3.7) \quad &\int_0^1 \int_\Omega \int_\Omega J(x-y)(u_\mu^*(y, t) - u_\mu^*(x, t))^2 dy dx dt \\ &= \frac{2}{\mu} \int_0^1 \int_\Omega f(x, t, u_\mu^*(x, t)) u_\mu^*(x, t) dx dt \\ &\leq \frac{2M^2 A |\Omega|}{\mu}. \end{aligned}$$

Let us assume $U(x, t) = u_\mu^*(x, t) - \bar{u}(t)$, where $\bar{u}(t) = \frac{1}{|\Omega|} \int_\Omega u_\mu^*(x, t) dx$. We get $\int_\Omega U(x, t) dx = 0$. Integrating (3.6) over Ω and substituting $U(x, t) = u_\mu^*(x, t) - \bar{u}(t)$ yield

$$\begin{aligned} \tau \bar{u}_t(t) &= \frac{1}{|\Omega|} \int_\Omega f(x, t, U(x, t) + \bar{u}(t)) dx \\ &= \frac{1}{|\Omega|} \int_\Omega f(x, t, \bar{u}(t)) dx + \frac{1}{|\Omega|} \int_\Omega (f(x, t, U(x, t) + \bar{u}(t)) - f(x, t, \bar{u}(t))) dx. \end{aligned}$$

Owing to assumption (F), we have

$$\begin{aligned} &\left| \int_\Omega (f(x, t, U(x, t) + \bar{u}(t)) - f(x, t, \bar{u}(t))) dx \right| \\ &\leq \left| \int_\Omega a(x, t) U(x, t) dx \right| \\ &\leq A |\Omega|^{\frac{1}{2}} \left(\int_\Omega U^2(x, t) dx \right)^{\frac{1}{2}} \end{aligned}$$

By [32, Page 1688, Formula (5.6)], there exists $C > 0$ such that

$$(3.8) \quad \int_\Omega U^2(x, t) dx \leq \frac{1}{2C} \int_\Omega \int_\Omega J(x-y)(U(y, t) - U(x, t))^2 dy dx \quad \text{for all } \mu \gg 1.$$

By (3.7) and (3.8), there holds

$$\int_0^1 \int_{\Omega} U^2(x, t) dx dt \leq \frac{M^2 A |\Omega|}{C \mu},$$

which implies that

$$\int_0^1 \left| \int_{\Omega} (f(x, t, U(x, t) + \bar{u}(t)) - f(x, t, \bar{u}(t))) dx \right| dt = O(\mu^{-\frac{1}{2}}) \quad \text{as } \mu \gg 1.$$

Thus, we have

$$\lim_{\mu \rightarrow \infty} u_{\mu}^*(x, t) = v^*(t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $v^*(t)$ is the unique positive 1-periodic solution or zero solution of the equation

$$\tau v_t(t) = \frac{1}{|\Omega|} \int_{\Omega} f(x, t, v(t)) dx.$$

Finally, we exclude that $v^*(t)$ is zero solution. Divide (3.6) by u_{μ}^* and integrate over $\Omega \times (0, 1)$ to obtain

$$\begin{aligned} \tau \int_0^1 \int_{\Omega} \frac{(u_{\mu}^*)_t(x, t)}{u_{\mu}^*(x, t)} dx dt &= \mu \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) \frac{u_{\mu}^*(y, t) - u_{\mu}^*(x, t)}{u_{\mu}^*(x, t)} dy dx dt \\ &\quad + \int_0^1 \int_{\Omega} \frac{f(x, t, u_{\mu}^*(x, t))}{u_{\mu}^*(x, t)} dx dt \quad \text{for all } \mu \in (\mu_1, \infty). \end{aligned}$$

Owing to $u_{\mu}^* \in \chi_{\Omega}^{++}$ and the symmetry of J , we get

$$\int_0^1 \int_{\Omega} \frac{f(x, t, u_{\mu}^*(x, t))}{u_{\mu}^*(x, t)} dx dt = -\frac{\mu}{2} \int_0^1 \int_{\Omega} \int_{\Omega} J(x - y) \frac{(u_{\mu}^*(y, t) - u_{\mu}^*(x, t))^2}{u_{\mu}^*(x, t) u_{\mu}^*(y, t)} dy dx dt$$

for all $\mu \in (\mu_1, \infty)$. This implies that

$$\int_0^1 \int_{\Omega} \frac{f(x, t, u_{\mu}^*(x, t))}{u_{\mu}^*(x, t)} dx dt \leq 0 \quad \text{for all } \mu \in (\mu_1, \infty).$$

If $v^*(t) \equiv 0$, then it follows from the above inequality and assumption (F) that

$$\int_0^1 \int_{\Omega} a(x, t) dx dt \leq 0,$$

which contradicts $\bar{a} > 0$. Therefore, v^* is non-zero and the proof is complete. $\square$

3.3. Effects of the dispersal spread.

This subsection is devoted to the proof of Theorem 1.9.

Proof of Theorem 1.9. (i) It follows from Theorem 1.3 and Lemma 1.6 that there exists $\sigma_1 > 0$ such that (1.3) admits a unique solution $u_{\sigma}^* \in \chi_{\Omega}^{++}$ that is globally asymptotically stable for all $\sigma \in (\sigma_1, \infty)$. Similar to the proof of Theorem 1.8 (i), we have

$$\lim_{\sigma \rightarrow \infty} u_{\sigma}^*(x, t) = v^*(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

(ii) It follows from Theorem 1.5 and Lemma 1.6 that there exists $\sigma_0 > 0$ such that (1.3) admits a unique solution $u_{\sigma}^* \in \chi_{\Omega}^{++}$ that is globally asymptotically stable for all $\sigma \in (0, \sigma_0)$. By the same proof of Theorem 1.8 (i), we can also obtain that

$$\lim_{\sigma \rightarrow 0^+} u_{\sigma}^*(x, t) = v^*(x, t) \quad \text{uniformly in } (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

The proof is complete. $\square$

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