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Data-driven Stabilization of Nonlinear Systems via Tree-Based Ensemble Learning

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Abstract— **We present an approach for the stabilization of an unknown nonlinear dynamical system when only data samples from its dynamics are available. Our approach is based on approximating the system dynamics with an ensemble of regression trees. As a result of our approximation, we obtain a model that is a piecewise-affine dynamical system defined over a partition of the state space. In general, the stabilization of the resulting piece-wise affine system requires, in the worst case, solving an exponential number of linear matrix inequalities (with respect to the state dimension). To overcome this computational limitation, we propose a stabilization procedure having a complexity that grows linearly with the number of partitions. This stabilization procedure explicitly exploits the fact that our model is described via an ensemble of regression trees. In addition, we derive probabilistic conditions under which the stabilization of the model implies that the original nonlinear system is also stabilized. Finally, we validate our approach by performing numerical simulations over trajectories of two coupled Van der Pol oscillators.**

I. Introduction

Standard model-based control techniques rely on the development of an explicit mathematical model of the dynamical system under consideration [1]. When dealing with complex phenomena, such as turbulent flows or biological processes, building such a model from first principles can be a cumbersome —sometimes impossible— task. On the other hand, in recent years, technological developments in sensors and data collection are making high-fidelity measurements more accessible to a wide variety of systems. In this setting, *data-driven* control techniques seek to design controllers using data obtained from system measurements in the absence of an explicit system model. One possible approach towards that goal is to use tools from statistical learning theory to obtain a model from data (as well as an uncertainty estimate), and design a controller using tools from robust control theory. In this direction, the work in [2] studies a data-driven linear-quadratic regulator (LQR) control problem for an unknown linear system, where both a nominal model and uncertainty bounds are estimated from a finite number of samples. In [3], a data-driven model predictive control (MPC) approach is proposed using information from repetitive trials to improve system performance while guaranteeing recursive feasibility. The work in [4] presents an MPC framework able to provide safety guarantees by exploiting the regularity assumptions on the dynamics in terms of a Gaussian process. In addition, recent approaches have also focused on extending system representations to include non-parametric models based on Gaussian processes [5], [6] and Dirichlet process mixtures of linear models [7].

A particularly relevant class of models for our work is the one of *piecewise-affine* systems [8]. In these models, the state space if partitioned into different regions and the dynamics in each region is modeled by an affine function. For example, using *regression trees* to model the dynamics of a system from data naturally results in a piece-wise constant dynamical system [9], [10]. Regression trees can be generalized to also consider affine, even polynomial functions, at each partition. Some examples of particular learning algorithms for regression trees are M5 [11], GUIDE [12], and model-based recursive partitioning [13]. Moreover, a powerful extension of regression trees can be obtained by considering *ensembles* of regression trees [14], [15], [16]. When compared to a single regression tree, ensembles of regression trees tend to dramatically reduce the approximation error [17]. In the control systems literature, ensembles of regression trees have been used in predictive control problems using switched affine models using historical data [18], [19].

We can find significant amount of work in the control community regarding the analysis and control of piecewise-affine systems. The reader is referred to [20] for a survey on discrete-time piece-wise affine systems, and to the seminal paper [21] for the continuous-time case. Some results in this direction have focused on checking stability using impact maps [22], non-monotonic Lyapunov functions [23], and sampling based methods [24]. Moreover, piecewise-linear controller *design* for piecewise-affine systems has also been well-studied in the literature. For the discrete-time case, a good overview can be found in [25]. For the continuous-time case, two approaches are given in [26] and [27]. In both cases, the problem of designing a linear feedback controller can be expressed via linear matrix inequalities (LMI's).

In this work, we consider the problem of designing controllers for nonlinear systems when an explicit model is not available. Instead, we have access to multiple samples of the system dynamics, from which we can build a data-driven model. In particular, we build our model using a regression tree ensemble [28], and propose a data-driven controller design framework to stabilize the system. First, we learn the system dynamics as an ensemble of regression trees, using the algorithm presented in [16]. Then, we propose a stabilization procedure that explicitly exploits the fact that

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our model is described via an ensemble of regression trees. The main advantage of our approach is that it requires the solution of a system of LMI's that grows *linearly* with the number of partitions. Secondly, we perform a probabilistic analysis to characterize conditions under which the controller designed with the tree-ensemble model can be used to stabilize the original nonlinear system with high probability.

II. Problem Formulation

Consider an input-affine, continuous-time nonlinear system with dynamics described by

$$\dot{x} = g(x,u) := f(x) + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, and $B \in \mathbb{R}^{n\times m}$ models the linear effect of the input on the state. Here, the function $f : \mathbb{R}^n \to \mathbb{R}^n$, which determines the autonomous dynamics of the state x, is assumed to be unknown. We seek to approximate f by an ensemble of T regression trees [9], where each tree represents a piecewise affine function over a partition of the state space. More precisely, the approximation $\hat{f}$ can be written as

$$\hat{f}(x) := \frac{1}{T}\sum_{k=1}^{T} \hat{h}_k(x), \tag{2}$$

with $\hat{h}_k$ being the piecewise affine function represented by the k-th regression tree. We denote the set of τ_k regions in the partition of the state space induced by the k-th regression tree by $\{\mathcal{R}_i^k\}_{i=1}^{\tau_k}$. Inside each region $\mathcal{R}_i^k$, we have an affine system $\dot{x} = A_i^k x + a_i^k$, where $A_i^k \in \mathbb{R}^{n\times n}$ and $a_i^k \in \mathbb{R}^n$. With these elements, the function defined by a regression tree $\hat{h}_k$ can be described as

$$\hat{h}_k(x) := \sum_{i=1}^{\tau_k} (A_i^k x + a_i^k)\mathbf{1}(x \in \mathcal{R}_i^k),$$

where $\mathbf{1}(\cdot)$ denotes the indicator function. Notice that in the case of regression trees, each region $\mathcal{R}_i^k$ is a polytope (more precisely, a hyper-rectangle). Without loss of generality, we represent each region as $\mathcal{R}_i^k = \{x : E_i^k x \geq e_i^k\}$. For a more compact representation of our model, we introduce the variables $\nu_k := \sum_{j=1}^{k} \tau_{j-1}$ with $\tau_0 = 0$, as well as $\mathcal{R}_{\nu_k+i} := \mathcal{R}_i^k$, $A_{\nu_k+i} := A_i^k$, and $a_{\nu_k+i} := a_i^k$. Therefore, since there are $\nu \equiv \nu_{T+1}$ regions in the ensemble, we can re-write (2) as

$$\hat{f}(x) = \frac{1}{T}\sum_{i=1}^{\nu} (A_i x + a_i)\mathbf{1}(x \in \mathcal{R}_i). \tag{3}$$

Using the representation in (3), we propose to approximate the original nonlinear system in (1) by

$$\hat{g}(x,u) := \hat{f}(x) + \hat{B}u, \tag{4}$$

where the parameters of the function $\hat{f}$ proposed in (3) can be estimated by existing regression tree boosting algorithms [16].

Broadly speaking, boosting algorithms train an ensemble of trees of increasing size by sequentially fitting each additional tree on the residuals of the previously trained ensemble. In particular, because of the higher approximation capability of ensemble models, we set the constant terms[1] $a_i = 0$ for $i = 1,\dots,\nu$. Hence, we can use the algorithm in [16] to obtain a set $\{(A_i, \mathcal{R}_i)\}_{i=1}^{\nu}$ representing $\hat{f}$. More precisely, we assume that we have r samples taken from *autonomous* trajectories of the system's dynamics in the form $\{(x_j, y_j)\}_{j=1}^{r}$, where x_j are samples of the system state, and y_j is an estimate of the corresponding derivative $f(x_j)$. For simplicity, we assume that $x_j \in \mathcal{S}$ for all j, where $\mathcal{S} := [-a,a]^n$. Given $\hat{f}$, we can obtain an approximation of the input-to-state matrix $\hat{B}$ by performing standard least-squares regression on a set of samples taken from *actuated* trajectories.

Next, equipped with the regression tree ensemble representation $\hat{g}$ of the original system g, we are ready to state the main problems we address in this paper.

Problem 1 (Stabilization of a regression tree ensemble)**.** *Find a linear state-feedback policy expressed in the form $u = \hat{K}_i x$ for $x \in \mathcal{R}_i$ (i.e., having a specific feedback gain $\hat{K}_i \in \mathbb{R}^{m\times n}$ for each region $\mathcal{R}_i$), such that the approximated system $\hat{g}$ is stabilized in closed-loop. The resulting controller is thus described by a set of pairs $\mathcal{K}_{\hat{g}} := \{(\hat{K}_i, \mathcal{R}_i)\}_{i=1}^{\nu}$.*

Provided with a solution to Problem 1, we can examine the probabilistic relation between g and the $\hat{g}$ to address the additional problem of finding a controller $\mathcal{K}_g := \{(K_i, \mathcal{R}_i)\}_{i=1}^{\nu}$ that stabilizes the original system, in the following sense:

Problem 2 (Probabilistic stabilization of the original system)**.** *Given a ball $\mathcal{X}_0 := \{x : \|x\|_2 \leq \eta_0\}$, find a controller of the form $\mathcal{K}_g$ such that the following two conditions are satisfied with high probability for any $x(0) \in \mathcal{X}_0$:*

(i) the trajectories of the system converge to a compact set $\mathcal{D} := \{x : x^\top P x \leq \eta_\infty\} \subset \mathcal{S}$, where $P \succ 0$;

(ii) the trajectories of the system do not leave the hypercube $\mathcal{S}$, i.e., $x(t) \in \mathcal{S}$ for all $t \geq 0$.

We seek to provide explicit conditions for the parameters η_0, η_∞, and P to satisfy Conditions (*i*) and (*ii*) from Problem 2. In particular, our results aim to find the 'smallest' set $\mathcal{D}$ for a given η_0.

III. Controller Design for the Regression Ensemble

In this section, we propose a method to address Problem 1 using a stabilization method based on Lyapunov functions. To that end, one could consider a direct approach to find a Lyapunov function using available methods in the literature [21], which would be applicable, in principle, for single-tree models. However, applying this method to an ensemble-tree model requires us to consider all the possible intersections over all the partitions defined by all the regression trees in the ensemble. Such an approach would involve a massive number of resulting intersections; more precisely, if each regression tree in the ensemble had $\bar{\tau}$ partitions ($\tau_k = \bar{\tau}$),

[1] Notice that, under this assumption, the origin of the approximation dynamics $\hat{f}$ is always an equilibrium point.

the resulting combined piecewise-affine system model could have up to $(\bar{\tau})^n$ partitions, resulting in a scalability issue.

To avoid this computational issue, we propose a method to stabilize the piecewise-affine dynamical model that avoids dealing with an exponential number of partitions. First, for each tree in the ensemble, we assign a controller gain matrix to each of the regions in the tree. In other words, we design a piece-wise linear controller for each tree in the ensemble by solving a collection of ν LMI's of size n. Then, we combine all these T controllers into a single controller by computing the average of all the T piece-wise linear controllers. As shown in Theorem 1, the resulting controller stabilizes the dynamics of the tree-ensemble model.

Theorem 1. *Consider the piecewise linear system:*

$$\dot{x} = \hat{g}(x,u) = \frac{1}{T}\sum_{i=1}^{\nu}(A_i x)\mathbf{1}(x \in \mathcal{R}_i) + \hat{B}u. \tag{5}$$

If there exist scalars $\alpha, \beta, \kappa > 0$, matrices $\{Y_i\}_{i=1}^{\nu} \in \mathbb{R}^{m\times n}$ and $Q \in \mathbb{R}^{n\times n}$ with $\alpha I_n \preceq Q \preceq \beta I_n$ such that, for all $i = 1, \ldots, \nu$

$$A_i Q + Q A_i^\top + \hat{B}Y_i + Y_i^\top \hat{B}^\top + \kappa Q \preceq 0, \tag{6}$$

then, the piecewise linear state-feedback control law

$$u = \frac{1}{T}\sum_{i=1}^{\nu}\hat{K}_i \mathbf{1}(x \in \mathcal{R}_i)x, \tag{7}$$

with $\hat{K}_i = Y_i Q^{-1}$, stabilizes every trajectory[2] $x(t) \in \mathcal{S}$ for all $t \geq 0$.

Proof. With the change of variables $Q = P^{-1}$, each LMI

$$(A_i + \hat{B}\hat{K}_i)^\top P + P(A_i + \hat{B}\hat{K}_i) + \kappa P \preceq 0, \tag{8}$$

is equivalent to $Q(A_i + \hat{B}\hat{K}_i)^\top + (A_i + \hat{B}\hat{K}_i)Q + \kappa Q \preceq 0$. Then, defining $Y_i = \hat{K}_i Q$ it follows that

$$Q A_i^\top + A_i Q + \hat{B}Y_i + Y_i^\top \hat{B}^\top + \kappa Q \preceq 0.$$

Now, consider a Lyapunov function of the form $V(x) = x^\top P x$. Using u from (7), the dynamics induced by (5) can be written as

$$\dot{x} = \frac{1}{T}\sum_{i=1}^{\nu} A_i \mathbf{1}(x \in \mathcal{R}_i)x + \frac{1}{T}\hat{B}\sum_{i=1}^{\nu}\hat{K}_i \mathbf{1}(x \in \mathcal{R}_i)x.$$

Further, the derivative of $V(x)$ can be written as

$$\begin{aligned}\dot{V}(x) &= \left(\phi(x) + \frac{1}{T}\hat{B}\sum_{i=1}^{\nu}\hat{K}_i \mathbf{1}(x \in \mathcal{R}_i)x\right)^\top Px \\ &\quad + x^\top P\left(\phi(x) + \frac{1}{T}\hat{B}\sum_{i=1}^{\nu}\hat{K}_i \mathbf{1}(x \in \mathcal{R}_i)x\right),\end{aligned} \tag{9}$$

where $\phi(x) := \frac{1}{T}\sum_{i=1}^{\nu} A_i \mathbf{1}(x \in \mathcal{R}_i)x$. Hence, (9) can be rewritten as

$$\dot{V}(x) = \frac{1}{T}\sum_{i\in\mathcal{Z}}[(A_i + \hat{B}\hat{K}_i)x]^\top Px + x^\top P\ [(A_i + \hat{B}\hat{K}_i)x], \tag{10}$$

where $\mathcal{Z} := \{k : \mathbf{1}(x \in \mathcal{R}_k) = 1\}$. Observing that the sum in (10) ranges over T elements for all x, we can combine (8) and (10) to obtain

$$\dot{V}(x) \leq -\kappa x^\top P x. \tag{11}$$

Since the derivative of $V(x)$ is strictly negative for all $x \neq 0$, it follows that the proposed control strategy (7) stabilizes $\hat{g}(x,u)$, as desired. □

A few comments are in order with respect to Theorem 1. The first is related to some possible conservatism of the controller design, arising from its lower complexity when compared to the combined-region case. In this latter case, if there exists controller gains $\{\hat{K}_i\}_{i=1}^{\nu}$ satisfying the conditions in Theorem 1, then we can always find matrices $\{\bar{K}_i\}_{i=1}^{h}$ that stabilize the combined system. On the other hand, if $\{\bar{K}_i\}_{i=1}^{h}$ exist for which the combined system is stable, this fact does not imply the existence of $\{\hat{K}_i\}_{i=1}^{\nu}$ such that (8) holds. Secondly, it can be seen from (6) that the required number of LMI's grows linearly with the number of partitions, as intended. Lastly, we see from (11) that the derivative $\dot{V}(x)$ is upper bounded by $-\kappa x^\top P x$. Hence, the parameter κ can be interpreted as a margin in the decay rate available to guarantee stabilization. In this regard, such a margin might be used to absorb occasional perturbations on the Lyapunov function that may be caused by differences in the approximated dynamics. Therefore, if we consider the modeling error between the approximation $\hat{g}$ and the original system g as the cause of such perturbations, we might seek to quantify how much modeling error can be tolerated as a function of κ. In the next section, we characterize this relationship.

IV. Stabilization with Probabilistic Guarantees

In this section, we perform a Lyapunov perturbation analysis to find a stabilization design criterion addressing Problem 2. Such a criterion is specified as a function of a parameter describing the difference between the original and approximated systems, which we estimate using probabilistic arguments. Then, using a controller designed with respect to such an estimate, we propose a validation strategy that produces a lower bound on the probability of achieving both conditions in Problem 2 with respect to the original system.

Considering a piecewise-linear controller designed according to Theorem 1, we now determine the set to which closed-loop trajectories converge.

Theorem 2. *Given a scalar $\gamma \geq 0$ and a locally Lipschitz $f(x)$ in $\mathcal{S}$, assume there exists a*

$$\theta^\star \in \{\theta \in \mathbb{R}_+ : \|f(x) - \hat{f}(x)\|_2 < \theta\|x\|_2 + \gamma,\ \forall x \in \mathcal{S}\},$$

as well as an approximation $\hat{B}$ of B such that $\|\hat{B} - B\|_2 \leq \epsilon_B$. Further, assume there exist $\kappa, \alpha, \beta > 0$ that satisfy the conditions in Theorem 1 such that

$$\Delta = -\kappa\beta^{-1} + 2\alpha^{-1}(\theta^\star + \epsilon_B\zeta) < 0,$$

[2] In this theorem, we adopt the definition of continuous piecewise $\mathcal{C}^1$ trajectories and solutions presented in [27].

with $\zeta := \max_i \|\hat{K}_i\|_2$. Let us also define the sets

$$\mathcal{G} := \left\{x \in \mathbb{R}^n : \|x\|_2 \leq -2\gamma\alpha^{-1}\Delta^{-1}\right\},$$
$$\mathcal{X}_0 = \left\{x \in \mathbb{R}^n : \|x\|_2 \leq a\sqrt{\alpha/\beta}\right\}.$$

Then, the following two results hold:

(i) If $\mathcal{G} \subset \mathcal{X}_0$ $\left(\text{i.e., } a\sqrt{\alpha/\beta}+2\gamma\alpha^{-1}\Delta^{-1} > 0\right)$ and the initial condition $x(0)$ lies inside the ball $\mathcal{X}_0$, then the trajectory $x(t)$ stays inside $\mathcal{S}$ for all $t \geq 0$.

(ii) If $\mathcal{G} \subset \mathcal{D} \subset \mathcal{S}$ (e.g., when $-2\gamma\alpha^{-1}\Delta^{-1} < \sqrt{\omega\beta^{-1}} \leq \sqrt{\omega\alpha^{-1}} < a$), then $V_o(x) = x^\top Q^{-1}x$, with Q satisfying the conditions in Theorem 1, is such that

(a) $V_o(x) > 0$ for any $x \neq 0$ with $V_o(0) = 0$, and

(b) $\dot{V}_o(x) < 0$ for any $x \in \mathcal{S} \setminus \mathcal{G}$,

when the input to the system in (1) is given by

$$u = \frac{1}{T}\sum_{i=1}^{\nu} \hat{K}_i \mathbf{1}(x \in \mathcal{R}_i)x.$$

Furthermore, for any $x_0 \in \mathcal{X}_0$, the trajectory[3] *of the closed-loop system converges to the ellipsoid $\mathcal{D} = \{x^T Px \leq \omega\}$.*

Proof. To prove *(ii)*, we write the dynamics g of the original system as

$$\dot{x} = \hat{f}(x) + \hat{B}u + z(x,u), \tag{12}$$

where $z(x,u) := f(x) + Bu - \hat{f}(x) - \hat{B}u$. Consider the Lyapunov function $V_o(x) = x^\top Px$. It follows from (12), that

$$\dot{V}_o(x) = \Psi(x) + z(x,u)^\top Px + x^\top Pz(x,u),$$

where $\Psi(x)$ is as in (9). This is equivalent to $\dot{V}_o(x) = \Psi(x) + 2x^\top Pz(x,u)$. From (11), it also holds that $\Psi(x) \leq -\kappa x^\top Px$. Using the fact that $x^\top Px \geq \lambda_{\min}(P)\|x\|_2^2$, and knowing that $\lambda_{\min}(P) \geq \beta^{-1}$ holds by construction from Theorem 1, it follows that $\Psi(x) \leq -\kappa\beta^{-1}\|x\|_2^2$. Now, we can write $\dot{V}_o(x)$ as

$$\dot{V}_o(x) \leq -\kappa\beta^{-1}\|x\|_2^2 + 2x^\top Pz(x,u), \tag{13}$$

whence it follows that

$$\begin{aligned}\|z(x,u)\|_2 &\leq \|f(x) - \hat{f}(x)\|_2 + \|B - \hat{B}\|_2\|u\|_2 \\ &\leq \theta^\star\|x\|_2 + \gamma + \epsilon_B\zeta\|x\|_2.\end{aligned}$$

Since $\lambda_{\max}(P) \leq \alpha^{-1}$, we have from (13) that

$$\dot{V}_o(x) \leq -\kappa\beta^{-1}\|x\|_2^2 + 2\alpha^{-1}\|x\|_2\left[(\theta^\star + \epsilon_B\zeta)\|x\|_2 + \gamma\right].$$

Then, the above leads to $\dot{V}_o(x) \leq \Delta\|x\|_2^2 + 2\alpha^{-1}\gamma\|x\|_2$. Since $\Delta < 0$, it is the case that

$$\Delta\|x\|_2^2 + 2\alpha^{-1}\gamma\|x\|_2 < 0 \;\text{ if }\; \|x\|_2 > \frac{-2\alpha^{-1}\gamma}{\Delta},$$

thus, it follows that $\dot{V}_o(x) < 0$ for all $x \in \mathcal{S} \setminus \mathcal{G}$. Hence, the trajectories converge to the smallest sub-level set of the Lyapunov function that contains $\mathcal{G}$.

To prove *(i)*, we simply note that, since $\mathcal{G} \subset \mathcal{X}_0$, the trajectories starting from $x_0 \in \mathcal{X}_0 \setminus \mathcal{G}$ stay inside the set $\mathcal{E} := \{x : x^\top Px \leq x_0^\top Px_0\}$. Therefore, it follows that $\mathcal{E} \subseteq \{x : \|x\|_2^2\beta^{-1} \leq \|x_0\|_2^2\alpha^{-1}\}$, and thus the trajectories are contained in the set $\mathcal{E} \subseteq \{x : \|x\|_2 \leq \sqrt{\beta/\alpha}\|x_0\|_2\}$. We conclude by noting that if $\sqrt{\beta/\alpha}\|x_0\|_2 \leq a$, then the trajectories stay in $\mathcal{S}$. □

From Theorem 2, we see that the stabilization condition relies on the negativity of Δ, which favors smaller $\theta^\star$ values. In addition, we observe that the parameter $\theta^\star$ in Theorem 2 also requires

$$\|f(x) - \hat{f}(x)\|_2 < \theta^\star\|x\|_2 + \gamma \tag{14}$$

to hold for all $x \in \mathcal{S}$. However, verifying such a condition exhaustively requires an impractical amount of samples due to the curse of dimensionality, even if the original system f was known. Alternatively, we propose to adopt a specific sampling mechanism in order to estimate a lower bound on the probability that any sampled point in $\mathcal{S}$ fulfills (14). Following a strategy similar to the one presented in [2], we assume that we obtain r independently and identically distributed (i.i.d) samples of pairs $\mathcal{M} := \{(x_j, y_j)\}_{j=1}^r$, where x_j is a random sample of the state, following a distribution σ_x, and y_j is a *noisy* sample of the derivative of the state. In other words, we have that

$$y_j = f(x_j) + w_j, \tag{15}$$

where we assume $w_j \in \mathcal{W} := \{w : \|w\|_2 \leq \gamma\}$ for all $j = 1, \ldots, r$. We partition the set of samples $\mathcal{M}$ into a training set $\mathcal{M}_s$ with r_s samples and a testing set $\mathcal{M}_t$ with r_t samples. Then, we apply a method similar to [29] using concentration inequalities [30] to obtain the following probabilistic relationship.

Theorem 3. *Define the set $\mathcal{T}(\theta)$, for $\theta \geq 0$, as*

$$\mathcal{T}(\theta) := \{x \in \mathbb{R}^n : \|f(x) - \hat{f}(x)\|_2 < \theta\|x\|_2 + \gamma\}.$$

Further, consider the empirical average

$$\hat{\mu}_{\mathcal{T}}(\theta) := (1/r_t)\sum_{(x_j,y_j)\in\mathcal{M}_t} \mathbf{1}(x_j \in \mathcal{T}(\theta)). \tag{16}$$

Also, define

$$\widetilde{\mathcal{T}}(\theta) := \{x : \exists w, \|w\|_2 \leq \gamma, \|f(x) + w - \hat{f}(x)\|_2 < \theta\|x\|_2\}.$$

Then, with probability at least $1 - \delta$, for all $\epsilon > 0$ and any new random sample $x_l \sim \sigma_x$, we have

$$\mathbb{P}\{\mathbf{1}(x_l \in \mathcal{T}(\theta)) = 1\} \geq \hat{\mu}_{\widetilde{\mathcal{T}}}(\theta) - \epsilon, \tag{17}$$

for $\delta := 2\exp(-2r_t\epsilon^2)$, $\hat{\mu}_{\widetilde{\mathcal{T}}}(\theta) := \frac{1}{r_t}\sum_{(x_j,y_j)\in\mathcal{M}_t} \mathbf{1}(x_j \in \widetilde{\mathcal{T}}(\theta))$.

Proof. We note that because the $\{x_j\}$ are i.i.d., then $\mathbf{1}(x_j \in \mathcal{T})$ are also i.i.d. random variables[4]. Since $0 \leq \mathbf{1}(x_j \in \mathcal{T}) \leq 1$, we can apply Hoeffding's inequality [31] to obtain,

$$\mathbb{P}\{|\hat{\mu}_{\mathcal{T}} - \mu_{\mathcal{T}}| \geq \epsilon\} \leq 2\exp(-2r_t\epsilon^2),$$

[3] We recall the definition of continuous piecewise $\mathcal{C}^1$ trajectories and solutions as presented in [27].

[4] The set $\mathcal{T}$ can be shown to be measurable, since it corresponds to a sub-level set of a measurable function. Then it follows that the indicator function $\mathbf{1}(x \in \mathcal{T})$ is a measurable function; hence the variables $\mathbf{1}(x_j \in \mathcal{T})$ are also i.i.d. random variables [30].

where $\mu_{\mathcal{T}} := \mathbb{P}\{x_l \in \mathcal{T}\}$. With probability of at least $1-\delta$ we have

$$\mathbb{P}\{x_l \in \mathcal{T}\} = \mu_{\mathcal{T}} \geq \hat{\mu}_{\mathcal{T}} - \epsilon. \tag{18}$$

Now, observe that for any $\tilde{x} \in \widetilde{\mathcal{T}}$ there exists w such that $\|w\|_2 \leq \gamma$ for which

$$\|f(\tilde{x}) + w - \hat{f}(\tilde{x})\|_2 < \theta\|\tilde{x}\|_2. \tag{19}$$

Using (19), it follows from the triangle inequality that $\|f(\tilde{x}) - \hat{f}(\tilde{x})\|_2 - \|w\|_2 < \theta\|\tilde{x}\|_2$. Then, using $\|w\|_2 \leq \gamma$, we have $\|f(\tilde{x}) - \hat{f}(\tilde{x})\|_2 < \theta\|\tilde{x}\|_2 + \gamma$. From the above, we have

$$x \in \widetilde{\mathcal{T}} \Rightarrow x \in \mathcal{T}, \ \forall x, \tag{20}$$

whence we have that $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$. It follows from (20) that $\hat{\mu}_{\mathcal{T}}(\theta) > \hat{\mu}_{\widetilde{\mathcal{T}}}(\theta)$. Using (18), we find that with probability $1-\delta$, $\mathbb{P}\{x_l \in \mathcal{T}\} = \mu_{\mathcal{T}} \geq \hat{\mu}_{\mathcal{T}} - \epsilon \geq \hat{\mu}_{\widetilde{\mathcal{T}}} - \epsilon$. □

Observing that $\hat{\mu}_{\widetilde{\mathcal{T}}}(\theta)$ increases monotonically with θ, Theorem 3 establishes a relationship between θ and a lower bound $\mu_l := \hat{\mu}_{\widetilde{\mathcal{T}}}(\theta) - \epsilon$ on the probability that a sampled point satisfies condition (14), required for Theorem 2. Then, for a desired lower bound $\bar{\mu}_l$ on μ_l, we can pick $\bar{\theta}$ such that $\bar{\mu}_l = \hat{\mu}_{\widetilde{\mathcal{T}}}(\bar{\theta}) - \epsilon$ holds. Finally, considering this $\bar{\theta}$, the next proposition provides a lower bound on the probability for Conditions *(i)* and *(ii)* from Problem 2 to be satisfied.

Proposition 1. *For given $\bar{\theta}$ and ϵ_B, consider a controller $\mathcal{K}_p := \{K_i, \mathcal{R}_i\}_{i=1}^{\nu}$ satisfying the conditions in Theorem 1 with $\Delta < 0$. Assume that the solution $\Phi(t, x(0))$ of the closed-loop system* (1) *at time t with initial condition $x(0)$ is measurable in $x(0)$ for every $t \geq 0$. Define the set*

$$X_k := \{x(t),\ t \in \{0, \ldots, \tilde{t}\} : x(0) = x_k \in \mathcal{X}_0, \dot{x} = g(x,u)\}$$

including discrete samples of the closed-loop dynamics starting from random i.i.d. initial points $x_k \sim \sigma_x$. Also, define the set $\mathcal{Q}_1$ of trajectories satisfying Condition (i) in Problem 2, as follows:

$$\mathcal{Q}_1 := \{X_k : t \in \{\bar{t}, \ldots, \tilde{t}\}, \min_{x^* \in \mathcal{D}} \|x(t) - x^*\| = 0\},$$

where we pick $\bar{t}$ as the time when the trajectory enters $\mathcal{D}$, and choose $\tilde{t}$ as large as desired. Further, introduce the set $\mathcal{Q}_2$ of trajectories that satisfy Condition (ii) in Problem 2, i.e.,

$$\mathcal{Q}_2 := \{X_k :\ t \in \{0, \ldots, \tilde{t}\},\ x(t) \in \mathcal{S}\}.$$

Consider the empirical mean $\hat{\mu}_{\mathcal{Q}} := \frac{1}{r_q}\sum_{x_j \in \mathcal{M}_q} \mathbf{1}(X_j \in \mathcal{Q})$, where $\mathcal{M}_q$ is a set containing r_q i.i.d. samples $x_j \sim \sigma_x$, and $\mathcal{Q} := \mathcal{Q}_1 \cap \mathcal{Q}_2$. Then, with probability at least $1-\delta_{\mathcal{Q}}$, for any $\epsilon_{\mathcal{Q}} > 0$ and any new random sample $x_b \sim \sigma_x$, we have $\mathbb{P}\{X_b \in \mathcal{Q}\} \geq \hat{\mu}_{\mathcal{Q}} - \epsilon_{\mathcal{Q}}$, where $\delta_{\mathcal{Q}} := 2\exp(-2r_q\epsilon_{\mathcal{Q}}^2)$.

Proof. Since $\{x_k\}$ are sampled i.i.d., it follows that $\{X_k\}$ are i.i.d. random variables[5] and $\mathbf{1}(X_k \in \mathcal{Q})$ are also i.i.d. random variables[6]. Since $0 \leq \mathbf{1}(X_k \in \mathcal{Q}) \leq 1$, we can apply Hoeffding's inequality [31] to obtain the result. □

[5]Since $\Phi(t,x)$ is a measurable function in x for every t, it follows that every corresponding element in different X_k's, sampled at time t, are i.i.d.; hence $\{X_k\}$ are also i.i.d..

[6]$\mathcal{Q}$ is measurable since it is an intersection of sub-level sets of measurable functions; so it follows that $\mathbf{1}(X_k \in \mathcal{Q})$ are also i.i.d..

Therefore, Proposition 1 can be used to validate a controller $\mathcal{K}_p$ designed as a function of the estimate $\bar{\theta}$. For that, one generates trajectories from random initial points x_k and computes a lower bound on the probability that a trajectory starting from a new initial point satisfies Conditions (*i*) and (*ii*) from Problem 2.

V. Computational Experiments

In this section, we present numerical simulations illustrating the use of the results developed in the previous sections to produce a stabilizing controller for an unstable dynamical system learned and represented as a regression tree ensemble. To that end, we consider a modified version of the coupled Van der Pol oscillators model [32]:

$$g(x,u) = \begin{cases} \dot{x}_1 &= x_2 - k_1 x_3 + u_1 \\ \dot{x}_2 &= k_2(1 - x_1^2)x_2 - k_3 x_1 + u_2 \\ \dot{x}_3 &= x_4 - k_4 x_1^3 - k_5 x_1 \\ \dot{x}_4 &= k_6(1 - x_3^2)x_4 - k_7 x_3 + u_3, \end{cases} \tag{21}$$

where we set $k_1 = 1$, $k_2 = k_4 = k_6 = 0.1$, $k_3 = 15$, $k_5 = 20$, and $k_7 = 5$. Here, a_1, k_4 and k_5 are the weights of the coupling terms. We generate $r = 2 \cdot 10^5$ i.i.d. samples of (21) from the set $\mathcal{S} = [-a, a]^n$ with $a = 20$, according to (15), where each entry j of the disturbance vector (w_i) has uniform distribution, i.e., $[w_i]_j \sim U[-0.5, 0.5]$. Then, we use the Adaboost.MRT algorithm [16] to create a regression tree ensemble with $T = 10$ regression trees, each having $\tau_k = \bar{\tau} = 1,024$ partitions, to generate the set of coefficients $\{A_i\}_{i=1}^{\nu}$ for $\nu = T\bar{\tau} = 10,240$. In particular, we assume the matrix B to be known exactly.

To design our stabilizing controller, we choose $\epsilon = 0.01$, $\bar{\theta} = 3$, and apply Theorem 3 using the samples $\{(x_j, y_j)\}_{j=1}^{r_t}$ where $r_t = 0.5r$. We get $\mathbb{P}\{|\hat{\mu}_{\mathcal{T}} - \mu_{\mathcal{T}}| \geq \epsilon\} \leq \delta$ from (17), with $\delta = 4 \cdot 10^{-9}$. Then, it follows that with probability $1-\delta$, we have $\mathbb{P}\{\mathbf{1}(x_l \in \mathcal{T}) = 1\} \geq \hat{\mu}_{\widetilde{\mathcal{T}}} - \epsilon$, where $\hat{\mu}_{\widetilde{\mathcal{T}}}(\theta) - \epsilon = 0.91$. Next, we design a controller solving the LMI conditions presented in (6) from Theorem 1, for which we pick $\kappa = 20$ and obtain $\alpha = 1$, $\beta = 3$, as well as the controller gains $\{K_i\}_{i=1}^{\nu}$ associated with partitions $\{\mathcal{R}_i\}_{i=1}^{\nu}$, achieving $\Delta = -0.67$. Then, from Theorem 2, we obtain the set of initial conditions that will stay inside $\mathcal{S}$ as $\mathcal{X}_0 = \{x : \|x\|_2 < 11.55\}$, and the set that the trajectories converge to as $\mathcal{D} = \{x : \sqrt{x^T P x} \leq 5.2\}$. In Figure 1, we present the envelope for trajectories generated from the $r_q = 10^5$ different initial points when the controller gains computed in this section are applied in closed-loop. It can be seen that all trajectories starting within $\mathcal{X}_0$ converge to $\mathcal{D}$, as desired. After that, we use the validation approach described in Proposition 1, as follows. We pick $\epsilon_{\mathcal{Q}} = 0.01$ and obtain that with probability $1-\delta_{\mathcal{Q}}$, we have $\mathbb{P}\{X_j \in \mathcal{Q}\} \geq \hat{\mu}_{\mathcal{Q}} - \epsilon_{\mathcal{Q}}$, where $\delta_{\mathcal{Q}} = 4 \cdot 10^{-9}$ and $\hat{\mu}_{\mathcal{Q}} - \epsilon_{\mathcal{Q}} = 0.99$. Thus, we can conclude that the proposed controller design method is successful with high probability for the coupled Van der Pol oscillator dynamics.

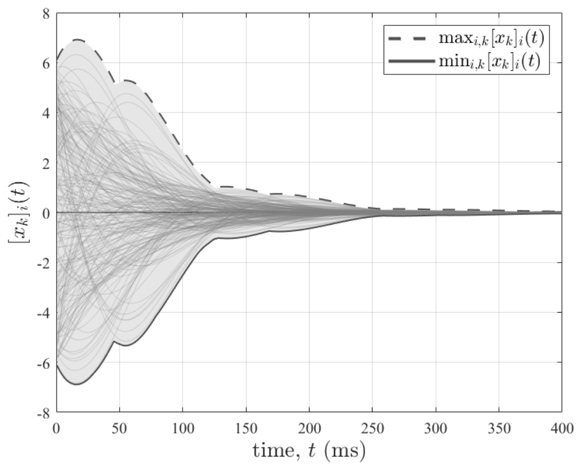

Fig. 1. Samples and envelopes of closed-loop trajectories obtained by applying the proposed controller design over $r_q = 10^5$ different initial points of a model of coupled Van der Pol oscillators.

VI. Conclusion

In this paper, we developed a method to stabilize nonlinear systems modeled via ensembles of regression trees. We proposed an input design based on Lyapunov function analysis to provide a stabilization procedure having a number of LMI's that grows linearly with the number of partitions and does not depend on the state dimension. Finally, we developed probabilistic guarantees relating the controller designed for the ensemble model to its performance on the original system.

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