On The Stability Region of the Layered Packet Erasure Broadcast Channel with Output Feedback

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Abstract—This paper studies the Layered Packet Erasure Broadcast Channel (LPE-BC) with Channel Output Feedback (COF), which is a high-SNR approximation of the fading Gaussian BC, proposed by Tse and Yates in 2012 for the case without COF. This model is also a multi-layer generalization of the Binary Erasure Channel (BEC). In a past work, the Authors derived inner and outer bounds to the *rate region* (set of achievable rates with backlogged arrivals) of the LPE-BC with COF; here, the *arrival region* (set of exogenous arrival rates for which packet arrival queues are stable) for the same model is analyzed. For the case of $K = 2$ users and $Q \geq 1$ layers, the known achievable rate region and the derived arrival region coincide; both strategically employ a network-coding based retransmission protocol. For the case of $Q = 2$ layers, sufficient conditions are given for the achievable arrival region to coincide with the known converse rate region, thus showing that in those cases the optimal rate and arrival regions coincide.

I. INTRODUCTION

In this paper, we study the *stability region* of the Layered Packet Erasure Broadcast Channel (LPE-BC) with Channel Output Feedback (COF), and compare it with the *capacity region*. The capacity region characterizes the largest set of simultaneously achievable message rates that can be reliably transmitted [1]. The capacity region assumes that all users, or nodes, have messages, or packets, to send at all times, that is, that the packet arrival queues are infinitely backlogged. The stability region instead assumes that packets arrive stochastically, and may be queued before transmission. The networked system is called stable if the packet queues are asymptotically finite, with finite packet delays. In [2], the stability region is defined as the closure of the set of all arrival rate vectors that can be stably supported by the network. In the following, *rate region* refers to an achievable message rate region, which can not be larger than the capacity region; similarly, *arrival region* refers to an achievable arrival rate region, which can not be larger than the stability region. While for some networks the capacity and stability regions are equivalent [3], in general the stability region in queueing theory is a subset of the capacity region in information theory [2]. A form of duality exists between the two regions [4], which is somewhat understood for multiple access channels [5], but general conditions under which the two regions coincide for general networks are not known.

The motivation to study the capacity and stability regions of the LPE-BC with COF is as follows. In wireless communications, the Additive White Gaussian Noise fading Broadcast Channel (AWGN-BC) models the downlink communication between one base-station and multiple users. While capacity is known in some settings [1], one notable exception is the fading AWGN-BC where the fading Transmitter Channel State Information (TxCSI) is not available. This model was studied in [6], where the Layered Packet Erasure Broadcast Channel (LPE-BC) model was proposed to approximate the AWGN-BC without TxCSI in the high SNR regime. In the LPE-BC, at each channel use, the base station sends a vector of inputs (or layers of packets) and each receiver receives a random number of layers. The missing layers are said to have been "erased." Once a layer is erased, all the layers with larger indices are also erased. The LPE-BC is a generalization of the Binary Erasure Broadcast Channel (BEC-BC). The BEC-BC has a single layer that is received or erased; the LPE-BC has multiple layers, which may be erased or received in the correlated fashion stated above. Note that COF enables the transmitter to know the number of layers that were erased at each receiver, that is, it becomes causally aware of the TxCSI.

Past Work: The capacity region of the LPE-BC without COF was determined in [6]. In [7], the capacity and stability regions of a 2-user BEC-BC with COF were derived and found to coincide. Several algorithms to achieve these regions were constructed, based on using the COF to determine which packets were received at the unintended receiver only, and then use this information to opportunistically and efficiently network code such packets. In [8], [9], the capacity region for the 3-user BEC-BC, as well as for two types of K-user symmetric and spatially independent BEC-BCs with one-sided fairness constraints, with COF were derived. In our prior work [10], we studied the general LPE-BC with COF and derived inner and outer bounds to the capacity region.

Contributions: In past work, only special cases of the LPE-BC were considered. In this paper, we derive an achievable stability region for LPE-BC with COF for $K = 2$ users and $Q \geq 1$ layers. *Our scheme uses network coding across all layers and all users in case retransmissions are needed.* Our inner bound to the stability region and outer bound to the capacity region are analytically and numerically compared. Conditions are given under which these two regions match for the case of $K = 2$ users and $Q = 2$ layers; hence, for such channels both the capacity region and the arrival region are fully characterized, and they coincide. Our proof techniques here (for the case of any number of layers) differs from that of [7] (for a single layer): we do not rely on a "Markov chain"-argument as [7] but rather on a "concentration to the mean

"-argument.

II. System Model and Main Results

For the rest of this paper, we use capital letters to represent the random variables (apart from rates which are capital R's which are not random, to conform with standard information theoretic notation) and small letters to indicate the fixed number or the expectation of the corresponding random variables.

The LPE-BC consists of one transmitter that communicates to K receivers. At each channel use (*slot*) the transmitter sends Q symbols (*packets, or layers*), where each symbol is from a finite input alphabet $\mathcal{X}$; the input is denoted as $X^Q := (X_1, \ldots, X_Q) \in \mathcal{X}^Q$. The LPE-BC is characterized by the random vector (*channel state*) $\mathbf{N} := (N_1, \ldots, N_K) \in [0 : Q]^K$, where N_k denotes how many layers have been successfully received by user $k \in [K]$. The channel output for user $k \in [K]$ is $Y_k := X^{N_k} = (X_1, \ldots, X_{N_k})$ for $N_k > 0$, that is, layers $(X_{N_k+1}, \ldots, X_Q)$ have been *erased*; if $N_k = 0$ then all layers have been erased and we set $Y_k = \mathsf{e}$ for some constant "erasure" symbol e. The channel state $\mathbf{N}$ is assumed to be independent and identically distributed (i.i.d.) across time slots, that is, the channel is memoryless. The case $Q = 1$ and $\mathcal{X} = \mathrm{GF}(2)$ is the well studied K-user BEC-BC.

A. Capacity Region – backlog arrivals

In this setup, the transmitter has K queues of packets, one per receiver, and all the queues have infinitely many packets. The transmitter must convey $|\mathcal{X}|^{nR_k}$ packets reliably to user $k \in [K]$ in n channel uses. Note that the rate R_k is measured in number of packets per channel use. Let $(W_1, \ldots, W_K)$ be the messages to be sent to the users. With COF the transmitter sends $X_t^Q(W_1, \ldots, W_K, \mathbf{N}^{t-1})$, where $X_t^Q(\cdot)$ is the encoding function at time $t \in [n]$. We assume that all receivers by time $t = n$ know $\mathbf{N}^n$. User k estimates $\widehat{W}_k(Y_k^n, \mathbf{N}^n), k \in [K]$. The probability of error is $P_e^{(n)} = \Pr[\cup_{k \in [K]} \widehat{W}_k \neq W_k]$. The *capacity region* is the convex closure of the set of all message rate-tuples $(R_1, \ldots, R_K) \in \mathbb{R}_+^K$ for which $\lim_{n \to \infty} P_e^{(n)} = 0$.

In our past work [10] we showed that with COF one has:

Theorem 1 (Bounds on the Capacity Region [10]). *For the LPE-BC with COF, let $\mathcal{C}^{\mathrm{out}}$ and $\mathcal{C}^{\mathrm{in}}$ denote outer and inner bounds to the capacity region, respectively. We have*

$$\mathcal{C}^{\mathrm{out}} = \Big\{ (R_1, \ldots, R_K) \in \mathbb{R}_+^K : \sum_{k \in [K]} \omega_k R_k \leq \quad (1)$$

$$\sum_{q \in [Q]} \max_{k \in [K]} \big(\omega_{\pi(k)} \Pr[\max(N_j : j \in \{\pi(k), \ldots, \pi(K)\}) \geq q] \big),$$

for all $(\omega_1, \ldots, \omega_K) \in \mathbb{R}_+^K$ and all permutations π of $[K]$ $\Big\}$,

$$\mathcal{C}^{in} = \{ (R_1, R_2) \in \mathbb{R}_+^2 : t^{(\mathrm{unc})} + t^{(\mathrm{NC})} \leq t$$

$$\text{for some } t \geq 0, k_{u,q} \geq 0, q \in [Q], u \in [2] \}, \quad (2)$$

$$R_u := \Big(\sum_{q \in [Q]} k_{u,q} \Big)/t, \forall u \in [2], \textit{ (rate)},$$

$$t^{(\mathrm{unc})} := \max_{q \in [Q]} \big(t_q^{(\mathrm{unc})} \big), \textit{ (duration of Phase1)},$$

$$t_q^{(\mathrm{unc})} := \frac{k_{1,q} + k_{2,q}}{\Pr[\max(N_1, N_2) \geq q]}, \forall q \in [Q],$$

$$t^{(\mathrm{NC})} := \max_{u \in [2]} \Big(\frac{k_u^{(\mathrm{rem})}}{\mathbb{E}[N_u]} \Big), \textit{ (duration of Phase2)},$$

$$k_u^{(\mathrm{rem})} := \Big[\sum_{q \in [Q]} k_{u,q}^{(\mathrm{rem})} - (t^{(\mathrm{unc})} - t_q^{(\mathrm{unc})}) \Pr[N_u \geq q] \Big]^+, \forall u \in [2].$$

$$k_{u,q}^{(\mathrm{rem})} := k_{u,q} \Big(1 - \frac{\Pr[N_u \geq q]}{\Pr[\max(N_1, N_2) \geq q]} \Big), \begin{array}{l} \forall q \in [Q], \\ \forall u \in [2]. \end{array}$$

Note: the outer bound $\mathcal{C}^{\mathrm{out}}$ in (1) is for any number of users, while the inner bound $\mathcal{C}^{\mathrm{in}}$ in (2) is for $K = 2$ users only. Extension of the scheme that attains $\mathcal{C}^{\mathrm{in}}$ to more than $K = 2$ users requires being able to track which subset of non-intended users has received a certain packet; this is the same stumbling block as in the single-layer case in [8] for $K \geq 4$.

B. Stability Region – random arrivals

In this setup, the transmitter maintains K packet queues, one per receiver, and exogenous packets arrive randomly at each queue. Let $A_{u,t}$ be the packets that arrived at the beginning of slot $t \in \mathbb{N}$ and are intended for user $u \in [K]$. Let $\mathbf{A}_t := (A_{1,t}, \ldots, A_{K,t})$ be the vector of exogenous arrivals, assumed to be i.i.d. over time, with average arrival rates $\lambda_u := \mathbb{E}[|A_{u,t}|], u \in [K]$, Let $Q_{u,t}$ be the queue that contains the packets that still need to be transmitted to user $u \in [K]$ at time $t \in \mathbb{N}$ (i.e., it includes the exogenous packets $A_{u,t}$, as well as those packets that were not yet delivered to user u at previous times, as described next). With COF, the transmitter sends $X_t^Q(\mathbf{Q}_t, \mathbf{N}^{t-1})$, for some encoding function $X_t^Q(\cdot)$. User $u \in [K]$ applies decoding function $D_{u,t}(Y_u^t, \mathbf{N}^t)$ that returns the packets that could be retrieved *error-free* by using all channel outputs and all channel states available to it up to time $t \in \mathbb{N}$. A successfully received packet is removed from its queue; this can be tracked at the transmitter thanks to COF. The evolution of the queue length over time is given by

$$|Q_{u,t}| = \big[|Q_{u,t-1}| + |A_{u,t}| - |D_{u,t}| \big]^+, \ t \in \mathbb{N}, u \in [K], \quad (3)$$

where $| \cdot |$ denotes the number of packets and $[v]^+$ denotes the positive part of v (i.e., $[v]^+ = \max\{0, v\}$). The *stability region* is the convex closure of the set of all arrival rate-tuples $(\lambda_1, \ldots, \lambda_K) \in \mathbb{R}_+^K$ for which the process of queue lengths $\{(|Q_{1,t}|, \ldots, |Q_{K,t}|)\}_{t \in \mathbb{N}}$ is stable[1].

In Appendix we shall show:

Theorem 2 (Achievable Stability Region (novel result)). *For the LPE-BC with COF and $K = 2$ users, the following region is an inner bound to the stability region*

$$\mathcal{S}^{\mathrm{in}} := \Big\{ (\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \lambda_k := \sum_{q \in [Q]} \lambda_{k,q}, k \in [2], \quad (4)$$

$$\frac{\lambda_{1,q} + \lambda_{2u,q}}{\Pr[\max(N_1, N_2) \geq q]} < 1, \forall q \in [Q],$$

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[1]From [7]: The process $\{\mathbf{X}_t\}_{t \in \mathbb{N}}$, where $\mathbf{X}_t = (X_{1,t}, \ldots, X_{K,t})$, is *stable* if the following holds at all points of continuity of some cumulative distribution function $F(\mathbf{x})$: $\lim_{t \to \infty} \Pr[\mathbf{X}_t \leq \mathbf{x}] = F(\mathbf{x})$ and $\lim_{\min(x_1, \ldots, x_K) \to \infty} F(\mathbf{x}) = 1$, where $\mathbf{x} := (x_1, \ldots, x_K)$ and $\mathbf{X}_t \leq \mathbf{x}$ means coordinate-wise inequalities. The process $\{\mathbf{X}_t\}_{t \in \mathbb{N}}$ is *substable* if $\lim_{\min(x_1, \ldots, x_K) \to \infty} \liminf_{t \to \infty} \Pr[\mathbf{X}_t \leq \mathbf{x}] = 1$. If the processes $\{X_{i,t}\}_{t \in \mathbb{N}}$ are substable for all $i \in [K]$, then the process $\{\mathbf{X}_t\}_{t \in \mathbb{N}}$ is substable. In our case, $\{\mathbf{X}_t\}_{t \in \mathbb{N}}$ will represent the process of queue lengths.

$$\sum_{q\in[Q]} \lambda_{u,q} + \lambda_{\bar{u},q} \frac{\Pr[N_u \geq q]}{\Pr[\max(N_1,N_2) \geq q]} < \mathbb{E}[N_u], \forall u \in [2],$$

$$(u,\bar{u}) \in [2]^2 : u \neq \bar{u}, \text{ for some } \lambda_{u,q} \geq 0, u \in [2], q \in [Q]\}.$$

Note: extension of $\mathcal{S}^{\text{in}}$ to more than two users incurs the same problem as discussed for $\mathcal{C}^{\text{in}}$ earlier. The region in (4) recovers the result in [7] when $Q = 1$.

C. Optimality

We have $\mathcal{C}^{\text{in}} \subseteq \mathcal{C} \subseteq \mathcal{C}^{\text{out}}$ from Theorem 1, and $\mathcal{S}^{\text{in}} \subseteq \mathcal{S}$ from Theorem 2, where $\mathcal{S}$ is the stability region and $\mathcal{C}$ the capacity region. We also know [2] that $\mathcal{S} \subseteq \mathcal{C}$. It can be easily shown that $\mathcal{C}^{\text{in}}$ can be written in the same form as $\mathcal{S}^{\text{in}}$, by replacing message rates with average arrival rates; the proof is not reported here for sake of space. Next, we find conditions under which $\mathcal{S}^{\text{in}} = \mathcal{C}^{\text{out}}$, for $Q = 2$ layers and $K = 2$ users, thus showing that under such conditions one has $\mathcal{S} = \mathcal{C}$. This result confirms the similarity between the capacity and stability regions already observed in [5], [7].

Let

$$A := \max_{q\in[2]} \left(\frac{\Pr[\max(N_1,N_2) \geq q]}{\Pr[N_1 \geq q]}, \right), \quad (5a)$$

$$B := \min_{q\in[2]} \left(\frac{\Pr[\max(N_1,N_2) \geq q]}{\Pr[N_1 \geq q]}, \right), \quad (5b)$$

$$C := \max_{q\in[2]} \left(\frac{\Pr[N_2 \geq q]}{\Pr[\max(N_1,N_2) \geq q]}, \right), \quad (5c)$$

$$D := \min_{q\in[2]} \left(\frac{\Pr[N_2 \geq q]}{\Pr[\max(N_1,N_2) \geq q]}, \right), \quad (5d)$$

where clearly $A \geq B \geq 1 \geq C \geq D \geq 0$. Rewrite the outer bound in (1) as

$$R_1 + \frac{R_2}{A} \leq \mathbb{E}[N_1], \quad (6a)$$

$$BR_1 + R_2 \leq \mathbb{E}[\max(N_1,N_2)], \quad (6b)$$

$$R_1 + \frac{R_2}{C} \leq \mathbb{E}[\max(N_1,N_2)], \quad (6c)$$

$$DR_1 + R_2 \leq \mathbb{E}[N_2]. \quad (6d)$$

We give the sufficient optimality conditions as follows.

Theorem 3. *The stability region inner bound in Theorem 2 coincides with capacity region outer bound in Theorem 1 for the LPE-BC with COF for the case of $K = 2$ users and $Q = 2$ layers when the following two conditions are verified: (C1) either bound* (6b) *or bound* (6c) *is redundant, and (C2) either $\frac{\Pr[N_1 \geq 2]}{\Pr[N_1 \geq 1]} \geq \frac{\Pr[\max(N_1,N_2) \geq 2]}{\Pr[\max(N_1,N_2) \geq 1]} \geq \frac{\Pr[N_2 \geq 2]}{\Pr[N_2 \geq 1]}$ or $\frac{\Pr[N_2 \geq 2]}{\Pr[N_2 \geq 1]} \geq \frac{\Pr[\max(N_1,N_2) \geq 2]}{\Pr[\max(N_1,N_2) \geq 1]} \geq \frac{\Pr[N_1 \geq 2]}{\Pr[N_1 \geq 1]}$.*

Intuition: The conditions in Theorem 3 may be interpreted as follows. User $u \in [2]$ is more likely to receive a packet from layer 1 than user $\bar{u} \in [2]$ where $\bar{u} \neq u$, while at the same time user $\bar{u}$ is more likely to receive a packet from layer 2 than user u. It is fairly straightforward to see that when $\frac{\Pr[N_1 \geq 2]}{\Pr[N_1 \geq 1]} = \frac{\Pr[N_2 \geq 2]}{\Pr[N_2 \geq 1]} = \frac{\Pr[\max(N_1,N_2) \geq 2]}{\Pr[\max(N_1,N_2) \geq 1]}$, both (6b) and (6c) are redundant and the outer bound becomes identical to the inner bound; under this condition we obtain the capacity region $\mathcal{C} = \{(R_1, R_2) \in \mathbb{R}^2_+ :$

TABLE I: Joint PMF $\Pr[(N_1, N_2) = (i,j)]$.

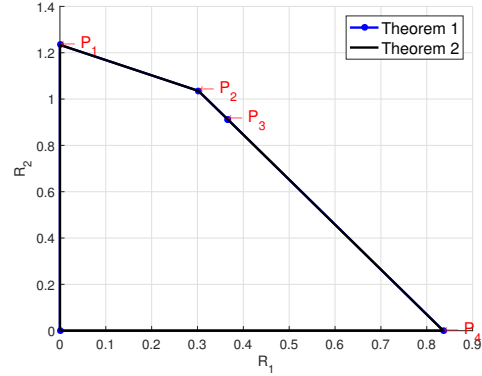|	$j = 0$	$j = 1$	$j = 2$	$\Pr[N_1 = i]$
$i = 0$	0.088	0.178	0.264	0.530
$i = 1$	0.011	0.018	0.075	0.104
$i = 2$	0.131	0.110	0.125	0.366
$\Pr[N_2 = j]$	0.230	0.306	0.464	

Fig. 1: Capacity and stability region for the channel in Table I.

$\max\left(\frac{R_1}{\mathbb{E}[N_1]} + \frac{R_2}{\mathbb{E}[\max(N_1,N_2)]}, \frac{R_2}{\mathbb{E}[\max(N_1,N_2)]} + \frac{R_2}{\mathbb{E}[N_2]} \right) \leq 1\}$ that has the same form as the capacity region derived in [7] for the single layer BEC-BC with COF; in other words, in this special case, the two-layer LPE-BC behaves as the one-layer BEC-BC where $1 - \varepsilon_1 = \Pr[N_1 \geq 1] + \Pr[N_1 \geq 2] = \mathbb{E}[N_1]$, $1 - \varepsilon_2 = \mathbb{E}[N_2]$, $1 - \varepsilon_{12} = \mathbb{E}[\max(N_1,N_2)]$ correspond to the notation in [7].

D. Numerical Evaluations

We conclude this section by giving an example where the achievable stability region in Theorem 2 coincides with the outer bound of capacity region in Theorem 1, i.e., the conditions in Theorem 3 are satisfied.

Consider the channel in Table I, in which both users have a more reliable look at layer 2 than at layer 1; here the channel states are correlated at each channel use. The outer bound in Theorem 1 is the convex-hull of the following rate pairs: $P_1 = (0, 1.234)$, $P_2 = (0.302, 1.035)$, $P_3 = (0.366, 0.912)$, $P_4 = (0.836, 0)$. If all four bounds in (6) were active, the outer bound would be a convex hull of at most 6 corner points (including the point $(0,0)$, two corner points on the R_1 and R_2 axes, and 3 other non-trivial corner points). Here, we only have two non-trivial corner points, points P_2, P_3. We know the bound in (6a) and the one in (6d) are always active. Hence, either the bound in (6b) or the one in (6c) is redundant. Here is a case where either (6b) or (6c) is redundant. For this channel, $\frac{\Pr[\max(N_1,N_2) \geq 1]}{\Pr[N_1 \geq 1]} > \frac{\Pr[\max(N_1,N_2) \geq 2]}{\Pr[N_1 \geq 2]}$, $\frac{\Pr[N_2 \geq 1]}{\Pr[\max(N_1,N_2) \geq 1]} > \frac{\Pr[N_2 \geq 2]}{\Pr[\max(N_1,N_2) \geq 2]}$ and $A = 1.940$, $B = 1.926$, $C = 0.844$, $D = 0.658$ in (5). This is an example where the erasures are correlated and for which we obtain the optimal capacity and stability regions.

III. CONCLUSIONS

This paper analyzed the stability region of the LPE-BC with COF. The LPE-BC extends the classical (single-layer) BEC-BC and approximates the fading AWGN-BC at high SNR. Our achievable stability region uses network coded retransmissions

of the packets received by the non-intended user only (a key element also for the single-layer binary erasure BC with COF) and across all layers. Conditions under which the obtained stability region inner bound coincides with the capacity region outer bound are given, thus establishing optimality. Future work includes determining a set of conditions under which the proposed scheme is optimal, extending the analysis to more than two users, and ultimately deriving constant gap approximations to the capacity of the fading AWGN-BC without TxCSIT but with COF.

APPENDIX

A. Protocol Description

We consider a protocol based on network coded retransmissions. Based on COF, the transmitter decides which packet to send next, and knows which packets have been successfully received by each user. The assumption of global state knowledge at all terminals allows users to keep track of which packets have been sent and which have been successfully received. Moreover, when a network coded packet is sent, we assume that the code (i.e., set of coefficients used for a linear combination) has been agreed upon in advance and is known to all terminals; every terminal knows the codebook.

The protocol works in epochs. During each epoch, a certain (random) number of packets have to be successfully delivered to the users by employing the coding scheme for the 'backlogged' case in Theorem 1. The beginning of a new epoch is a *renewal event* for the system. Epoch $m \in \mathbb{N}$ starts at time $T[m]$ and ends at time $T[m + 1]$. Denote $L[m] := T[m + 1] - T[m]$ as the number of time slots in epoch m, where each packet is transmitted in one slot. Epoch $m + 1$ starts at time $T[m + 1]$ (right after the end of epoch m), employing the same procedure as in epoch m. At the beginning of epoch m, $K_u[m] := \sum_{t \in L[m-1]} |A_{u,t}|$ new exogenous packets need to be transmitted to user $u \in [2]$. Epoch m ends when all $K_u[m]$ packets have been delivered successfully to user u, thus $L[m]$ are random.

The transmitter maintains $\mathsf{Q} + 2$ queues, denoted by $Q_{01}, Q_{02}, \cdots, Q_{0\mathsf{Q}}, Q_1, Q_2$. For each user u, the average arrival rate $\lambda_u = \mathbb{E}[A_{u,t}]$ is expressed as $\lambda_u = \sum_{q \in [\mathsf{Q}]} \lambda_{u,q} = \sum_{q \in [\mathsf{Q}]} \mathbb{E}[A_{u,q;t}]$ where $A_{u,q;t}$ is the number of exogenous packets assigned for user u on layer q at slot t, for some $\lambda_{u,q} \geq 0$ and $q \in [\mathsf{Q}]$. At the beginning of epoch m, no packet is assigned to queue Q_u (i.e., there are no overheard packets at the start of an epoch as the previous epoch ends after all packets are delivered), and each of the $K_u[m]$ packets is assigned independently at random with probability $\lambda_{u,q}/\lambda_u$ to queue Q_{0q}. Let $K_{u,q}[m]$ be the (random) number of packets that are assigned to queue Q_{0q}, and destined to user u.

The protocol works as follows. All packets, whether they are destined to user 1 or 2, are transmitted on a first-come-first-served policy from Q_{0q}. If a packet transmitted from Q_{0q} is successfully received by at least one of the users, the packet leaves Q_{0q}; otherwise, it goes back to Q_{0q}; if the packet from Q_{0q} is erased at the intended user u and received by the other user, that packet is placed in Q_u. At any time slot t:

- all queues are empty: this epoch ends;

- all Q_{0q}'s are non-empty: a packet from Q_{0q} is transmitted on layer q.
- some Q_{0q}'s are empty, and all Q_u's are non-empty: if $\exists q, p \in [\mathsf{Q}] : p \neq q, Q_{0q} = \emptyset, Q_{0p} \neq \emptyset$ and $Q_u \neq \emptyset, \forall u \in [2]$, then we transmit a network coded packet (by using linear network coding with packets from Q_1 and Q_2) on layer q, and a packet from Q_{0p} on layer p;
- some Q_{0q}'s are empty, and some Q_u's are empty: if $\exists q, p \in [\mathsf{Q}] : p \neq q, Q_{0q} = \emptyset, Q_{0p} \neq \emptyset$ and $\exists u \neq \bar{u} : Q_u \neq \emptyset, Q_{\bar{u}} = \emptyset$, then we transmit an uncoded packet from Q_u on layer q, and an uncoded packet from Q_{0p} on layer p;
- all Q_{0q}'s are empty, and some Q_u's are empty: if $Q_{0q} = \emptyset, \forall q \in [\mathsf{Q}]$ and either $Q_1 \neq \emptyset$ or $Q_2 \neq \emptyset$, then we transmit uncoded packets from all the non-empty Q_u's on all layers.

B. Stability Analysis

In this section, we use Lyapunov drift analysis and show that if the arrival rates are within the region $\mathcal{S}^{\text{in}}$, then the Markov chain $\{K_{u,q}[m], u \in [2], q \in [\mathsf{Q}]\}_{m \in \mathbb{N}}$ is ergodic. The ergodicity implies that there exists a stationary distribution, which implies that the stochastic process $\{|Q_{01;t}|, |Q_{02;t}|, \ldots, |Q_{0\mathsf{Q};t}|, |Q_{1;t}|, |Q_{2;t}|\}_{t \in \mathbb{N}}$ characterizing the number of packets in queues is stable. Let us define a Lyapunov function

$$
\begin{aligned}
v(\mathbf{k}) := \max \Bigg\{ & \frac{k_{1,q} + k_{2,q}}{\Pr[\max(N_1, N_2) \geq q]}, \forall q \in [\mathsf{Q}], \\
& \sum_{q \in [\mathsf{Q}]} \frac{k_{1,q}}{\mathbb{E}[N_1]} + \sum_{q \in [\mathsf{Q}]} \frac{k_{2,q}}{\mathbb{E}[N_1]} \frac{\Pr[N_1 \geq q]}{\Pr[\max(N_1, N_2) \geq q]}, \\
& \sum_{q \in [\mathsf{Q}]} \frac{k_{2,q}}{\mathbb{E}[N_2]} + \sum_{q \in [\mathsf{Q}]} \frac{k_{1,q}}{\mathbb{E}[N_2]} \frac{\Pr[N_2 \geq q]}{\Pr[\max(N_1, N_2) \geq q]} \Bigg\},
\end{aligned} \quad (7)
$$

where $\mathbf{k}$ is the vector containing all the $k_{u,q}$'s, $k_{u,q}$ is the number of packets for user u in queue Q_{0q} at the beginning of epoch m. Let $|\mathbf{k}| := \sqrt{(\sum_{q \in [\mathsf{Q}]} k_{1,q})^2 + (\sum_{q \in [\mathsf{Q}]} k_{2,q})^2}$, $\mathbf{K}[m] := (K_{u,q}[m], \forall u, \forall q), m \in \mathbb{N}$. From [7, Theorem 6], to show the ergodicity of the Markov chain we need that

$$
\mathbb{E}\left[v(\mathbf{K}[m + 1]) \big| \mathbf{K}[m] = \mathbf{k}\right] < \infty \quad (8)
$$

holds for $\mathbf{k}$ inside a bounded region, and for some $\epsilon > 0$

$$
\mathbb{E}\left[v(\mathbf{K}[m + 1]) \big| \mathbf{K}[m] = \mathbf{k}\right] \leq (1 - \epsilon)v(\mathbf{k}) \quad (9)
$$

holds for $\mathbf{k}$ outside a bounded region for all $m \in \mathbb{N}$.

Next, we characterize $v(\mathbf{K}[m+1])$. Denote by $\hat{G}_{u,q}(\mathbf{k})$ the number of packets destined to receiver u on layer q that arrived to the system during epoch m, given that there are $\mathbf{k}$ packets at the beginning of epoch m. Let $\hat{\mathbf{G}}(\mathbf{k}) := (\hat{G}_{u,q}(\mathbf{k}), \forall u, \forall q)$. Considering the definitions and the operation of the scheme, we obtain

$$
K_{u,q}[m + 1] = \hat{G}_{u,q}(\mathbf{k}) = \sum_{l \in L[m]} |A_{u,q;l}|, u \in [2], q \in [\mathsf{Q}],
$$

$$
v(\mathbf{K}[m + 1]) = v(\hat{\mathbf{G}}(\mathbf{k})).
$$

Now that we showed $v(\mathbf{K}[m+1]) = v(\hat{\mathbf{G}}(\mathbf{k}))$, our next step is to characterize $\frac{\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))]}{v(\mathbf{k})}$. However, we should first define some important limits.

We showed in [10] that $L[m]$, the time needed to successfully complete the transmission of $\mathbf{k}$ packets, has a sharp concentration to its mean value $\mathbb{E}[L[m]] = v(\mathbf{k})$ when $|\mathbf{k}| \to \infty$. Since we assume that the number of packets to be transmitted is large enough at each epoch, also $L[m]$ has a sharp concentration at $v(\mathbf{k})$, i.e., for every $\varepsilon > 0$, we have

$$\lim_{m \to \infty} \Pr[\cup_{i=m}^{\infty} \{|L[m] - v(\mathbf{k})| > \varepsilon\}] = 0. \quad (10)$$

Based on [11, Proposition 1.1], we have $L[m] \xrightarrow{\text{a.s.}} v(\mathbf{k})$, as $m \to \infty$. Thus,

$$\lim_{|\mathbf{k}| \to \infty} \frac{L[m]}{v(\mathbf{k})} = 1, \qquad \lim_{|\mathbf{k}| \to \infty} \frac{\mathbb{E}[L[m]]}{v(\mathbf{k})} = 1. \quad (11)$$

As $|\mathbf{k}| \to \infty, L[m] \to \infty$, using the strong law of large numbers, we have $\lim_{|\mathbf{k}| \to \infty} \frac{\sum_{l \in L[m]} |A_{u,q;l}|}{L[m]} = \lambda_{u,q}$.

Now we return to our original goal of characterizing $\frac{\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))]}{v(\mathbf{k})}$. By Wald's equation [12, Theorem 12],

$$\mathbb{E}[\hat{G}_{u,q}(\mathbf{k})] = \mathbb{E}[\sum_{l \in L[m]} |A_{u,q;l}|] = \lambda_{u,q} \mathbb{E}[L[m]] \quad (12)$$

and considering (11), we have

$$\lim_{|\mathbf{k}| \to \infty} \frac{\hat{G}_{u,q}(\mathbf{k})}{v(\mathbf{k})} = \lim_{|\mathbf{k}| \to \infty} \frac{\hat{G}_{u,q}(\mathbf{k})}{L[m]} \frac{L[m]}{v(\mathbf{k})} = \lambda_{u,q}, \quad (13)$$

$$\lim_{|\mathbf{k}| \to \infty} \frac{\mathbb{E}[\hat{G}_{u,q}(\mathbf{k})]}{v(\mathbf{k})} = \lim_{|\mathbf{k}| \to \infty} \frac{\lambda_{u,q} \mathbb{E}[L[m]]}{v(\mathbf{k})} = \lambda_{u,q}. \quad (14)$$

According to [11, Corollary 4.1.], (14) implies that the sequence $\left\{ \frac{\hat{G}_{u,q}(\mathbf{k})}{v(\mathbf{k})}, u \in [2], q \in [\mathsf{Q}] \right\}$ is uniformly integrable. Moreover, $\frac{v(\hat{\mathbf{G}}(\mathbf{k}))}{v(\mathbf{k})}$ is uniformly integrable since the sum and the maximum of uniformly integrable functions are also uniformly integrable. Let $\boldsymbol{\lambda} := (\lambda_{u,q}, \forall u \in [2], q \in [\mathsf{Q}])$. By (13), we can write

$$\lim_{|\mathbf{k}| \to \infty} \frac{v(\hat{\mathbf{G}}(\mathbf{k}))}{v(\mathbf{k})} = \lim_{|\mathbf{k}| \to \infty} \max \Big\{$$

$$\frac{\hat{G}_{1,q}(\mathbf{k}) + \hat{G}_{2,q}(\mathbf{k})}{\Pr[\max(N_1, N_2) \geq q] v(\mathbf{k})}, \forall q \in [\mathsf{Q}], \sum_{q \in [\mathsf{Q}]} \frac{\hat{G}_{1,q}(\mathbf{k})}{\mathbb{E}[N_1] v(\mathbf{k})} +$$

$$\sum_{q \in [\mathsf{Q}]} \frac{\hat{G}_{2,q}(\mathbf{k})}{\mathbb{E}[N_1] v(\mathbf{k})} \frac{\Pr[N_1 \geq q]}{\Pr[\max(N_1, N_2) \geq q]}, \sum_{q \in [\mathsf{Q}]} \frac{\hat{G}_{2,q}(\mathbf{k})}{\mathbb{E}[N_2] v(\mathbf{k})}$$

$$+ \sum_{q \in [\mathsf{Q}]} \frac{\hat{G}_{1,q}(\mathbf{k})}{\mathbb{E}[N_2] v(\mathbf{k})} \frac{\Pr[N_2 \geq q]}{\Pr[\max(N_1, N_2) \geq q]} \Big\} = v(\boldsymbol{\lambda})$$

and since $\frac{v(\hat{\mathbf{G}}(\mathbf{k}))}{v(\mathbf{k})}$ is uniformly integrable, we have

$$\lim_{|\mathbf{k}| \to \infty} \frac{\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))]}{v(\mathbf{k})} = v(\boldsymbol{\lambda}).$$

Next, for some $\epsilon > 0$, pick $k(\epsilon)$ large enough such that $|\mathbf{k}| > k(\epsilon)$; pick $\boldsymbol{\lambda}$ in $\mathcal{S}^{\text{in}}$, so that $v(\boldsymbol{\lambda}) \leq 1 - \epsilon$. As a result,

$$\frac{\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))]}{v(\mathbf{k})} \leq v(\boldsymbol{\lambda}) + \frac{\epsilon}{2} \leq 1 - \epsilon + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2}. \quad (15)$$

The inequality in (15) shows that the condition in (9) is satisfied. Let us now focus on the condition in (8), and characterize $\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))]$. It is quite straightforward to show that $\mathbb{E}[L[m]] < \infty$ when $|\mathbf{k}| \leq k(\epsilon)$. By (12), we have $\mathbb{E}[\hat{G}_{u,q}(\mathbf{k})] < \infty$. This indicates that the sequence $\left\{ \hat{G}_{u,q}(\mathbf{k}), u \in [2], q \in [\mathsf{Q}] \right\}$ is uniformly integrable by [11, Corollary 4.1], and $v(\hat{\mathbf{G}}(\mathbf{k}))$ is uniformly integrable since the sum and the maximum of uniformly integrable functions are also uniformly integrable. Thus, $\mathbb{E}[v(\hat{\mathbf{G}}(\mathbf{k}))] < \infty$, and this concludes that the condition in (8) holds.

Now that we showed that both conditions in (8) and (9) are satisfied, we can conclude that the Markov Chain $\{K_{u,q}[m], u \in [2], q \in [\mathsf{Q}]\}_{m \in \mathbb{N}}$ is geometrically ergodic by following [7, Theorem 6]. Also, $\{|Q_{01;t}|, |Q_{02;t}|, \ldots, |Q_{0\mathsf{Q};t}|, |Q_{1;t}|, |Q_{2;t}|\}_{t \in \mathbb{N}}$ is regenerative concerning the renewal process characterizing the time needed for successive returns of the process $\{(K_{u,q}[m], u \in [2], q \in [\mathsf{Q}])\}_{m \in \mathbb{N}}$ to the all-zero state. The renewal process is nonlattice and the regenerative process is right-continuous and has left-hand limits. This implies that there exists a distribution function $F(\mathbf{x})$ satisfying the conditions in definition such that $\{|Q_{01;t}|, |Q_{02;t}|, \ldots, |Q_{0\mathsf{Q};t}|, |Q_{1;t}|, |Q_{2;t}|\}_{t \in \mathbb{N}}$ converges in distribution to it by [12, Theorem 20]. Finally, we conclude that if the arrival rates are in the interior of the region $\mathcal{S}^{\text{in}}$, then the stochastic process $\{|Q_{01;t}|, |Q_{02;t}|, \ldots, |Q_{0\mathsf{Q};t}|, |Q_{1;t}|, |Q_{2;t}|\}_{t \in \mathbb{N}}$ representing the length of queues is stable. This concludes the proof.

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