More nonabelian mirrors and some two-dimensional dualities

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In this paper we extend the nonabelian mirror proposal of two of the authors from two-dimensional gauge theories with connected gauge groups to the case of $O(k)$ gauge groups with discrete theta angles. We check our proposed extension by counting and comparing vacua in mirrors to known dual two-dimensional $(S)O(k)$ gauge theories. The mirrors in question are Landau-Ginzburg orbifolds, and for mirrors to $O(k)$ gauge theories, the critical loci of the mirror superpotential often intersect fixed-point loci, so that to count vacua, one must take into account twisted sector contributions. This is a technical novelty relative to mirrors of gauge theories with connected gauge groups, for which critical loci do not intersect fixed-point loci and so no orbifold twisted sector contributions are pertinent. The vacuum computations turn out to be a rather intricate test of the proposed mirrors, in particular as untwisted sector states in the mirror to one theory are often exchanged with twisted sector states in the mirror to the dual. In cases with nontrivial IR limits, we also check that central charges computed from the Landau-Ginzburg mirrors match those expected for the IR SCFTs.

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1 Introduction

In the paper [1], two of the authors proposed a description of mirrors to nonabelian two-dimensional gauge theories, extending work on mirrors to abelian two-dimensional gauge theories in [2]. The paper [1] only considered mirrors to gauge theories with connected gauge groups: $SO(k)$ but not $O(k)$, for example. The later work [3] also only considered mirrors of theories with connected gauge groups. The purpose of this paper is to remedy this omission by extending the nonabelian mirror proposal of [1] to mirrors to gauge theories with non-connected gauge group $O(k)$, which we will check by comparing mirrors to dual gauge theories described in [4]. The dualities described there relate, for example, theories with gauge group $SO(k)$ to theories with gauge group $O(k)$ with various mod two theta angles and $\mathbb{Z}_2$ orbifold actions, denoted $O_{\pm}(k)$ in [4], and so by studying the mirrors to those duals we can get some nontrivial tests of our extension of the nonabelian mirror proposal of [1].

The dual gauge theories discussed in [4] are Seiberg dual: equivalent in the IR, but not necessarily in the UV. We will compute mirrors to gauge theories on each side of the dualities in [4], and compare the number of vacua in the mirrors to either side. This is a rather intricate computation, and we will see that in every case, our (extended) nonabelian mirror construction correctly duplicates both the number of vacua of the original gauge theory as well as the number of vacua of the dual.

On a more technical level, this paper also extends the nonabelian mirror proposal of [1] in another way. The nonabelian mirrors constructed are Landau-Ginzburg orbifolds, but for mirrors to theories with connected gauge groups, the orbifold group acts freely on the non-excluded critical loci, so that there are no twisted sector contributions to low-energy analyses. By contrast, mirrors to $O_{\pm}(k)$ theories will often involve orbifolds whose fixed point locus does intersect the critical locus, and so it will be necessary to consider twisted sector contributions. Furthermore, the mirrors to the dualities of [4] will often relate untwisted sector vacua on one side of the duality, to a combination of untwisted and twisted sector vacua on the other side of the duality. Global symmetries in one theory become quantum symmetries [5] in the other. This is part of the reason why we described the vacuum counting and comparisons as a rather intricate test of the nonabelian mirror proposal: to get the number of vacua right is a very sensitive test of whether we have utilized the correct orbifold in the mirror theory.

We begin in section 2 by reviewing the nonabelian mirrors proposal, and describing its extension to gauge theories with gauge groups $O_{\pm}(k)$. We also very briefly review the notion of regularity, defined in [4], as the dualities we will test later relate regular gauge theories in that sense, and we include a table (taken from [4]) listing the number of vacua expected in each (regular) gauge theory pertinent to the dualities explored here. Later in this paper we will see mirror computations reproducing every entry of that table.

In section 3, we then test the dualities described in [4] relating SO and O_+ gauge theories,

by computing the number of vacua in the mirrors to the gauge theories on either side, and verifying that the number of vacua in the mirrors match on either side of the duality, and also match the number of vacua in the original gauge theory. (As the original gauge theories are related in the IR, so too do we expect the mirrors to match in the IR.) In the mirrors to the $O_\pm$ gauge theories, the fixed-point locus will often intersect the critical locus of the superpotential, so that there can be twisted sector contributions to the vacua, resulting in a rather intricate computation. To assist the reader, rather than just giving general arguments, we also examine in detail a prototypical special case: the mirror to the duality $O_+(2) \leftrightarrow SO(2)$. It is our hope that the details in this special case are easier to follow explicitly, and so we include it to improve readability of the paper.

In addition to counting vacua, for cases in which the theory is expected to flow to a nontrivial SCFT, we also check that the central charge one computes from the Landau-Ginzburg mirror matches that one expects for the IR SCFT. That central charge comparison applies to a variety of cases, so we only describe it once.

In section 4, we perform the analogous computations and verifications for the gauge dualities in [4] relating O_- gauge theories to other O_- gauge theories. In each case, we compute mirrors to the gauge theories on each side of the duality, and verify that the number of vacua in the mirror matches the number of vacua of the original gauge theory as well as in the mirror to the dual gauge theory. Here also, the fixed-point locus of the Landau-Ginzburg orbifold of the mirror can intersect the critical locus, and so again this results in an intricate intermixing between untwisted and twisted sector contributions.

In sections 5, 6 we check some additional claims about gauge theories in [4] using the nonabelian mirror construction, such as supersymmetry breaking in cases with too few vectors, and dualities between gauge theories with symplectic gauge groups. These computations do not specifically involve $O_\pm$ mirrors, and for the most part can be performed using just the original nonabelian mirror construction for connected gauge groups [1]. We include them here as this seems the natural place to discuss mirrors of these other results from [4].

Finally, in appendix A we briefly make some observations about mirrors to chirals rings.

2 Proposal

2.1 Basics

In [1], it was proposed that the mirror to a (2,2) supersymmetric GLSM with connected gauge group G, with matter in some representation $\mathcal{R}$ and with twisted masses $\tilde{m}_i$, is a Weyl-group-orbifold of a Landau-Ginzburg model with fields

- $(\dim \mathcal{R})$ chiral superfields Y_i, each of periodicity $2\pi i$, mirror to the matter fields,
- $(\dim G - \mathrm{rank}\, G)$ chiral superfields $X_{\tilde{\mu}}$,
- $(\mathrm{rank}\, G)$ chiral superfields σ_a,

and a superpotential

$$\begin{aligned} W &= \sum_{a=1}^{r} \sigma_a \left(\sum_{i=1}^{N} \rho_i^a Y_i - \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) \\ &\quad - \sum_{i=1}^{N} \tilde{m}_i Y_i + \sum_{i=1}^{N} \exp(-Y_i) + \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}}, \end{aligned} \tag{2.1}$$

generalizing the construction of mirrors to abelian GLSMs in [2], where the ρ_i^a are the weight vectors for the representation $\mathcal{R}$, and the $\alpha_{\tilde{\mu}}^a$ are root vectors for the Lie algebra of G.

The purpose of this paper is to extend this proposal to cases in which the group G is not connected. Specifically, in this section we will propose the form of mirrors to theories with gauge group $O_\pm(n)$, which we will check by comparing mirrors of the dual SO, $O_\pm$ theories discussed in [4]. (In addition, we will also discuss a handful of other results in [4] which do not involve $O_\pm$ mirrors, but which will enable us to test the nonabelian mirror proposal of [1].)

The mirrors to theories with gauge group $O_\pm(n)$ for n even will have a somewhat different form from mirrors to $O_\pm(n)$ gauge theories with n odd. We discuss each case in turn below.

2.2 $O_\pm$(even)

First, let us recall the details of the Weyl group orbifold action on the mirror that already exists for the mirror of $SO(2k)$. The Weyl group W of $SO(2k)$ is an extension

$$1 \longrightarrow K \longrightarrow W \longrightarrow S_k \longrightarrow 1, \tag{2.2}$$

where K is the subset of $(\mathbb{Z}_2)^k$ with an even number of generators. The action on fields is as follows (in the notation of [1][sections 9, 10]):

- Elements of S_k permute the σ_a and also permute corresponding blocks $\{Y_{i,2a-1}, Y_{i,2a}\}$ (in mirrors to vector-valued fields),
- Elements of K act by sign flips on σ's:

$$\sigma_a \mapsto \epsilon_a \sigma_a, \tag{2.3}$$

where $\epsilon_a \in \{\pm 1\}$, and $\epsilon_1 \epsilon_2 \cdots \epsilon_k = +1$. At the same time, for each sign flip on a σ_a, one also exchanges

$$\begin{aligned} Y_{i,2a-1} &\leftrightarrow Y_{i,2a}, & (2.4) \\ X_{\mu,2a-1} &\leftrightarrow X_{\mu,2a} \quad \text{for } \mu < 2a-1, & (2.5) \\ X_{2a-1,\nu} &\leftrightarrow X_{2a,\nu} \quad \text{for } \nu > 2a. & (2.6) \end{aligned}$$

We propose that the mirror to $O(2k)$ is defined by an additional $\mathbb{Z}_2$ orbifold. We can think of this as one that acts as $\sigma_k \mapsto -\sigma_k$, with a corresponding action $Y_{i,2k-1} \leftrightarrow Y_{i,2k}$ (for mirrors to vector-valued fields), as well as $X_{\mu,2k-1} \leftrightarrow X_{\mu,2k}$. More invariantly, we can think of combining this with the Weyl group of $SO(2k)$ to form a W' orbifold where

$$1 \longrightarrow (\mathbb{Z}_2)^k \longrightarrow W' \longrightarrow S_k \longrightarrow 1. \tag{2.7}$$

In essence, we drop the constraint on K, and allow all elements of $(\mathbb{Z}_2)^k$.

Now, strictly speaking, there are two different $O(2k)$ gauge theories, distinguished by whether the extra $\mathbb{Z}_2$ orbifold is τ or $\tau(-)^F$, and this choice distinguishes $O_+(2k)$ from $O_-(2k)$, in the notation of [4]. We will carefully define and distinguish τ from $\tau(-)^F$ in section 2.6, when we discuss details of twisted sector contributions. Briefly, in our conventions, for a single free chiral superfield, τ flips the sign of the RR Fock vacuum, whereas $\tau(-)^F$ leaves it invariant.

Now, we need to specify which of these two extra $\mathbb{Z}_2$ orbifolds, τ or $\tau(-)^F$, appears in the mirror. To further confuse matters, when one speaks of Hori-Vafa-type mirror constructions, sometimes one takes the mirror superpotential to be a twisted superpotential depending upon twisted chirals, and sometimes one takes it to be an ordinary superpotential depending upon ordinary chirals, and these two matters are linked, essentially because this distinguishes complex from twisted masses, which can have different numbers of twisted sector ground states. Our proposal for the extra $\mathbb{Z}_2$ orbifold above is as follows:

- In conventions in which the mirror superpotential is an ordinary superpotential, depending upon ordinary chiral superfields, we take the extra orbifold appearing in the mirror to be $\tau(-)^F$.
- In conventions in which the mirror superpotential is a twisted superpotential, depending upon twisted chiral superfields, we take the mirror of the orbifold to be just τ.

(It may be helpful for the reader to recall that T-duality performs an analogous swap of $\mathbb{Z}_2$s, see for example [6].)

In this paper, we will take the mirror superpotential to an $O_+(2k)$ gauge theory to be an ordinary superpotential, a function of ordinary chiral superfields. In this convention, the

extra $\mathbb{Z}_2$ will be $\tau(-)^F$, in the notation of [4]. Similarly, we take the mirror to the $O_-(2k)$ gauge theory to be an ordinary superpotential, depending upon ordinary chiral superfields with the extra $\mathbb{Z}_2$ given by τ.

2.3 $O_\pm$(odd)

We have just argued that in the mirror of an $O(2k)$ gauge theory, we essentially orbifold by the Weyl group of $SO(2k+1)$. This then begs the question, how do we describe the mirror of $O(2k+1)$?

Another puzzle in describing this case arises from the fact that it includes $O(1) = \mathbb{Z}_2$ as a special case. As discussed in [7–11], orbifolds with trivially-acting $\mathbb{Z}_2$'s 'decompose' into two copies of one underlying theory (a notion that generalizes to other two-dimensional gauge theories whose gauge groups have nontrivial finite centers). Our mirror proposal for O(odd) gauge theories needs to account for the possibility of such decompositions.

Finally, the difference between $O_\pm$(odd) gauge theories depends upon both a choice of orbifold (τ or $\tau(-)^F$), as well as the number N of vectors in the gauge theory.

With the questions above in mind, we propose that the mirror of an $O_\pm(k)$ gauge theory for k odd, for generic twisted masses, is either one or two copies of the mirror to the corresponding $SO(k)$ gauge theory (with the same matter), depending upon the $\pm$ as well as upon whether N is even or odd. The precise dictionary we give in the table below:

	N even	N odd
O_+(odd)	2 copies	1 copy
O_-(odd)	1 copy	2 copies

As a consistency check, this correctly reproduces relations between vacua in $O_\pm$(odd) and SO(odd) gauge theories listed in [4][table (4.20)], computed for generic nonzero twisted masses, and we will check extensively that it is also consistent with (mirrors to) dualities of two-dimensional theories later in this paper.

In particular, for O_+(odd) gauge theories with N even, and O_-(odd) gauge theories with N odd, our proposal is that the mirror decomposes into two disjoint theories, which suggests that, at least for generic nonzero masses, the original gauge theories also decompose into a disjoint union of simpler theories, much as in [11]. We will not pursue this predicted decomposition of certain O(odd) gauge theories here, but instead leave it for future work.

2.4 Regularity

The dual gauge theories described in [4] are all assumed to be 'regular.' This term is defined in that reference, and for completeness, we repeat that definition here. Briefly, an $(S)O(k)$ gauge theory with N chirals in the vector representation is regular when

- $N-k$ is odd and the mod 2 theta angle is turned off, or
- $N-k$ is even and the mod 2 theta angle is turned on.

2.5 Number of vacua

In much of this paper we will duplicate results for the number of vacua in $SO(k)$ and $O_\pm(k)$ gauge theories via mirror computations. Results for the original gauge theories are listed in [4][table (4.20)]. Because we will refer to this table extensively, to make this paper self-contained, and also to visually illustrate the complexity of the vacuum computations we are duplicating in our mirror construction, we reproduce here the vacuum counts of [4][table (4.20)].

Specifically, consider a regular $SO(k)$ or $O_\pm(k)$ gauge theory with N vectors. For $N \geq k-1$, the number of vacua is as listed in table 1. We will reproduce the listed number of vacua in each case in mirror computations later in this paper. (For non-regular theories, vacuum counts are not listed.)

	k	N	Number of vacua	k	N	Number of vacua
$O_+(k)$	even	even	$\binom{N/2}{k/2}$	odd	even	$2\binom{N/2}{(k-1)/2}$
$O_+(k)$	even	odd	$\binom{(N-1)/2}{k/2} + 2\binom{(N-1)/2}{(k/2)-1}$	odd	odd	$\binom{(N-1)/2}{(k-1)/2}$
$O_-(k)$	even	even	$\binom{N/2}{k/2}$	odd	even	$\binom{N/2}{(k-1)/2}$
$O_-(k)$	even	odd	$\binom{(N+1)/2}{k/2}$	odd	odd	$2\binom{(N-1)/2}{(k-1)/2}$
$SO(k)$	even	even	$2\binom{N/2}{k/2}$	odd	even	$\binom{N/2}{(k-1)/2}$
$SO(k)$	even	odd	$2\binom{(N-1)/2}{k/2} + \binom{(N-1)/2}{(k/2)-1}$	odd	odd	$\binom{(N-1)/2}{(k-1)/2}$

Table 1: Results for number of vacua of regular $SO(k)$ and $O_\pm(k)$ gauge theories, taken from [4][table (4.20)].

2.6 Orbifold vacua

One of the important results we will use later concerns the number of vacua in various orbifolds in which the orbifold group only acts on massive matter, not massless matter. Despite the fact that it only acts on massive fields, it nevertheless can alter the vacuum structure of the theory, as discussed in [7–11]. The precise version we need here is in [4][section 2.2]. Applying it, one should bear in mind that the twisted masses $\tilde{m}_i$ of the original gauge theory, are complex masses in the mirror, as they arise as superpotential terms.

As we will frequently use these results in this paper, let us quickly review the results of [4][section 2.2]. Consider a single chiral superfield $(\phi, \psi_\pm)$ of some nonzero mass, either complex (appearing in a superpotential) or twisted. Following [4], we distinguish two $\mathbb{Z}_2$ orbifolds:

- τ,
- $\tau(-1)^F$.

When acting on a single chiral superfield with components $(\phi, \psi_\pm)$, they both map

$$(\phi, \psi_\pm) \mapsto (-\phi, -\psi_\pm),$$

and are distinguished by their action on the vacuum.

In the RR sector, let $|\Omega\rangle_{RR}$ denote the untwisted ground state of the theory of a single chiral superfield with complex mass m, and $|\tilde{\Omega}\rangle_{RR}$ denote the untwisted ground state of the theory of a single chiral superfield with twisted mass $\tilde{m}$, so that [4][section 2.2]:

$$|\Omega\rangle_{RR} = |0\rangle_0 + \frac{\overline{m}}{|m|}\overline{\psi}_{+0}\overline{\psi}_{-0}|0\rangle_0, \tag{2.8}$$

$$|\tilde{\Omega}\rangle_{RR} = \overline{\psi}_{+0}|0\rangle_0 + \frac{\tilde{m}}{|\tilde{m}|}\overline{\psi}_{-0}|0\rangle_0. \tag{2.9}$$

Under the action of the orbifolds above, we define and distinguish τ and $\tau(-)^F$ as follows:

- τ flips the sign of $|\Omega\rangle_{RR}$, but leaves $|\tilde{\Omega}\rangle_{RR}$ invariant,
- $\tau(-)^F$ leaves $|\Omega\rangle_{RR}$ invariant, but flips the sign of $|\tilde{\Omega}\rangle_{RR}$.

The twisted sector ground state is invariant under both orbifolds (since all fields are half-integrally moded, so that there are no zero modes). As a result, in the RR sector, ground states exist as follows:

	Complex mass	Twisted mass
τ	Twisted sector only	Untwisted and twisted sector
$\tau(-)^F$	Untwisted and twisted sector	Twisted sector only

In a NS-NS sector, the analysis is precisely reversed: the untwisted sector ground state is invariant for both complex and twisted masses, since the fields are half-integrally moded. In the twisted sector, on the other hand, the fields have zero modes, and then the analysis is identical to RR sector untwisted sectors. As a result, ground states in the NS-NS sector exist as follows:

	Complex mass	Twisted mass
τ	Untwisted sector only	Untwisted and twisted sector
$\tau(-)^F$	Untwisted and twisted sector	Untwisted sector only

Following [4][section 2.2], we can derive similar results for multiple massive fields by tensoring together the various ground states. In a system with n massive fermions, if the ground state of a one-fermion system is not invariant under a given $\mathbb{Z}_2$ orbifold but n is even, the ground state of the n-fermion system will be invariant. For example, in a $\tau(-)^F$ orbifold of a system of n chiral multiplets with nonzero complex mass and m chiral multiplets with nonzero twisted mass,

- if m is even, there will be an invariant ground state in both the twisted and untwisted sectors for both RR and NS-NS,
- if m is odd, only the RR twisted sector and NS-NS untwisted sector will have an invariant ground state.

By contrast, in a τ orbifold,

- if n is even, there will be an invariant ground state in both the twisted and untwisted sectors for both RR and NS-NS,
- if n is odd, only the RR twisted sector and NS-NS untwisted sector will have an invariant ground state.

3 SO-O duality

In [4][section 4.6], it was proposed that for $N \geq k$, there exist IR dualities

$$O_+(k) \leftrightarrow SO(N-k+1), \tag{3.1}$$
$$SO(k) \leftrightarrow O_+(N-k+1), \tag{3.2}$$
$$O_-(k) \leftrightarrow O_-(N-k+1), \tag{3.3}$$

where in each case

- the theory on the left has N massless vectors $x_1, \cdots, x_N$, with twisted masses $\tilde{m}_i$, and
- the theory on the right has N vectors $\tilde{x}^1, \cdots, \tilde{x}^N$, of twisted masses $\tilde{m}_i$, along with $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$, $1 \leq i, j \leq N$, of twisted mass $-\tilde{m}_i - \tilde{m}_j$, and a superpotential

$$W = \sum_{i,j} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{3.4}$$

The mesons in the two theories are related by

$$s_{ij} = x_i \cdot x_j. \tag{3.5}$$

The dualities are only claimed to exist when all the theories in question are regular, in the sense of [4] (and as reviewed in section 2.4), which constrains the discrete theta angle.

In this section, we will examine the mirrors to each side of the first two dualities above, using an extension of the nonabelian mirrors proposal of [1], and argue that the mirrors of the duals are equivalent.

The general case of the $O_+(k) \leftrightarrow SO(N-k+1)$ duality will be discussed in subsection 3.2. However, this section is rather technical, so to improve readability, we begin with detailed discussions of the simplest case, the mirror to $O_+(2)$ with $N = 3$ and its dual, in subsection 3.1, and include a detailed review of pertinent orbifold twisted sector contributions that will be utilized later in this paper.

The mirror to the $SO(k) \leftrightarrow O_+(N-k+1)$ duality can be treated in a fairly similar fashion. Because singlets appear asymmetrically in the duality, this is not quite identical to the $O_+(k) \leftrightarrow SO(N-k+1)$ duality, but it is sufficiently similar that we omit any detailed analysis of this variation in this paper.

We perform the analogous analysis for the $O_- - O_-$ duality in section 4.

3.1 Prototype: $O_+(2) \leftrightarrow SO(2)$

In this section we will study the simplest duality proposed in [4][section 4.6], namely a duality between

- an $O_+(2)$ gauge theory with three chiral multiplets x_i^a in the doublet representation (i a flavor index, $i \in \{1,2,3\}$, a a color index, $a \in \{1,2\}$), with twisted masses $\tilde{m}_i$,

- an $SO(2)$ gauge theory with three chiral multiplets $\tilde{x}^{ia}$ in the doublet representation, of twisted mass $-\tilde{m}_i$ and R-charge 1, six singlets $s_{ij} = +s_{ji}$, of twisted mass $\tilde{m}_i + \tilde{m}_j$ and R-charge 0, and a superpotential

$$W \;=\; \sum_{ij} s_{ij} \tilde{x}^{ia} \tilde{x}^{jb} \delta_{ab}. \tag{3.6}$$

Our analysis of the mirror to this duality will serve as a prototype for our analysis of mirrors to more general dualities. (Note also that since the groups are abelian, this is an exercise in abelian mirrors [2] rather than nonabelian mirrors [1], though it will serve as a prototype for work with the latter.)

3.1.1 Mirror to $O_+(2)$ gauge theory

Let us first consider the mirror to the $O_+(2)$ gauge theory with $N = 3$ doublets. In principle, this is an orbifold of a $U(1)$ gauge theory, but we will follow the same analysis here as for $O(k)$ gauge theories for larger k, so we will interpret the matter as being in a representation of the complexification of the gauge group. As a result, we will speak of doublets under $SO(2)$, pairs of chiral fields in which one has charge $+1$ and the other has charge -1.

The mirror Landau-Ginzburg orbifold has six fields Y_a^i as well as one σ, with a superpotential

$$W \;=\; \sigma \left(-\sum_{i=1}^{3} Y_1^i + \sum_{i=1}^{3} Y_2^i \;+\; t \right) \;-\; \sum_{i=1}^{3} \tilde{m}_i \left(Y_1^i + Y_2^i \right) \;+\; \sum_{i=1}^{3} \exp\left(-Y_1^i \right) \;+\; \sum_{i=1}^{3} \exp\left(-Y_2^i \right). \tag{3.7}$$

As discussed in section 2.2, the disconnected component of $O_+(2)$ is realized in the mirror by a $\mathbb{Z}_2$ orbifold that acts as

$$\sigma \;\mapsto\; -\sigma, \quad Y_1^i \;\leftrightarrow Y_2^i. \tag{3.8}$$

(Note that although the original gauge theory is an abelian gauge theory, the $\mathbb{Z}_2$ orbifold above restricts allowed values of the Fayet-Iliopoulos parameter to $\{0, \pi i\}$. For this example, regularity in the sense of [4] implies that t should vanish, but we include the possibility of a nonzero t to better illustrate certain features of other $O_+(k)$ cases.)

Integrating out the σ field, we get the constraint

$$\sum_i Y_1^i \;=\; \sum_i Y_2^i \;+\; t. \tag{3.9}$$

At this point, it is conceptually useful to change variables to

$$Y_+^i \;=\; \frac{1}{2} \left(Y_1^i \;+\; Y_2^i \right), \tag{3.10}$$

$$Y_-^i \;=\; \frac{1}{2} \left(-Y_1^i \;+\; Y_2^i \right), \tag{3.11}$$

so that the constraint arising from the σ field is

$$\sum_i Y_-^i \;=\; -t/2. \tag{3.12}$$

Note that the periodicity of $Y_\pm^i$ is a bit different from the original Y fields: under $Y_1^i \mapsto Y_1^i + 2\pi i$, for example, $Y_\pm^i \mapsto Y_\pm^i + \pi i$, which effectively eliminates the square-root branch cut ambiguity implied implicitly in "$t/2$."

Since we are changing variables, we should also check Jacobians, at least formally, to determine if the true fundamental fields have changed, as in [1]. Here the pertinent Jacobian is of the form

$$\det \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \;=\; 1/2, \tag{3.13}$$

a constant. Since it is constant, we can ignore it – there is no meaningful change in the fundamental fields – but in the next section, it will play a more important role.

Using this constraint to eliminate[1] Y_-^3 as

$$Y_-^3 \;=\; -Y_-^1 \;-\; Y_-^2 \;-\; t/2, \tag{3.14}$$

the superpotential (3.7) can be rewritten as

$$\begin{aligned} W \;&=\; -2\sum_{i=1}^3 \tilde{m}_i Y_+^i \;+\; \sum_{i=1}^3 \exp\left(-Y_+^i + Y_-^i\right) \;+\; \sum_{i=1}^3 \exp\left(-Y_+^i - Y_-^i\right), && (3.15) \\ &=\; -2\sum_{i=1}^3 \tilde{m}_i Y_+^i \;+\; \sum_{i=1}^2 \exp\left(-Y_+^i + Y_-^i\right) \;+\; \sum_{i=1}^2 \exp\left(-Y_+^i - Y_-^i\right) \\ &\quad +\; \exp\left(-Y_+^3 - Y_-^1 - Y_-^2 - t/2\right) \;+\; \exp\left(-Y_+^3 + Y_-^1 + Y_-^2 + t/2\right). && (3.16) \end{aligned}$$

The (untwisted) critical locus of this superpotential is defined by the following equations:

$$\begin{aligned} &\exp\left(-Y_+^i + Y_-^i\right) \;+\; \exp\left(-Y_+^3 + Y_-^1 + Y_-^2 + t/2\right) \\ &\quad =\; \exp\left(-Y_+^i - Y_-^i\right) \;+\; \exp\left(-Y_+^3 - Y_-^1 - Y_-^2 - t/2\right) \quad \text{for } i = 1, 2, \end{aligned}$$

$$\begin{aligned} \exp\left(-Y_+^i + Y_-^i\right) \;+\; \exp\left(-Y_+^i - Y_-^i\right) \;&=\; -2\tilde{m}_i \quad \text{for } i = 1, 2, \\ \exp\left(-Y_+^3 - Y_-^1 - Y_-^2 - t/2\right) \;+\; \exp\left(-Y_+^3 + Y_-^1 + Y_-^2 + t/2\right) \;&=\; -2\tilde{m}_3. \end{aligned}$$

[1] We are eliminating a Y_- field, which is antiinvariant under $\mathbb{Z}_2$, instead of an invariant field because we are also integrating out σ, which also picks up a sign under $\mathbb{Z}_2$. The combination of σ and a Y_- can form a $\mathbb{Z}_2$-invariant combination, which is surely what we want to integrate out of an orbifold.

Along the critical locus, define

$$X = \frac{1}{2}\left(\exp\left(-Y_+^i + Y_-^i\right) - \exp\left(-Y_+^i - Y_-^i\right)\right) \quad \text{for } i = 1, 2, \tag{3.17}$$

$$= \frac{1}{2}\left(\exp\left(-Y_+^3 - Y_-^1 - Y_-^2 - t/2\right) - \exp\left(-Y_+^3 + Y_-^1 + Y_-^2 + t/2\right)\right), \tag{3.18}$$

where we have used the first constraint above. It is then straightforward to verify that in terms of X, along the critical locus,

$$\prod_{i=1}^{3}(X - \tilde{m}_i) = q^{-1}\prod_{i=1}^{3}(-X - \tilde{m}_i), \tag{3.19}$$

$$= q^{-1/2}\prod_{i=1}^{3}\exp\left(-Y_+^i\right), \tag{3.20}$$

where $q = \exp(-t)$. (Note that for the allowed values of t, $q = q^{-1}$.) For the remainder of this section, we will restrict to the regular case, which in this case means $q = +1$.

Now, equation (3.19) is a cubic polynomial in X, that is symmetric under $X \mapsto -X$. It has roots at 0 and $\pm X_0$ where

$$X_0^2 = -\sum_{i<j}\tilde{m}_i\tilde{m}_j. \tag{3.21}$$

For simplicity, we will assume that $X_0 \neq 0$, so that these three roots are distinct.

It is straightforward to check that the $\mathbb{Z}_2$ orbifold (3.8) acts as $X \mapsto -X$, exchanging the two nonzero roots above, but leaving the zero root fixed. As a result, in this model, the critical locus of the mirror superpotential does intersect an orbifold fixed point, namely at $X = 0$.

Expanding about that point, we find that all of the Y fields are massive. Specificaly, we find that after evaluating along the critical locus, at $X = 0$, the only nonzero second derivatives of the superpotential are

$$\frac{\partial^2 W}{\partial Y_+^j \partial Y_+^i} = -2\tilde{m}_i\delta_{ij}, \tag{3.22}$$

$$\frac{\partial^2 W}{\partial Y_+^3 \partial Y_+^3} = -2\tilde{m}_3, \tag{3.23}$$

$$\frac{\partial^2 W}{\partial Y_-^j \partial Y_-^i} = -2\tilde{m}_i\delta_{ij} - 2\tilde{m}_3, \tag{3.24}$$

where $i, j \in \{1, 2\}$. As a result, for generic twisted masses, the five Y fields are all massive.

Now, we need to compute the number of twisted sector states. To make this more interesting, the $\mathbb{Z}_2$ orbifold is only acting on the two Y_-^i fields, which are massive. The

reader might well ask, how should an orbifold that acts only on massive states modify the number of ground states?

This matter is addressed in section 2.6. Applying to the present case, the mirror of the $O_+(2)$ gauge theory, described by ordinary chiral superfields and a $\tau(-)^F$ orbifold, we find that for our orbifold acting on chiral superfields of nonzero complex mass (mirror to chirals of nonzero twisted mass), we see that there is an invariant ground state in both the twisted and untwisted sectors for both RR and NS-NS, independent of the number of massive chirals. As a result, there is a total of $2 + 1 = 3$ vacua, in agreement with the number of vacua in the original gauge theory, as listed in [4][table (4.20)] and table 1.

3.1.2 Mirror to $SO(2)$ gauge theory

Next, we consider the mirror to the $SO(2)$ gauge theory with three doublets, six singlets, and a superpotential. This mirror has six chiral multiplets $W^{ia} = \exp\left(-(1/2)\tilde{Y}^{ia}\right)$, six chiral multiplets T_{ij}, and a superpotential

$$\begin{aligned} W &= -\sum_{i=1}^{3} \sigma\left(-\ln(W^{i1})^2 + \ln(W^{i2})^2\right) - \sum_i \tilde{m}_i \left(\ln(W^{i1})^2 + \ln(W^{i2})^2\right) \\ &\quad + \sum_i (W^{i1})^2 + \sum_i (W^{i2})^2 \\ &\quad - \sum_{i \le j} \left(\tilde{m}_i + \tilde{m}_j\right) T_{ij} + \sum_{i \le j} \exp\left(-T_{ij}\right). \end{aligned} \tag{3.25}$$

Because the original six fields $\tilde{x}^{ia}$ each have R-charge 1, the mirror has a $(\mathbb{Z}_2)^6$ orbifold, in which each $\mathbb{Z}_2$ maps $W^{ia} \mapsto -W^{ia}$.

In principle, since the critical locus is defined by vanishing of a first derivative, and the superpotential (when written in terms of $\tilde{Y}^{ia}$) has the same form as for the $O_+(2)$ model, so the critical locus in the untwisted sector should have the same form as there. Nevertheless, let us briefly walk through the highlights in the new variables W^{ia}.

First, integrating out σ gives the constraint

$$\sum_{i=1}^{3} \ln\left(W^{i1}\right)^2 = \sum_{i=1}^{3} \ln\left(W^{i2}\right)^2, \tag{3.26}$$

or more simply,

$$\prod_{i=1}^{3} \left(\frac{W^{i1}}{W^{i2}}\right)^2 = 1. \tag{3.27}$$

We define

$$W_+^i \equiv W^{i1}W^{i2}, \tag{3.28}$$

$$W_-^i \equiv \frac{W^{i1}}{W^{i2}}, \tag{3.29}$$

in terms of which, the constraint from σ becomes

$$\prod_{i=1}^{3} \left(W_-^i\right)^2 = 1. \tag{3.30}$$

We eliminate W_-^3 as

$$\left(W_-^3\right)^2 = \frac{1}{\left(W_-^1 W_-^2\right)^2}, \tag{3.31}$$

which has solutions

$$W_-^3 = \pm \frac{1}{W_-^1 W_-^2}. \tag{3.32}$$

In principle, we also need to compute the Jacobian of the coordinate transformation from (W^{i1}, W^{i2}) to (W_+^i, W_-^i), to determine the true fundamental fields, as explained in [1]. Here, the Jacobian is

$$\det \begin{bmatrix} W^{i2} & W^{i1} \\ 1/W^{i2} & -W^{i1}/(W^{i2})^2 \end{bmatrix} = -2W_-^i, \tag{3.33}$$

hence the fundamental fields are slightly different. We can write

$$(-4)^{-1} dW_+^i d\ln(W_-^i)^2 = dW^{i1} dW^{i2}, \tag{3.34}$$

and so after changing variables, we can take the fundamental fields to be W_+^i and $\ln(W_-^i)^2$. This will not change the critical locus or vacuum computation, which we will leave in terms of $W_\pm^i$, but will be important for e.g. central charge computations.

In principle, we should solve for critical loci for each of these two roots separately. For the moment, we shall focus on the + root, for which

$$W_-^3 = +\frac{1}{W_-^1 W_-^2}. \tag{3.35}$$

The superpotential (3.25) becomes

$$\begin{aligned} W = & -\sum_{i=1}^{2} \tilde{m}_i \ln\left(W_+^i\right)^2 - \tilde{m}_3 \ln\left(W_+^3\right)^2 + \sum_{i=1}^{2} W_+^i W_-^i + \sum_{i=1}^{2} \frac{W_+^i}{W_-^i} \\ & + \frac{W_+^3}{W_-^1 W_-^2} + W_+^3 W_-^1 W_-^2 \\ & - \sum_{i \le j} \left(\tilde{m}_i + \tilde{m}_j\right) T_{ij} + \sum_{i \le j} \exp\left(-T_{ij}\right). \end{aligned} \tag{3.36}$$

The critical locus is defined by the equations

$$W_+^i W_-^i + \frac{W_+^i}{W_-^i} = 2\tilde{m}_i \quad \text{for } i = 1, 2, \tag{3.37}$$

$$\frac{W_+^3}{W_-^1 W_-^2} + W_+^3 W_-^1 W_-^2 = 2\tilde{m}_3, \tag{3.38}$$

$$W_+^i W_-^i - \frac{W_+^i}{W_-^i} = \frac{W_+^3}{W_-^1 W_-^2} - W_+^3 W_-^1 W_-^2 \quad \text{for } i = 1, 2, \tag{3.39}$$

$$\exp(-T_{ij}) = -(\tilde{m}_i + \tilde{m}_j). \tag{3.40}$$

The last equation completely determines $\exp(-T_{ij})$ on the critical locus. For the others, on the critical locus, define

$$X = \frac{1}{2}\left(W_+^i W_-^i - \frac{W_+^i}{W_-^i}\right) \quad \text{for } i = 1, 2, \tag{3.41}$$

$$= \frac{1}{2}\left(\frac{W_+^3}{W_-^1 W_-^2} - W_+^3 W_-^1 W_-^2\right). \tag{3.42}$$

(On the critical locus, these various definitions all match.) Then, it is straightforward to show that

$$\prod_{i=1}^{3}(X - \tilde{m}_i) = \prod_{i=1}^{3}(-X - \tilde{m}_i). \tag{3.43}$$

This has all been for the + root of the solution for W_-^3. If we pick the other root,

$$W_-^3 = -\frac{1}{W_-^1 W_-^2}, \tag{3.44}$$

then after a very similar analysis, for

$$W = \frac{1}{2}\left(W_+^i W_-^i - \frac{W_+^i}{W_-^i}\right) \quad \text{for } i = 1, 2, \tag{3.45}$$

$$= -\frac{1}{2}\left(\frac{W_+^3}{W_-^1 W_-^2} - W_+^3 W_-^1 W_-^2\right), \tag{3.46}$$

we again find

$$\prod_{i=1}^{3}(X - \tilde{m}_i) = \prod_{i=1}^{3}(-X - \tilde{m}_i)$$

describes the critical loci.

Now, we have to take into account the $\mathbb{Z}_2$ orbifolds that act by signs on each W^i. Those orbifolds exchange the + and − roots of the solution for W_-^3, so we see that we do not need to consider the other root in detail.

Furthermore, note that for generic twisted masses, the critical loci cannot intersect the fixed points of these $\mathbb{Z}_2$ orbifolds. For example, from (3.37), we see that $W_+^i \neq 0$ along the critical locus, so long as $\tilde{m}_i \neq 0$. Similarly, from (3.38), we see that $W_+^3 \neq 0$ along the critical locus, so long as $\tilde{m}_3 \neq 0$. Similarly, from (3.38), neither W_-^i can vanish so long as $\tilde{m}_3$ is finite. As a result, since the critical loci do not intersect the fixed points of the $\mathbb{Z}_2$ orbifolds, there should not be any twisted sector ground states to consider.

To summarize, in this theory, the mirror to the $SO(2)$ theory, the ground states are defined by the solutions to the equation

$$\prod_{i=1}^{3} (X - \tilde{m}_i) \;=\; \prod_{i=1}^{3} (-X - \tilde{m}_i)\,,$$

much as in the mirror to the $O_+(2)$ theory. This has three solutions, at $X = 0$ and $X = \pm X_0$, where

$$X_0^2 \;=\; -\sum_{i<j} \tilde{m}_i \tilde{m}_j.$$

Although there are six $\mathbb{Z}_2$ orbifolds, they leave X invariant. For example, from the definitions (3.28), (3.29), W_+^i will be anti-invariant if and only if W_-^i is antiinvariant, hence combinations such as $W_+^i W_-^i$ and W_+^i / W_-^i are invariant.

3.1.3 Comparison of vacua

Now, let us compare the vacua in the mirror to the $O_+(2)$ theory and the mirror to the dual $SO(2)$ theory. They both have the same quantum cohomology relation,

$$\prod_{i=1}^{3} (X - \tilde{m}_i) \;=\; \prod_{i=1}^{3} (-X - \tilde{m}_i)\,, \tag{3.47}$$

but in one case there is an orbifold whose fixed points intersect critical loci and generates a twisted sector state, whereas in the other, orbifolds leave the vacua invariant, and their fixed points do not intersect critical loci.

- In the mirror to the $O_+(2)$ theory, we have two untwisted sector states, corresponding to $X = 0$ and $X = \pm X_0$. The two nonzero solutions are exchanged by the $\mathbb{Z}_2$ orbifold. Sitting over $X = 0$ is one twisted sector state.
- In the mirror to the dual $SO(2)$ theory, we have three untwisted sector states, corresponding to $X = 0$ and $X = \pm X_0$. There is a global symmetry sending $X \mapsto -X$, but this is not gauged, it does not correspond to an orbifold group action.

In addition, there is a global $\mathbb{Z}_2$ symmetry in both theories:

- In the mirror to the $O_+(2)$ theory, there is a $\mathbb{Z}_2$ quantum symmetry [5], which acts by a sign flip on the twisted sector state. Under the quantum symmetry, the states

$$|X = 0, \text{untwisted}\rangle \ \pm \ |X = 0, \text{twisted}\rangle$$

  are exchanged.

- In the mirror to the dual $SO(2)$ theory, the critical locus is invariant under a global $\mathbb{Z}_2$ that maps $X \mapsto -X$. This symmetry exchanges the states $X = \pm X_0$.

Based on the global symmetries above, we believe that the ground states

$$|X = 0, \text{untwisted}\rangle \ \pm \ |X = 0, \text{twisted}\rangle$$

of the $O_+(2)$ theory are dual to the $X = \pm X_0$ states of the $SO(2)$ theory, and that the single $\pm X_0$ state of the $O_+(2)$ theory is dual to the $X = 0$ state of the $SO(2)$ theory. Thus, in this example, the proposed mirror is compatible with the gauge theory duality.

3.2 $O_+(k) \leftrightarrow SO(N-k+1)$ duality

In this section, we will discuss mirrors to each side of the duality relating

- an $O_+(k)$ gauge theory with N vectors $x_1, \cdots, x_N$ of twisted masses $\tilde{m}_i$, and
- an $SO(N-k+1)$ gauge theory with N vectors $\tilde{x}^1, \cdots, \tilde{x}^N$ of twisted masses $\tilde{m}_i$, along with $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$ of twisted masses $-\tilde{m}_i - \tilde{m}_j$, and a superpotential

$$W \ = \ \sum_{ij} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{3.48}$$

We will compute the mirrors to either side of the duality, and check that the mirrors have the same number of vacua as the original theories, and each other. We will also compare central charges of the corresponding IR SCFTs in the special case that the twisted masses all vanish.

3.2.1 Mirror to $O_+(k)$ gauge theory

In this section we will compute the mirror to an $O_+(k)$ gauge theory with $N \geq k$ massless vectors $x_1, \cdots, x_N$ with twisted masses $+\tilde{m}_i$. This mirror will be closely related to the mirror of $SO(k)$ – for k even, it will have a slightly more complicated orbifold group, and for k odd,

it will be either one or two copies of the $SO(k)$ mirror, as discussed in sections 2.2, 2.3. We will therefore begin by describing the mirror to the $SO(k)$ gauge theory with $N \geq k$ massless vectors and twisted masses as above.

Basics for k both even and odd

Following [1][sections 9, 10], the mirror is an orbifold of a Landau-Ginzburg model with fields

- $Y_{i\alpha}$, $i \in \{1, \cdots, N\}$, $\alpha \in \{1, \cdots, k\}$,
- $X_{\mu\nu} = \exp(-Z_{\mu\nu})$, $X_{\mu\nu} = X_{\nu\mu}^{-1}$, $\mu\nu \in \{1, \cdots, k\}$ (excluding $X_{2a-1,2a}$),
- σ_a, $a \in \{1, \cdots, M\}$ where for k even, $k = 2M$, and for k odd, $k = 2M+1$,

with superpotential

$$\begin{aligned} W &= \sum_{a=1}^{M} \sigma_a \left(\sum_{i\alpha\beta} \rho^a_{i\alpha\beta} Y_{i\beta} + \sum_{\mu<\nu;\mu',\nu'} \alpha^a_{\mu\nu,\mu'\nu'} Z_{\mu'\nu'} - t \right) \\ &\quad + \sum_{i\alpha} \exp(-Y_{i\alpha}) + \sum_{\mu<\nu} X_{\mu\nu} \\ &\quad - \sum_{i,\alpha} \tilde{m}_i Y_{i\alpha}. \end{aligned} \tag{3.49}$$

There is no continuous FI parameter, but we retain t to allow for the possibility of a discrete theta angle. In the superpotential above, we take

$$\rho^a_{i\alpha\beta} = \delta_{\alpha,2a-1}\delta_{\beta,2a} - \delta_{\beta,2a-1}\delta_{\alpha,2a}, \tag{3.50}$$

$$\begin{aligned} \alpha^a_{\mu\nu,\mu'\nu'} &= \delta_{\nu\nu'}\left(\delta_{\mu,2a-1}\delta_{\mu',2a} - \delta_{\mu,2a}\delta_{\mu',2a-1}\right) \\ &\quad + \delta_{\mu\mu'}\left(\delta_{\nu,2a-1}\delta_{\nu',2a} - \delta_{\nu,2a}\delta_{\nu',2a-1}\right). \end{aligned} \tag{3.51}$$

We can analyze the untwisted sector of this theory in the same fashion as [1][sections 9, 10]. For completeness, we outline the highlights here. Integrating out the σ_a, we have the constraints

$$\sum_{i=1}^{N} \left(Y_{i,2a} - Y_{i,2a-1}\right) + \sum_{\nu>2a} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right) + \sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) = t. \tag{3.52}$$

To eliminate fields in a fashion somewhat consistent with the orbifold, we define

$$Y^+_{i,a} \equiv \frac{1}{2}\left(Y_{i,2a} + Y_{i,2a-1}\right), \tag{3.53}$$

$$Y^-_{i,a} \equiv \frac{1}{2}\left(Y_{i,2a} - Y_{i,2a-1}\right), \tag{3.54}$$

and then use the constraints above to eliminate $Y^-_{N,a}$:

$$Y^-_{N,a} \;=\; -\sum_{i=1}^{N-1} Y^-_{i,a} \;-\; \frac{1}{2}\sum_{\nu>2a} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right) \;-\; \frac{1}{2}\sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) \;+\; \frac{t}{2}. \tag{3.55}$$

We define

$$\begin{aligned} \Upsilon_a \;&\equiv\; \exp\left(-Y^-_{N,a}\right), && (3.56)\\ &=\; q^{1/2}\left(\prod_{i=1}^{N-1}\exp\left(+Y^-_{i,a}\right)\right)\cdot \\ &\qquad \cdot\left(\prod_{\nu>2a}\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right)^{1/2}\left(\prod_{\mu<2a-1}\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right)^{1/2}, && (3.57)\end{aligned}$$

where $q = \exp(-t)$. As in the previous section, the square root branch cut ambiguities are absorbed by the periodicities of $Y^{\pm}$.

The superpotential then reduces to

$$\begin{aligned} W \;=\; &\sum_{i=1}^{N-1}\sum_{a=1}^{M}\left(\exp\left(-Y^+_{i,a}-Y^-_{i,a}\right) \;+\; \exp\left(-Y^+_{i,a}+Y^-_{i,a}\right)\right)\\ &+\sum_{a=1}^{M}\exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a+\Upsilon_a^{-1}\right)\\ &+\sum_{\mu<\nu}X_{\mu\nu} \;-\; \sum_{i=1}^{N}\sum_{a=1}^{M}2\tilde{m}_i Y^+_{i,a}\\ &+\begin{cases} \sum_{i=1}^{N}\exp\left(-Y_{i,2M+1}\right) \;-\; \sum_{i=1}^{N}\tilde{m}_i Y_{i,2M+1} & k \text{ odd},\\ 0 & \text{else}. \end{cases} \end{aligned} \tag{3.58}$$

The critical locus is then given by

$$
\begin{aligned}
Y^+_{i,a}: \quad & \exp\left(-Y^+_{i,a} - Y^-_{i,a}\right) + \exp\left(-Y^+_{i,a} + Y^-_{i,a}\right) = -2\tilde{m}_i \quad \text{for } i < N, && (3.59)\\
Y^+_{N,a}: \quad & \exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a + \Upsilon_a^{-1}\right) = -2\tilde{m}_N, && (3.60)\\
Y^-_{i,a}: \quad & \exp\left(-Y^+_{i,a} - Y^-_{i,a}\right) - \exp\left(-Y^+_{i,a} + Y^-_{i,a}\right) \\
& \qquad = \exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a - \Upsilon_a^{-1}\right) \quad \text{for } i < N, && (3.61)\\
Y_{i,2M+1}: \quad & \exp\left(-Y_{i,2M+1}\right) = -\tilde{m}_i \quad \text{for } k \text{ odd}, && (3.62)\\
X_{\mu\nu}: \quad & 2X_{2a,2b} = \exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a - \Upsilon_a^{-1}\right) + \exp\left(-Y^+_{N,b}\right)\left(\Upsilon_b - \Upsilon_b^{-1}\right), && (3.63)\\
& 2X_{2a,2b-1} = \exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a - \Upsilon_a^{-1}\right) + \exp\left(-Y^+_{N,b}\right)\left(-\Upsilon_b + \Upsilon_b^{-1}\right), && (3.64)\\
& 2X_{2a-1,2b} = \exp\left(-Y^+_{N,a}\right)\left(-\Upsilon_a + \Upsilon_a^{-1}\right) + \exp\left(-Y^+_{N,b}\right)\left(\Upsilon_b - \Upsilon_b^{-1}\right), && (3.65)\\
& 2X_{2a-1,2b-1} = \exp\left(-Y^+_{N,a}\right)\left(-\Upsilon_a + \Upsilon_a^{-1}\right) \\
& \qquad + \exp\left(-Y^+_{N,b}\right)\left(-\Upsilon_b + \Upsilon_b^{-1}\right), && (3.66)\\
& 2X_{2a,2M+1} = \exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a - \Upsilon_a^{-1}\right) \quad \text{for } k \text{ odd}, && (3.67)\\
& 2X_{2a-1,2M+1} = \exp\left(-Y^+_{N,a}\right)\left(-\Upsilon_a + \Upsilon_a^{-1}\right) \quad \text{for } k \text{ odd}. && (3.68)
\end{aligned}
$$

On the critical locus, define

$$
\begin{aligned}
\sigma_a \equiv & \frac{1}{2}\exp\left(-Y^+_{i,a} - Y^-_{i,a}\right) - \frac{1}{2}\exp\left(-Y^+_{i,a} + Y^-_{i,a}\right) \quad \text{for any } i < N, && (3.69)\\
= & \frac{1}{2}\exp\left(-Y^+_{N,a}\right)\left(\Upsilon_a - \Upsilon_a^{-1}\right). && (3.70)
\end{aligned}
$$

(The fact that these expressions are all equal is a consequence of critical locus equation (3.61).) Since $\exp(-Y) \neq 0$, we have from the first two critical locus equations that

$$\sigma_a \neq \tilde{m}_i, \tag{3.71}$$

for all i. Since $X_{\mu\nu} \neq 0$, we find for $a \neq b$ that

$$\sigma_a \neq \pm\sigma_b, \tag{3.72}$$

and in the special case that k is odd,

$$\sigma_a \neq 0. \tag{3.73}$$

It is also straightforward to demonstrate that on the critical locus,

$$\prod_{i=1}^{N}\left(\sigma_a - \tilde{m}_i\right) = (-)^k q \prod_{i=1}^{N}\left(-\sigma_a - \tilde{m}_i\right). \tag{3.74}$$

Before moving on, we should also observe that the operator mirror map described in [1] relates the quantity we have labelled "σ_a" in the algebra above, to the σ_a of the original theory; our notation was deliberately chosen to encode the operator mirror map.

Now that we have reviewed the basics of the mirror to an $SO(k)$ gauge theory, we will discuss the mirror of $O_+(k)$.

k even:

Let us first consider the case that k is even, $k = 2M$. In this case, the orbifold group for the mirror of $O_+(k)$ is effectively the Weyl group of $SO(2M+1)$. This has a subgroup corresponding to the Weyl group of $SO(2M)$, which as discussed in [1][section 9], has fixed points which do not intersect the critical loci, and so does not generate any twisted sector contributions to ground states. The remainder is generated by a $\mathbb{Z}_2$ which can be taken to act as

$$\sigma_M \mapsto -\sigma_M, \quad Y_{i,2M-1} \leftrightarrow Y_{i,2M}, \quad X_{\mu,2M-1} \leftrightarrow X_{\mu,2M}. \tag{3.75}$$

(This also encodes the action on the critical locus of the map $Y_{i,2M} \leftrightarrow Y_{i,2M-1}$.) This orbifold has a fixed point locus at

$$\sigma_a = 0, \tag{3.76}$$

and since k is even, this is not on the excluded locus. Furthermore, for k even, depending upon the number N of vectors and the value of q, this fixed point locus can also be on the critical locus, hence we have twisted sector contributions to the ground states. In fact, after combining this $\mathbb{Z}_2$ with other elements of the Weyl group of $SO(2M)$, for $q = +1$ and k even, we have intersections of orbifold fixed points and critical loci where

$$\sigma_a = 0 \tag{3.77}$$

for all a. If any one of these combinations vanishes, for any one value of a, then other combinations for other a cannot also vanish, from the excluded locus condition (3.72). Furthermore, since the S_M in the Weyl group of $SO(2M)$ exchanges these quantities, there is at most one vacuum defined by such an intersection of the orbifold fixed point with the critical locus.

Let us back up a step and count the number of possible vacua more systematically.

First, consider the case that $q = +1$. For N even (and even k), the Coulomb branch relation (3.74) reduces, for even k, to the polynomial

$$\left(\sum_i \tilde{m}_i \right) \sigma^{N-1} + \left(\sum_{i_1 < i_2 < i_3} \tilde{m}_{i_1} \tilde{m}_{i_2} \tilde{m}_{i_3} \right) \sigma^{N-3} + \cdots = 0. \tag{3.78}$$

This equation is symmetric in $\sigma \mapsto -\sigma$, so all nonzero roots will come in positive/negative pairs. This equation has $N-1$ roots, of the form

$$\sigma = 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-2)/2}, \tag{3.79}$$

with (for generic $\tilde{m}_i$) none vanishing. In terms of vacua, any two signs $\pm\tilde{\sigma}$ are related by the action of one of the $\mathbb{Z}_2$ subgroups of the orbifold group, and so define the same vacuum.

Since there are $M = k/2$ σ_a, and the corresponding σ are all distinct (including signs) from condition (3.72), exchanged by the action of an S_k orbifold, we see that in this case, there are

$$\left(\begin{array}{c} (N-2)/2+1 \\ M \end{array} \right) = \left(\begin{array}{c} N/2 \\ k/2 \end{array} \right) \tag{3.80}$$

untwisted vacua in the system. (Since $N \geq k = 2M$, we see that $N/2 \geq M$, and so the expression above is well-defined.) The root at $\sigma = 0$ intersects the fixed-point locus of the orbifold, and so there can be additional ground states arising in the twisted sector.

As mentioned in section 2.2, our convention is to treat the mirror superpotential as an ordinary superpotential, depending upon ordinary chiral superfields, and so the extra $\mathbb{Z}_2$ is realized in the mirror by $\tau(-)^F$. In this case, since the fields have complex masses (instead of twisted masses), there will be invariant ground states in both the untwisted and twisted sectors for both RR and NS-NS, regardless of the number of massive fields, following the analysis of section 2.6 and [4].

We can combine a twisted sector ground state with untwisted sector vacua corresponding to nonzero roots, to get a total of

$$\left(\begin{array}{c} (N-2)/2 \\ k/2-1 \end{array} \right) \tag{3.81}$$

twisted sector vacua, and a grand total of

$$\left(\begin{array}{c} N/2 \\ k/2 \end{array} \right) + \left(\begin{array}{c} (N-2)/2 \\ k/2-1 \end{array} \right) \tag{3.82}$$

untwisted and twisted sector vacua.

For completeness, using the identity

$$\left(\begin{array}{c} a \\ b \end{array} \right) + \left(\begin{array}{c} a \\ b-1 \end{array} \right) = \left(\begin{array}{c} a+1 \\ b \end{array} \right), \tag{3.83}$$

it is easy to show that the number of vacua above equals

$$\left(\begin{array}{c} (N-2)/2 \\ k/2 \end{array} \right) + 2 \left(\begin{array}{c} (N-2)/2 \\ k/2-1 \end{array} \right). \tag{3.84}$$

Finally, this particular theory (the mirror of $O_+(k)$ with k even, N even, and $q = +1$, is not regular in the sense of [4], and so is not listed in [4][table (4.20)] or table 1.

For N odd (and $q = +1$), the Coulomb branch relation (3.74) reduces, for even k, to the polynomial

$$\sigma^N + \left(\sum_{i<j} \tilde{m}_i \tilde{m}_j \right) \sigma^{N-2} + \left(\sum_{i_1<i_2<i_3<i_4} \tilde{m}_{i_1} \tilde{m}_{i_2} \tilde{m}_{i_3} \tilde{m}_{i_4} \right) \sigma^{N-4} + \cdots = 0. \tag{3.85}$$

As before, this equation is symmetric in $\sigma \mapsto -\sigma$, so all nonzero roots will come in positive/negative pairs. The Coulomb branch relation has N roots of the form

$$\sigma \; = \; 0, \; \pm\tilde{\sigma}_1, \; \cdots \; , \; \pm\tilde{\sigma}_{(N-1)/2} \tag{3.86}$$

with (for generic $\tilde{m}_i$) none of the $\tilde{\sigma}$ vanishing (except the first root in the list above). For each nonzero root $\tilde{\sigma}_p$, the two signs $\pm\tilde{\sigma}_p$ are exchanged by a $\mathbb{Z}_2$ subgroup of the orbifold group, and so define the same vacuum. The root at $\sigma = 0$ is more interesting. This root intersects the fixed-point locus of the orbifold, and so there can be an additional ground state arising in the twisted sector.

Including the zero, the number of untwisted sector ground states is

$$\left(\begin{array}{c} (N+1)/2 \\ M \end{array} \right) \; = \; \left(\begin{array}{c} (N+1)/2 \\ k/2 \end{array} \right). \tag{3.87}$$

Next, we consider twisted sector states. As mentioned earlier, in our conventions, the mirror of an $O(2M)$ gauge theory involves a Landau-Ginzburg orbifold by $\tau(-)^F$ (combined with the Weyl group orbifold). In the same conventions, we are treating the mirror superpotential as a function of ordinary chiral superfields, with complex masses, so there are invariant ground states in both the untwisted and twisted sectors for both RR and NS-NS, regardless of the number of massive fields, following section 2.6 and [4]. We can combine a twisted sector ground state with untwisted sector vacua corresponding to nonzero roots, to get

$$\left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array} \right) \tag{3.88}$$

twisted sector vacua.

Putting this together, for N odd and $q = +1$, there are

$$\left(\begin{array}{c} (N+1)/2 \\ k/2 \end{array} \right) \; + \; \left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array} \right) \; = \; \left(\begin{array}{c} (N-1)/2 \\ k/2 \end{array} \right) \; + \; 2 \left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array} \right) \tag{3.89}$$

untwisted and twisted vacua. This theory is regular in the sense of [4], and the result above matches an entry in [4][table (4.20)] and table 1.

Now let us turn to the case $q = -1$, for k even.

For $N-1$ even (N odd), the Coulomb branch relation (3.74) reduces to the degree $N-1$ polynomial (3.78). This equation has $N-1$ roots, of the form

$$\sigma \; = \; \pm\tilde{\sigma}_1, \; \cdots \; , \; \pm\tilde{\sigma}_{(N-1)/2}, \tag{3.90}$$

with (for generic $\tilde{m}_i$) none of the $\tilde{\sigma}$ vanishing. In terms of vacua, as before, any two signs $\pm\tilde{\sigma}$ are related by the action of one of the $\mathbb{Z}_2$ subgroups of the orbifold group, and so define

the same vacuum. Since there are M σ_a, and the corresponding σ are all distinct (including signs) from condition (3.72), exchanged by the action of an S_k orbifold, we see that in this case, there are

$$\left(\begin{array}{c} (N-1)/2 \\ M \end{array} \right) = \left(\begin{array}{c} (N-1)/2 \\ k/2 \end{array} \right) \tag{3.91}$$

vacua in the system, all in untwisted sectors. Since $N \geq k = 2M$ and N is odd, we have that $N - 1 \geq 2M$, hence $(N-1)/2 \geq M$, and so the expression above is well-defined. This theory is not regular in the sense of [4].

Finally, we turn to the case that $N-1$ is odd (N even), again for $q = -1$ and k even. In this case, the Coulomb branch relation reduces to the degree N polynomial (3.85). It has N roots, of the form

$$\sigma \; = \; \pm\tilde{\sigma}_1, \; \cdots \; , \; \pm\tilde{\sigma}_{N/2}, \tag{3.92}$$

where (for generic $\tilde{m}_i$) the $\tilde{\sigma}$ are all nonzero. None of these roots intersect the fixed-point locus of the orbifold. As before, for each nonzero root $\tilde{\sigma}_p$, the two signs $\pm\tilde{\sigma}_p$ are exchanged by a $\mathbb{Z}_2$ subgroup of the orbifold group, and so define the same vacuum. We count

$$\left(\begin{array}{c} N/2 \\ M \end{array} \right) = \left(\begin{array}{c} N/2 \\ k/2 \end{array} \right) \tag{3.93}$$

vacua in this case. This theory is regular in the sense of [4], and the number of vacua counted above matches the number given in [4][table (4.20)] and table 1 for this case.

k odd:

Now that we have discussed the mirror to $O_+(k)$ for k even, we turn to the case that k is odd, $k = 2M+1$. As discussed in section 2.3, if the number of vectors N is even, we take the mirror to be two copies of the $SO(k)$ mirror, and if N is odd, we take the mirror to be one copy of the $SO(k)$ mirror.

First, consider the case that $q = +1$. For N even (and odd k), the Coulomb branch relation (3.74) reduces to a degree N polynomial, with roots

$$\sigma \; = \; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{3.94}$$

The mirror to $SO(k)$ with N even therefore has

$$\left(\begin{array}{c} N/2 \\ M \end{array} \right) = \left(\begin{array}{c} N/2 \\ (k-1)/2 \end{array} \right) \tag{3.95}$$

vacua, and since the mirror of $O_+(k)$ with an even number of vectors is two copies of the mirror of $SO(k)$, we find that in this case the mirror to the $O_+(k)$ gauge theory has

$$2 \left(\begin{array}{c} N/2 \\ (k-1)/2 \end{array} \right) \tag{3.96}$$

vacua. This theory is regular in the sense of [4], and the number of vacua computed above matches the corresponding entry in [4][table (4.20)] and table 1.

Next, consider the case that $q = +1$ and N is odd. In this case, the Coulomb branch relation (3.74) reduces to a degree $N-1$ polynomial, with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{3.97}$$

The mirror to $SO(k)$ with N odd therefore has

$$\left(\begin{array}{c} (N-1)/2 \\ M \end{array} \right) \;=\; \left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right) \tag{3.98}$$

vacua, and since the mirror of $O_+(k)$ with an odd number of vectors is the same as the mirror of the $SO(k)$ gauge theory, we find that in this case the mirror to the $O_+(k)$ gauge theory has

$$\left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right) \tag{3.99}$$

vacua. This theory is not regular in the sense of [4].

Next, consider the case that $q = -1$. For N even (and odd k), the Coulomb branch relation (3.74) reduces to a degree $N-1$ polynomial, with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-2)/2}. \tag{3.100}$$

For the mirror to $SO(k)$, the zero root lies along the excluded locus, so we see that the $SO(k)$ mirror for N even has

$$\left(\begin{array}{c} (N-2)/2 \\ (k-1)/2 \end{array} \right) \tag{3.101}$$

vacua. The mirror to $O_+(k)$ with an even number of vectors is two copies of the mirror to the corresponding $SO(k)$ gauge theory, so we see that the mirror to $O_+(k)$ has

$$2\left(\begin{array}{c} (N-2)/2 \\ (k-1)/2 \end{array} \right) \tag{3.102}$$

vacua. This theory is not regular in the sense of [4].

Finally, consider the case that $q = -1$ and N is odd. In this case, the Coulomb branch relation (3.74) is a degree N polynomial, with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{3.103}$$

The zero root lies along the excluded locus, so we see that the $SO(k)$ mirror for N odd and $q = -1$ has

$$\left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right) \tag{3.104}$$

vacua. The mirror to $O_+(k)$ with N odd is one copy of the mirror to $SO(k)$, so we see that the $O_+(k)$ mirror in this case has

$$\begin{pmatrix} (N-1)/2 \\ (k-1)/2 \end{pmatrix} \tag{3.105}$$

vacua also. This theory is regular in the sense of [4], and the number of vacua matches the corresponding entry in [4][table (4.20)] and table 1.

3.2.2 Mirror to $SO(N-k+1)$ gauge theory

In this section we will compute the mirror to an $SO(N-k+1)$ gauge theory with $N \geq k$ vectors $\tilde{x}^1, \cdots, \tilde{x}^N$ of R-charge 1 and twisted mass $+\tilde{m}_i$, $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$ of R-charge 0 and twisted mass $-\tilde{m}_i - \tilde{m}_j$, and a superpotential

$$W = \sum_{i \leq j} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{3.106}$$

If $N-k+1$ is even, $N-k+1 = 2M$, then the mirror [1] is given by a Landau-Ginzburg orbifold with fields

- $W^{i\alpha} = \exp\left(-(1/2)\tilde{Y}^{ia}\right)$, $i \in \{1, \cdots, N\}$, $\alpha \in \{1, \cdots, N-k+1\}$,
- $X_{\mu\nu} = \exp(-Z_{\mu\nu})$, $X_{\nu\mu} = X_{\mu\nu}^{-1}$, $\mu, \nu \in \{1, \cdots, N-k+1\}$ (excluding $X_{2j-1,2j}$),
- σ_a, $a \in \{1, \cdots, M\}$,
- $T_{ij} = +T_{ji}$,

and superpotential

$$\begin{aligned} W &= \sum_{a=1}^{M} \sigma_a \left(-\sum_{i,\alpha,\beta} \rho^a_{i\alpha\beta} \ln \left(W^{i\beta}\right)^2 - \sum_{\mu<\nu;\mu',\nu'} \alpha^a_{\mu\nu;\mu'\nu'} \ln X_{\mu\nu} - \tilde{t} \right) \\ &\quad + \sum_{i\alpha} \left(W^{i\alpha}\right)^2 + \sum_{\mu<\nu} X_{\mu\nu} + \sum_{i\leq j} \exp\left(-T_{ij}\right) \\ &\quad + \sum_{i\alpha} 2\tilde{m}_i \ln W^{i\alpha} + \sum_{i\leq j} \left(\tilde{m}_i + \tilde{m}_j\right) T_{ij}, \end{aligned} \tag{3.107}$$

where

$$\rho^a_{i\alpha\beta} = \delta_{\alpha,2a-1}\delta_{\beta,2a} - \delta_{\beta,2a-1}\delta_{\alpha,2a}, \tag{3.108}$$

$$\begin{aligned} \alpha^a_{\mu\nu,\mu'\nu'} &= \delta_{\nu\nu'}\left(\delta_{\mu,2a-1}\delta_{\mu',2a} - \delta_{\mu,2a}\delta_{\mu',2a-1}\right) \\ &\quad + \delta_{\mu\mu'}\left(\delta_{\nu,2a-1}\delta_{\nu',2a} - \delta_{\nu,2a}\delta_{\nu',2a-1}\right). \end{aligned} \tag{3.109}$$

(Our description is modelled on the closely related mirror described in [1][section 9].) Although there is no continuous FI parameter, we retain t above to allow for the possibility of a discrete theta angle.

We orbifold this model by the Weyl group W of $SO(2M)$, where W is the extension

$$1 \longrightarrow K \longrightarrow W \longrightarrow S_M \longrightarrow 1, \tag{3.110}$$

where K is the subgroup of $(\mathbb{Z}_2)^M$ with an even number of nontrivial generators. We also orbifold by additional $\mathbb{Z}_2$s, one for each $W^{i\alpha}$, mapping $W^{i\alpha} \mapsto -W^{i\alpha}$.

The case $N-k+1$ odd is described in a formally identical fashion (compare e.g. [1][section 10]). Here, we define M by $N-k+1 = 2M+1$, and in the description of the Weyl group, the kernel K is taken to be all of $(\mathbb{Z}_2)^M$. Its action on the fields is identical to the case $N-k+1$ even – meaning, for example, that $W^{i,N-k+1} = W^{i,2M+1}$ is invariant under the Weyl group orbifold.

Proceeding as in [1][sections 9, 10], we integrate out the σ_a to get constraints

$$2\sum_{i=1}^{N} \ln\left(\frac{W^{i,2a-1}}{W^{i,2a}}\right) + \sum_{\nu>2a} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right) + \sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) = \tilde{t}. \tag{3.111}$$

We define

$$W_+^{i,a} \equiv W^{i,2a-1}W^{i,2a}, \tag{3.112}$$

$$W_-^{i,a} \equiv \frac{W^{i,2a-1}}{W^{i,2a}}, \tag{3.113}$$

in terms of which the constraint becomes

$$\sum_{i=1}^{N} \ln\left(W_-^{i,a}\right)^2 + \sum_{\nu>2a} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right) + \sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) = \tilde{t}. \tag{3.114}$$

We use this constraint to eliminate $W_-^{N,a}$:

$$\ln\left(W_-^{N,a}\right)^2 = -\sum_{i=1}^{N-1} \ln\left(W_-^{i,a}\right)^2 - \sum_{\nu>2a} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\right) - \sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) + \tilde{t}. \tag{3.115}$$

(In principle, there are two roots for $W_-^{N,a}$, but as we saw previously in the discussion of the $SO(2)$ mirror, there is a $\mathbb{Z}_2$ orbifold that relates the two signs.) Since we will no longer treat $W_-^{N,a}$ as a propagating field, we shall rename it to Υ_a, following the pattern of the discussion of the $O_+(k)$ mirror:

$$\Upsilon_a \equiv W_-^{N,a}, \tag{3.116}$$

$$= \tilde{q}^{-1/2}\left(\prod_{i=1}^{N-1} W_-^{i,a}\right)^{-1}\left(\prod_{\nu>2a} \frac{X_{2a,\nu}}{X_{2a-1,\nu}}\right)^{1/2}\left(\prod_{\mu<2a-1} \frac{X_{\mu,2a}}{X_{\mu,2a-1}}\right)^{1/2}, \tag{3.117}$$

where $\tilde{q} = \exp(-\tilde{t})$.

In passing, just as in the previous discussion of the $SO(2)$ mirror, if we check the pertinent Jacobian, we find that the fundamental fields are $W_+^{i,a}$ and $\ln W_-^{i,a}$. This will not modify the vacuum computation, but will be relevant for the central charge computation.

The superpotential can now be written

$$\begin{aligned} W &= \sum_{i=1}^{N-1}\sum_{a=1}^{M} W_+^{i,a}\left(W_-^{i,a} + \frac{1}{W_-^{i,a}}\right) + \sum_{a=1}^{M} W_+^{N,a}\left(\Upsilon_a + \Upsilon_a^{-1}\right) \\ &+ \sum_{\mu<\nu} X_{\mu\nu} + \sum_{i=1}^{N}\sum_{a=1}^{M} \tilde{m}_i \ln\left(W_+^{i,a}\right)^2 \\ &+ \sum_{i\leq j} \exp\left(-T_{ij}\right) + \sum_{i\leq j}\left(\tilde{m}_i + \tilde{m}_j\right) T_{ij} \\ &+ \begin{cases} \sum_{i=1}^{N}\left(W^{i,2M+1}\right)^2 + \sum_{i=1}^{N}\tilde{m}_i \ln\left(W^{i,2M+1}\right)^2 & N-k+1 \text{ odd}, \\ 0 & \text{else}. \end{cases} \end{aligned} \tag{3.118}$$

The critical locus is then given by

$$W_+^{i,a}\left(W_-^{i,a} + \frac{1}{W_-^{i,a}}\right) = -2\tilde{m}_i \quad \text{for } i < N, \tag{3.119}$$

$$W_+^{N,a}\left(\Upsilon_a + \Upsilon_a^{-1}\right) = -2\tilde{m}_N, \tag{3.120}$$

$$W_+^{i,a}\left(W_-^{i,a} - \frac{1}{W_-^{i,a}}\right) = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) \quad \text{for } i < N, \tag{3.121}$$

$$\exp\left(-T_{ij}\right) = \tilde{m}_i + \tilde{m}_j, \tag{3.122}$$

$$2X_{2a,2b} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right) + W_+^{N,b}\left(-\Upsilon_b + \Upsilon_b^{-1}\right), \tag{3.123}$$

$$2X_{2a,2b-1} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right) + W_+^{N,b}\left(\Upsilon_b - \Upsilon_b^{-1}\right), \tag{3.124}$$

$$2X_{2a-1,2b} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) + W_+^{N,b}\left(-\Upsilon_b + \Upsilon_b^{-1}\right), \tag{3.125}$$

$$= -2X_{2a,2b-1}, \tag{3.126}$$

$$2X_{2a-1,2b-1} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) + W_+^{N,b}\left(\Upsilon_b - \Upsilon_b^{-1}\right), \tag{3.127}$$

$$= -2X_{2a,2b}, \tag{3.128}$$

and for $N-k+1$ odd,

$$\left(W^{i,2M+1}\right)^2 = -\tilde{m}_i, \tag{3.129}$$

$$2X_{2a,2M+1} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right), \tag{3.130}$$

$$2X_{2a-1,2M+1} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right), \tag{3.131}$$

$$= -2X_{2a,2M+1}. \tag{3.132}$$

On the critical locus, define

$$\sigma_a \equiv \frac{1}{2} W_+^{i,a} \left(W_-^{i,a} - \frac{1}{W_-^{i,a}} \right) \quad \text{for any } i < N, \tag{3.133}$$

$$= \frac{1}{2} W_+^{N,a} \left(\Upsilon_a - \Upsilon_a^{-1} \right). \tag{3.134}$$

(These expressions match due to expression (3.121).) Furthermore, from the operator mirror map, as discussed in [1], it is straightforward to see that the "σ_a" defined above is mirror to the σ_a of the original theory. We chose the notation to implicitly reflect the operator mirror map.

It is straightforward to verify that on the critical locus, one has the Coulomb branch relation

$$\prod_{i=1}^{N} (\sigma_a - \tilde{m}_i) = (-)^{N-k+1} \tilde{q} \prod_{i=1}^{N} (-\sigma_a - \tilde{m}_i). \tag{3.135}$$

The excluded locus computed from the critical locus relations for the W fields above is then

$$\sigma_a \neq \pm \tilde{m}_i. \tag{3.136}$$

From the X fields, for $a \neq b$,

$$\sigma_a \neq \pm \sigma_b, \tag{3.137}$$

and for $N - k + 1$ odd, we also have

$$\sigma_a \neq 0. \tag{3.138}$$

Now, let us quickly review the orbifold group actions, to assist us in counting vacua later. Briefly, we claim the orbifold group will exchange roots, but the critical locus will not intersect the fixed point locus of the orbifold in this mirror, so in this particular mirror we do not have to consider any twisted sector ground states.

For $N - k + 1$ odd, the Weyl orbifold group will act on σs by signs, exchanging positive and negative roots, and leaving zero roots fixed. However, from the excluded locus condition (3.138) in this case, we see that no zero roots are allowed. It will also interchange the various solutions for different σ_a.

For $N - k + 1$ even, the Weyl group orbifold will exchange signs of pairs (σ_a, σ_b), $a \neq b$, and so on the fixed-point locus of this part of the orbifold, $\sigma_a = \sigma_b = 0$ for $a \neq b$, which is disallowed by excluded locus condition (3.137). Any one σ_a can vanish along the critical locus, but not multiple σ_a simultaneously. The fact that signs of σ's are flipped only in pairs means there is a two-fold degeneracy relative to the case of $N - k + 1$ odd. For example, for $SO(2)$, given two nonzero roots $\pm\sigma_0$, there is no second element to exchange, so both

roots count separately. For $SO(4)$, the orbifold group identifies elements of two equivalence classes of nonzero σ:

$$\begin{aligned}(+\sigma_1, +\sigma_2) &\sim (-\sigma_1, -\sigma_2),\\ (+\sigma_1, -\sigma_2) &\sim (-\sigma_1, +\sigma_2).\end{aligned}$$

For $SO(6)$, again we have two equivalence classes of nonzero roots under the orbifold, which we illustrate schematically below:

$$\begin{aligned}(+,+,+) &\sim (+,-,-) \sim (-,+,-) \sim (-,-,+),\\ (+,+,-) &\sim (+,-,+) \sim (-,+,+) \sim (-,-,-).\end{aligned}$$

It is straightforward to argue that the same pattern holds for any group $SO(2M)$: the orbifold group action (flipping pairs of signs, not individual signs) always results in precisely two inequivalent nonzero σ solutions. As before, the orbifold also interchanges the various solutions for different σ_a.

Finally, in both cases, there are multiple additional $\mathbb{Z}_2$ orbifolds, each acting as $W^{i,\alpha} \mapsto -W^{i,\alpha}$. However, all $W^{i,\alpha} \neq 0$ (excluded loci), hence these orbifolds do not have any fixed points.

Now that we have established that there are no twisted sector ground states, we will next count vacua, as distinct roots of the Coulomb branch relation.

k **even:**

First, consider the case that

$$(-)^{N-k+1}\tilde{q} \;=\; +1, \tag{3.139}$$

for k even.

If $N-k+1$ is even, then for k even, N is odd, and the Coulomb branch relation (3.135) reduces to the degree N polynomial

$$\sigma^N \;+\; \left(\sum_{i<j} \tilde{m}_i \tilde{m}_j\right) \sigma^{N-2} \;+\; \left(\sum_{i_1<i_2<i_3<i_4} \tilde{m}_{i_1}\tilde{m}_{i_2}\tilde{m}_{i_3}\tilde{m}_{i_4}\right) \sigma^{N-4} \;+\; \cdots \;=\; 0, \tag{3.140}$$

where $\sigma = \Pi_a + \tilde{m}_N$. This equation is symmetric in $\sigma \mapsto -\sigma$, so all nonzero roots will come in positive/negative pairs. The N roots are of the form

$$\sigma \;=\; 0,\, \pm\tilde{\sigma}_1,\, \cdots,\, \pm\tilde{\sigma}_{(N-1)/2} \tag{3.141}$$

with (for generic $\tilde{m}_i$) none of the $\tilde{\sigma}$ vanishing (except the first root in the list above). For $N-k+1$ even, one zero root is allowed by the excluded locus conditions, and so we find the number of vacua is

$$\left(\begin{array}{c}(N+1)/2\\ M\end{array}\right) \;+\; \left(\begin{array}{c}(N-1)/2\\ M-1\end{array}\right) \tag{3.142}$$

(including possible zero roots), all in untwisted sectors. The second term above reflects the two-fold degeneracy among nonzero roots arising from the fact that the Weyl orbifold only flips signs of pairs of σ_a. (Since $N \geq k$, and $N - k + 1 = 2M$, it is straightforward to see that the expression above is well-defined.) The original gauge theory in this case is regular in the sense of [4].

Next we turn to the case that $N - k + 1$ is odd, so that (since k is even), N is even. In this case, the Coulomb branch relation (3.135) reduces to the degree $N - 1$ polynomial

$$\left(\sum_i \tilde{m}_i \right) \sigma^{N-1} \, + \, \left(\sum_{i_1 < i_2 < i_3} \tilde{m}_{i_1} \tilde{m}_{i_2} \tilde{m}_{i_3} \right) \sigma^{N-3} \, + \, \cdots \, = \, 0, \tag{3.143}$$

where again $\sigma = \Pi_a + \tilde{m}_N$. As before, this equation is symmetric in $\sigma \mapsto -\sigma$, so all nonzero roots will come in positive/negative pairs.

Just as in the analysis of the last section, for N even, this equation has N roots, of the form

$$\sigma \, = \, \pm \tilde{\sigma}_1, \, \cdots \, , \, \pm \tilde{\sigma}_{N/2}, \tag{3.144}$$

with (for generic $\tilde{m}_i$) none vanishing. Given that the orbifold group will flip signs of σs, and that absolute values of different σ_a must differ, and finally given the S_{N-k+1} portion of the orbifold, we count

$$\left(\begin{array}{c} N/2 \\ M \end{array} \right) \, = \, \left(\begin{array}{c} N/2 \\ (N-k)/2 \end{array} \right) \tag{3.145}$$

vacua, all in the untwisted sector. Since $N \geq k$ and $N - k + 1 = 2M + 1$, $N/2 \geq M$, and so the expression above is well-defined.

Now, we turn to the case that

$$(-)^{N-k+1} \tilde{q} \, = \, -1. \tag{3.146}$$

Consider the case that $N - k + 1$ is even, so that (since k is even), N is odd. In this case, the Coulomb branch relation (3.135) reduces to the degree $N - 1$ polynomial (3.143). The $N - 1$ roots are of the form

$$\sigma \, = \, \pm \tilde{\sigma}_1, \, \cdots \, , \, \pm \tilde{\sigma}_{(N-1)/2}, \tag{3.147}$$

with (for generic $\tilde{m}_i$) none of the $\tilde{\sigma}$ vanishing. As before, we count

$$2 \left(\begin{array}{c} (N-1)/2 \\ M \end{array} \right) \, = \, 2 \left(\begin{array}{c} (N-1)/2 \\ (N-k+1)/2 \end{array} \right) \tag{3.148}$$

vacua. (The factor of 2 arises because the Weyl group orbifold only flips signs of pairs of σ_a.)

Finally, we turn to the case that $N-k+1$ is odd, so that (since k is even), N is even. In this case, the Coulomb branch relation (3.135) is a degree N polynomial, with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}, \tag{3.149}$$

where (for generic $\tilde{m}_i$) the $\tilde{\sigma}$ are all nonzero. The number of vacua is

$$\left(\begin{array}{c} N/2 \\ M \end{array} \right) \;=\; \left(\begin{array}{c} N/2 \\ (N-k)/2 \end{array} \right). \tag{3.150}$$

The original gauge theory in this case is regular in the sense of [4].

k odd:

So far we have discussed results for k even. Now, we turn to k odd.

First, consider the case that

$$(-)^{N-k+1}\tilde{q} \;=\; +1 \tag{3.151}$$

for k odd.

If $N-k+1$ is even, then for k odd, N is even, and the Coulomb branch relation (3.135) reduces to a degree $N-1$ polynomial, with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-2)/2}. \tag{3.152}$$

The number of vacua is

$$\left(\begin{array}{c} N/2 \\ (N-k+1)/2 \end{array} \right) + \left(\begin{array}{c} (N-2)/2 \\ (N-k+1)/2 \end{array} \right), \tag{3.153}$$

all in untwisted sectors. The second term reflects the two-fold degeneracy among nonzero roots, arising from the fact that the Weyl group acts by pairs of sign flips. This theory is not regular in the sense of [4].

If $N-k+1$ is odd, then for k odd, N is odd, and the Coulomb branch relation (3.135) reduces to a degree N polynomial, with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{3.154}$$

Since $N-k+1$ is odd, the zero root is on the excluded locus. The number of vacua is

$$\left(\begin{array}{c} (N-1)/2 \\ (N-k)/2 \end{array} \right), \tag{3.155}$$

using the fact that the Weyl orbifold now flips individual signs, not just by pairs. This theory is regular in the sense of [4], and the number of vacua matches that given in [4][table (4.20)] and table 1.

Next, consider the case that

$$(-)^{N-k+1}\tilde{q} \;=\; -1 \tag{3.156}$$

for k odd.

If $N-k+1$ is even, then for k odd, N is even, and the Coulomb branch relation (3.135) reduces to a degree N polynomial, with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{3.157}$$

Since $N-k+1$ is even, the Weyl group acts by even numbers of sign flips, leading to a two-fold degeneracy, so the number of vacua is

$$2\left(\begin{array}{c} N/2 \\ (N-k+1)/2 \end{array}\right). \tag{3.158}$$

This theory is regular in the sense of [4], and the number of vacua matches that given in [4][table (4.20)] and table 1.

Finally, if $N-k+1$ is odd, then for k odd, N is odd, and the Coulomb branch relation (3.135) reduces to a degree $N-1$ polynomial, with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{3.159}$$

Since $N-k+1$ is odd, the Weyl group acts by flipping individual signs, so the number of vacua is

$$\left(\begin{array}{c} (N-1)/2 \\ (N-k)/2 \end{array}\right). \tag{3.160}$$

This theory is not regular in the sense of [4].

3.2.3 Comparison of vacua

Now, we shall compare vacua in the mirrors to the $O_+(k)$ and $SO(N-k+1)$ gauge theories. We shall compare regular theories (in the sense of [4], and will see that mirrors to dual regular theories have the same number of vacua, as expected. Non-regular theories are not related by the duality in the same fashion, and so do not exhibit matching vacua.

First, we shall assume k even.

Consider the $O_+(k)$ gauge theory with N odd and $q=+1$. We have seen that the mirror has

$$\left(\begin{array}{c} (N+1)/2 \\ k/2 \end{array}\right) + \left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array}\right) \tag{3.161}$$

vacua, the same number as the mirror of the corresponding $SO(N-k+1)$ gauge theory with $\tilde{q}=+1$. In the $O_+(k)$ mirror, the second set of vacua arise in twisted sectors, whereas in the $SO(N-k+1)$ mirror, all of the vacua arise in an untwisted sector.

Much as in the prototype case, there is a $\mathbb{Z}_2$ symmetry on both sides. In the $O_+(k)$ mirror, there is a quantum $\mathbb{Z}_2$ symmetry, acting by a sign on the

$$\begin{pmatrix} (N-1)/2 \\ k/2-1 \end{pmatrix}$$

states in twisted sectors. In the $SO(N-k+1)$ mirror, there is an ordinary global $\mathbb{Z}_2$ symmetry, flipping the sign of a single σ, and the corresponding set of states (reflecting a two-fold degeneracy amongst nonzero roots, as the Weyl orbifold only acts on pairs of σs) are again anti-invariant.

This case specializes to the $O_+(2) \leftrightarrow SO(2)$ prototype matching, for $k=2$, $N=3$, and $q=\tilde{q}=+1$. In this case,

$$\begin{pmatrix} (N+1)/2 \\ k/2 \end{pmatrix} + \begin{pmatrix} (N-1)/2 \\ k/2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3. \tag{3.162}$$

The comparison of $\mathbb{Z}_2$ symmetries also specializes.

Next, for k even, consider the $O_+(k)$ gauge theory with N even and $q=-1$. We have seen that the mirror has

$$\begin{pmatrix} N/2 \\ k/2 \end{pmatrix} \tag{3.163}$$

vacua, the same number as the mirror of the corresponding $SO(N-k+1)$ gauge theory with $\tilde{q}=+1$. In both cases, all of the vacua arise in an untwisted sector; in neither case are there any twisted sector ground states.

The other theories with k even are not regular, and it is easy to check that the number of vacua in mirrors to corresponding $O_+(k)$ and $SO(N-k+1)$ gauge theories do not match. The lesson we take is that the duality is only meant to apply to regular theories, where regularity determines the value of the discrete theta angle on either side of the duality.

Next, consider the case k odd. We have seen that the mirror of the $O_+(k)$ gauge theory with N odd and $q=-1$ has

$$\begin{pmatrix} (N-1)/2 \\ (k-1)/2 \end{pmatrix} \tag{3.164}$$

vacua, the same number as the mirror of the corresponding $SO(N-k+1)$ gauge theory with $\tilde{q}=-1$. In this case, all of the vacua arise in untwisted sectors.

Finally, for k odd, consider the $O_+(k)$ gauge theory with N even and $q=+1$. We have seen that the mirror has

$$2\begin{pmatrix} N/2 \\ (k-1)/2 \end{pmatrix} \tag{3.165}$$

vacua, the same number as the mirror of the corresponding $SO(N-k+1)$ gauge theory with $\tilde{q} = -1$.

To summarize, we have verified that our mirror construction correctly reproduces the

$$O_+(k) \;\leftrightarrow\; SO(N-k+1)$$

duality between regular theories described in [4], at least at the level of vacua.

3.2.4 Central charges

In the case that all of the twisted masses vanish, these dual gauge theories are believed to flow to a nontrivial SCFT in the IR. The central charge of that IR SCFT is computed in [4][equ'n (4.28)] to be

$$\frac{c}{3} \;=\; Nk \;-\; \frac{1}{2}k(k-1). \tag{3.166}$$

In this section, we will describe how the same result can be derived from the mirror theories, as another consistency check on the mirrors.

Our computation is based on the fact that for a Landau-Ginzburg model with a quasi-homogeneous potential, one can determine the central charge. Specifically, if the superpotential is quasi-homogeneous in the form

$$W\left(\lambda^{q_i}\Phi_i\right) \;=\; \lambda^2 W\left(\Phi\right), \tag{3.167}$$

then it is believed that there is a nontrivial SCFT in the IR with central charge [12]

$$\frac{c}{3} \;=\; \sum_i \left(1 - q_i\right). \tag{3.168}$$

As a consistency check, if a Landau-Ginzburg model has a massive field, then the superpotential is of the form $W = x^2$, which is quasi-homogeneous with $q = 1$, but since it is massive, it should not survive to the IR, and indeed from the formula above, its contribution to the central charge is $1 - 1 = 0$.

First, consider the mirror to the $O_+(k)$ theory, described in section 3.2.1. The superpotential (3.58), in the special case that all twisted masses $\tilde{m}_i$, vanish, is quasi-homogeneous under the following symmetry:

$$Y \;\mapsto\; Y - \ln\lambda^2, \tag{3.169}$$

$$X \;\mapsto\; \lambda^2 X. \tag{3.170}$$

This determines multiplicative charges q and central charge contributions as follows:

Field	q	$c/3 = 1 - q$
Y	0	1
X	2	-1

To compute the central charge, we need to count the number of each type of field:

- $Y_{i,2a}$: $(N-1)M$ fields,
- $Y_{i,2a-1}$: NM fields,
- $Y_{i,2M+1}$: N fields, in the case that k is odd,
- $X_{\mu\nu}$: $(1/2)k(k-1) - M$ fields.

Adding up these contributions, we get the following results for the central charge. First, for k even, so that $M = k/2$, we get

$$\begin{aligned} \frac{c}{3} &= (N-1)M + NM - \frac{1}{2}k(k-1) + M, && (3.171) \\ &= 2NM - \frac{1}{2}k(k-1), && (3.172) \\ &= Nk - \frac{1}{2}k(k-1), && (3.173) \end{aligned}$$

matching [4][equ'n (4.28)]. For k odd, so that $M = (k-1)/2$, we get

$$\begin{aligned} \frac{c}{3} &= (N-1)M + NM + N - \frac{1}{2}k(k-1) + m, && (3.174) \\ &= 2NM + N - \frac{1}{2}k(k-1), && (3.175) \\ &= Nk - \frac{1}{2}k(k-1), && (3.176) \end{aligned}$$

again matching [4][equ'n (4.28)].

Now, let us turn to the mirror to the dual $SO(N-k+1)$ gauge theory, described in section 3.2.2, again in the special case that all twisted masses vanish. Here, the fundamental fields, their charges, and their contributions to the central charge are as follows:

Field	R-symmetry	q	$c/3 = 1 - q$	Number of fields
$W_+^{i,a}$	$W \mapsto \lambda^2 W$	2	-1	NM
$\ln W_-^{i,a}$	invariant	0	1	$(N-1)M$
$W^{i,2M+1}$	$W \mapsto \lambda W$	1	0	N if k is odd, 0 else
$X_{\mu\nu}$	$X \mapsto \lambda^2 X$	2	-1	$(1/2)(N-k+1)(N-k) - M$
T_{ij}	$T \mapsto T + \ln \lambda^2$	0	1	$(1/2)N(N+1)$

Summing the contributions, we get for the total central charge,

$$\frac{c}{3} \ = \ Nk - \frac{1}{2}k(k-1), \tag{3.177}$$

which matches the result for the dual theory, as well as [4][equ'n (4.28)], as expected.

4 $O_- - O_-$ duality

In this section, we will study mirrors to both sides of the duality [4][section 4.6]

$$O_-(k) \ \leftrightarrow \ O_-(N-k+1) \tag{4.1}$$

for $N \geq k$. As in the $SO - O$ dualities,

- the theory on the left has N massless vectors $x_1, \cdots, x_N$, with twisted masses $\tilde{m}_i$, and
- the theory on the right has N vectors $\tilde{x}^1, \cdots, \tilde{x}^N$, of twisted masses $\tilde{m}_i$, along with $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$, $1 \leq i, j \leq N$, of twisted mass $-\tilde{m}_i - \tilde{m}_j$, and a superpotential

$$W \ = \ \sum_{i,j} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{4.2}$$

The mesons in the two theories are related by

$$s_{ij} \ = \ x_i \cdot x_j. \tag{4.3}$$

The dualities are only claimed to exist when all the theories in question are regular, in the sense of [4] (and as reviewed in section 2.4), which constrains the discrete theta angle.

4.1 Prototype: $O_-(2) \leftrightarrow O_-(N-1)$

In this section, we will consider mirrors to the $O_- - O_-$ duality in the special case $k = 2$, relating

- an $O_-(2)$ gauge theory with N chiral multiplets in the doublet representation, of twisted mass $\tilde{m}_i$, and
- an $O_-(N-1)$ gauge theory with N chiral multiplets $\tilde{x}^1, \cdots, \tilde{x}^N$ in the vector representation, of twisted mass $\tilde{m}_i$, plus $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$, $1 \leq i, j \leq N$, of twisted mass $-\tilde{m}_i - \tilde{m}_j$, and a superpotential

$$W \ = \ \sum_{ij} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{4.4}$$

4.1.1 Mirror of $O_-(2)$ gauge theory

Here we consider the mirror to an $O_-(2)$ gauge theory with N chirals in the doublet representation. This will be closely related to the $O_+(2)$ mirror discussed in section 3.1.1, but with N not necessarily the same as 3. We will assume N is odd.

As the mirror superpotential is essentially identical to that of section 3.1.1, for brevity we refer the reader there for a derivation of the critical locus, which is given by

$$\prod_{i=1}^{N} (\sigma - \tilde{m}_i) \;=\; \prod_{i=1}^{N} (-\sigma - \tilde{m}_i)\,. \tag{4.5}$$

In this section we focus on the special case N is odd, so that the theory is regular for $q = +1$. When N is odd, equation (4.5) is a degree N polynomial, that is symmetric under $\sigma \mapsto -\sigma$. It has roots

$$0, \pm\tilde{\sigma}_1, \pm\tilde{\sigma}_2, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{4.6}$$

The orbifold relates $\sigma \sim -\sigma$, so there are only $(N+1)/2$ distinct roots of the Coulomb branch relation, in the untwisted sector. The root at $\sigma = 0$ intersects the fixed point locus of the orbifold, and so there can be twisted sector ground states.

Here, we have N fields with complex masses (Y_-^i) that are acted upon by the orbifold (including Y_-^N, which we previously integrated out). (We do not include the σ field in this counting, as in the $e^2 \to \infty$ limit, it is merely an auxiliary field.) From section 2.6 and [4], in a τ orbifold, we know that since N is odd, only the RR twisted sector and NS-NS untwisted sector will have an invariant ground state. As a result, the total number of vacua in this theory, for N odd, both untwisted and twisted, is

$$(N-1)/2 \;+\; 1 \;=\; (N+1)/2. \tag{4.7}$$

This matches the result for a regular O_-(even) theory with N odd given in [4][table (4.20)] and table 1.

4.1.2 Mirror of $O_-(N-1)$ gauge theory

In this section, we discuss the mirror to the dual $O_-(N-1)$ gauge theory with N chiral multiplets of R-charge 1 and twisted mass $\tilde{m}_i$ and $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$ of twisted mass $-\tilde{m}_i - \tilde{m}_j$.

As in the $O_-(2)$ discussion, we assume that N is odd. For N odd, the mirror theory is an orbifold of a Landau-Ginzburg model with fields

- $W^{i\alpha}$, $i \in \{1, \cdots, N\}$, $\alpha \in \{1, \cdots, N-1\}$,

- $T_{ij} = +T_{ji}$,
- $X_{\mu\nu} = X_{\nu\mu}^{-1}$, $\mu, \nu \in \{1, \cdots, N-1\}$,
- σ_a, $a \in \{1, \cdots, M = (N-1)/2\}$,

with superpotential

$$\begin{aligned} W \;=\; & \sum_{a=1}^{M} \sigma_a \Bigg(- \sum_{i=1}^{N} \ln \left(W^{i,2a}\right)^2 \;+\; \sum_{i=1}^{N} \ln \left(W^{i,2a-1}\right)^2 \;+\; \sum_{\nu>2a} \ln \left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}} \right)^2 \\ & \qquad + \sum_{\mu<2a-1} \ln \left(\frac{X_{\mu,2a-1}}{X_{\mu,2a}} \right) \;-\; \tilde{t} \Bigg) \\ & + \sum_{i\alpha} \left(W^{i\alpha}\right)^2 \;+\; \sum_{\mu<\nu} X_{\mu\nu} \;+\; \sum_{i\leq j} \exp\left(-T_{ij}\right) \\ & + \sum_{i\alpha} \tilde{m}_i \ln \left(W^{i,\alpha}\right)^2 \;+\; \sum_{i\leq j} \left(\tilde{m}_i + \tilde{m}_j\right) T_{ij}. \end{aligned} \tag{4.8}$$

The first part of the orbifold group is the extension of the Weyl group discussed in section 2.2. Here, to describe the O_- mirror rather than the O_+ theory, we take the extra $\mathbb{Z}_2$ to be τ, in the conventions of that section. In addition, for each $W^{i,\alpha}$, there is an additional $\mathbb{Z}_2$ orbifold that maps $W^{i,\alpha} \mapsto -W^{i,\alpha}$.

The untwisted sector analysis then closely follows the same pattern as in section 3.2.2 for the mirror to an $SO(N-k+1)$ gauge theory, for $N-k+1$ even. We only summarize the pertinent results here. First, the Coulomb branch relation is

$$\prod_{i=1}^{N} \left(\sigma_a - \tilde{m}_i\right) \;=\; (-)^{N-1} \tilde{q} \prod_{i=1}^{N} \left(-\sigma_a - \tilde{m}_i\right), \tag{4.9}$$

where the σ_a describe the critical locus, subject to the following excluded loci:

$$\sigma_a \;\neq\; \pm\tilde{m}_i, \tag{4.10}$$

$$\sigma_a \;\neq\; \pm\sigma_b \quad \text{for } a \neq b. \tag{4.11}$$

Repeating the analysis of section 3.2.2, if N is odd and $(-)^{N-1}\tilde{q} = +1$, or more simply $\tilde{q} = +1$, then the Coulomb branch equation has N roots of the form

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}, \tag{4.12}$$

Precisely one zero root is allowed by the excluded locus conditions, which may or may not have twisted sector contributions. For a τ orbifold acting on n fields with nonzero complex

masses, as discussed in section 2.6, if n is even, there will be invariant ground states in both untwisted and twisted sectors, in both RR and NS-NS, whereas if n is odd, only the RR twisted sector and NS-NS untwisted sector will have an invariant ground state.

In the present case, if we take the extra $\mathbb{Z}_2$ extending Weyl to act on the ath σ, then it also acts as

$$W^{i,2a-1} \leftrightarrow W^{i,2a}, \tag{4.13}$$

$$X_{\mu,2a-1} \leftrightarrow X_{\mu,2a} \quad \text{for } \mu < 2a-1, \tag{4.14}$$

$$X_{2a-1,\nu} \leftrightarrow X_{2a,\nu} \quad \text{for } \nu > 2a, \tag{4.15}$$

As a result, there are N W fields of eigenvalue -1 under the $\mathbb{Z}_2$ (possibly including the previously integrated-out $W^{i,2M}$), and $2(M-1)$ X fields of eigenvalue -1. Since N is odd, $N + 2(M-1)$ is always odd.

As a consequence, if N is odd, then we only get contributions from RR twisted and NS-NS untwisted sectors, hence there are

$$\begin{pmatrix} (N+1)/2 \\ (N-1)/2 \end{pmatrix} = (N+1)/2 \tag{4.16}$$

vacua. This particular case (N odd, $k = N-1$ even, $\tilde{q} = +1$) is regular in the sense of [4], and the result above for the number of vacua in the mirror correctly matches an entry in [4][table (4.20)] and table 1. As expected, this matches the result for the number of vacua in the $O_-(2)$ mirror for N odd, as discussed in the last section, and so again we have confirmed that the mirror proposal is consistent with the dualities in [4].

4.2 $O_-(k) \leftrightarrow O_-(N-k+1)$ duality

In this section, we will compute the mirrors to either side of the duality between

- an $O_-(k)$ gauge theory with N massless vectors $x_1, \cdots, x_N$, with twisted masses $\tilde{m}_i$, and
- an $O_-(N-k+1)$ gauge theory with N vectors $\tilde{x}^1, \cdots, \tilde{x}^N$ with twisted masses $\tilde{m}_i$, singlets $s_{ij} = +s_{ji}$ of twisted masses $-\tilde{m}_i - \tilde{m}_j$, and the superpotential

$$W = \sum_{i,j} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{4.17}$$

As before, since the original gauge theories are only equivalent in the IR, we only expect the mirrors to be equivalent in the IR, so we compare vacua for generic twisted masses, and verify that the mirrors have the same number of vacua as one another and as the original gauge theories.

4.2.1 Mirror to $O_-(k)$ gauge theory

In this section we will give the mirror to an $O_-(k)$ gauge theory with $N \geq k$ vectors.

Now, the mirror to the $O_-(k)$ theory is closely related to the mirror of an $SO(k)$ theory with the same matter. From sections 2.2, 2.3:

- If k is even, the $O_-(k)$ mirror is nearly the same as the $O_+(k)$ mirror, in which the Weyl group has been extended by $\mathbb{Z}_2$, specifically a τ orbifold.
- If k is odd, the $O_-(k)$ mirror will be one or two copies of the $SO(k)$ mirror: one copy if N is even, two copies if N is odd.

In any event, the starting point is the mirror to an $SO(k)$ gauge theory with N vectors. This is discussed in [1][sections 9, 10] and also reviewed in section 3.2.1, so we will focus on vacua, referring readers to those references for further details. For the mirror to $SO(k)$, the critical loci of the mirror superpotential are defined by the Coulomb branch relation

$$\prod_{i=1}^{N} (\sigma_a - \tilde{m}_i) \;=\; (-)^k q \prod_{i=1}^{N} (-\sigma_a - \tilde{m}_i)\,, \tag{4.18}$$

where $a \in \{1, \cdots, M\}$ for $k = 2M$ if even or $2M+1$ if odd, and with excluded loci

$$\sigma_a \;\neq\; \pm\tilde{m}_i, \tag{4.19}$$

$$\sigma_a \;\neq\; \pm\sigma_b \quad \text{for } a \neq b, \tag{4.20}$$

and if k is odd, then in addition,

$$\sigma_a \;\neq\; 0. \tag{4.21}$$

Next, we shall count vacua, in the case k is even. For brevity, we restrict to regular cases in the sense of [4], meaning that $q = (-)^{N+1}$ for k even.

If N is even (and k is even), then in the regular case, the Coulomb branch relation (4.18) reduces to a degree N polynomial, with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{4.22}$$

The Weyl orbifold group, for $O_-(k)$, is extended to that of $SO(k+1)$, and so acts by flipping the sigms of each σ individually, and by rearrangement of the σ_a. As a result, taking into account the excluded locus requirement that different σ_a must be distinct, and the S_k quotient, we see that there are

$$\begin{pmatrix} N/2 \\ k/2 \end{pmatrix} \tag{4.23}$$

vacua, all in the untwisted sector, which matches the counting in [4][table (4.20)] and table 1.

If N is odd (and k is even), then in the regular case, the Coulomb branch relation (4.18) reduces to a degree N polynomial, with roots

$$\sigma \ = \ 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{4.24}$$

The Weyl orbifold group, for $O_-(k)$, is extended to that of $SO(k+1)$, and so acts by flipping the signs of each σ individually, as well as by rearrangement of the σ_a. However, since k is even, the zero root is not excluded, and is on the fixed-point locus of the orbifold.

Specifically, the zero root lies in the fixed-point locus of the extra $\mathbb{Z}_2$ orbifold, which from section 2.2 is a τ orbifold. As reviewed in section 2.6, the number of ground states that survive a τ orbifold projection depends upon whether the number of chiral multiplets with nonzero complex mass is even or odd. In this case, if we think of the extra $\mathbb{Z}_2$ as acting on the 'last' component[2], then for generic $\tilde{m}_i$, there are $k-2$ massive X fields which are acted upon nontrivially (the antiinvariant parts of $X_{\mu,k-1} \leftrightarrow X_{\mu,k}$), which for k even is always even, and N massive Y fields which are acted upon nontrivially, which is odd. (We include the Y field we integrated out, as it is an example of a massive field, but omit the σ fields, which in the limit $e^2 \to \infty$ are auxiliary fields.)

As a result, the number of fields with a complex mass in this (mirror) theory is odd, so from the analysis of section 2.6, only the RR twisted sector and NS-NS untwisted sector will have an invariant ground state. Combining twisted and untwisted sector ground states, corresponding to different roots, we find a total of

$$\left(\begin{array}{c} (N-1)/2 \\ k/2 \end{array} \right) + \left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array} \right) = \left(\begin{array}{c} (N+1)/2 \\ k/2 \end{array} \right) \tag{4.25}$$

vacua, which matches the appropriate entry in [4][table (4.20)] and table 1.

Next, we turn to the case that k is odd. For brevity, we restrict to regular cases in the sense of [4], which means that $q = (-)^N$. The mirror to $O_-(k)$ for k odd will be

- one copy of the $SO(k)$ mirror if N is even,
- two copies of the $SO(k)$ mirror if N is odd,

so we first analyze the $SO(k)$ mirror.

If N is even (and k odd), the Coulomb branch relation (4.18) reduces to a degree N polynomial, with roots

$$\sigma \ = \ \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{4.26}$$

[2] The Weyl group makes this choice equivalent to any other.

Since k is odd, the Weyl orbifold group acts by rearrangement and individual sign flips, so the number of vacua of the $SO(k)$ mirror is

$$\left(\begin{array}{c} N/2 \\ (k-1)/2 \end{array} \right). \tag{4.27}$$

Since the $O_-(k)$ mirror is one copy of the $SO(k)$ mirror, we see that the $O_-(k)$ mirror also has

$$\left(\begin{array}{c} N/2 \\ (k-1)/2 \end{array} \right) \tag{4.28}$$

vacua, which matches the appropriate entry in [4][table (4.20)] and table 1.

If N is odd (and k odd), the Coulomb branch relation (4.18) reduces to a degree N polynomial, with roots

$$\sigma \; = \; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{4.29}$$

Since k is odd, the Weyl orbifold group acts by rearrangement and by individual sign flips, and furthermore the zero root lies on the excluded locus. As a result, the number of vacua of the $SO(k)$ mirror is

$$\left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right). \tag{4.30}$$

The $O_-(k)$ mirror is two copies of the $SO(k)$ mirror since N is odd, so we see that the $O_-(k)$ mirror has

$$2 \left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right) \tag{4.31}$$

vacua, which matches [4][table (4.20)] and table 1.

Next, we will analyze the mirror of the dual, and compare vacua.

4.2.2 Mirror to $O_-(N-k+1)$ gauge theory

In this section, we will compute the mirror to the $O_-(N-k+1)$ gauge theory with $N \geq k$ vectors $\tilde{x}^1, \cdots, \tilde{x}^N$ of twisted masses $\tilde{m}_i$ and $(1/2)N(N+1)$ singlets $s_{ij} = +s_{ji}$ of twisted mass $-\tilde{m}_i - \tilde{m}_j$, with superpotential

$$W \; = \; \sum_{ij} s_{ij} \tilde{x}^i \cdot \tilde{x}^j. \tag{4.32}$$

Now, the mirror to an O_-(even) theory is merely the mirror to SO(even) with an enlarged Weyl orbifold, and the mirror to an O_-(odd) theory is merely copies of the mirror to SO(odd). As a result, we can re-use the results of section 3.2.2 on mirrors to closely related $SO(N-k+1)$ gauge theories.

In section 3.2.2, briefly, it was argued that the critical locus of the mirror superpotential is given by the Coulomb branch relation

$$\prod_{i=1}^{N} (\sigma_a - \tilde{m}_i) \;=\; (-)^{N-k+1} \tilde{q} \prod_{i=1}^{N} (-\sigma_a - \tilde{m}_i)\,, \tag{4.33}$$

with excluded locus

$$\sigma_a \;\neq\; \pm \tilde{m}_i, \tag{4.34}$$

$$\sigma_a \;\neq\; \pm \sigma_b \quad \text{for } a \neq b, \tag{4.35}$$

and if $N-k+1$ is odd, then in addition,

$$\sigma_a \;\neq\; 0. \tag{4.36}$$

We shall restrict to cases in which the original gauge theory is regular, meaning $\tilde{q} = (-)^k$.

Now, we shall count vacua, beginning with the case that k is even.

If N is even (and k is even), then $N-k+1$ is odd, and in the regular case, the Coulomb branch relation (4.33), reduces to a degree N polynomial with roots

$$\sigma \;=\; \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{4.37}$$

The Weyl group orbifold of $SO(N-k+1)$ acts by flipping the signs of each σ individually, so as a result, in the $SO(N-k+1)$ mirror, there are

$$\left(\begin{array}{c} N/2 \\ (N-k)/2 \end{array} \right) = \left(\begin{array}{c} N/2 \\ k/2 \end{array} \right) \tag{4.38}$$

vacua, all in the untwisted sector. Since $N-k+1$ is odd, and N is even, the mirror of the $O_-(N-k+1)$ guage theory is one copy of the $SO(N-k+1)$ gauge theory, from section 2.3, and so we see that the $O_-(N-k+1)$ gauge theory has

$$\left(\begin{array}{c} N/2 \\ k/2 \end{array} \right) \tag{4.39}$$

vauca. This precisely matches the result for N even and k even for the $O_-(k)$ mirror, as expected.

If N is odd (and k is even), then $N-k+1$ is even, and in the regular case, the Coulomb branch relation (4.33) reduces to a degree N polynomial with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{4.40}$$

The orbifold group for the $O_-(N-k+1)$ mirror extends the Weyl group orbifold of $SO(N-k+1)$ to act by, for example, flipping signs of all σ, not just pairs, in addition to rearrangment

of the σ_a. In addition, the zero root is not excluded (since $N-k+1$ is even), and so represents an intersection of the critical locus with the fixed-point locus, hence implies possible twisted sector contributions.

For generic $\tilde{m}_i$, there should be $N-k-1$ massive X fields which are acted upon nontrivially by the extra $\mathbb{Z}_2$ (the antiinvariant part of $X_{\mu,N-k} \leftrightarrow X_{\mu,N-k+1}$), which is always even, and N massive Y fields which are acted upon nontrivially (the antiinvariant part of $Y_{i,N-k} \leftrightarrow Y_{i,N-k+1}$), which is odd. (As before, we include the Y field we integrated out, as it is a massive field, but omit the σ fields, which are auxiliary in the limit.)

As a result, there is an odd number of fields with nonzero complex mass which are acted upon by the τ orbifold in the mirror to the $O_-(N-k+1)$ theory, so from the analysis of section 2.6 there will only be an invariant ground state in the RR twisted sector and NS-NS twisted sector. Combining twisted and untwisted sector ground states, corresponding to different roots, we find a total of

$$\left(\begin{array}{c} (N-1)/2 \\ k/2 \end{array} \right) + \left(\begin{array}{c} (N-1)/2 \\ k/2-1 \end{array} \right) = \left(\begin{array}{c} (N+1)/2 \\ k/2 \end{array} \right) \tag{4.41}$$

vacua, which matches the result for N odd and k even for the $O_-(k)$ mirror, as expected.

Next, we turn to the case that k is odd.

If N is even (and k odd), then $N-k+1$ is even, and in the regular case, the Coulomb branch relation (4.33) reduces to a degree N polynomial, with roots

$$\sigma = \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{N/2}. \tag{4.42}$$

The Weyl group orbifold of $O_-(N-k+1)$ acts by flipping signs of individual σ (a $\mathbb{Z}_2$ extension of the Weyl group of $SO(N-k+1)$), so in the $O_-(N-k+1)$ mirror, there are

$$\left(\begin{array}{c} N/2 \\ (N-k+1)/2 \end{array} \right) = \left(\begin{array}{c} N/2 \\ (k-1)/2 \end{array} \right) \tag{4.43}$$

vacua. This precisely matches the result for N even and k odd for the $O_-(k)$ mirror, as expected.

Finally, if N is odd (and k is odd), then $N-k+1$ is odd, and in the regular case, the Coulomb branch relation (4.33) reduce to a degree N polynomial, with roots

$$\sigma = 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{4.44}$$

Since $N-k+1$ is odd, the zero root is excluded, and for $SO(N-k+1)$, the Weyl group acts by signs on individual σs, so the number of vacua in the $SO(N-k+1)$ mirror is

$$\left(\begin{array}{c} (N-1)/2 \\ (k-1)/2 \end{array} \right). \tag{4.45}$$

Since $N-k+1$ is odd and N is odd, the mirror of the $O_-(N-k+1)$ gauge theory is two copies of the $SO(N-k+1)$ mirror, from section 2.3, so we see that the number of vacua in the mirror to the $O_-(N-k+1)$ gauge theory is

$$2\left(\begin{array}{c}(N-1)/2\\(k-1)/2\end{array}\right). \tag{4.46}$$

This precisely matches the number of vacua of the mirror of the dual $O_-(k)$ gauge theory, as expected.

Thus, we see that the mirror of the dual $O_-(N-k+1)$ theory has the same number of vacua as the mirror to the $O_-(k)$ theory, confirming our mirror construction.

5 Other properties of SO theories

5.1 Supersymmetry breaking: $N \leq k-2$

It was argued in [4][section 4.4] that for $SO(k)$ gauge theories with $N \leq k-2$ vectors, supersymmetry is broken. We can see that explicitly in the mirror. Briefly, applying the $SO(k)$ mirrors described in [1][sections 9, 10] and reviewed in section 3.2.1, the perturbative vacua, in the untwisted sector, are defined by solutions to the Coulomb branch relation

$$\prod_{i=1}^{N}(\sigma_a - \tilde{m}_i) \;=\; (-)^q \prod_{i=1}^{N}(-\sigma_a - \tilde{m}_i)\,, \tag{5.1}$$

subject to excluded loci

$$\sigma_a \;\neq\; \pm\tilde{m}_i, \tag{5.2}$$

$$\sigma_a \;\neq\; \pm\sigma_b \quad \text{for } a \neq b, \tag{5.3}$$

and for k odd,

$$\sigma_a \;\neq\; 0. \tag{5.4}$$

Put briefly, for $N \leq k-2$, no solutions can be found that are not excluded, hence there are no supersymmetric vacua.

Depending upon the value of q and whether N is even or odd, the Coulomb branch relation above will have N roots, at most one of which will vanish, for which a vacuum is defined as a set of either $k/2$ (k even) or $(k-1)/2$ (k odd) distinct values of σ coinciding with roots. Below is a table of possibilities for mirrors to $SO(k)$, culled from the results in section 3.2.1:

	$q=+1$		$q=-1$	
	N even	N odd	N even	N odd
k even	0, $(N-2)/2$	0, $(N-1)/2$	$N/2$	$(N-1)/2$
k odd	$N/2$	$(N-1)/2$	0, $(N-2)/2$	0, $(N-1)/2$

Entries including a '0' have a zero root; the other number gives the number of nonzero entries, up to signs.

For k even, to have a supersymmetric vacuum, we need at least $k/2$ distinct (after signs) roots, to be consistent with the excluded locus conditions. However, since $N \leq k-2$, $N/2 \leq (k/2)-1$, it is straightforward to see that not enough distinct σ can be found, hence no supersymmetric vacua exist in the $SO(k)$ mirror.

For k odd, to have a supersymmetric vacuum, we need at least $(k-1)/2$ distinct (after signs) roots, to be consistent with the excluded locus conditions. However, since $N \leq k-2$, we have $N/2 \leq (k/2)-1$, and examining the table above we again see that not enough distinct roots exist, hence no supersymmetric vacua exist in the $SO(k)$ mirror.

Thus, for $N \leq k-2$, the mirror to an $SO(k)$ gauge theory with N vectors does not admit a supersymmetric vacuum, and so supersymmetry is broken.

5.2 $N = k-1$ and free field theories

Consider an $SO(k)$ gauge theory with N chirals in the vector representation. For $N = k-1$, according to [4], these theories flow in the IR to a free theory of $(1/2)k(k-1)$ mesons on the Higgs branch. In this section, we will perform some consistency checks of this claim in the mirror: we will find the free field mirrors in the structure of $SO(2)$ and $SO(3)$ theories, and will check central charge computations for more general k. We will leave detailed verifications for general $SO(k)$ to future work.

5.2.1 Prototype: $SO(2)$, $N=1$

As a simple model of later analyses, we consider the case that $k=2$, namely the mirror to an $SO(2)$ gauge theory with $N=1$ doublet. The mirror is described by three fields, σ, Y_1, Y_2, with superpotential

$$W \;=\; \sigma\left(Y_2 - Y_1 - t\right) \;+\; \exp\left(-Y_1\right) \;+\; \exp\left(-Y_2\right). \tag{5.5}$$

(We assume all twisted masses vanish.) As we are interested in generalizations to other $SO(k)$ groups, we take $t \in \{0, \pi i\}$.

Integrating out σ gives the constraint

$$Y_2 \;=\; Y_1 + t, \tag{5.6}$$

hence the resulting superpotential

$$W \;=\; \exp\left(-Y_1\right) \;+\; e^{-t}\exp\left(-Y_1\right) \;=\; \left(1+e^{-t}\right)\exp\left(-Y_1\right). \tag{5.7}$$

In the case that the original gauge theory is regular, $t = 0$, and the superpotential $W \propto \exp\left(-Y_1\right)$, which is the same as the mirror to a single free chiral multiplet [2, 13].

Thus, in this case, we see that the mirror to a regular $SO(2)$ gauge theory with $N = 1$ doublet is the same as the mirror to a free field theory of one chiral multiplet, consistent with the expectation that the gauge theory flows to a theory of $(1/2)k(k-1) = 1$ chirals.

5.2.2 Prototype: $SO(3)$, $N = 2$

As a slightly more complicated example, we next turn to the case that $k = 3$, the mirror to an $SO(3)$ gauge theory with $N = 2$ vectors. This theory is predicted to have $(1/2)k(k-1) = 3$ free fields in the IR. The mirror is described by the fields Y_a^i ($a \in \{1,2,3\}$, $i \in \{1,2\}$), X_{13}, X_{23}, and σ, and has the superpotential

$$W \;=\; \sigma\left(\sum_{i=1}^{2}\left(Y_2^i - Y_1^i\right) \;-\; \ln\left(\frac{X_{23}}{X_{13}}\right)\right) \;+\; X_{13} \;+\; X_{23} \;+\; \sum_{i=1}^{2}\sum_{a=1}^{3}\exp\left(-Y_a^i\right). \tag{5.8}$$

We assume the theory is regular, so we have set $t = 0$ for simplicity.

Now, we will first integrate out the X fields. Doing so will generate a measure factor we should compute. It is straightforward to show that

$$\frac{\partial^2 W}{\partial X_{13}^2} \;=\; -\frac{\sigma}{X_{13}^2}, \quad \frac{\partial^2 W}{\partial X_{23}^2} \;=\; +\frac{\sigma}{X_{23}^2}, \quad \frac{\partial^2 W}{\partial X_{13}\partial X_{23}} \;=\; 0, \tag{5.9}$$

so

$$\det \partial^2 W \;=\; -\frac{\sigma^2}{X_{13}^2 X_{23}^2}, \tag{5.10}$$

which after taking into account the operator mirror map relation

$$X_{23} \;=\; +\sigma \;=\; -X_{13}, \tag{5.11}$$

implies that

$$\det \partial^2 W \;=\; -\frac{1}{\sigma^2}, \tag{5.12}$$

and so we get the measure factor

$$\left(\det \partial^2 W\right)^{-1} \;=\; -\sigma^2. \tag{5.13}$$

After integrating out the two X fields, the superpotential becomes

$$W \;=\; \sigma\left(\sum_{i=1}^{2}\left(Y_2^i - Y_1^i\right) \;-\; \pi i\right) \;+\; \sum_{i=1}^{2}\sum_{a=1}^{3}\exp\left(-Y_a^i\right). \tag{5.14}$$

Next, we will integrate out two Y fields, so as to generate a measure factor that will cancel out the one obtained from integrating out X fields. Specifically, we will integrate out Y_1^2 and Y_2^2. We find, for the superpotential above,

$$\frac{\partial W}{\partial Y_1^2} \;=\; -\sigma - \exp\left(-Y_1^2\right), \quad \frac{\partial W}{\partial Y_2^2} \;=\; +\sigma - \exp\left(-Y_2^2\right), \tag{5.15}$$

so that along the critical locus we can identify

$$\exp\left(-Y_1^2\right) \;=\; -\sigma, \quad \exp\left(-Y_2^2\right) \;=\; +\sigma. \tag{5.16}$$

To get the measure factor, note that

$$\frac{\partial^2 W}{\partial Y_1^2 \partial Y_1^2} \;=\; +\exp\left(-Y_1^2\right), \quad \frac{\partial^2 W}{\partial Y_2^2 \partial Y_2^2} \;=\; +\exp\left(-Y_2^2\right), \quad \frac{\partial^2 W}{\partial Y_1^2 \partial Y_2^2} \;=\; 0, \tag{5.17}$$

from which we compute

$$\det \partial^2 W \;=\; \exp\left(-Y_1^2\right)\exp\left(-Y_2^2\right) \;=\; -\sigma^2, \tag{5.18}$$

which precisely cancels out the measure factor resulting from integrating out the Y fields.

Now, solving for $Y_{1,2}^2$ along the critical locus, we find

$$Y_1^2 \;=\; \ln\left(-\sigma\right), \quad Y_2^2 \;=\; \ln\left(+\sigma\right), \tag{5.19}$$

so after integrating out these two Y fields, the superpotential becomes

$$\begin{aligned} W \;&=\; \sigma\left(Y_2^1 - Y_1^1 + \ln\left(\frac{-\sigma}{\sigma}\right) - \pi i\right) \;+\; \sigma \;-\sigma \;+\; \exp\left(-Y_1^1\right) \;+\; \exp\left(-Y_2^1\right) \\ &\quad +\; \exp\left(-Y_3^1\right) \;+\; \exp\left(-Y_3^2\right), \end{aligned} \tag{5.20}$$

$$\begin{aligned} &=\; \sigma\left(Y_2^1 - Y_1^1\right) \;+\; \exp\left(-Y_1^1\right) \;+\; \exp\left(-Y_2^1\right) \\ &\quad +\; \exp\left(-Y_3^1\right) \;+\; \exp\left(-Y_3^2\right). \end{aligned} \tag{5.21}$$

From the last line of the equation above, we see that Y_3^1 and Y_3^2 already are visible as the mirrors to two free fields, so it remains to make explicit the third.

Next, we integrate out σ, which gives the constraint

$$Y_2^1 - Y_1^1 \;=\; 0. \tag{5.22}$$

Define

$$Y_+^i \;=\; \frac{1}{2}\left(Y_1^i + Y_2^i\right), \tag{5.23}$$

$$Y_-^i \;=\; \frac{1}{2}\left(-Y_1^i + Y_2^i\right), \tag{5.24}$$

then the constraint is simply

$$Y_-^1 \;=\; 0. \tag{5.25}$$

Eliminating Y_-^1, the superpotential becomes

$$W \;=\; 2\exp\left(-Y_+^1\right) \;+\; \exp\left(-Y_3^1\right) \;+\; \exp\left(-Y_3^2\right). \tag{5.26}$$

Modulo a field redefinition to rescale the first term, this is precisely the mirror superpotential for three free fields, as expected for an $SO(3)$ theory with $N = 2$.

5.2.3 Central charges

In this section, we will briefly perform a consistency check by computing the central charge predicted by the mirror theory. Our analysis will be very similar to the central charge computation in section 3.2.4, and so we will be very brief.

If one sets all twisted masses to zero, the mirror to an $SO(k)$ theory, as discussed in [1][sections 9, 10], has a superpotential which is quasi-homogeneous with respect to the symmetry

$$x_{\mu\nu} \;\mapsto\; \lambda^2 X_{\mu\nu}, \quad Y_{i\alpha} \;\mapsto\; Y_{i\alpha} - \ln\lambda^2. \tag{5.27}$$

As a result, these fields contribute to the central charge of a possible nontrivial IR limit as follows:

Field	q	$c/3 = 1 - q$
Y	0	1
X	2	-1.

Suppose first that k is even. In that case, there are

$$\frac{1}{2}k(k-1) \;-\; \frac{1}{2}k \;=\; \frac{1}{2}k(k-2)$$

X fields, and

$$(N-1)\frac{k}{2} \ + \ N\frac{k}{2} \ = \ \frac{1}{2}k(k-2) \ + \ \frac{1}{2}k(k-1)$$

Y fields, so the total predicted central charge is

$$\frac{c}{3} \ = \ \frac{1}{2}k(k-1), \tag{5.28}$$

which matches the predicted number of free mesons in the IR.

Next, suppose that k is odd. In this case, there are

$$\frac{1}{2}k(k-1) \ - \ \frac{1}{2}(k-1) \ = \ \frac{1}{2}(k-1)^2$$

X fields, and

$$(N-1)\frac{(k-1)}{2} \ + \ N\frac{(k-1)}{2} \ + \ N \ = \ \frac{1}{2}(k-1)(k-2) \ + \ \frac{1}{2}(k-1)^2 \ + \ k-1$$

Y fields, so the total predicted central charge is

$$\frac{c}{3} \ = \ \frac{1}{2}(k-1)(k-2) \ + \ (k-1) \ = \ \frac{1}{2}k(k-1), \tag{5.29}$$

again matching the predicted number of free mesons in the IR.

6 Symplectic-symplectic duality

In [4][section 5.6], it was proposed for that N odd, $N \geq 2k+3$, the following two-dimensional (2,2) supersymmetric gauge theories are dual in the IR:

- $Sp(2k)$ gauge theory with N fundamentals $x_1, \cdots, x_N$, with twisted masses $\tilde{m}_i$,
- $Sp(N-2k-1)$ gauge theory with N fundamentals $\tilde{x}^1, \cdots \tilde{x}^N$, with twisted masses $\tilde{m}_i$, and $(1/2)N(N-1)$ singlets $a_{ij} = -a_{ji}$, $1 \leq i, j \leq N$, with twisted masses $-\tilde{m}_i - \tilde{m}_j$ and superpotential
$$W \ = \ \sum_{i,j} a_{ij}[\tilde{x}^i \tilde{x}^j]. \tag{6.1}$$

(In the notation used here, $Sp(2) = SU(2)$.) The mesons in the two theories are related as

$$[x_i x_j] \ = \ a_{ij}. \tag{6.2}$$

In the expressions above, the bracket notation indicates Sp-invariants:

$$[\tilde{x}^i \tilde{x}^j] \;=\; J^{ab} \tilde{x}^i_a \tilde{x}^j_b, \tag{6.3}$$

where J^{ab} is the antisymmetric Sp symplectic form.

In this section we will compute the number of vacua of the mirrors to each side of the duality above, and check that they match. Unlike previous cases, here on both sides of the duality the gauge group is connected, so this will not be a test of mirrors of gauge theories with non-connected gauge groups; nevertheless, as this is another duality in [4], this paper does seem the appropriate place to test it.

6.1 Mirror to $Sp(2k)$ gauge theory

The mirror to the first theory is discussed in [1][section 11]. For brevity, we refer the reader to that reference for details of the mirror and its analysis.

It was shown in [1][section 11] that the critical loci are defined by the equation

$$\prod_{i=1}^{N} (\sigma_a \;-\; \tilde{m}_i) \;=\; \prod_{i=1}^{N} (-\sigma_a \;-\; \tilde{m}_i)\,, \tag{6.4}$$

and the excluded loci are defined by [1][section 11]

$$\sigma_a \;\neq\; 0, \pm \tilde{m}_i, \tag{6.5}$$

$$\sigma_a \;\neq\; \pm \sigma_b \quad \text{for } a \neq b. \tag{6.6}$$

The Weyl group orbifold W has the same form as for $SO(2k+1)$: it is an extension

$$1 \;\longrightarrow\; (\mathbb{Z}_2)^k \;\longrightarrow\; W \;\longrightarrow\; S_k \;\longrightarrow\; 1, \tag{6.7}$$

where the symmetric group S_k acts by exchanging σ_a, and each $\mathbb{Z}_2$ acts by flipping signs of σ_a's.

The Coulomb branch relation (6.4) can be rewritten as

$$\sigma_a^N \;+\; \left(\sum_{i<j} \tilde{m}_i \tilde{m}_j \right) \sigma_a^{N-2} \;+\; \left(\sum_{i_1<i_2<i_3<i_4} \tilde{m}_{i_1} \tilde{m}_{i_2} \tilde{m}_{i_3} \tilde{m}_{i_4} \right) \sigma_a^{N-4} \;+\; \cdots \;=\; 0, \tag{6.8}$$

which is symmetric under $\sigma \mapsto -\sigma$.

In this theory, N is always odd, so this has roots

$$\sigma \;=\; 0, \pm \tilde{\sigma}_1, \cdots, \pm \tilde{\sigma}_{(N-1)/2}, \tag{6.9}$$

where for generic $\tilde{m}_i$, none of the $\tilde{\sigma}_i$ vanish. The zero root is not an allowed solution, as it sits on the excluded locus. The Weyl group orbifold acts on each σ_a by sign flips, and also exchanges different σ_a, so as a result, since there are no twisted sector contributions, we see there are

$$\left(\begin{array}{c} (N-1)/2 \\ k \end{array} \right) \tag{6.10}$$

vacua.

6.2 Mirror to $Sp(N-2k-1)$ gauge theory

The mirror to the second theory is a Weyl-group orbifold of a Landau-Ginzburg model with fields

- $W_\mu^i = \exp(-Y_\mu^i/2)$, $1 \leq i \leq N$, $1 \leq \mu \leq N-2k-1$, mirror to the fundamentals $\tilde{x}^i$, which we take to have R-charge 1,
- $A_{ij} = -A_{ji}$, mirror to the singlets a_{ij}, which we take to have R-charge 0,
- $X_{\mu\nu}$, $\mu \leq \nu$, $1 \leq \mu, \nu \leq N-2k-1$, excluding $X_{2a-1,2a}$ (which would be mirror to the Cartan subalgebra),
- σ_a, $1 \leq a \leq (1/2)(N-2k-1)$,

with superpotential

$$\begin{aligned} W \;=\; & \sum_a \sigma_a \Bigg(-2 \sum_{i,\mu} \rho_{i\mu}^a \ln W_\mu^i \;-\; \sum_\mu \alpha_{\mu\mu}^a \ln X_{\mu\mu} \\ & \qquad - \sum_{b<c} \big(\alpha_{2b,2c}^a \ln X_{2b,2c} + \alpha_{2b-1,2c-1}^a \ln X_{2b-1,2c-1} \\ & \qquad\qquad + \alpha_{2b-1,2c}^a \ln X_{2b-1,2c} + \alpha_{2b,2c-1}^a \ln X_{2b,2c-1} \big) \Bigg) \\ & + \sum_{i\mu} \left(W_\mu^i \right)^2 \;+\; 2 \sum_{i\mu} \tilde{m}_i \ln W_\mu^i \\ & + \sum_\mu X_{\mu\mu} \;+\; \sum_{a<b} \left(X_{2a,2b} + X_{2a-1,2b-1} + X_{2a-1,2b} + X_{2a,2b-1} \right) \\ & + \sum_{i<j} \exp(-A_{ij}) \;+\; \sum_{i<j} \left(\tilde{m}_i + \tilde{m}_j \right) A_{ij}, \end{aligned} \tag{6.11}$$

where

$$\rho^a_{i\mu} = \delta_{\mu,2a} - \delta_{\mu,2a-1}, \tag{6.12}$$

$$\alpha^a_{\mu\nu} = \delta_{\mu,2a} - \delta_{\mu,2a-1} + \delta_{\nu,2a} - \delta_{\nu,2a-1}. \tag{6.13}$$

In addition to the Weyl orbifold, there is an orbifold by $(\mathbb{Z}_2)^{N(N-k-1)}$, where the generator of each $\mathbb{Z}_2$ acts as

$$W^i_\mu \mapsto -W^i_\mu. \tag{6.14}$$

The extra superpotential term of the dual theory defines R-charges for the fields, but does not otherwise appear in the mirror.

Integrating out the σ_a gives the constraints

$$\ln\left(\prod_{i=1}^N \frac{W^i_{2a-1}}{W^i_{2a}}\right)^2 + \ln\left(\prod_{\mu\le\nu} \frac{X_{2a-1,\nu}}{X_{2a,\nu}}\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) = 0. \tag{6.15}$$

We define

$$W^{i,a}_+ \equiv W^{i,2a-1}W^{i,2a}, \tag{6.16}$$

$$W^{i,a}_- \equiv \frac{W^{i,2a-1}}{W^{i,2a}}, \tag{6.17}$$

in terms of which the constraint becomes

$$\ln\left(\prod_{i=1}^N W^{i,a}_-\right)^2 + \ln\left(\prod_{\mu\le\nu} \frac{X_{2a-1,\nu}}{X_{2a,\nu}}\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right) = 0. \tag{6.18}$$

We use this constraint to eliminate $W^{N,a}_-$:

$$\ln\left(W^{N,a}_-\right)^2 = -\sum_{i=1}^{N-1} \ln\left(W^{i,a}_-\right)^2 - \sum_{\mu\le\nu} \ln\left(\frac{X_{2a-1,\nu}}{X_{2a,\nu}}\frac{X_{\mu,2a-1}}{X_{\mu,2a}}\right). \tag{6.19}$$

To make our notation consistent with earlier sections, define

$$\Upsilon_a \equiv W^{N,a}_-, \tag{6.20}$$

$$= \left(\prod_{i=1}^{N-1} W^{i,a}_-\right)^{-1} \left[\prod_{\mu\le\nu} \frac{X_{2a,\nu}}{X_{2a-1,\nu}}\frac{X_{\mu,2a}}{X_{\mu,2a-1}}\right]^{1/2}. \tag{6.21}$$

The superpotential now becomes

$$\begin{aligned} W &= \sum_{i=1}^{N-1}\sum_a W_+^{i,a}\left(W_-^{i,a} + \frac{1}{W_-^{i,a}}\right) + \sum_a W_+^{N,a}\left(\Upsilon_a + \Upsilon_a^{-1}\right) \\ &+ \sum_{i=1}^{N}\sum_a \tilde{m}_i \ln\left(W_+^{i,a}\right)^2 \\ &+ \sum_\mu X_{\mu\mu} + \sum_{a<b}\left(X_{2a,2b} + X_{2a-1,2b-1} + X_{2a-1,2b} + X_{2a,2b-1}\right) \\ &+ \sum_{i<j}\exp(-A_{ij}) + \sum_{i<j}\left(\tilde{m}_i + \tilde{m}_j\right)A_{ij}. \end{aligned} \tag{6.22}$$

The critical locus is then given by

$$W_-^{ia}: \quad W_+^{i,a}\left(W_-^{i,a} + \frac{1}{W_-^{i,a}}\right) = -2\tilde{m}_i \quad \text{for } i < N, \tag{6.23}$$

$$W_+^{N,a}: \quad W_+^{N,a}\left(\Upsilon_a + \Upsilon_a^{-1}\right) = -2\tilde{m}_N, \tag{6.24}$$

$$W_-^{i,a}: \quad W_+^{i,a}\left(W_-^{i,a} - \frac{1}{W_-^{i,a}}\right) = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right), \quad \text{for } i < N, \tag{6.25}$$

$$A_{ij}: \quad \exp\left(-A_{ij}\right) = \tilde{m}_i + \tilde{m}_j, \tag{6.26}$$

$$X_{\mu\nu}: \quad X_{2a,2a} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right), \tag{6.27}$$

$$X_{2a-1,2a-1} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) = -X_{2a,2a}, \tag{6.28}$$

$$2X_{2a,2b} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) + W_+^{N,b}\left(\Upsilon_b - \Upsilon_b^{-1}\right) \quad \text{for } a < b, \tag{6.29}$$

$$2X_{2a,2b-1} = W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right) + W_+^{N,b}\left(-\Upsilon_b + \Upsilon_b^{-1}\right), \tag{6.30}$$

$$2X_{2a-1,2b} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right) + W_+^{N,b}\left(\Upsilon_b - \Upsilon_b^{-1}\right) \tag{6.31}$$

$$= -2X_{2a,2b-1}, \tag{6.32}$$

$$2X_{2a-1,2b-1} = W_+^{N,a}\left(-\Upsilon_a + \Upsilon_a^{-1}\right) + W_+^{N,b}\left(-\Upsilon_b + \Upsilon_b^{-1}\right) \quad \text{for } a < b, \tag{6.33}$$

$$= -2X_{2a,2b}. \tag{6.34}$$

On the critical locus, define

$$\sigma_a \equiv \frac{1}{2}W_+^{i,a}\left(W_-^{i,a} - \frac{1}{W_-^{i,a}}\right) \quad \text{for } i < N, \tag{6.35}$$

$$= \frac{1}{2}W_+^{N,a}\left(\Upsilon_a - \Upsilon_a^{-1}\right). \tag{6.36}$$

These expressions match due to the third critical locus equation above.

It is straightforward to verify the Coulomb branch relation

$$\prod_{i=1}^{N} (\sigma_a - \tilde{m}_i) \;=\; \prod_{i=1}^{N} (-\sigma_a - \tilde{m}_i) \,. \tag{6.37}$$

The excluded locus is given by

$$\sigma_a \;\neq\; \pm\tilde{m}_i, \tag{6.38}$$

$$\sigma_a \;\neq\; \pm\sigma_b \quad \text{for } a \leq b, \tag{6.39}$$

hence, for example, $\sigma_a \neq 0$.

Now, by assumption, N is odd, so the Coulomb branch relation above reduces to a degree N polynomial with roots

$$\sigma \;=\; 0, \pm\tilde{\sigma}_1, \cdots, \pm\tilde{\sigma}_{(N-1)/2}. \tag{6.40}$$

The zero root is excluded, and as the Weyl orbifold group exchanges vacua and flips signs of individual σ_a, we find that the total number of vacua is

$$\left(\begin{array}{c} (N-1)/2 \\ k \end{array} \right), \tag{6.41}$$

all in the untwisted sector, which matches our previous result for the number of vacua in the mirror to the corresponding $Sp(2k)$ gauge theory.

Thus, we see that at the level of vacua, the nonabelian mirror proposal is consistent with the symplectic-symplectic duality of [4].

7 Conclusions

In this paper we have described an extension of the nonabelian mirrors proposal of [1], from mirrors of two-dimensional (2,2) supersymmetric gauge theories with connected gauge groups to include the gauge groups $O_\pm$. We have checked the proposal by comparing mirrors to each side of the gauge theories dualities of [4]. Since the original gauge duality relates gauge theories in the IR, the proposal here relates the mirrors in the IR, and so we compared the number of vacua. For the groups $O_\pm$, unlike connected gauge groups [1], the orbifold group acting on the Landau-Ginzburg mirror often has fixed points intersecting the critical loci of the mirror superpotential, and so one must take into account twisted sectors. The mirror dualities often relate untwisted sector vacua to twisted sector contributions, making for a rather intricate test of the nonabelian mirror proposal. In cases with nontrivial IR limits, we have also checked central charges, and in addition we have also checked mirrors to some other unrelated properties discussed in [4].

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A Higgs branches and chiral rings

Most of this paper and previous work [1,3] has focused on understanding Coulomb branches of gauge theories in mirror constructions. Briefly, we suggest in this section that the mirror to a Higgs branch might be encoded in asymptotic limits of the fields, and based on that idea, we will outline a proposal for mirrors to the chiral rings discussed in [4].

Gauge-invariant operators in a (2,2) GLSM can roughly be characterized into two types:

- Products of σs, along a Coulomb branch. These correspond to equivariant cohomology of the ambient space, as discussed in detail in e.g. [14][appendix A].
- Gauge-invariant combinations of matter fields, along a Higgs branch.

For a GLSM describing a compact space such as a hypersurface, the gauge-invariant matter field combinations typically corresopnd to algebraic complex structure deformations. For example, for the quintic hypersurface in $\mathbb{P}^4$, the gauge-invariant field combinations have the form

$$pf_5(\phi),$$

and products thereof, where $f_5(\phi)$ is a degree five polynomial in the charge-one matter fields. If the target is a Calabi-Yau threefold, these can be identified with elements of $H^{2,1}$, modulo the usual subtleties of overcounting and non-algebraic deformations.

For a GLSM with vanishing superpotential, the same gauge-invariant field combinations merely correspond to functions on the target space. For example, for the GLSM with gauge group $U(1)$ and fields

	x_0	x_1	p
$U(1)$	1	1	-2

the gauge-invariant field combinations are

$$pf_2(x_0, x_1),$$

which simply correspond to the invariant function ring on $\mathbb{C}^2/\mathbb{Z}_2$: if we identify

$$x \equiv px_0^2, \quad y \equiv px_1^2, \quad z \equiv px_0x_1, \tag{A.1}$$

then the ring of gauge-invariant polynomials is

$$\mathbb{C}[x,y,z]/(xy=z^2). \tag{A.2}$$

As the GLSM itself is describing a resolution of $\mathbb{C}^2/\mathbb{Z}_2$, this is entirely appropriate.

In any event, due to the fact that for GLSMs describing compact spaces, the rings of gauge invariant operators corresond to algebraic complex structure deformations, this ring of GLSM operators is sometimes referred to as the GLSM "(c,c)" ring, generalizing the notion of chiral rings in (2,2) SCFTs [15].

In the remainder of this section, we will briefly outline a suggestion for how to see such chiral rings for $(S)O(k)$ theories, from [4][section 4.7], in the mirror presented here, in the special case $k = 2$. As discussed there, the chiral ring in the untwisted sector of an $O(k)$ theory is

$$\mathbb{C}[(x_ix_j)]/J_1, \tag{A.3}$$

and the chiral ring of an $SO(k)$ theory is

$$\mathbb{C}[(x_ix_j),[x_{i_1}\cdots x_{i_k}]]/(J_1,J_2,J_3), \tag{A.4}$$

where J_1 denotes relations of the form

$$\det \begin{bmatrix} (x_{i_0}x_{j_0}) & \cdots & (x_{i_0}x_{j_k}) \\ \vdots & & \vdots \\ (x_{i_k}x_{j_0}) & \cdots & (x_{i_k}x_{j_k}) \end{bmatrix}, \tag{A.5}$$

J_2 denotes relations of the form

$$[x_{i_1}\cdots x_{i_k}][x_{j_1}\cdots x_{j_k}] \;=\; \det \begin{bmatrix} (x_{i_1}x_{j_1}) & \cdots & (x_{i_1}x_{j_k}) \\ \vdots & & \vdots \\ (x_{i_k}x_{j_1}) & \cdots & (x_{i_k}x_{j_k}) \end{bmatrix}, \tag{A.6}$$

and J_3 denotes relations of the form

$$\sum_{p=0}^{k}(-)^p[x_{i_0}\cdots\widehat{x_{i_p}}\cdots x_{i_k}](x_{i_p}x_j) \;=\; 0, \tag{A.7}$$

where

$$(x_ix_j) \;=\; \sum_{ab} x_i^a x_j^b \delta_{ab}, \tag{A.8}$$

$$[x_{i_1}\cdots x_{i_k}] \;=\; \det \begin{bmatrix} x_{i_1}^1 & x_{i_2}^1 & \cdots & x_{i_k}^1 \\ x_{i_1}^2 & x_{i_2}^2 & \cdots & x_{i_k}^k \\ \vdots & & & \vdots \\ x_{i_1}^k & x_{i_2}^k & \cdots & x_{i_k}^k \end{bmatrix}. \tag{A.9}$$

Formally, we will consider a region where all the $Y^i \to \infty$. As a result, defining

$$Y_-^i = \frac{1}{2}\left(Y_2^i - Y_1^i\right), \tag{A.10}$$

$$Y_+^i = \frac{1}{2}\left(Y_2^i + Y_1^i\right), \tag{A.11}$$

we see that each $Y_+^i \to \infty$, but we will take limits in such a way that each Y_-^i remains finite.

First, consider an $O(2)$ theory. We make the ansatz that the mirror of $(x_i x_j)$ is

$$(x_i x_j) \leftrightarrow \exp\left(-Y_-^i\right) + \exp\left(-Y_-^j\right) + \exp\left(+Y_-^i\right) + \exp\left(+Y_-^j\right). \tag{A.12}$$

It is straightforward to show that these quantities have relations in the ideal J_1, which here is generated by relations of the form

$$\det \begin{bmatrix} (x_{i_0}x_{j_0}) & (x_{i_0}x_{j_1}) & (x_{i_0}x_{j_2}) \\ (x_{i_1}x_{j_0}) & (x_{i_1}x_{j_1}) & (x_{i_1}x_{j_2}) \\ (x_{i_2}x_{j_0}) & (x_{i_2}x_{j_1}) & (x_{i_2}x_{j_2}) \end{bmatrix}. \tag{A.13}$$

It is straightforward to check that the mirror operators have vanishing determinants of 3×3 matrices of the form above, meaning for example

$$\det \begin{bmatrix} x_1 + x_1^{-1} + y_1 + y_1^{-1} & x_1 + x_1^{-1} + y_2 + y_2^{-1} & x_1 + x_1^{-1} + y_3 + y_3^{-1} \\ x_2 + x_2^{-1} + y_1 + y_1^{-1} & x_2 + x_2^{-1} + y_2 + y_2^{-1} & x_2 + x_2^{-1} + y_3 + y_3^{-1} \\ x_3 + x_3^{-1} + y_1 + y_1^{-1} & x_3 + x_3^{-1} + y_2 + y_2^{-1} & x_3 + x_3^{-1} + y_3 + y_3^{-1} \end{bmatrix} = 0,$$

for $x_i = \exp(-Y_-^i)$, $y_j = \exp(-Y_-^j)$, but for example corresponding 2×2 matrices do not have vanishing determinant:

$$\det \begin{bmatrix} x_1 + x_1^{-1} + y_1 + y_1^{-1} & x_1 + x_1^{-1} + y_2 + y_2^{-1} \\ x_2 + x_2^{-1} + y_1 + y_1^{-1} & x_2 + x_2^{-1} + y_2 + y_2^{-1} \end{bmatrix} \neq 0,$$

exactly as expected for the relations to be encoded in the ideal J_1.

Next, consider an $SO(2)$ theory. We claim the mirror map acts on chiral ring generators as

$$(x_i x_j) \leftrightarrow \exp\left(-Y_-^i\right) + \exp\left(-Y_-^j\right), \tag{A.14}$$

$$[x_i x_j] \leftrightarrow i\left(\exp\left(-Y_-^i\right) - \exp\left(-Y_-^j\right)\right). \tag{A.15}$$

It is straightforward to check that the mirror operators obey the relations in the ideals J_1, J_2, J_3. For example, for $x_i = \exp(-Y_-^i)$, $y_j = \exp(-Y_-^j)$, it is straightforward to show that

$$\det \begin{bmatrix} x_1 + y_1 & x_1 + y_2 & x_1 + y_3 \\ x_2 + y_1 & x_2 + y_2 & x_2 + y_3 \\ x_3 + y_1 & x_3 + y_2 & x_3 + y_3 \end{bmatrix} = 0,$$

but they do not obey relations encoded in determinants of smaller matrices, for example

$$\det \begin{bmatrix} x_1 + y_1 & x_1 + y_2 \\ x_2 + y_1 & x_2 + y_2 \end{bmatrix} \neq 0.$$

Hence, the mirror operators are in the ideal J_1, but do not obey a smaller subset of those relations. Similarly, it is straightforward to show that

$$-(x_1 - x_2)(y_1 - y_2) = \det \begin{bmatrix} x_1 + y_1 & x_1 + y_2 \\ x_2 + y_1 & x_2 + y_2 \end{bmatrix},$$

which generates the ideal J_2, and also

$$i(x_1 - x_2)(x_0 + y) - i(x_0 - x_2)(x_1 + y) + i(x_0 - x_1)(x_2 + y) = 0,$$

which generates the ideal J_3.

We leave proposals for mirrors to chiral ring structures in $(S)O(k)$ theories for $k > 2$ to future work.

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