# Undoing decomposition

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In this paper we discuss gauging one-form symmetries in two-dimensional theories. The existence of a global one-form symmetry in two dimensions typically signals a violation of cluster decomposition – an issue resolved by the observation that such theories decompose into disjoint unions, a result that has been applied to, for example, Gromov-Witten theory and gauged linear sigma model phases. In this paper we describe how gauging one-form symmetries in two-dimensional theories can be used to select particular elements of that disjoint union, effectively undoing decomposition. We examine such gaugings explicitly in examples involving orbifolds, nonsupersymmetric pure Yang-Mills theories, and supersymmetric gauge theories in two dimensions. Along the way, we learn explicit concrete details of the topological configurations that path integrals sum over when gauging a one-form symmetry, and we also uncover 'hidden' one-form symmetries.

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# Contents

# 1 Introduction

This paper is devoted to gauging one-form symmetries in two-dimensional theories. Two-dimensional theories with global one-form symmetries have been studied for a number of years, see for example [1–5], which discussed a variety of examples in orbifolds and gauge theories, including, for example, analogues of the supersymmetric $\mathbb{P}^n$ model, and how these theories are different from ordinary theories via theta angle periodicities, massless spectra, partition functions, anomalies, quantum cohomologies, and mirrors. (Lattice gauge theories with analogous properties had been studied even earlier.) In particular, any two-dimensional orbifold or gauge theory in which a finite subgroup of the gauge group acts trivially can exhibit a one-form symmetry, under which one modifies any gauge bundle by tensoring in a bundle whose structure group is in the trivially-acting subgroup to get a different nonperturbative sector that is symmetric with respect to the original one. (These theories can also be understood as sigma models on generalized spaces known as gerbes, which geometrically admit one-form symmetries, though we shall not emphasize that perspective in this paper.)

One of the properties of these theories is that they violate cluster decomposition (as can be seen, for example, from the multiplicity of dimension zero operators), but they do so in the mildest possible way. Specifically, such theories 'decompose' into disjoint unions of theories, a result described in [4] as the 'decomposition conjecture.'

The decomposition conjecture has been checked in a wide variety of ways and in numerous two-dimensional examples. We list here a few highlights:

- In orbifolds, decomposition reproduces multiloop partition functions, correlation functions, and massless spectra [2, 4].
- In gauged linear sigma models (GLSMs), decomposition reproduces quantum cohomology rings and is manifestly visible in mirrors [1, 4, 6–8].
- In two-dimensional nonsupersymmetric pure Yang-Mills theories, decomposition reproduces partition functions and correlation functions of Wilson loops [9].
- In supersymmetric two-dimensional gauge theories, decomposition reproduces partition functions via splitting lattices [9].
- In K theory, decomposition reproduces the structure of the K theory groups, and visibly illustrates how D-brane charges split into charges for two distinct summands [4]. Derived categories decompose similarly [4]. Ext groups also are only nonzero between complexes corresponding to the same component, corresponding to the fact that open strings endpoints must lie on the same connected component in a disjoint union.

Other examples of decomposition and computations checking decomposition in two-dimensional theories are outlined in [4, 9].

The decomposition conjecture [4] makes a prediction for Gromov-Witten theory, namely that Gromov-Witten invariants of gerbes should be equivalent to Gromov-Witten invariants of disjoint unions of ordinary spaces. This was checked and proven rigorously in the mathematics literature, see e.g. [10–15], reproducing expectations from physics.

Another application of decomposition was to understand phases of certain gauged linear sigma models [5]. Briefly, in certain theories, locally in a Born-Oppenheimer approximation one has a $\mathbb{Z}_2^{(1)}$ one-form symmetry, so that the theory decomposes. This one-form symmetry is broken along a codimension-one locus, about which there are nontrivial Berry phases. This results in a geometric interpretation as a branched double cover. This trick has been utilized since in for example [16–20], and see also [21] for a recent summary.

A discussion of decomposition in two-dimensional theories as a limit of dualities in three-dimensional theories is given in [22].

In this paper, we will see explicitly how gauging such discrete one-form symmetries in two-dimensional theories can 'undo' decomposition, by projecting onto components of the decomposition. Some highlights include:

- We will make a prediction for the topological classes that path integrals should sum over when gauging one-form symmetries. Specifically, it appears that path integrals only sum over 'banded' gerbes, not more general gerbes, when gauging one-form symmetries. We will also see explicitly how gauge theories in a sector of a nontrivial gerbe are modified.
- We will uncover 'hidden' one-form symmetries. Specifically, we will see in examples that in addition to the 'obvious' one-form symmetries that arise in gauge theories with matter that is invariant under a subgroup of the gauge group, there can be additional one-form symmetries that do not correspond to trivially-acting gauge subgroups.

We should mention that there has recently been a great deal of interest in gauging one-form symmetries in other dimensions, see for example [23–26].

We begin in section 2 by reviewing work on decomposition of two-dimensional theories with one-form symmetries. In section 3 we review how sigma models on disjoint unions of spaces naturally admit one-form symmetries, to help illustrate the connection between decomposition and one-form symmetries in two-dimensional theories. In section 4 we turn to one-form symmetries in two-dimensional orbifolds, and discuss how discrete torsion and various modified group actions can obstruct the existence of a one-form symmetry that would otherwise arise from a trivial group action. In section 5 we explicitly describe the gauging of one-form symmetries in orbifolds, and discuss how such one-form gauging can 'undo decomposition' by projecting onto particular factors in the decomposition. In section 6 we discuss a number of explicit orbifold examples, to make clear both the decomposition of a two-dimensional theory with a one-form symmetry, as well as the detailed structure of gauging that one-form symmetry and how the decomposition summands are recovered via gauging. We conclude that discussion of orbifold examples by outlining two-dimensional Dijkgraaf-Witten theory in this language, which is just a theory of orbifolds of a point.

In section 7 we turn our attention to gauge theories with non-finite gauge groups, beginning with the case of pure Yang-Mills theory in two dimensions, for which there exists exact expressions for partition functions and correlation functions. We review how decomposition arises in two-dimensional pure Yang-Mills theories with center symmetry, and how the corresponding one-form symmetry can be explicitly gauged, recovering partition functions of the decomposition summands. In section 8 we discuss two-dimensional (2,2) supersymmetric gauge theories. In particular, we discuss the effect of gauging one-form symmetries in a family of gerby generalizations of the supersymmetric $\mathbb{P}^n$ model, and how various physical features of those theories and their one-form gauging can be seen in their mirrors, mirrors to both abelian and nonabelian gauge theories. We also discuss partition functions for such supersymmetric gauge theories obtained via supersymmetric localization, and explicitly illustrate the one-form-symmetry gauging at the level of such partition functions, verifying that the effect is to select summands in decomposition. In section 9 we discuss how these matters can be seen in open string charges as computed in K theory.

Finally, in appendix A we discuss how these phenomena can be understood geometrically in terms of generalized spaces known as stacks. In particular, although we will not emphasize this point of view, the theories described in this paper also form examples of sigma models on special stacks known as gerbes, which geometrically admit one-form symmetries. This is one geometric way of understanding the presence of global one-form symmetries in these two-dimensional theories: just as a sigma model on a space with an action of an ordinary group $G$ itself has a global symmetry group $G$, so too do sigma models on generalized spaces with actions of $BG = G^{(1)}$ admit global $BG = G^{(1)}$ symmetries. It is also worth noting that the results in this paper implicitly are making predictions for the Gromov-Witten invariants of higher stacks.

# 2 Review of decomposition

In orbifolds and two-dimensional gauge theories with a discrete one-form symmetry, cluster decomposition is typically violated. This issue was studied in [4], where it was argued that the theories are equivalent to disjoint unions, which also violate cluster decomposition, but do so in a controllable fashion.

The paper [4] focused on orbifolds and gauged sigma models in which a subgroup of the orbifold or gauge group acts trivially (hence, the theory admits a discrete one-form symmetry). Specifically, suppose one has a $G$ orbifold or $G$-gauged sigma model of a space $X$ in which a finite normal subgroup $K \subseteq G$ acts trivially. The theory then has[1] a $BZ(K) = Z(K)^{(1)}$ one-form symmetry, where $Z(K)$ denotes the center of $K$. (The one-form symmetry acts by tensoring any gauge bundle with a $BZ(K)$ bundle – if all the matter is invariant under $Z(K)$, then, this symmetrically permutes nonperturbative sectors, defining a one-form symmetry, barring obstructions defined by giving non-symmetric phases to different sectors related in this fashion, as in section 4.)

Let $\hat{K}$ denote the set of irreducible representations of $K$. $\hat{K}$ admits a natural action of the group $H \equiv G/K$. Then, the central claim of [4] is that the $G$ orbifold or gauged sigma model is equivalent to a sigma model on $Y \equiv [(X \times \hat{K})/H]$.

The space $Y$ will have multiple disjoint components, as many as orbits of $H$ on $\hat{K}$. If $H_1, \cdots, H_n$ are the stabilizers in $H$ of the various orbits, then

$$Y \;=\; \coprod_i [X/H_i], \tag{2.1}$$

and each component has a natural $B$ field, as described in [4]. See e.g. [9] for analogous statements for other two-dimensional gauge theories, not necessarily with a geometric interpretation. Such theories also admit a decomposition into disjoint QFTs. The result above has a number of applications, e.g. to Gromov-Witten theory and to understanding phases of gauged linear sigma models, as was outlined in the introduction.

Now, this picture can be simplified. Quotients in which a nontrivial subgroup acts trivially define generalized spaces known as gerbes, Gerbes are better known in connection with $B$ fields, but can also be interpreted as analogues of spaces, special cases of stacks, admitting metrics, spinors, bundles, gauge fields, and so forth, just like an ordinary manifold, and more to the point, admit one-form symmetries, just as an ordinary space might admit an ordinary group of symmetries. In any event, there are special classes of gerbes known as 'banded gerbes,' for which decomposition simplifies. Briefly, a $G$ gerbe over a space $M$ is said to be banded if it is classified by $H^2(M, C^\infty(G))$. (More general gerbes have a more complicated classification.) For examples, if $G = U(1)$, it is the banded gerbes whose connections are $B$ fields. (See [4, section 3] for more information on the distinction between banded and non-banded gerbes.)

In the special case that the quotient $[X/G]$ defines a banded $K$-gerbe, the description of decomposition simplifies. In this case, the $H$ action on $\hat{K}$ is trivial, hence $[(X \times \hat{K})/H] \cong [X/H] \times \hat{K}$, and so there are as many components as elements of $\hat{K}$. Furthermore, the flat $B$ field on each component is determined by the image of the characteristic class of the gerbe

[1] One-form symmetries are only defined for abelian groups. If $K$ is abelian, then the theory has a $K^{(1)}$ one-form symmetry. If $K$ is not abelian, then $Z(K)^{(1)}$ acts on the fibers of the $K$-gerbe.

under a map defined by the corresponding irreducible representation $\rho \in \hat{K}$:

$$H^2([X/H], Z(K)) \xrightarrow{\rho} H^2([X/H], U(1)). \tag{2.2}$$

Briefly, in this paper we will argue that decomposition can be undone by gauging the corresponding one-form symmetries. There can be multiple such one-form symmetry actions on a given theory: we can weight different sectors in the path integral by phases, and use those phases to select amongst the different components. As we shall see in more detail in examples later, when we gauge a one-form symmetry, the path integral sums over banded gerbes on the worldsheet. We can weight the different gerbe sectors by an analogue of a discrete theta angle, a phase factor in the path integral of the form $\exp(w_2(\xi))$, where $\xi$ is the gerbe on the worldsheet appearing in the sum, and $\exp(w_2(\xi))$ is a phase determined by the characteristic class of the gerbe on the worldsheet. In terms of decomposition, for any irreducible representation $\rho$ of $G$, we can assign a phase to a banded $G$ gerbe on the worldsheet $\Sigma$ as the image of

$$H^2(\Sigma, G) \xrightarrow{\rho} H^2(\Sigma, U(1)). \tag{2.3}$$

We shall see that for this phase factor, we recover the component corresponding to $\rho$. (In this paper, we focus on gauging one-form symmetries in banded gerbes, and leave more general discussions for later work.)

# 3 One-form symmetries in disjoint unions

In this section we will discuss how sigma models on disjoint unions admit (global) discrete one-form symmetries, acting on discrete Fourier transforms of projection operators and domain walls.

Suppose $X$ is a disjoint union of $k$ Calabi-Yau's, denoted $X_i$:

$$X = \coprod_{i=0}^{k-1} X_i. \tag{3.1}$$

A sigma model on $X$ will admit a collection of projection operators, projecting onto states associated with each summand $X_i$. These are constructed as linear combinations of the dimension zero operators in the theory (of which there will be one for every connected component $X_i$.) (See e.g. [4] for further details.)

Let $\Pi_i$ denote a projection operator in the CFT of a sigma model on $X$, projecting onto states corresponding to $X_i$, where $i \in \{0, \cdots, k-1\}$. Discrete Fourier transforms of these projection operators form the elements of the group $\mathbb{Z}_k$. For example,

$$1 \;=\; \sum_{j=0}^{k-1} \Pi_j, \tag{3.2}$$

as projecting onto all possibilites is the same as the identity operator. A general discrete Fourier transform takes the form

$$A(p) \;=\; \sum_{j=0}^{k-1} \Pi_j \, \xi^{jp}, \tag{3.3}$$

where

$$\xi \;\equiv\; \exp\left( \frac{2\pi i}{k} \right). \tag{3.4}$$

Note that if we define

$$z \;\equiv\; A(1) \;=\; \sum_{j=0}^{k-1} \Pi_j \, \xi^{j}, \tag{3.5}$$

then it is straightforward to show that

$$z^p \;=\; A(p), \tag{3.6}$$

using the property

$$\Pi_m \Pi_n \;=\; \left\{ \begin{array}{cc} 0 & m \neq n, \\ \Pi_m & m = n \end{array} \right. \tag{3.7}$$

of projection operators. Furthermore, these linear combinations have the multiplications of the group $\mathbb{Z}_k$. For example, it is straightforward to compute that

$$z^p z^q \;=\; \left( \sum_{j=0}^{k-1} \Pi_j \, \xi^{pj} \right) \left( \sum_{\ell=0}^{k-1} \Pi_\ell \, \xi^{q\ell} \right), \tag{3.8}$$

$$=\; \sum_{j=0}^{k-1} \Pi_j \, \xi^{(p+q)j}, \tag{3.9}$$

$$=\; z^{p+q}, \tag{3.10}$$

and also that $z^0 = 1$. As a result, these linear combinations of projection operators form the elements of the group $\mathbb{Z}_k$. Under an $SL(2,\mathbb{Z})$ transformation, these projection operators become, in effect, domain walls on the worldsheet.

Now, consider the case that the worldsheet of the sigma model is $T^2$, for simplicity. One can have domain walls along both the spacelike and timelike directions, giving rise to boundary conditions in close analogy with orbifolds. We have just seen that linear combinations of projection operators form elements of the group $\mathbb{Z}_k$, so we can think of a pair of domain walls on the worldsheet as linear combinations of sectors closely analogous to orbifold twisted sectors, of the form

$$ z^m \underset{z^n}{\square} . \tag{3.11} $$

An element

$$ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL(2,\mathbb{Z}) \tag{3.12} $$

acts on such a sector by mapping

$$ (z^m, z^n) \mapsto (z^{am+bn}, z^{cm+dn}). \tag{3.13} $$

More pertinently, to each such twisted sector, we can associate a $\mathbb{Z}_k$ bundle on the worldsheet.

Now, we are ready to discuss the $\mathbb{Z}_k^{(1)}$ one-form symmetry. Given a $\mathbb{Z}_k$ bundle, we simply tensor that bundle with the bundle corresponding to a linear combination of projection operators/domain walls, to get another such. For example, consider the case of $\mathbb{Z}_2$, and take the worldsheet to be $T^2$. The $\mathbb{Z}_2$ bundle corresponding to

$$ z \underset{z}{\square} \tag{3.14} $$

maps, for example,

$$ 1 \underset{1}{\square} \mapsto z \underset{z}{\square}, \tag{3.15} $$

$$ 1 \underset{z}{\square} \mapsto z \underset{1}{\square}, \tag{3.16} $$

$$ z \underset{z}{\square} \mapsto 1 \underset{1}{\square} . \tag{3.17} $$

Thus, we see in this case that there is a well-defined $\mathbb{Z}_2^{(1)}$ one-form action, which permuts the various projection operators and domain walls of the sigma model on the disjoint union.

In passing, note we have not assumed any symmetry between the various connected components $X_i$ of $X$, only that each is nonempty. This formal argument applies to any space with multiple disconnected (nonempty) components, regardless of the components. Later in this paper we will study two-dimensional theories obtained by gauging groups with trivially-acting subgroups, which will be equivalent to disjoint unions of theories but with components related by various symmetries, a special case of the picture presented in this section.

So far, we have established that a sigma model on a space $X$ with $k$ connected components has a $\mathbb{Z}_k^{(1)}$ one-form symmetry, an action of $\mathbb{Z}_k^{(1)}$. We should pause at this point to observe that this action is not unique. Suppose for example $X$ decomposes into four components $X_i$, so that, from the analysis above, a sigma model on $X$ admits an action of $\mathbb{Z}_4^{(1)}$. Let us illustrate two different sets of ways to get actions of different one-form groups:

- First, group the four $X_i$ into two pairs $Y_0$, $Y_1$, so that $X = \coprod Y_a$ and each $Y_a$ itself can be decomposed. Applying the same argument as above, there is a $\mathbb{Z}_2^{(1)}$ action, that acts by interchanging discrete Fourier transforms of domain walls and projectors onto those two components $Y_0$, $Y_1$. By further decomposing the $Y_a$ and repeating, one sees that $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ also acts on a sigma model on $X$.

- Second, write $Y = X_2 \coprod X_3$ and consider the decomposition
$$X \;=\; X_0 \coprod X_1 \coprod Y. \tag{3.18}$$
From the analysis above, we see that there is an action of $\mathbb{Z}_3^{(1)}$ that interchanges discrete Fourier transforms of domain walls and projectors onto $X_0$, $X_1$, and $Y$.

Thus, for a sigma model on $X$ with four disconnected components, we have derived actions of $\mathbb{Z}_4^{(1)}$, $\mathbb{Z}_2^{(1)}$, $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$, and $\mathbb{Z}_3^{(1)}$. Thus, for theories of this form, one-form symmetry groups are not unique, and which to use will vary depending upon the application.

In passing, note that this non-uniqueness is not specific to one-form symmetries, and also arises in theories with ordinary group symmetries. Consider for example an $SO(3)$ WZW model in two dimensions. It admits a (symmetric) action of $SO(3)$, but also admits a (symmetric) action of $SU(2)$, as the $\mathbb{Z}_2$ center simply factors out. For that matter, there is also a (symmetric) action of $U(1)$, as $U(1)$ is a maximal torus of $SO(3)$.

Next, we shall consider gauging one-form symmetries. We will just outline basics in this section, and will consider this in more detail in later sections. Briefly, we claim that by gauging the one-form symmetries described above, one can pick out particular summands, particular connected components, in a decomposition. (Our discussion will anticipate and cite methodology that will be justified later in section 5. As a result, readers may wish to skip the remainder of this section for the moment and only return after reading section 5.)

To this end, we return to the example of a sigma model on a disjoint union of $k$ spaces $X_n$, which has a $B\mathbb{Z}_k = \mathbb{Z}_k^{(1)}$ one-form symmetry. It is straightforward to show that
$$\Pi_n \;=\; \frac{1}{k}\sum_{j=0}^{k-1} z^j \xi^{-nj}, \tag{3.19}$$
and any given summand in the disjoint union has partition function
$$\mathrm{CFT}(X_n) \;=\; {}_{\Pi_n}\underset{\Pi_n}{\square} \;=\; \frac{1}{k^2}\sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \xi^{-pn}\xi^{-qn}\, {}_{z^p}\underset{z^q}{\square}. \tag{3.20}$$
We claim that this partition function, for a given component, can be obtained by gauging the $\mathbb{Z}_k^{(1)}$ one-form symmetry.

When we gauge a $G^{(1)}$ one-form symmetry, in the partition function,

1. we first sum over banded[2] $G$-gerbes,
2. then in each $G$-gerbe sector, we sum over field configurations twisted by the gerbe, multiplied by a gerbe-dependent phase.

Here, this means that the partition function schematically has the form

$$Z \;=\; \frac{1}{k}\sum_{z\in\mathbb{Z}_k} \epsilon(z)\cdots\,, \tag{3.21}$$

where $z$ defines the characteristic class of the banded gerbe, $\epsilon(z)$ is the gerbe-dependent phase, and the $\cdots$ indicates the path integral over gerbe-twisted field configurations. When constructing the partition function of an orbifold on worldsheet $T^2$, as we shall describe in detail in section 5, in a $z$-gerbe twisted sector, instead of summing over commuting pairs of group elements, one sums over pairs $(g,h)$ obeying $gh = hgz$. Here, since $\mathbb{Z}_k$ is abelian, all group elements commute, so $gh = hgz$ only has solutions when $z = 1$, so the $z$-gerbe twisted sectors are empty except in the case $z = 1$. Furthermore, for $z = 1$, $\epsilon(z) = 1$, as we argue later. As a result, all that is left, aside from the factor of $1/k$, is an ordinary $\mathbb{Z}_k$ orbifold partition function for the $\cdots$ in the schematic description above. That $\mathbb{Z}_k$ orbifold is equivalent to the quantum symmetry orbifold described in [27, section 8.5]. Adding the extra factor of $1/k$ (from the remnant of the sum over $\mathbb{Z}_k$ gerbes), we get that the partition function of a sigma model on the disjoint union after gauging the $\mathbb{Z}_k^{(1)}$ one-form symmetry is

$$\frac{1}{k^2}\sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \xi^{-pn}\xi^{-qn}\, {}_{z^p}\underset{z^q}{\square}, \tag{3.22}$$

which precisely matches the partition function of a component $X_n$ as above in equation (3.20).

# 4 Existence of one-form symmetries in orbifolds

We have just discussed one set of theories with global one-form symmetries, namely, sigma models on disjoint unions. As previously mentioned, another common source of theories with one-form symmetries is a gauge theory (or orbifold) in which a subgroup of the gauge group acts trivially on the matter fields. In such a theory, there is a one-form symmetry, which acts by permuting the nonperturbative sectors (gauge bundles) by tensoring in bundles whose structure group lies in the trivially-acting subgroup. (These theories are related to disjoint unions via decomposition, as reviewed in section 2.)

[2] We will justify summing only over banded gerbes, and not more general gerbes, later in this paper.

However, the one-form symmetry in such a gauge theory can be broken, by giving different phases to nonperturbative sectors permuted by the one-form symmetry, and when the one-form symmetry is broken, so too is the decomposition prediction of [4]. In this section we will discuss specific examples in which the one-form symmetries arising in gauge theories with trivially-acting subgroups are broken.

## 4.1 Discrete torsion

One way to break one-form symmetries is to add phases which are asymmetric between sectors related by tensoring by bundles associated with the one-form symmetry. Such phases are constrained by modular invariance, and a classic example in orbifolds is that of discrete torsion [28, 29]. In this subsection, we will look at some examples of how turning on discrete torsion breaks one-form symmetries (and also decomposition), taken from [4, section 10].

First, consider the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$, where the first $\mathbb{Z}_2$ acts trivially but the second $\mathbb{Z}_2$ acts nontrivially on $X$. This was studied in [4, section 10.1]. Since

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \;=\; \mathbb{Z}_2, \tag{4.1}$$

this orbifold does admit the possibility of turning on (exactly one choice of) discrete torsion.

If we do not turn on discrete torsion, then the one-loop twisted sectors are invariant under tensoring with arbitrary $\mathbb{Z}_2$ bundles, which simply permuts the various twisted sectors. As a result, the theory admits a $\mathbb{Z}_2^{(1)}$ one-form symmetry, and as argued in [4], the theory decomposes into a sigma model on a disjoint union of two copies of $[X/\mathbb{Z}_2]$.

Now, let us modify this orbifold by turning on discrete torsion, which will weight the various one-loop twisted sectors differently and thereby break the one-form symmetry above.

Let $a$ denote the generator of the first (trivial) $\mathbb{Z}_2$, and $b$ the generator of the second. Discrete torsion acts by multiplying the $T^2$ twisted sectors

$$a \underset{b}{\square}, \quad a \underset{ab}{\square}, \quad b \underset{ab}{\square} \tag{4.2}$$

by $-1$.

First, the

$$1 \underset{1}{\square} \tag{4.3}$$

one-loop sector in the effective $[X/\mathbb{Z}_2]$ orbifold emerges from any of the sectors

$$1,a \underset{1,a}{\square}, \tag{4.4}$$

and so appears with multiplicity four. The

$$ {}^{1}\square_{b} \tag{4.5} $$

sector in the effective $[X/\mathbb{Z}_2]$ orbifold arises from any of the

$$ {}^{1,a}\square_{b,ab} \tag{4.6} $$

sectors; however, because of discrete torsion, the sectors

$$ {}^{1}\square_{b,ab}, \quad {}^{a}\square_{b,ab} \tag{4.7} $$

contribute with opposite signs, and so cancel out of the one-loop partition function in the $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ orbifold. Similarly, the

$$ {}^{b}\square_{b}, \quad {}^{ab}\square_{ab} \quad \text{and} \quad {}^{ab}\square_{b}, \quad {}^{b}\square_{ab} \tag{4.8} $$

one-loop sectors contribute with opposite signs because of discrete torsion, and so cancel out. As a result, the one-loop partition function of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ (with discrete torsion) matches that of a sigma model on (one copy of ) $X$, not a disjoint union.

Implicit in the analysis above, discrete torsion introduces an asymmetry which breaks the $\mathbb{Z}_2^{(1)}$ one-form symmetry. Recall that this symmetry acts by tensoring existing bundles by $\mathbb{Z}_2$ bundles. Thus, in order for the symmetry to exist, one would need, for example, the

$$ {}^{1}\square_{b,ab} \tag{4.9} $$

sectors to enter the partition function symmetrically with the

$$ {}^{a}\square_{b,ab} \tag{4.10} $$

sectors, as they differ by tensor product with a $\mathbb{Z}_2$ bundle given by

$$ {}^{a}\square_{1}. \tag{4.11} $$

Since discrete torsion weights those sectors differently, the $\mathbb{Z}_2^{(1)}$ one-form symmetry is necessarily broken in this model with discrete torsion in the noneffectively-acting 'directions.'

This analysis trivially extends to the case of the orbifold $[X/\mathbb{Z}_k \times \mathbb{Z}_k]$, where the first $\mathbb{Z}_k$ acts trivially and the second, nontrivially. (This example was discussed in [4, section 10.2].) Here, $H^2(\mathbb{Z}_k \times \mathbb{Z}_k, U(1)) = \mathbb{Z}_k$, there are $k$ possible values for discrete torsion.

If one does not turn on discrete torsion at all, then all the twisted sectors are symmetric with respect to one another, and tensoring in $\mathbb{Z}_k \times \mathbb{Z}_k$ bundles simply permutes them. In this case, the theory admits a $\mathbb{Z}_k^{(1)}$ one-form symmetry. In this case, with no discrete torsion, the theory is equivalent to a sigma model on a disjoint union of $k^2$ copies of $X$.

If we turn on discrete torsion, then the symmetry between the one-loop sectors is broken, exactly as we saw in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case, so there is no longer a $\mathbb{Z}_k^{(1)}$ one-form symmetry, and hence the theory need not decompose. Indeed, computations in [4, section 10.2] indicated that in this example, for any nonzero value of discrete torsion, the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ is equivalent to a sigma model on one copy of $X$.

In terms of decomposition of two-dimensional theories with one-form symmetries, turning on discrete torsion breaks the one-form symmetry and so breaks the decomposition. This is discussed in [4, section 10], which discusses explicitly how decomposition is broken in cases where the one-form symmetry is broken.

## 4.2 $(-)^F$

Another, more subtle, way to break naive one-form symmetries is to modify a $\mathbb{Z}_2$ orbifold with a $(-)^F$ factor, as implicit in the analysis of [18, section 2]. (See also [21] for an excellent recent review and overview of the application to gauged linear sigma models [5].) We will see momentarily that the choice of orbifold can 'make or break' the existence of a $\mathbb{Z}_2^{(1)}$ one-form symmetry.

Consider for example a $\mathbb{Z}_2$ orbifold, acting on a two-dimensional (2,2) supersymmetric theory with matter. There are two different $\mathbb{Z}_2$ orbifolds, one of which incorporates an action of $(-)^F$. Curiously, as discussed in [18, section 2], even if all of the matter is massive, the IR behavior is sensitive to the orbifold, in a manner that depends upon the number of massive fields (which are assumed to be acted upon nontrivially by the orbifold).

If all the matter on which the orbifold acts is massive, then at low energies, below the scale of that matter, one naively expects the theory to have a $\mathbb{Z}_2^{(1)}$ one-form symmetry. If that symmetry is indeed present, the decomposition predicts the number of vacua should be even. However, it was shown in [18, section 2] that whether the number of vacua is even depends upon the number of massive fields upon which the orbifold acts, as well as the type of orbifold and whether the masses are complex masses or twisted masses. In particular, if the number of vacua is odd, then a $\mathbb{Z}_2^{(1)}$ one-form symmetry cannot be present in the theory.

Briefly, in a $\mathbb{Z}_2$ orbifold in which the number of chirals with complex masses, acted upon by the orbifold, is even, or in a $\mathbb{Z}_2(-)^F$ orbifold in which the number of chirals with twisted masses, acted upon by the orbifold, is even, the number of vacua is even, consistent with the existence of a $\mathbb{Z}_2^{(1)}$ one-form symmetry. In other cases, one would not expect a one-form symmetry, based on the number of vacua. (See for example [8] for a more detailed overview of the vacuum counting in these theories.)

So far our analysis has been based solely on state counting. We can also outline the obstruction at the level of orbifold twisted sectors. For example, multiplying in

$$ {}_{1}\square_{z} \tag{4.12} $$

maps between twisted and untwisted sector states, so, briefly, existence of a one-form symmetry implies a symmetry between the two, and from the computations in [18, section 2] that symmetry is only present in certain cases.

In passing, the even/odd distinction here is reminiscent of analogous even/odd distinctions in existence and behavior of one-form symmetries discussed in [30].

# 5 Gauging one-form symmetries in orbifolds

In this section we discuss gauging one-form symmetries in orbifolds, setting up the technology we will use in specific examples in the next section. When we discuss specific examples, we will see that this gauging has the effect of projecting onto particular summands in the decomposition of the two-dimensional theory with a one-form symmetry.

In an ordinary orbifold $[X/H]$, we sum over principal $H$-bundles on the worldsheet, each of which defines a twisted sector of the orbifold. For example, the partition function for the case that the worldsheet is $T^2$ has the form

$$ Z \;=\; \frac{1}{|H|} \sum_{gh=hg} {}_{g}\square_{h}, \tag{5.1} $$

where we have used the notation

$$ {}_{g}\square_{h} \tag{5.2} $$

to indicate the contribution to the partition function from a sector with boundary conditions / branch cuts determined by commuting element $g, h \in H$. Each such commuting pair $gh = hg$ defines a principle $H$ bundle on the worldsheet, so the partition function sums over principal $H$-bundles.

For an orbifold $[X/H]$ to have a $BG = G^{(1)}$ one-form symmetry means that $G \subset H$ is a subgroup of the orbifold group that acts trivially on the theory [1–3]. Since $G$ acts trivially, for example, partition functions are invariant under permutations defined by tensoring with $G$ bundles, which defines the action of the one-form symmetry. For example, if $G$ acts trivially on the theory, and $z \in G$, then we can map

$$ {}_{a}\square_{b} \;\mapsto\; {}_{a}\square_{zb}, \quad {}_{za}\square_{b}, \tag{5.3} $$

defined by the $G$ bundles

$$ {}_{1}\underset{z}{\square}\,, \quad {}_{z}\underset{1}{\square}\,, \qquad (5.4) $$

respectively. This is a symmetry of the theory, since the sectors which are exchanged make the same contribution to the partition function – because $G$ acts trivially.

In an orbifold $[X/H]$ with a one-form symmetry $BG = G^{(1)}$, to gauge[3] the one-form symmetry means to

1. sum over banded $G$-gerbes on the worldsheet (i.e., $BG = G^{(1)}$ bundles), and

2. for each $G$-gerbe, sum over $H$ bundles which have been twisted by the $G$ gerbe.

Finally, the sum over gerbes can be weighted by a gerbe-dependent phase factor $\epsilon(z)$, one phase for all contributions to that gerbe sector. In a two-dimensional theory, this phase factor will take the form

$$ \epsilon(z) \; \equiv \; \exp\left( i \langle \xi, z \rangle \right), \qquad (5.5) $$

where $z \in H^2(\Sigma, C^\infty(G))$ is the characteristic class of the banded gerbe on the worldsheet $\Sigma$, and $\xi$ is a fixed (gerbe-independent) class.

Let us take a moment to explain a few details of this procedure. First, we have specified a sum over 'banded' gerbes. For a group $G$, these are the $G$-gerbes on a space $M$ classified by $H^2(M, C^\infty(G))$, and given on triple overlaps by cocycles $h_{\alpha\beta\gamma}$. If $G = U(1)$, these are the gerbes whose connection is a two-form $B$ field – and so summing over banded gerbes naturally correlates with performing a path integral over $B$ fields. However, banded gerbes are special cases of a more general class of gerbes, which are also $BG = G^{(1)}$ bundles. (See for example [4, section 3] for a more detailed discussion of the differences.) Connections on the more general class involve both a $B$ field as well as an ordinary gauge field on an associated outer automorphism bundle, and so summing over nonbanded gerbes as well as banded gerbes would introduce an extra field beyond what has been previously discussed in the literature. In any event, we shall see explicitly in examples that restricting the path integral to banded gerbes, instead of more general gerbes, seems to be the physically consistent choice, and we will leave more thorough treatments of why gauging one-form symmetries involves path integrals that only sum over banded gerbes instead of general gerbes for future work.

[3] One might ask whether there can be gauge anomalies in gauged one-form symmetries. Certainly this can happen in theories in odd dimensions. Consider for example an abelian Chern-Simons theories in three dimensions, whose action can be recast as an integral of $F \wedge F$ over a bounding four-manifold. Gauging a one-form symmetry, schematically, means replacing $F$ by $F - B$, resulting in a term involving an integral of $B \wedge B$ over a bounding four-manifold, interpretable in terms of an anomaly inflow to cancel what must be a one-form gauge anomaly in three dimensions. For a two-form $B$, one can only get such interactions by integrating wedge powers of $B$ in even dimensions, suggesting that one-form gauge anomalies only arise in odd dimensions, and hence are not relevant for this paper, which focuses on two-dimensional theories. We would like to thank Y. Tanizaki and M. Ünsal for this observation.

By 'twisting the $H$ bundle,' we mean that the transition functions of the $H$ bundle on the worldsheet only close on triple overlaps up to the transition functions of the $G$ gerbe, i.e.

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} \;=\; h_{\alpha\beta\gamma} \tag{5.6}$$

for $g_{\alpha\beta}$ the transition functions on double overlaps defining the $H$ bundle, and $h_{\alpha\beta\gamma}$ the transition functions on triple overlaps defining the banded gerbe. (This is the same twisting one encounters when describing Chan-Paton factors in nontrivial $B$ field backgrounds; see for example [31] for a more detailed discussion of twisted bundles on gerbes.) For $H$ bundles on a $T^2$, this means that instead of defining a bundle by a commuting pair $g, h \in H$, we now work with almost-commuting pairs $g, h$, where

$$gh \;=\; hgz \tag{5.7}$$

for $z \in G$. We will denote contributions to partition functions satisfying these boundary conditions by

$$g \underset{h}{\boxed{\mathrm{z}}}. \tag{5.8}$$

As a result, the partition function of a theory in which one gauges a one-form action $BG = G^{(1)}$ on an orbifold $[X/H]$, for the case that the worldsheet is $T^2$, takes the form

$$Z \;=\; \frac{1}{|G|} \frac{1}{|H|} \sum_{z \in G} \sum_{gh = hgz} g \underset{h}{\boxed{\mathrm{z}}}. \tag{5.9}$$

(The sum over $z \in G$ is a sum over characteristic classes of banded $G$ gerbes, since for abelian $G$ on an oriented compact Riemann surface $\Sigma$, $H^2(\Sigma, G) = G$.) Furthermore, we can turn on an analogue of a discrete theta angle or discrete torsion, to weight gerbe-twisted sectors by phases $\epsilon(z)$:

$$Z \;=\; \frac{1}{|G|} \frac{1}{|H|} \sum_{z \in G} \epsilon(z) \sum_{gh = hgz} g \underset{h}{\boxed{\mathrm{z}}}. \tag{5.10}$$

The phases $\epsilon(z)$ are the same phases described earlier – they determine different $G^{(1)}$ one-form group actions on the theory, and in the case of orbifolds, are consistent with modular invariance and factorization constraints as we shall see later. Also note that from the general expression given earlier, $\epsilon(+1) = +1$.

In passing, note that in this fashion one can recover all elements of the original group $H$: for any $g, h \in H$, $gh = hgz$ for $z = (hg)^{-1}(gh) = g^{-1}h^{-1}gh$.

Next, let us consider the action of the modular group, to check that the expression above is well-defined, and to understand what constraints are imposed by modular invariance. $SL(2, \mathbb{Z})$ always acts on $T^2$ twisted sectors with boundary conditions defined by $(g, h)$ by mapping

$$(g, h) \;\mapsto\; (g^a h^b, g^c h^d). \tag{5.11}$$

In an ordinary orbifold, this maps twisted sectors to twisted sectors: since $g$ and $h$ commute, so too do $g^a h^b$ and $g^c h^d$. Here, consider a gerbe-twisted sector, twisted by $z \in G$. Assume for the moment that $z$ lies in the center of $G$. We will argue that, for $z$ in the center of $G$, $SL(2,\mathbb{Z})$ again maps $z$-twisted sectors to $z$-twisted sectors. It is straightforward to show that

$$g^a h^b \;=\; h^b g^a z^{ab}, \tag{5.12}$$

hence

$$g^a h^b g^c h^d \;=\; g^c h^d g^a h^b z^{ad-bc} \;=\; g^c h^d g^a h^b z, \tag{5.13}$$

(since for an $SL(2,\mathbb{Z})$ matrix, $ad - bc = 1$,) hence we propose that modular transformations map

$$g \, \boxed{\text{z}} \, h \tag{5.14}$$

to

$$g^a h^b \, \boxed{\text{z}} \, g^c h^d, \tag{5.15}$$

remaining within the $z$-gerbe-twisted sector.

In general, for $z$ not necessarily in the center, $SL(2,\mathbb{Z})$ transformations will exchange different gerbe characteristic classes. Consider for example the $SL(2,\mathbb{Z})$ transformation

$$\left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]. \tag{5.16}$$

This modular transformation maps the pair $(g,h) \mapsto (gh, h)$. Now, if $gh = hgz$, then

$$ghh \;=\; hgzh \;=\; hgh(h^{-1}zh), \tag{5.17}$$

and so we see that the original $z$-gerbe sector is mapped to an $h^{-1}zh$-gerbe sector. (We will see a more detailed example of this form in section 6.4.)

As a quick consistency check, let us quickly note that $h^{-1}zh$ is still in the group defining the gerbe. Suppose $K$ is any normal subgroup of $H$:

$$1 \longrightarrow K \longrightarrow H \stackrel{f}{\longrightarrow} G \longrightarrow 1. \tag{5.18}$$

Let $z \in K$. This implies that $f(z) = 1$. Now,

$$\begin{aligned} f(h^{-1}zh) &= f(h)^{-1} f(z) f(h), && (5.19) \\ &= f(h)^{-1} f(h), && (5.20) \\ &= 1, && (5.21) \end{aligned}$$

and so we see that $h^{-1}zh \in K$ also.

As a result, in these theories, modular invariance requires us to sum over all gerbes (in addition to bundles), in the same fashion that in ordinary orbifolds, modular invariance requires us to sum over all twisted sectors defined by bundles.

Now, let us turn to the phases $\epsilon(z)$ and constraints imposed by modular invariance. In general, modular transformations map $z$ to some conjugacy class within the orbifold group, so $\epsilon(z)$ must be constant on those conjugacy classes – it must be the restriction of a class function on the orbifold group to the gerbe group $G$, in other words. If for example the orbifold group is abelian, then the set of $z$-gerbe twisted sectors close into themselves under $SL(2,\mathbb{Z})$, and so no constraints are imposed upon the phases $\epsilon(z)$. (In fact, if the orbifold group is abelian, then the only sectors that arise in a partition function are those for which $z = 1$, so $\epsilon(z)$ for $z \neq 1$ is irrelevant.)

Modular invariance is not the only criterion that orbifold phase factors must obey; they must also be consistent with factorization (see e.g. [29, section 4.3]). This says that if a multiloop diagram can degenerate into a product of diagrams with fewer loops, separated by long thin tubes, then since phase factors are moduli independent, the phase factor associated to the multiloop diagram should be the product of the phase factors associated to the diagrams with fewer loops.

In figure 1, we have schematically illustrated an orbifold two-loop diagram, and figure 2 we have illustrated its factorization.

$g$ $h$ $b$

$h$ $a$

$g$ $a$ $b$

Figure 1: A schematic illustration of an orbifold two-loop diagram, i.e. a contribution to an orbifold partition function on a genus two surface. We have assumed that the group elements are such that the diagram factorizes into a product of two one-loop diagrams, in which effectively the dashed line shrinks to a point.

$h$ $b$

$g$ $a$

Figure 2: A two-loop diagram, factored into a pair of one-loop diagram.

If the two one-loop diagram factors are in sectors twisted by gerbe characteristic classes $z_1$, $z_2$, then, in the case that $G$ lies in the center of $H$, the two-loop diagram twisted by gerbe characteristic class $z_1 z_2$. In two dimensional theories on a space $\Sigma$, since $H^2(\Sigma)$ is one-dimensional, independent of the genus of $\Sigma$, the phases associated to two-loop diagrams are the same as the phases associated to one-loop diagrams, and so we find, again for the case that $G$ is in the center of the orbifold group, that

$$\epsilon(z_1 z_2) \;=\; \epsilon(z_1)\epsilon(z_2). \tag{5.22}$$

Thus, formally[4], factorization implies that $\epsilon : G \to U(1)$ is a group homomorphism, which is consistent with its first-principles description, given earlier, as a phase factor derived from the characteristic class of the banded gerbe on the worldsheet.

We leave a discussion of factorization in non-banded gerbes, in which $G$ does not lie in the center of $H$, for future work.

So far we have discussed partition functions. Next, we shall briefly turn our attention to massless spectra. Of course, once one has computed the partition function, one can read off the massless spectrum, so a separate discussion of massless spectra in orbifolds in which a one-form symmetry has been gauged is somewhat redundant. As a result, we will be brief. In a nutshell, because the one-form symmetry permutes twisted sectors, e.g. schematically

$$a \underset{b}{\square} \;\mapsto\; za \underset{b}{\square}, \quad a \underset{zb}{\square}, \tag{5.23}$$

the effect of the one-form-symmetry-gauging in massless spectra is very closely related to massless spectra in orbifolds in which one gauges a quantum symmetry, as in e.g. [27, section 8.3]. We note that if one gauges the $\mathbb{Z}_k$ quantum symmetry of a $\mathbb{Z}_k$ orbifold, one recovers the original unorbifolded theory, which is closely related to the observation of this work that gauging one-form symmetries selects out summands of decomposition. In any event, since massless spectra can be read off from partition functions, we shall focus on partition functions in our discussions of orbifolds.

# 6 Examples in orbifolds

In this section we will describe apply the technology set up in the previous section to particular examples of orbifolds with global one-form symmetries. We will see that gauging the one-form symmetry has the effect of projecting onto particular summands in the decomposition of the original theory.

## 6.1 Trivial $\mathbb{Z}_k$ gerbe

Let us begin with the simplest possible example, the orbifold $[X/\mathbb{Z}_k]$ where all of $\mathbb{Z}_k$ acts trivially on $X$. (The details of $X$ are irrelevant; we assume it is Calabi-Yau so that this

[4]Not every gerbe twisted sector might be nonempty, so for some values of $z$, there may be no corresponding twisted sectors. As a result, this argument should be interpreted as giving a consistency test on existing results.

theory is a CFT, for simplicity.) This theory has a global $\mathbb{Z}_k^{(1)} = B\mathbb{Z}_k$ one-form symmetry, which on the partition function

$$Z \;=\; \frac{1}{|\mathbb{Z}_k|} \sum_{g,h} {\scriptstyle g}\underset{h}{\square} \tag{6.1}$$

simply permutes the various twisted sectors. As discussed in [4, section 5.1], decomposition predicts that this theory is equivalent to a disjoint union of $k$ copies of $X$, one copy for each irreducible representation of $\mathbb{Z}_k$. Furthermore, since the $\mathbb{Z}_k$ gerbe is trivial, there is no difference in $B$ fields – each copy of $X$ in the decomposition has the same $B$ field.

Now, we shall gauge the $\mathbb{Z}_k^{(1)} = B\mathbb{Z}_k$ one-form symmetry. Including possible phases $\epsilon(z)$, the partition function of the new theory has the form

$$Z \;=\; \frac{1}{|\mathbb{Z}_k|}\frac{1}{|\mathbb{Z}_k|} \sum_{z\in\mathbb{Z}_k} \epsilon(z) \sum_{gh=hgz} {\scriptstyle g}\boxed{\scriptstyle z}{\scriptstyle h}. \tag{6.2}$$

However, in this theory, only when $z = 1$ are there any solutions to $gh = hgz$. If $z \neq 1$, then there are no $g, h$ satisfying that equation, because all elements of the group commute. As a result, using $\epsilon(1) = 1$, the partition function simplifies to

$$Z \;=\; \frac{1}{|\mathbb{Z}_k|}\frac{1}{|\mathbb{Z}_k|} \sum_{gh=hg} {\scriptstyle g}\underset{h}{\square}, \tag{6.3}$$

and since $\mathbb{Z}_k$ acts trivially,

$${\scriptstyle g}\underset{h}{\square} \;=\; {\scriptstyle 1}\underset{1}{\square} \;=\; Z(X), \tag{6.4}$$

hence

$$Z \;=\; \frac{1}{|\mathbb{Z}_k|}\frac{1}{|\mathbb{Z}_k|}(k^2) Z(X) \;=\; Z(X). \tag{6.5}$$

Thus, we see that gauging the one-form symmetry in the trivially-acting orbifold returns the partition function of $X$, as expected.

## 6.2 First nontrivial banded example

In this section, we will consider the first banded example discussed in [4, section 5.2]. This is an orbifold $[X/D_4]$, where the $\mathbb{Z}_2$ center of the eight-element group $D_4$ acts trivially. As before, the details of $X$ are not important; we assume $X$ is Calabi-Yau and admits an effective action of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, or more elegantly,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1. \tag{6.6}$$

The elements of $D_4$ are

$$\{1, z, a, b, az, bz, ab, ba = abz\}, \tag{6.7}$$

where $z$ generates the $\mathbb{Z}_2$ center, $a^1 = 1$, $b^2 = z$. We denote elements of the coset $\mathbb{Z}_2 \times \mathbb{Z}_2$ as

$$\{\overline{1}, \overline{a}, \overline{b}, \overline{ab}\}, \tag{6.8}$$

where $\overline{1} = \{1, z\}$, $\overline{a} = \{a, az\}$, and so forth.

In [4], it was argued that this orbifold decomposed into a disjoint union of two copies of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$, one copy without discrete torsion, one copy with. (In this example, [4] also computed multiloop partition functions and massless spectra to double-check the claimed decomposition.)

Because the $\mathbb{Z}_2$ acts trivially, the orbifold $[X/D_4]$ is a $\mathbb{Z}_2$ gerbe, which is to say, it has a global $\mathbb{Z}_2$ one-form symmetry, also denoted $\mathbb{Z}_2^{(1)} = B\mathbb{Z}_2$.

We can gauge that $\mathbb{Z}_2$ one-form symmetry following the procedure described above. At one-loop, the partition function has the form

$$Z \; = \; \frac{1}{|\mathbb{Z}_2|} \frac{1}{|D_4|} \sum_{z \in \mathbb{Z}_2} \sum_{gh = hgz} \epsilon(z) \, {}_{g}\boxed{\mathrm{z}}_{h}. \tag{6.9}$$

Now, for the trivial gerbe, when $z = 1$, the allowed one-loop twisted sectors (defined by commuting $g$, $h$) are

$$ {}_{1,z}\mathop{\square}\limits_{1,z}, \quad {}_{1,z}\mathop{\square}\limits_{a,az}, \quad {}_{1,z}\mathop{\square}\limits_{b,bz}, \quad {}_{1,z}\mathop{\square}\limits_{ab,ba}, \quad {}_{a,az}\mathop{\square}\limits_{a,az}, \quad {}_{b,bz}\mathop{\square}\limits_{b,bz}, \quad {}_{ab,ba}\mathop{\square}\limits_{ab,ba}. \tag{6.10}$$

As $D_4$ is nonabelian, only commuting pairs are in the list above. Projecting to the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, this list omits

$$ {}_{\overline{a}}\mathop{\square}\limits_{\overline{b}}, \quad {}_{\overline{a}}\mathop{\square}\limits_{\overline{ab}}, \quad {}_{\overline{b}}\mathop{\square}\limits_{\overline{ab}} \tag{6.11}$$

twisted sectors, as these do not lift to commuting elements of $D_4$. In the language of decomposition, each of these $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds is also multiplied by a sign when discrete torsion is turned on, so adding theories with and without discrete torsion effectively cancels these out of partition functions.

Next, for the nontrivial gerbe, when $z \neq 1$, the only possible sectors are

$$ {}_{a,az}\boxed{\mathrm{z}}_{b,bz}, \quad {}_{a,az}\boxed{\mathrm{z}}_{ab,ba}, \quad {}_{b,bz}\boxed{\mathrm{z}}_{ab,ba}. \tag{6.12}$$

Note that when we project to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, these fill in precisely the twisted sectors that were missing from the $D_4$ orbifold.

Putting this together, the one-loop partition function of the theory after gauging $B\mathbb{Z}_2$ is

$$Z = \frac{1}{|\mathbb{Z}_2|}\frac{1}{|D_4|}\Bigg[ {}_{1,z}\mathop{\square}\limits_{1,z} + {}_{1,z}\mathop{\square}\limits_{a,az} + {}_{1,z}\mathop{\square}\limits_{b,bz} + {}_{1,z}\mathop{\square}\limits_{ab,ba} + {}_{a,az}\mathop{\square}\limits_{a,az} + {}_{b,bz}\mathop{\square}\limits_{b,bz} + {}_{ab,ba}\mathop{\square}\limits_{ab,ba}$$
$$+ \epsilon(z)\left( {}_{a,az}\boxed{\scriptstyle Z}{}_{b,bz} + {}_{a,az}\boxed{\scriptstyle Z}{}_{ab,ba} + {}_{b,bz}\boxed{\scriptstyle Z}{}_{ab,ba} \right)\Bigg], \tag{6.13}$$

and since $z$ acts trivially, we can reduce this to $\mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sectors as

$$Z = \frac{1}{4}\Bigg[ {}_{\overline{1}}\mathop{\square}\limits_{\overline{1}} + {}_{\overline{1}}\mathop{\square}\limits_{\overline{a}} + {}_{\overline{1}}\mathop{\square}\limits_{\overline{b}} + {}_{\overline{1}}\mathop{\square}\limits_{\overline{ab}} + {}_{\overline{a}}\mathop{\square}\limits_{\overline{a}} + {}_{\overline{b}}\mathop{\square}\limits_{\overline{b}} + {}_{\overline{ab}}\mathop{\square}\limits_{\overline{ab}} + \cdots$$
$$+ \epsilon(z)\left( {}_{\overline{a}}\mathop{\square}\limits_{\overline{b}} + {}_{\overline{a}}\mathop{\square}\limits_{\overline{ab}} + {}_{\overline{b}}\mathop{\square}\limits_{\overline{ab}} \right)\Bigg]. \tag{6.14}$$

When $\epsilon(z) = +1$, this is the one-loop partition function of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ without discrete torsion, one factor in the decomposition, and when $\epsilon(z) = -1$, this is the one-loop partition function of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ with discrete torsion, the other factor in the decomposition.

A closely related example is discussed in [4, section 5.3]. That example is a $[X/\mathbb{H}]$ orbifold, where $\mathbb{H}$ is the eight-element group of quaternions, and the $\mathbb{Z}_2$ center is taken to act trivially on $X$. Just as in the $D_4$ example above, the theory decomposes into a disjoint union of two $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ orbifolds, one with discrete torsion, and one without. The analysis of both the original theory as well as the gauging of the $B\mathbb{Z}_2$ symmetry runs very closely parallel to the $D_4$ example just discussed, so we do not repeat the details here.

## 6.3 A non-banded example

Next we consider an example discussed in [4, section 5.4]. Specifically, this is a $[X/\mathbb{H}]$ orbifold, where $\mathbb{H}$ is the eight-element group of quaternions

$$\{\pm 1, \pm i, \pm j, \pm k\}, \tag{6.15}$$

and the non-central subgroup $\langle i \rangle = \{\pm 1, \pm i\} \cong \mathbb{Z}_4$ acts trivially on $X$. In this case, $\mathbb{H}/\langle i \rangle = \mathbb{Z}_2$. This is a non-banded example, and for the most part in this paper we focus on banded examples, but we include this and other non-banded examples to illustrate further complexities that arise.

The decomposition conjecture [4] predicts that this orbifold decomposes into three disjoint components as

$$[X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \coprod X, \tag{6.16}$$

which was checked in [4, section 5.4] by computing one-loop and two-loop partition functions, and by examining operators in the theory.

This theory has a $B\mathbb{Z}_4$ symmetry, which at the level of the partition function corresponds to interchanging the twisted sectors. Let us consider gauging that $B\mathbb{Z}_4$ symmetry. Following the procedure described earlier, the one-loop partition function of the new theory, the orbifold with the gauged $B\mathbb{Z}_4$, has the form

$$Z \;=\; \frac{1}{|\mathbb{Z}_4|}\frac{1}{|\mathbb{H}|}\sum_{z\in\mathbb{Z}_4}\sum_{gh=hgz}\epsilon(z)\; {}_{g}\boxed{z}_{h}, \tag{6.17}$$

where the $\epsilon(z)$ are possible phases.

For the $z=1$ gerbe, there are the following contributions:

$$ {}_{\pm 1,\pm i}\underset{\pm 1,\pm i}{\square}\,, \quad {}_{\pm 1}\underset{\pm j,\pm k}{\square}\,, \quad {}_{\pm j,\pm k}\underset{\pm 1}{\square}, \quad {}_{\pm j}\underset{\pm j}{\square}, \quad {}_{\pm k}\underset{\pm k}{\square}. \tag{6.18}$$

The only contribution, the only twisted bundles, in sectors with a nontrivial gerbe are in the case $z=-1$:

$$ {}_{\pm i}\boxed{-1}_{\pm j,\pm k}, \quad {}_{\pm j,\pm k}\boxed{-1}_{\pm i}, \quad {}_{\pm j}\boxed{-1}_{\pm k}, \quad {}_{\pm k}\boxed{-1}_{\pm j}. \tag{6.19}$$

Putting this together, and writing this in terms of the twisted sectors of an effectively-acting $\mathbb{Z}_2$ orbifold, with group elements $\{1,\xi\}$, $\xi^2=1$, we have that the one-loop partition function is

$$\begin{aligned} Z \;=\; & \frac{1}{|\mathbb{Z}_4|}\frac{1}{|\mathbb{H}|}\Bigg[16\,{}_{1}\underset{1}{\square} + 8\,{}_{1}\underset{\xi}{\square} + 8\,{}_{\xi}\underset{1}{\square} + 4\,{}_{\xi}\underset{\xi}{\square} + 4\,{}_{\xi}\underset{\xi}{\square} \\ & \qquad + \epsilon(-1)\left(8\,{}_{1}\underset{\xi}{\square} + 8\,{}_{\xi}\underset{1}{\square} + 4\,{}_{\xi}\underset{\xi}{\square} + 4\,{}_{\xi}\underset{\xi}{\square}\right)\Bigg]. \end{aligned} \tag{6.20}$$

In the case that $\epsilon(-1)=+1$,

$$Z \;=\; \frac{1}{4}\frac{16}{8}\left({}_{1}\underset{1}{\square} + {}_{1}\underset{\xi}{\square} + {}_{\xi}\underset{1}{\square} + {}_{\xi}\underset{\xi}{\square}\right), \tag{6.21}$$

$$\;=\; \frac{1}{2}\left({}_{1}\underset{1}{\square} + {}_{1}\underset{\xi}{\square} + {}_{\xi}\underset{1}{\square} + {}_{\xi}\underset{\xi}{\square}\right) \;=\; Z([X/\mathbb{Z}_2]), \tag{6.22}$$

the one-loop partition function of a $\mathbb{Z}_2$ orbifold.

In the case that $\epsilon(-1)=-1$,

$$Z \;=\; \frac{1}{4}\frac{16}{8}\left({}_{1}\underset{1}{\square}\right) \;=\; (1/2)Z(X), \tag{6.23}$$

proportional to the one-loop partition function of $X$ itself.

## 6.4 Another non-banded example

Next, consider the orbifold $[X/A_4]$, where $A_4$ is the twelve-element nonabelian group of alternating permutations of four elements. This group has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we will take to act trivially. It can be shown that $A_4/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_3$. Explicitly, in terms of permutations, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ normal subgroup is generated by

$$\alpha \equiv (14)(23), \quad \beta \equiv (13)(24), \quad \gamma \equiv (12)(34), \tag{6.24}$$

and elements of the $\mathbb{Z}_3$ cosets are given by

$$\{1, \alpha, \beta, \gamma\}, \quad \{(123), (142), (243), (134)\}, \quad \{(132), (143), (124), (234)\}. \tag{6.25}$$

This example was considered in [4, section 5.5], [2, section 2.0.5] where it was shown that this theory decomposes into

$$[X/\mathbb{Z}_3] \coprod X. \tag{6.26}$$

This theory has a $B(\mathbb{Z}_2 \times \mathbb{Z}_2)$ symmetry, which we will gauge. Proceeding as before, the one-loop partition function of the orbifold with $B(\mathbb{Z}_2 \times \mathbb{Z}_2)$ gauged has the form

$$Z = \frac{1}{|\mathbb{Z}_2 \times \mathbb{Z}_2|} \frac{1}{|A_4|} \sum_{z \in \mathbb{Z}_2 \times \mathbb{Z}_2} \sum_{gh=hgz} \epsilon(z) \, {}_g\boxed{z}_h . \tag{6.27}$$

Let $\xi$ denote the generator of $\mathbb{Z}_3$. As noted in [2, section 2.0.5], the

$${}_1\underset{1}{\square} \tag{6.28}$$

sector of the $\mathbb{Z}_3$ orbifold arises from 16 sectors of the $[X/A_4]$ orbifold (with trivial gerbe), and the other $\mathbb{Z}_3$ twisted sectors arise from only 4 sectors of the $[X/A_4]$ orbifold (with trivial gerbe). For example, the

$${}_1\underset{\xi}{\square} \tag{6.29}$$

sector of the $\mathbb{Z}_3$ orbifold arises from the

$${}_1\underset{(123)}{\square}, \quad {}_1\underset{(142)}{\square}, \quad {}_1\underset{(243)}{\square}, \quad {}_1\underset{(134)}{\square} \tag{6.30}$$

one-loop twisted sectorss of the $[X/A_4]$ orbifold.

Related contributions arise in gerbe-twisted sectors. For example, the $\gamma$-twisted sectors include

$${}_\alpha\boxed{\gamma}_{(123)}, \quad {}_\alpha\boxed{\gamma}_{(142)}, \quad {}_\alpha\boxed{\gamma}_{(243)}, \quad {}_\alpha\boxed{\gamma}_{(134)}, \tag{6.31}$$

which also project to the

$$ {}_{1}\underset{\xi}{\square} \tag{6.32} $$

sector of the $\mathbb{Z}_3$ orbifold.

In this example, the action of $SL(2,\mathbb{Z})$ will rotate between different gerbes. To see this, first note that

$$ \alpha(123) \;=\; (123)\alpha\gamma, \tag{6.33} $$

as can be seen explicitly as follows:

$$ \alpha(123): \quad 1234 \;\stackrel{(123)}{\longrightarrow}\; 2314 \;{}^{\alpha}\; 3241, \tag{6.34} $$

$$ (123)\alpha: \quad 1234 \;\stackrel{\alpha}{\longrightarrow}\; 4321 \;{}^{(123)}\; 4132, \tag{6.35} $$

and the two outputs are related by $\gamma$. As a result,

$$ (123)^{-1}\alpha(123) \;=\; \alpha\gamma \;=\; \beta. \tag{6.36} $$

In addition:

$$ (123)^{-1}\beta(123) \;=\; \beta\alpha \;=\; \gamma, \tag{6.37} $$

$$ (123)^{-1}\gamma(123) \;=\; \gamma\beta \;=\; \alpha, \tag{6.38} $$

From the analysis of section 5, we see therefore that modular transformations will relate all gerbes of characteristic class $z \neq 1$, in addition to twisted sectors.

As a result, to add phases $\epsilon(z)$ consistent with modular invariance, $\epsilon$ must be constant on $z \neq 1$.

Finally, we can write the partition function (6.27) in terms of $\mathbb{Z}_3$ orbifold twisted sectors as follows:

$$ Z \;=\; \frac{1}{|\mathbb{Z}_2 \times \mathbb{Z}_2|}\frac{1}{|A_4|}\left( 16\, {}_{1}\underset{1}{\square} + 4(1+3\epsilon)\, {}_{1}\underset{\xi}{\square} + 4(1+3\epsilon)\, {}_{1}\underset{\xi^2}{\square} + 4(1+3\epsilon)\, {}_{\xi}\underset{\xi^2}{\square} + \cdots \right), \tag{6.39} $$

where we have taken $\epsilon(1) = 1$ and $\epsilon(z) = \epsilon$ for $z \neq 1$.

In the case $\epsilon = +1$, this can be simplified to

$$ Z \;=\; \frac{1}{3}\left( {}_{1}\underset{1}{\square} + {}_{1}\underset{\xi}{\square} + {}_{1}\underset{\xi^2}{\square} + {}_{\xi}\underset{\xi^2}{\square} + \cdots \right), \tag{6.40} $$

which is the partition function of $[X/\mathbb{Z}_3]$.

If $\epsilon = -1/3$, the partition function is proportional to that of $X$, the other component of the decomposition.

## 6.5 A nonabelian gerbe

Consider the orbifold $[X/D_4]$, where now all of $D_4$ acts trivially on $X$. This is a $D_4$ gerbe over $X$, a generalized space with fibers $BD_4$. However, the one-form symmetry is determined by the center of $D_4$, which is $\mathbb{Z}_2$. The two-group $B\mathbb{Z}_2$ acts on $BD_4$, and acts on $[X/D_4]$.

From [4], the decomposition of this theory has as many components as irreducible representation of $D_4$, of which there are five: four one-dimensional representations and one two-dimensional representation. This theory is therefore equivalent to a sigma model on a disjoint union of five copies of $X$.

Now, let us consider gauging the $B\mathbb{Z}_2 = \mathbb{Z}_2^{(1)}$ one-form symmetry of this orbifold. Since we can only[5] gauge the action of $\mathbb{Z}_2^{(1)}$ on the $D_4$ gerbe, we should not expect gauging alone to select out each individual component, but at best merely groups of the five components of the decomposition, and indeed that is what we shall find.

Proceeding as before, the partition function on $T^2$ takes the form

$$Z \;=\; \frac{1}{|\mathbb{Z}_2|}\frac{1}{|D_4|}\sum_{z\in\mathbb{Z}_2}\sum_{gh=hgz}\epsilon(z)\, {}_{g}\boxed{z}_{h}. \tag{6.41}$$

Proceeding as in section 6.2, the one-loop partition function of the theory after gauging $B\mathbb{Z}_2$ is

$$\begin{aligned} Z \;=\; \frac{1}{|\mathbb{Z}_2|}\frac{1}{|D_4|}\Bigg[ & {}_{1,z}\underset{1,z}{\square} + {}_{1,z}\underset{a,az}{\square} + {}_{1,z}\underset{b,bz}{\square} + {}_{1,z}\underset{ab,ba}{\square} + {}_{a,az}\underset{a,az}{\square} + {}_{b,bz}\underset{b,bz}{\square} + {}_{ab,ba}\underset{ab,ba}{\square} \\ & + \epsilon(z)\left( {}_{a,az}\boxed{z}_{b,bz} + {}_{a,az}\boxed{z}_{ab,ba} + {}_{b,bz}\boxed{z}_{ab,ba} \right)\Bigg], \end{aligned} \tag{6.42}$$

and since all group elements act trivially, this reduces to

$$Z \;=\; \frac{1}{16}\left(40 + 24\epsilon(-1)\right) {}_{1}\underset{1}{\square}. \tag{6.43}$$

When $\epsilon(-1) = +1$,

$$Z \;=\; 4Z(X), \tag{6.44}$$

and when $\epsilon(-1) = -1$,

$$Z \;=\; Z(X). \tag{6.45}$$

It appears that one choice of phase $\epsilon$ selects four copies of $X$, and the other choice selects one copy of $X$, for a total of five copies, reproducing the decomposition of $[X/D_4]$, and also mimicking the structure of the irreducible representation of $D_4$: four irreducible representations of dimension one, and one of dimension two.

[5] $D_4$ is nonabelian, but one-form symmetries correlate to abelian groups. Hence, there is an action of a center one-form symmetry (here, $\mathbb{Z}_2^{(1)}$) on the $D_4$ gerbe, but we cannot define a $D_4^{(1)}$ symmetry.

### 6.6 Dijkgraaf-Witten theory in two dimensions

Two-dimensional Dijkgraaf-Witten theory [32] is defined by a two-dimensional quantum field theory which is essentially an orbifold of a point. The theory [point/$G$] clearly is associated with a gerbe, and is essentially a degenerate limit of the orbifolds discussed previously in this section. It can be twisted by turning on discrete torsion, but as we observed in section 4.1, turning on discrete torsion typically breaks two-group symmetries, so we will only consider theories without discrete torsion.

As our analysis amounts to an application of ideas discussed extensively elsewhere in this paper, we shall be brief. Following decomposition [4], the two-dimensional orbifold [point/$G$] ($G$ finite) is equivalent to a sigma model on as many points as irreducible representations of $G$. If $G$ is abelian, the theory has a $BG = G^{(1)}$ one-form symmetry, and gauging that one-form symmetry can be used to select particular summands in that decomposition, as has been discussed elsewhere in this paper. If $G$ is nonabelian, then only a $BK = K^{(1)}$ one-form symmetry is manifest, for $K$ the center of $G$; however, if the number of irreducible representations of $G$ is greater than $|K|$, then the theory will have additional, less obvious, one-form symmetries which can be used to select each summand in the decomposition.

## 7 Nonsupersymmetric pure Yang-Mills theories in two dimensions

So far, we have discussed orbifolds with one-form symmetries, and the gauging of those one-form symmetries. In this and the next several sections, we turn our attention to two-dimensional gauge theories, and perform similar analyses. Specifically, we will describe several families of two-dimensional gauge theories with one-form symmetries and their gauging. Just as in the case of orbifolds, decomposition [4,9] predicts that two-dimensional gauge theories with one-form symmetries will decompose into disjoint unions of field theories, and we will argue that by turning on suitable phases in different gerbe sectors, gauging those one-form symmetries will project onto various components of the decomposition. (In passing, see also [33] for another recent work on one-form symmetries and pure Yang-Mills theories in two dimensions.)

Consider pure Yang-Mills theories in two dimensions. Specifically, consider a nonsupersymmetric pure Yang-Mills theory with gauge group $G$. Classically, if $K$ denotes the center of $G$, this theory has a $BK = K^{(1)}$ one-form symmetry, which is preserved in the quantum theory [34]. As a result, since the two-dimensional quantum theory has a discrete one-form symmetry, one expects that the theory will decompose, and indeed, the decomposition of this theory into $|K|$ gauge theories with discrete theta angles is discussed in [9, section 2.4].

In this section, after briefly reviewing that decomposition, we will discuss the gauging of that one-form symmetry, so as to derive various summands in the decomposition, namely $G/K$ gauge theories with various discrete theta angles. theory.

First, recall that the partition function of pure bosonic two-dimensional Yang-Mills theory with gauge group $G$ is known exactly [35–37] and is of the form [38, equ'n (3.20)], [39, equ'n (2.51)]

$$Z \;=\; \sum_R (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), \tag{7.1}$$

where the sum is over representations $R$ of the gauge group $G$, $g$ is the genus of the two-dimensional spacetime, and $A$ its area.

If $G$ is not simply-connected, the theory admits discrete theta angles. Turning on such discrete theta angles corresponds to modifying the sum over representations [40]. For example, let us compare the partition functions of $SU(2)$ and $SO(3)_\pm$. For the $SU(2)$ partition function, one sums over all representations of $SU(2)$, and for the $SO(3)_+$ partition function, one sums over all representations of $SO(3)$ – meaning, all $SU(2)$ representations that are invariant under the center. The $SO(3)_-$ partition function is given by [40] a sum over all $SU(2)$ representations that are not representations of $SO(3)$.

For the $SU(2)$, $SO(3)_\pm$ examples, decomposition is more or less clear: the $SU(2)$ partition function is precisely a sum of the $SO(3)_+$ and $SO(3)_-$ partition functions,

$$Z(SU(2)) \;=\; Z\left(SO(3)_+\right) \;+\; Z\left(SO(3)_-\right). \tag{7.2}$$

This picture generalizes to any pure two-dimensional $G$-gauge theory with center $K$, as discussed in [9, section 2.4]. Briefly, let $w : \mathit{Rep} \to \hat{K}$ denote a map that computes the $n$-ality of any given representation. (It is valued in the characters $\hat{K}$ of $K$, rather than $K$ itself, as it gives the phase picked up by a representation of $K \subset G$, hence, is a character of $K$.) Then, the partition function of the corresponding $G/K$ gauge theory with discrete theta angle $\lambda \in \hat{K}$ is

$$Z\left((G/K)_\lambda\right) \;=\; \sum_{R, w(R)=\lambda} (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), \tag{7.3}$$

where the sum runs over representations $R$ of $G$ of suitable $n$-ality.

In particular, the partition function of the $G$ gauge theory can now be written

$$\begin{aligned} Z(G) \;&=\; \sum_R (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), &(7.4)\\ &=\; \sum_{\lambda \in \hat{K}} \sum_{R, w(R)=\lambda} (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), &(7.5)\\ &=\; \sum_{\lambda \in \hat{K}} Z\left((G/K)_\lambda\right), &(7.6) \end{aligned}$$

consistent with decomposition of this two-dimensional theory.

Now, let us consider gauging the center one-form symmetry. In principle, the partition function has a sum over $K$-gerbes, weighted by phases, of the form

$$Z(G/BK, \lambda) \; = \; \frac{1}{|K|} \sum_{z \in K} \exp\left(-i \lambda(z)\right) \cdots , \tag{7.7}$$

where the $\cdots$ gives the contribution to the partition function from sectors that are twisted by the banded $K$ gerbe with characteristic class $z$. We will determine those contributions next.

Recall that in an orbifold, in a gerbe-twisted sector, the usual boundary conditions were twisted: on $T^2$, for example, instead of summing over pairs of group elements $g, h$ such that $gh = hg$, we instead summed over pairs of group elements such that $gh = hgz$. Similarly, here, we twist by twisting a cap, when we construct the partition function by gluing together along a triangulation. In other words, ordinarily a cap contributes (see e.g. [38, section 3.7])

$$Z_{\rm cap}(U) \; = \; \sum_R (\dim R) \chi_R(U) \exp(-A C_2(R)), \tag{7.8}$$

where $A$ is the area of the cap, and $U$ is the Wilson line around the edge. Here, in a gerbe-$z$-twisted sector, we instead use a twisted cap, in which the Wilson line on the edge is twisted by $z$:

$$Z_{\rm cap, twisted}(U, z) \; = \; \sum_R (\dim R) \chi_R(zU) \exp(-A C_2(R)), \tag{7.9}$$

for $z \in K$, the center of the gauge group $G$. By definition of the function $w$,

$$\chi_R(zU) \; = \; \exp\left( i w(R)(z) \right) \chi_R(U), \tag{7.10}$$

so we can write this as

$$Z_{\rm cap, twisted}(U, z) \; = \; \sum_R (\dim R) \chi_R(U) \exp\left( i w(R)(z) \right) \exp(-A C_2(R)). \tag{7.11}$$

We can then compute partition functions in the gerbe-twisted sector by gluing the twisted cap above into an otherwise normal triangulation. For example, the partition function of the ordinary (untwisted) theory on $S^2$ is

$$\begin{aligned}
Z_{S^2} & = \int dU \, \overline{Z}_{\rm cap}(U) Z_{\rm cap}(U), & (7.12) \\
& = \int dU \sum_{R,R'} (\dim R')(\dim R) \overline{\chi_{R'}(U)} \chi_R(U) \exp\left(-A({\rm cap}) C_2(R)\right) \exp\left(-A({\rm cap}') C_2(R')\right), & \\
& = \sum_{R,R'} (\dim R')(\dim R) \delta_{R,R'} \exp\left(-A({\rm cap}) C_2(R)\right) \exp\left(-A({\rm cap}') C_2(R')\right), & (7.13) \\
& = \sum_R (\dim R)^2 \exp\left( -A(S^2) C_2(R) \right), & (7.14)
\end{aligned}$$

the usual partition function in the genus zero case. In a sector twisted by the banded $K$ gerbe with characteristic class $z$, on the other hand, we have instead

$$
\begin{aligned}
Z_{S^2}(z) &= \int dU\, \overline{Z}_{\rm cap}(U) Z_{\rm cap,twisted}(U,z), && (7.15)\\
&= \sum_R (\dim R)^2 \exp\left(i w(R)(z)\right) \exp\left(-A(S^2) C_2(R)\right), && (7.16)
\end{aligned}
$$

following a similar computation, and more generally, on a genus $g$ Riemann surface,

$$
Z \;=\; \sum_R (\dim R)^{2-2g} \exp\left(i w(R)(z)\right) \exp\left(-A C_2(R)\right). \tag{7.17}
$$

Returning to the partition function of the $BK$-gauged $G$-gauge theory, and putting the pieces above together, the resulting partition function is then

$$
\begin{aligned}
Z(G/BK, \lambda) &= \frac{1}{|K|} \sum_{z \in K} \exp\left(-i \lambda(z)\right) Z(z), && (7.18)\\
&= \frac{1}{|K|} \sum_{z \in K} \exp\left(-i\lambda(z)\right) \sum_R \exp\left(i w(R)(z)\right) (\dim R)^{2-2g} \exp\left(-A C_2(R)\right), && (7.19)
\end{aligned}
$$

where the sum over representations $R$ runs over all representations of $G$. Now, it is straightforward to check that

$$
Z\left((G/K)_\lambda\right) \;=\; \frac{1}{|K|} \sum_{z \in K} \sum_R \exp\left(i (w(R) - \lambda)(z)\right) (\dim R)^{2-2g} \exp\left(-A C_2(R)\right), \tag{7.20}
$$

as the sum over $z \in K$ effectively performs a projection onto representations $R$ such that $w(R) = \lambda$. As a result, we see that

$$
Z(G/BK, \lambda) \;=\; Z\left((G/K)_\lambda\right), \tag{7.21}
$$

and so, at least at the level of partition functions, gauging the center $BK$ one-form symmetry selects one of the summands in the decomposition of the original two-dimensional $G$ gauge theory.

# 8 Supersymmetric gauge theories

## 8.1 Multiplet for two-form field

To gauge the one-form symmetry, one needs to introduce a supersymmetrization of a two-form potential. A natural candidate superfield is a twisted chiral superfield, as for example in an abelian gauge theory in two dimensions, the twisted chiral $\Sigma = \overline{D}_+ D_- V$ depends upon the field strength of the gauge field, but not the gauge field itself. For our purposes, following [45, section 2], we take

$$\begin{aligned}\Sigma^{(2)} &= \sigma - i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^-\lambda_- + \sqrt{2}\theta^+\overline{\theta}^-(D - iB_{01}) - i\overline{\theta}^-\theta^-\partial_-\sigma - i\theta^+\overline{\theta}^+\partial_+\sigma \\ &\quad +\sqrt{2}\overline{\theta}^-\theta^+\theta^-\partial_-\overline{\lambda}_+ + \sqrt{2}\theta^+\overline{\theta}^-\overline{\theta}^+\partial_+\lambda_- - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\partial_+\partial_-\sigma, \end{aligned} \tag{8.1}$$

where $B_{01}$ is the two-form field, and $\sigma, \lambda_\pm, D$ are other fields introduced for the (2,2) supersymmetric completion. The supersymmetry variations of the components are then given by

$$\delta B_{01} = i\overline{\epsilon}_+\partial_-\lambda_+ - i\overline{\epsilon}_-\partial_+\lambda_- + i\epsilon_+\partial_-\overline{\lambda}_+ - i\epsilon_-\partial_+\lambda_-, \tag{8.2}$$

$$\delta\sigma = -i\sqrt{2}\overline{\epsilon}_+\lambda_- - i\sqrt{2}\epsilon_-\overline{\lambda}_+, \tag{8.3}$$

$$\delta\overline{\sigma} = -i\sqrt{2}\epsilon_+\overline{\lambda}_- - i\sqrt{2}\overline{\epsilon}_-\lambda_+, \tag{8.4}$$

$$\delta D = -\overline{\epsilon}_+\partial_-\lambda_+ - \overline{\epsilon}_-\partial_+\lambda_- + \epsilon_+\partial_-\overline{\lambda}_+ + \epsilon_-\partial_+\overline{\lambda}_-, \tag{8.5}$$

$$\delta\lambda_+ = i\epsilon_+ D + \sqrt{2}\partial_+\overline{\sigma}\epsilon_- - B_{01}\epsilon_+, \tag{8.6}$$

$$\delta\lambda_- = i\epsilon_- D + \sqrt{2}\partial_-\sigma\epsilon_+ + B_{01}\epsilon_-, \tag{8.7}$$

$$\delta\overline{\lambda}_+ = -i\overline{\epsilon}_+ D + \sqrt{2}\partial_+\sigma\overline{\epsilon}_- - B_{01}\overline{\epsilon}_+, \tag{8.8}$$

$$\delta\overline{\lambda}_- = -i\overline{\epsilon}_- D + \sqrt{2}\partial_-\overline{\sigma}\overline{\epsilon}_+ + B_{01}\overline{\epsilon}_-. \tag{8.9}$$

Gauge transformations act as $\Sigma^{(2)} \mapsto \Sigma^{(2)} + \overline{D}_+ D_- V$, where

$$\begin{aligned}\overline{D}_+ D_- V &= \sigma' - i\sqrt{2}\theta^+\overline{\lambda}'_+ - i\sqrt{2}\overline{\theta}^-\lambda'_- + \sqrt{2}\theta^+\overline{\theta}^-(D' - i(dv)_{01}) - i\overline{\theta}^-\theta^-\partial_-\sigma' - i\theta^+\overline{\theta}^+\partial_+\sigma' \\ &\quad +\sqrt{2}\overline{\theta}^-\theta^+\theta^-\partial_-\overline{\lambda}'_+ + \sqrt{2}\theta^+\overline{\theta}^-\overline{\theta}^+\partial_+\lambda'_- - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\partial_+\partial_-\sigma', \end{aligned} \tag{8.10}$$

where

$$\delta(dv)_{01} = i\overline{\epsilon}_+\partial_-\lambda_+ - i\overline{\epsilon}_-\partial_+\lambda_- + i\epsilon_+\partial_-\overline{\lambda}_+ - i\epsilon_-\partial_+\lambda_-,$$

so we see that gauge transformations can be used to eliminate all components of $\Sigma^{(2)}$ other than $B_{01}$, which undergoes $B \mapsto B + dv$ for $v$ a one-form. (This is closely analogous to Wess-Zumino gauge for ordinary vector superfields, in which gauge transformations can be used to eliminate some of the fields in the multiplet.)

## 8.2 Abelian examples

Now, let us apply this to an example. The prototype for supersymmetric two-dimensional abelian gauge theories with a $\mathbb{Z}_k^{(1)} = B\mathbb{Z}_k$ one-form symmetry is the supersymmetric $\mathbb{P}^n$ model in which all fields have charge $k > 1$, as discussed in [1–3]. (See also [46] for a recent related discussion of the charge $q$ Schwinger model.)

Let us briefly review some basic features of this theory, beginning with the fact that this description can be consistent with two different theories. After all, the perturbative physics of the theory above and the ordinary supersymmetric $\mathbb{P}^n$ model are identical – rescaling charges makes no difference. The difference between these two theories is nonperturbative in nature, and can be summarized as follows:

- If the charges can be rescaled to 1 to get the ordinary supersymmetric $\mathbb{P}^n$ model, then the theta angle has periodicity $2\pi k$ (in the desciption as a charge-$k$ model), the axial anomaly breaks $U(1)_A$ to $\mathbb{Z}_{2(n+1)}$, the quantum cohomology ring is $\mathbb{C}[x]/(x^{n+1} - q)$, and there is no $\mathbb{Z}_k^{(1)}$ one-form symmetry,
- In the more interesting case, the theta angle has periodicity $2\pi$, the axial anomaly breaks $U(1)_A$ to $\mathbb{Z}_{2k(n+1)}$, the quantum cohomology ring is $\mathbb{C}[x]/(x^{k(n+1)} - q)$, and there is a $\mathbb{Z}_k^{(1)}$ one-form symmetry.

Now, let us consider gauging that $\mathbb{Z}_k^{(1)}$ one-form symmetry. The gauge curvature $F_{01}$ appears in the twisted chiral superfield $\Sigma$, which appears in the action in two terms:

$$\int d^4\theta \, \Sigma \overline{\Sigma}, \quad t \int d^2\tilde{\theta} \, \Sigma, \tag{8.11}$$

i.e. the gauge kinetic term and the FI term. We can add the two-form gauge field by replacing $\Sigma$ with $\Sigma - \Sigma^{(2)}$ in each of the terms above:

$$\int d^4\theta \, \left(\Sigma - \Sigma^{(2)}\right) \left(\overline{\Sigma} - \overline{\Sigma}^{(2)}\right), \quad t \int d^2\tilde{\theta} \left(\Sigma - \Sigma^{(2)}\right). \tag{8.12}$$

Now, so as to only gauge the $\mathbb{Z}_k^{(1)}$ symmetry, and not entirely eliminate the $U(1)$ gauge field, we must also constraint the two-form field, for which we add the term

$$k \int d^2\tilde{\theta} \, \Sigma^{(2)} \Lambda, \tag{8.13}$$

where $\Lambda$ is a twisted chiral superfield we introduce with no kinetic term, to act as a Lagrange multiplier. This is just a supersymmetrization of a $BF$-type term

$$k\phi B_{01}, \tag{8.14}$$

for a scalar $\phi$.

The reader will note that we have not modified the matter kinetic terms, which contain the gauge field $A_\mu$. Briefly, this will be consistent so long as the theory admits a $\mathbb{Z}_k^{(1)}$ one-form symmetry. When we gauge the one-form symmetry, the bundles appearing in the path integral are 'twisted,' which concretely for the gauge field means that across coordinate patches,

$$A_\mu \mapsto \partial_\mu \phi + \Lambda_\mu. \tag{8.15}$$

In other words, in addition to the usual gauge transformation term $\partial_\mu \phi$ for some function $\phi$, the gauge field can also pick up an affine translation $\Lambda_\mu$, determined by the $\mathbb{Z}_k$ gerbe responsible for the twisting. In the present case, because we are gauging a $\mathbb{Z}_k^{(1)}$ one-form symmetry, the $\Lambda_\mu$ are invisible to the charge-$k$ matter fields, and so the kinetic terms are unmodified. (Put another way, after gauging the $\mathbb{Z}_k^{(1)}$, the original gauge field $A$ is no longer a well-defined ordinary gauge field, but $kA$ is, and because the matter fields all have charge $k$, only the product $kA$ appears in the kinetic terms.)

If the matter was not invariant under that $\mathbb{Z}_k^{(1)}$ one-form symmetry, then we would need to add explicit couplings to the matter kinetic energies to make them well-defined across coordinate patches, because of those affine translations $\Lambda_\mu$. Put another way, only in a theory with a $\mathbb{Z}_k^{(1)}$ one-form symmetry does it suffice to only modify the terms in the supersymmetric action involving $\Sigma$.

Note in passing that the worldsheet theta angle flows in the IR to the pullback of of the target-space $B$ field, which is distinct from the worldsheet $B$ field we introduce to gauge the one-form symmetry. As a result, we cannot gauge the one-form symmetry by merely promoting the worldsheet theta angle to an axion.

Finally, to allow for the gerbes in the path integral to contribute with different phases, we also add a term

$$\alpha \int d^2\tilde{\theta}\, \Sigma^{(2)}. \tag{8.16}$$

Since $\Sigma^{(2)}$ is constrained to define a $\mathbb{Z}_k$ BF theory, $\alpha$ is only meaningful mod $k$.

In particular, in the analogue of Wess-Zumino gauge,

$$\int d^2\tilde{\theta}\, \Sigma^{(2)} \;\propto\; B_{01}, \tag{8.17}$$

and so adding this term to the Lagrangian introduces a gerbe-dependent phase into the path integral, weighting different worldsheet gerbes differently.

Briefly, in the case of the worldsheet $\mathbb{P}^n$ model with charges $k > 1$, gauging the one-form symmetry as above has the effect of changing the size of the $U(1)$ by a factor of $k$. One way to understand this is in terms of its effect on the theta angle periodicity. The theory with the $\mathbb{Z}_k^{(1)}$ one-form symmetry has a theta angle that is $2\pi$ periodic. When we gauge the one-form symmetry, we sum over $\mathbb{Z}_k$ gerbes, which twist the bundle to a 'fractional' bundle. Such fractional bundles have fractional Chern classes; for example, over $\mathbb{P}^1_{[k,k]}$, one can have line bundles

$$\mathcal{O}(m) \;\longrightarrow\; \mathbb{P}^1_{[k,k]}, \tag{8.18}$$

for any integer $m$, including integers less than $k$, and such a line bundle has $c_1 = m/k$. (See e.g. [31] for a discussion of bundles on gerbes.) As a result, in the new theory, the theta angle multiplies a factor $\int F$ which is valued in multiples of $1/k$ instead of integers, hence in this theory the theta angle is now $2\pi k$ periodic. Such $2\pi k$ periodicity signals that the resulting theory is equivalent to the ordinary $\mathbb{P}^n$ model.

Alternatively, one can think of gauging the one-form symmetry as a change in the 'size' of the $U(1)$, replacing $U(1)$ by $U(1)/\mathbb{Z}_k$, for which charge $k$ fields become charge 1 fields, restoring the original theory.

The axial $U(1)_A$ anomaly is of the form

$$[D\psi] \;\mapsto\; [D\psi] \exp\left( i2k(n+1)\omega Q_{\rm top} \right), \tag{8.19}$$

where

$$Q_{\rm top} \;=\; \frac{1}{2\pi} \int d^2x F_{01}, \tag{8.20}$$

and for the original $\mathbb{P}^n$ model, $Q_{\rm top} \in \mathbb{Z}$. As a result, there is an anomaly-free $\mathbb{Z}_{2k(n+1)} \subset U(1)_A$ subgroup corresponding to phases in which $\omega \in 2\pi\mathbb{Z}/(2k(n+1))$. If we gauge the $\mathbb{Z}_k^{(1)}$ one-form symmetry, then $Q_{\rm top} \in \mathbb{Z}/k$. Now, if we restrict to the anomaly-free subgroup of $U(1)_A$ of the original theory, before gauging $\mathbb{Z}_k^{(1)}$, then

$$\omega \;=\; \frac{2\pi m}{2k(n+1)} \tag{8.21}$$

for $m \in \mathbb{Z}$, and in the new theory, after gauging the $\mathbb{Z}_k^{(1)}$,

$$[D\psi] \;\mapsto\; [D\psi] \exp\left( 2\pi i \frac{m}{k} \right). \tag{8.22}$$

Thus, since the original anomaly-free $\mathbb{Z}_{2k(n+1)} \subset U(1)_A$ now becomes anomalous after gauging the one-form symmetry, we say that there is a mixed 0-form / 1-form anomaly, mixing the $\mathbb{Z}_{2k(n+1)}$ axial 0-form symmetry and the $\mathbb{Z}_k^{(1)}$ one-form symmetry of the original theory. Specifically, the anomaly is given by the phase

$$\exp\left( 2\pi i/k \right). \tag{8.23}$$

Another interpretation of this mixed 0/1-form anomaly is as follows. Note that in the new theory, under an axial $U(1)_A$ rotation,

$$[D\psi] \;\mapsto\; [D\psi] \exp\left( i2(n+1)\omega m \right), \tag{8.24}$$

where $m \in \mathbb{Z}$, $m = kQ_{\rm top}$. As a result, after gauging the one-form symmetry of the original theory, the axial anomaly of the new theory has a nonanomalous $\mathbb{Z}_{2(n+1)}$ subgroup, matching the standard result for the axial anomaly in the ordinary $\mathbb{P}^n$ model.

This is only one generalization of the supersymmetric $\mathbb{P}^n$ model. A more general family can be described as follows [1]. (See also e.g. [1] for analogous generalizations of GLSMs for toric varieties.) Consider a GLSM with gauge group $U(1)^2$ and matter fields with charges as follows:

|  | $x_0$ | $\cdots$ | $x_n$ | $z$ |
|---|---|---|---|---|
| $U(1)_\lambda$ | 1 | $\cdots$ | 1 | $-m$ |
| $U(1)_\mu$ | 0 | $\cdots$ | 0 | $k$ |

Here, we interpret $x_{0,\cdots,n}$ in terms of homogeneous coordinates on $\mathbb{P}^n$, as before. We interpret $z$ as an analogous coordinate on the total space of $\mathcal{O}(-m)$; however, from the D-term associated to $U(1)_\lambda$ for nonzero FI parameter, we omit the zero locus, and then gauge out $\mathbb{C}^\times$ rotations.

Geometrically, this GLSM describes a family of $\mathbb{Z}_k$ gerbes over $\mathbb{P}^n$, i.e., generalized spaces with $\mathbb{Z}_k^{(1)}$ one-form symmetries. Such gerbes are classified by a characteristic class in $H^2(\mathbb{P}^n, \mathbb{Z}_k)$, and GLSMs of the form above describe all possible $\mathbb{Z}_k$ gerbes over $\mathbb{P}^n$. The GLSM with the charges shown describes the gerbe of characteristic class $-m \bmod k$, for example, and the previous generalization of the $\mathbb{P}^n$ model corresponds to the gerbe of characteristic class $-1 \bmod k$. The characteristic class is reflected in the quantum cohomology ring, for example, which is given by

$$\mathbb{C}[x, y]/\langle y^k - 1, x^{n+1} y^{-m} - q \rangle. \tag{8.25}$$

Gauging the one-form symmetry acts here much as in the previous case: the charge $k$ field becomes, in effect, a charge 1 field, the global $\mathbb{Z}_k^{(1)}$ disappears, and the theory reduces to a copy of the ordinary $\mathbb{P}^n$ model.

Other examples of gerbe structures arising in GLSMs are described in [1]. Briefly, gauging one-form symmetries there follows the same pattern as outlined here for these various analogues of the supersymmetric $\mathbb{P}^n$ model.

In passing, it is also worth mentioning that gauging the flavor symmetry in these models can also be of interest, see [41, section 2.4] and [42–44].

## 8.3 Mirrors to two-dimensional gauge theories

So far we have discussed one-form symmetries in two-dimensional gauge theories, next we shall discuss how the one-form symmetries are visible in the mirrors, following [1, 4].

### 8.3.1 Abelian mirrors

For our first generalization of the supersymmetric $\mathbb{P}^n$ model, discussed in section 8.2, a $U(1)$ gauge theory with charges $k > 1$, the mirror was computed in [1] and is a Landau-Ginzburg model with a superpotential of the form

$$W = \exp(-Y_1) + \cdots + \exp(-Y_n) + q\Upsilon \prod_{i=1}^{n} \exp(+Y_i), \tag{8.26}$$

where $\Upsilon$ is a $\mathbb{Z}_k$-valued field. (This is clearly equivalent to a disjoint union of Landau-Ginzburg models, and is one way to understand decomposition in this example.) For more general gerbes over $\mathbb{P}^n$, mirrors were also computed in [1], and have the form

$$W = \exp(-Y_1) + \cdots + \exp(-Y_n) + q\Upsilon^{-m} \prod_{i=1}^{n} \exp(+Y_i). \tag{8.27}$$

We can see the axial anomaly as follows. First, for simplicity, consider the mirror to the ordinary supersymmetric $\mathbb{P}^n$ model, given by a Landau-Ginzburg model with superpotential

$$W = \exp(-Y_1) + \cdots + \exp(-Y_n) + q \prod_{i=1}^{n} \exp(+Y_i). \tag{8.28}$$

The mirror to the axial symmetry of the original theory is the translation

$$Y_i \mapsto Y_i - 2i\alpha, \tag{8.29}$$

where $\alpha$ parametrizes the $U(1)_R$. Under this translation, the superpotential terms $\exp(-Y_i)$ have weight 2, as needed for an R-symmetry. The anomaly is visible in the last term. The product

$$\prod_{i=1}^{n} \exp(+Y_i) \tag{8.30}$$

has weight $-2n$, so in order for the corresponding term to have the same weight as the other terms, we must require that $q$ have weight $2 + 2n = 2(n+1)$. This reflects the fact that $q$ encodes the theta angle of the original theory, which will effectively shift under an anomalous chiral rotation. We can read off the anomaly from the theta angle shift, and see in particular there there is a nonanomalous $\mathbb{Z}_{2(n+1)}$ subgroup.

It is straightforward to modify this computation to apply to the gerbe mirror case. There, again the mirror of the axial symmetry is the translation (8.29), and so the terms $\exp(-Y_i)$ have weight two, as needed for an R-symmetry. To understand the action on the remaining terms, first recall that for the first gerbe, $\mathbb{P}^n_{[k,k,\cdots,k]}$, the theta angle of the original theory is encoded in $\tilde{q}$, which satisfies

$$\tilde{q} = \exp(-kY_{n+1}) \prod_{i=1}^{n} \exp(-kY_i), \tag{8.31}$$

and so transforms as

$$\tilde{q} \mapsto \tilde{q} \exp\left(+2k(n+1)i\alpha\right). \tag{8.32}$$

The combination $q\Upsilon = \tilde{q}^{1/k}$, where the $\Upsilon$ reflects the ambiguity by $k$th roots of unity, and so we see that under the mirror of the axial symmetry,

$$q\Upsilon \mapsto q\Upsilon \exp\left(+2(n+1)i\alpha\right). \tag{8.33}$$

This describes the mirror of the axial anomaly. The more general gerbe mirror can be analyzed similarly.

Let us now turn our attention to gauging the $\mathbb{Z}_k^{(1)}$ one-form symmetry in the mirror. In fact, this is extremely easy: since the one-form symmetry is realized just by changing $\Upsilon$ from one $k$th root of unity to another, gauging the one-form symmetry gauges 'translations' of this discrete-valued field, effectively removing it from the theory. In this language, it is also easy to see the mixed 0/1-form anomaly in the mirror. In the original mirror, from equation (8.32), there is an anomaly-free $\mathbb{Z}_{2k(n+1)}$ subgroup of the axial $U(1)_A$, defined by

$$\alpha \in \frac{2\pi\mathbb{Z}}{2k(n+1)}, \tag{8.34}$$

which leaves $\tilde{q}$ invariant. However, after gauging the $\mathbb{Z}_k^{(1)}$ (meaning, removing $\Upsilon$ from the mirror), such $\alpha$ shifts $q$ by a factor of

$$\exp\left(2(n+1)\frac{2\pi i\mathbb{Z}}{2k(n+1)}\right) = \exp\left(\frac{2\pi i\mathbb{Z}}{k}\right), \tag{8.35}$$

and so we see that there is a mixed 0/1-form anomaly, given by the phase $\exp(2\pi i/k)$, matching the result for the original theory. The more general gerbe mirror can be analyzed similarly.

### 8.3.2 Nonabelian mirrors

An extension of the Hori-Vafa ansatz for mirrors to abelian GLSMs was proposed in [6–8,47]. For example, [6, section 12] discussed pure (2,2) supersymmetric $SU(k)$ gauge theories in two dimensions, which have a $\mathbb{Z}_k^{(1)}$ one-form center symmetry.

Let us first review the structure of those mirrors. The mirror to the pure supersymmetric $SU(2)$ theory was described there by a Landau-Ginzburg orbifold with superpotential

$$W = 2\Sigma \ln\left(\frac{X_{12}}{X_{21}}\right) + X_{12} + X_{21}, \tag{8.36}$$

the mirror to the pure $SO(3)_+$ theory was a closely-related Landau-Ginzburg orbifold with superpotential

$$W \;=\; \Sigma \ln \left( \frac{X_{12}}{X_{21}} \right) \;+\; X_{12} \;+\; X_{21}, \tag{8.37}$$

and the mirror to the pure $SO(3)_-$ theory was the Landau-Ginzburg orbifold with superpotential

$$W \;=\; \Sigma \ln \left( \frac{X_{12}}{X_{21}} \right) \;+\; \pi i \Sigma \;+\; X_{12} \;+\; X_{21}. \tag{8.38}$$

When solving for the vacua in these mirrors, one finds a continuum of vacua in both the $SU(2)$ and $SO(3)_-$ theories, corresponding to $X_{12} = -X_{21}$, but no vacua at all in the $SO(3)_+$ theory, consistent with the expected decomposition [9]

$$SU(2) \;=\; SO(3)_+ \;+\; SO(3)_-, \tag{8.39}$$

and analogous decompositions in other pure supersymmetric gauge theories.

Ultimately, the presence of the $\mathbb{Z}_2^{(1)}$ one-form symmetry in the mirror to the $SU(2)$ theory is reflected in the factor of 2 in the first term in the superpotential, just as the mirror to the $\mathbb{Z}_k^{(1)}$ one-form symmetry in the abelian examples of the last section was reflected in a superpotential term of the form $k\Sigma Y$, which forced $\exp(-Y)$ to be a $k$th root of unity. Here, because of the factor of 2, one requires

$$\left( \frac{X_{12}}{X_{21}} \right)^2 \;=\; 1, \tag{8.40}$$

instead of merely

$$\left( \frac{X_{12}}{X_{21}} \right) \;=\; \pm 1, \tag{8.41}$$

and so in the $SU(2)$ mirror one sums over the two roots of the equation above, which are easily seem to correspond to the two $SO(3)$ mirrors for either discrete theta angle.

Briefly, much as in the abelian case, gauging the $\mathbb{Z}_2^{(1)}$ of the original theory corresponds to removing that choice of roots – removing the factor of 2 in the $SU(2)$ mirror superpotential – and possibly adding a phase, to distinguish the $SO(3)_+$ from $SO(3)_-$ mirrors.

## 8.4 Partition functions of abelian and nonabelian theories

Partition functions of two-dimensional supersymmetric gauge theories on $S^2$ were computed in [48, 49], and have the form [48, equ'n (3.34)]

$$Z_{S^2} \;=\; \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \left( \prod_j \frac{d\sigma_j}{2\pi} \right) Z_{\rm class}(\sigma, \mathfrak{m}) Z_{\rm gauge}(\sigma, \mathfrak{m}) \prod_{\Phi} Z_{\Phi}(\sigma, \mathfrak{m}; \tau, \mathfrak{n}), \tag{8.42}$$

where [48, equ'n (3.35)]

$$
\begin{aligned}
Z_{\rm class}(\sigma, \mathfrak{m}) & = e^{-4\pi i \xi {\rm Tr}\, \sigma - i \theta {\rm Tr}\, \mathfrak{m}} \exp\left( 8 \pi i r {\rm Re}\, \tilde{W}(\sigma/r + i \mathfrak{m}/(2r)) \right), \\
Z_{\rm gauge}(\sigma, \mathfrak{m}) & = \prod_{\alpha \in G} \left( \frac{|\alpha(\mathfrak{m})|}{2} \: + \: i \alpha(\sigma) \right) \: = \: \prod_{\alpha > 0} \left( \frac{\alpha(\mathfrak{m})^2}{4} \: + \: \alpha(\sigma)^2 \right), \\
Z_{\Phi}(\sigma, \mathfrak{m}; \tau, \mathfrak{n}) & = \prod_{\rho \in R_{\Phi}} \frac{ \Gamma\left( \frac{R[\Phi]}{2} \: - \: i \rho(\sigma) \: - \: i f^a[\Phi] \tau_a \: - \: \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \right) }{ \Gamma\left( 1 \: - \: \frac{R[\Phi]}{2} \: + \: i \rho(\sigma) \: + \: i f^a[\Phi] \tau_a \: - \: \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \right) }.
\end{aligned}
$$

Briefly, in the notation of [48], $R[\Phi]$ is the R-charge of a chiral multiplet $\Phi$, $f^a[\Phi]$ a non-R-charge, $R_\Phi$ denotes the gauge group representation in which $\Phi$ appears, $\mathcal{W}$ is the Weyl group of the gauge group, and $\tau = (\tau_a)$ and $\mathfrak{n} = (n_a)$ define twisted masses for the chiral superfield. The $\mathfrak{m}$ are elements of the cocharacter (dual weight) lattice for the gauge group, meaning for any weight $\rho$ in the weight lattice, $\rho(\mathfrak{m}) \in \mathbb{Z}$.

Decomposition of two-dimensional supersymmetric gauge theories with center-invariant matter at the level of such partition functions was discussed in [9]. Let us briefly review that analysis here. (We will implicitly assume in this section that the center one-form symmetry is unbroken by any sort of anomalies.)

For simplicity, we begin with a comparison of two-dimensional supersymmetric $SU(2)$ gauge theories with center-invariant matter to corresponding $SO(3)_\pm$ gauge theories. As discussed in [9], the cocharacter lattice of $SU(2)$ is twice as large as that of $SO(3)$, so if for $SO(3)$, $\mathfrak{m}$ varies over all integers, then for $SU(2)$, $\mathfrak{m}$ varies over even integers. Furthermore, the discrete theta angle of the $SO(3)$ theories is encoded in the partition function lattice sum as a factor

$$
\exp\left( - i \pi \mathfrak{m} \right) \: = \: (-)^{\mathfrak{m}}. \tag{8.43}
$$

At the level of partition functions, decomposition can now be understood as follows. If we write the partition function of a $SO(3)_+$ gauge theory with center-invariant matter in the form

$$
Z\left( SO(3)_+ \right) \: = \: \frac{1}{2} \sum_{\mathfrak{m} \in \mathbb{Z}} A(\mathfrak{m}), \tag{8.44}
$$

for some function $A(\mathfrak{m})$ given by the supersymmetric localization formulas earlier, then for the corresponding $SO(3)_-$ gauge theory,

$$
Z\left( SO(3)_- \right) \: = \: \frac{1}{2} \sum_{\mathfrak{m} \in \mathbb{Z}} (-)^{\mathfrak{m}} A(\mathfrak{m}), \tag{8.45}
$$

and for the corresponding $SU(2)$ gauge theory,

$$
Z\left( SU(2) \right) \: = \: \sum_{\mathfrak{m} \in 2 \mathbb{Z}} A(\mathfrak{m}) \: = \: Z\left( SO(3)_+ \right) \: + \: Z\left( SO(3)_- \right). \tag{8.46}
$$

In particular, the form of the expressions from supersymmetric localization is the same for the three theories $SO(3)_\pm$ and $SU(2)$, hence the function $A(\mathfrak{m})$ is necessarily the same in each case; only the lattice sum, and a possible phase factor, can differ.

In this spirit, we can now gauge the $\mathbb{Z}_2^{(1)}$ one-form symmetry at the level of partition functions as follows. As before, we sum over banded $\mathbb{Z}_2$ gerbes on the worldsheet. For the trivial $\mathbb{Z}_2$ gerbe, we then sum over the lattice of the original $SU(2)$ gauge theory (meaning, $\mathfrak{m} \in 2\mathbb{Z}$). For the nontrivial $\mathbb{Z}_2$ gerbe, we sum over $SU(2)$ bundles twisted by the $\mathbb{Z}_2$ gerbe, which is to say, $SO(3)$ bundles which are not also $SU(2)$ bundles, and hence we sum over odd $\mathfrak{m}$. We add a gerbe-dependent phase factor $\epsilon$. Thus, using "$SU(2)/B\mathbb{Z}_2$" to denote the $SU(2)$ gauge theory with the action of $B\mathbb{Z}_2 = \mathbb{Z}_2^{(1)}$ gauged, we have

$$Z\left(SU(2)/B\mathbb{Z}_2\right) \;=\; \frac{1}{2}\epsilon(1)\sum_{\mathfrak{m}\in 2\mathbb{Z}} A(\mathfrak{m}) \;+\; \frac{1}{2}\epsilon(-1)\sum_{\mathfrak{m}\in 2\mathbb{Z}+1} A(\mathfrak{m}), \tag{8.47}$$

where we take $\epsilon(1) = 1$. In the case that $\epsilon(-1) = 1$, we have

$$Z\left(SU(2)/B\mathbb{Z}_2\right) \;=\; \frac{1}{2}\sum_{\mathfrak{m}\in\mathbb{Z}} A(\mathfrak{m}) \;=\; Z\left(SO(3)_+\right). \tag{8.48}$$

In the case that $\epsilon(-1) = -1$, we have

$$Z\left(SU(2)/B\mathbb{Z}_2\right) \;=\; \frac{1}{2}\sum_{\mathfrak{m}\in\mathbb{Z}} (-)^{\mathfrak{m}} A(\mathfrak{m}) \;=\; Z\left(SO(3)_-\right). \tag{8.49}$$

Thus, we see that the two different gaugings of $B\mathbb{Z}_2 = \mathbb{Z}_2^{(1)}$ yield the two different discrete theta angles, the two theories $SO(3)_\pm$.

Now, we shall generalize this to more general semisimple gauge groups $G$, following [9, section 2.6]. Let $K$ denote the center of $G$ and $M_G$ the cocharacter lattice of $G$, so that $M_G \subset M_{G/K}$ and $M_{G/K}/M_G$ has as many element sas $K$. The integral of the analogue of the second Stiefel-Whitney class is encoded in the map $w$ in

$$1 \longrightarrow M_G \longrightarrow M_{G/K} \stackrel{w}{\longrightarrow} K \longrightarrow 1. \tag{8.50}$$

Then,

$$\frac{1}{|K|}\sum_{\mu\in\hat{K}} \exp\left(i\mu(w(\mathfrak{m}))\right) \tag{8.51}$$

is a projection operator that projects $M_{G/K}$ onto $M_G$. In this language, the partition function of a two-dimensional supersymmetric $G$-gauge theory can be expressed in the form

$$\begin{aligned} Z(G) &= \sum_{\mathfrak{m}\in M_G} A(\mathfrak{m}), && (8.52)\\ &= \frac{1}{|K|}\sum_{\lambda\in\hat{K}}\sum_{\mathfrak{m}\in M_{G/K}} \exp\left(i\lambda(w(\mathfrak{m}))\right) A(\mathfrak{m}), && (8.53)\\ &= \sum_{\lambda\in\hat{K}} Z\left((G/K)_\lambda\right), && (8.54) \end{aligned}$$

where

$$Z\left((G/K)_\lambda\right) = \frac{1}{|K|}\sum_{\mathfrak{m}\in M_{G/K}} \exp\left(i\lambda(w(\mathfrak{m}))\right) A(\mathfrak{m}) \tag{8.55}$$

is the partition function of the corresponding $G/K$ gauge theory with discrete theta angle $\lambda$. In this fashion, we see how, at the level of partition functions, a two-dimensional supersymmetric $G$ gauge theory with center-invariant matter decomposes into a disjoint union of $G/K$ gauge theories with discrete theta angles.

Now, let us gauge the $BK = K^{(1)}$ one-form symmetry of this theory. Proceeding as before, to gauge the action of $BK$ on the $G$-gauge theory, the partition function is a sum over $K$ gerbes

$$\frac{1}{|K|}\sum_{z\in K}\cdots \tag{8.56}$$

in which for each $K$ gerbe, one sums over gerbe-twisted $G$ bundles, which in this case means $G/K$ bundles defined by cocharacters $\mathfrak{m} \in M_{G/K}$ which are not also $G$ bundles, meaning $\mathfrak{m} \not\in M_G \subset M_{G/K}$. Furthermore, to be twisted by $z$ specifically, as opposed to a random element of $K$, we also need to require that the bundles twisted by the $K$-gerbe with characteristic class $z$ have $w(\mathfrak{m}) = z$.

Putting this together, we see that the partition function of the $G$ gauge theory after gauging by $BK$ has the form

$$Z(G/BK) = \frac{1}{|K|}\sum_{z\in K}\epsilon(z)\sum_{\mathfrak{m}\in M_{G/K}, w(\mathfrak{m})=z} A(\mathfrak{m}), \tag{8.57}$$

where $\epsilon(z)$ represent phases introduced for each gerbe sector. We can relate $\epsilon(z)$ to a particular character $\lambda \in \hat{K}$ by taking

$$\epsilon(z) = \exp\left(i\lambda(z)\right). \tag{8.58}$$

Then, corresponding to that character $\lambda$, we have the partition function

$$Z(G/BK, \lambda) = \frac{1}{|K|} \sum_{z \in K} \sum_{\mathfrak{m} \in M_{G/K}, w(\mathfrak{m}) = z} \exp\left(i\lambda(w(\mathfrak{m}))\right) A(\mathfrak{m}), \tag{8.59}$$

$$= \frac{1}{|K|} \sum_{\mathfrak{m} \in M_{G/K}} \exp\left(i\lambda(w(\mathfrak{m}))\right) A(\mathfrak{m}), \tag{8.60}$$

$$= Z\left((G/K)_\lambda\right). \tag{8.61}$$

Thus, we see that the partition function of the $G$-gauge theory, with gauged $BK$ action determined by $\lambda \in \hat{K}$, matches the partition function of the corresponding $G/K$ gauge theory with discrete theta angle determined by $\lambda$, as expected.

# 9 K theory

One of the consistency checks of decomposition applied in [4] involved computing D-brane charges in gauge theories with trivially-acting subgroups – namely, K theory classes. Although a subgroup of the gauge (or orbifold) group may act trivially on the underlying space, it might still act nontrivially on bundles over the space, which leads to a decomposition of K theory classes parametrized by irreducible representations of the trivially-acting subgroup, or put another way, a decomposition of K theory into K theory groups of the various summands appearing in decomposition. Thus, briefly, D-brane charges decompose in the fashion predicted by decomposition; schematically,

$$K = K\left(\coprod_i X_i\right). \tag{9.1}$$

We can similarly understand how gauging a one-form symmetry can select out summands in K theory. To be clear, consider a $G$-gauged nonlinear sigma model on $X$, where $K \subset G$ acts trivially, a $K$ gerbe on $[X/H]$ for $H = G/K$. For simplicity, assume that the $K$ gerbe is banded, and $K$ abelian. When we gauge the $BK = K^{(1)}$ one-form symmetry, we will see momentarily that it acts in the worldsheet theory by multiplying the various K theory elements by phases.

Physically, we can see the details of this operation as follows. Consider an open string disk diagram. The 2-group $BK$ on the Chan-Paton factors by 'tensoring' the corresponding bundle by a principal $K$ bundle. If the disk diagram has a $K$-twist field in its bulk, as illustrated in figure 3.

Because the disk is contractible, principal $K$ bundles on the disk are completely determined by their boundary conditions, here by their holonomy around the edge of the disk.

$z$

Figure 3: Open string disk diagram with bulk twist field insertion.

A bulk twist field in the interior creates a 'branch cut' by $z \in K$, hence the worldsheet bundle on the disk diagram shown in figure 3. When acting on the Chan-Paton factors on the boundary of this disk, the action of $BK$ on a K theory class in a component of the decomposition indexed by character $\rho \in \hat{K}$, is to multiply it by a phase $\rho(z)$.

Putting these pieces together, if in the bulk of the disk one inserts a projection operator of the form

$$\sum_{z \in K} \epsilon(z) \mathcal{O}_z, \tag{9.2}$$

where $\mathcal{O}_z$ is a twist field corresponding to $z$, the effect is to make the corresponding disk amplitude vanish if the Chan-Paton factors are not associated with a particular character of $K$. For example, if the phases $\epsilon(z)$ are all 1, then the effect is to project onto K theory classes that are associated with the trivial representation of $K$. The result is consistent with the picture presented elsewhere in this paper, that gauging one-form symmetries in two-dimensional theories projects onto components of decomposition.

# 10 Conclusions

In this paper we have discussed the gauging of one-form symmetries in two-dimensional theories, and how this selects out summands in the decomposition of such two-dimensional theories. We have in particular tracked through the examples of two-dimensional theories with one-form symmetries discussed in [1–3, 9], a mix of orbifolds and two-dimensional nonsupersymmetric and supersymmetric gauge theories, and explicitly demonstrated that gauging the one-form symmetries reverses decomposition, by selecting out one component in the summand. In so doing, we have given a very concrete description of the topological configurations over which path integrals sum when gauging one-form symmetries, and also discussed the available patterns of one-form symmetries in disjoint unions.

We have primarily focused on two-dimensional theories corresponding to banded gerbes. We have outlined some results for nonbanded and nonabelian gerbes, but leave a thorough examination of those cases for future work.

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# A Stacks and 2-stacks

An elegant way to understand gauge theories and quotients by ordinary groups is in terms of a stack, a generalization of a space, for which the prototypical example is $[X/G]$, for $G$ acting on a space $X$. Mathematical discussions of these can be found in e.g. [51–58]. Stacks admit metrics, spinors, and so forth, and so one can reasonably expect that one can define a nonlinear sigma model on a stack, see e.g. [1–3] for details.

To similarly make sense of gauging one-form symmetries, in principle one should appeal to 2-stacks.

Now, an ordinary stack can be defined by its incoming maps from other spaces (a description that is very relevant for sigma models). For example, for $G$ finite, maps $\Sigma \to [X/G]$ are defined by pairs consisting of

- a principal $G$ bundle $E \to \Sigma$,
- a $G$-equivariant map $\mathrm{Tot}(E) \to X$,

which can straightforwardly be seen to correspond to the twisted sectors one sums over in a path integral description of orbifolds.

In principle, a 2-stack $[\mathfrak{X}/BG]$, where $\mathfrak{X}$ is an ordinary stack admitting an action of $BG$. A map $\Sigma \to [\mathfrak{X}/BG]$ should be again defined by data including a $G$-gerbe on $\Sigma$.

In this language, we can get a geometric picture of what gauging a one-form symmetry is accomplishing. Suppose we start with a $G$-gerbe on a space $X$, for $G$ abelian. The $G$-gerbe is the total space of a $BG$ bundle on $X$. (As such, a gerbe is a 'generalized space' with a one-form symmetry, which is the basic reason why sigma models on gerbes have global one-form symmetries.) Gauging $BG$ is just quotienting those $BG$ fibers, leaving (modulo details of two-group actions) the underlying space $X$. For example, for a trivial $G$-gerbe on

$X$, the gerbe is $[X/G] = X \times BG$, and so gauging the one-form symmetry $G^{(1)} = BG$ is just the quotient

$$\left[ \frac{X \times BG}{BG} \right] \cong X. \tag{A.1}$$

This simple formal observation gives a mathematical prototype for the constructions described in this paper.

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