Stability and Noise in Frequency Combs: Harnessing the Music of the Spheres

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with

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and with other principal contributors

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and with thanks to

S. T. Cundiff, N. R. Newbury, P. St. J. Russell, and AMRDEC/DARPA

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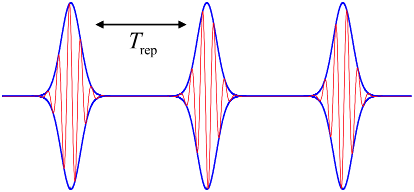
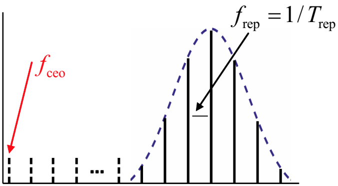
In 1999–2000: Frequency combs were invented

> *“The excitement surrounding the rapid evolution in these fields since 1999 gives us a hint of what it must have been like after 1927 when the first ideas of quantum mechanics were introduced. . .”*
>
> — *J. L. Hall and T. W. Hänsch, 2005 Nobel prize winners in Physics*

Time Domain



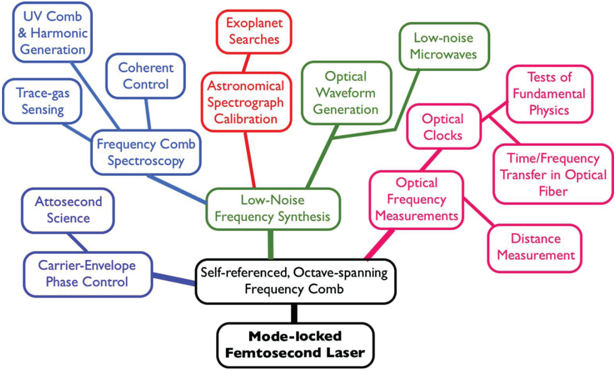
Frequency Domain



The key advance was electronically locking f_{ceo} ***and*** f_{rep}***!***

Background

- Frequency and time measurement was revolutionized
- Many new applications opened up[1]



[1] S. Diddams, J. Opt. Soc. Am. B **27**, 51 (2010).

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Motivation

- Advances have all been through “cut-and-try” experimentation
- Theoretical tools for analyzing and designing frequency combs are primitive
  - “Brute force” simulations or rough analytical approximations
    - ★ Adequate for post-hoc analysis; inadequate for design
    - ★ yield limited insight into the sources of instability

Key theoretical questions:

- Where in the adjustable parameter space are combs stable?
- What is the noise performance?
- How can we optimize the comb?
  - high output power and/or large bandwidth and/or low noise

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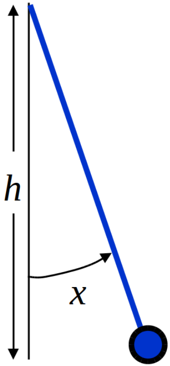
Approach

Combine: 400 years of dynamical systems theory

$+$

modern computers and algorithms

(linear algebra $+$ root-finding)



Evolutionary approach vs. Dynamical approach

A simple example: The damped pendulum

Nonlinear evolution equation:

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -A \sin x - \alpha y$$

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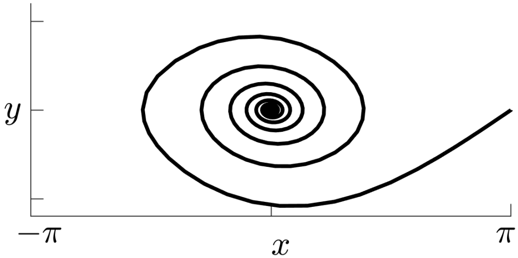
The Damped Pendulum: Evolutionary Approach

- Pick an initial condition
- Solve the evolution equation
- Look for convergence to steady-state
  $\Rightarrow$ Stable stationary solution exists



Our example:

Solution converges to $(0, 0)$
$\Rightarrow$ $(0, 0)$ is a stable stationary solution

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The Damped Pendulum: Dynamical Approach

- Find stationary solution directly (root-finding problem)

  Our example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y = 0; \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -A\sin x - \alpha y = 0$$

$\Rightarrow$ both $(0,0)$ and $(\pi,0)$ are stationary solutions

- Linearize the evolution equations.

  Our example:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}0 & 1\\ \mp A & \alpha\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} \equiv \mathcal{L}\mathbf{v}; \qquad \begin{bmatrix}(0,0)\to -A;\\ (\pi,0)\to A\end{bmatrix}$$

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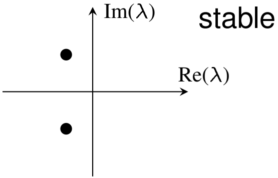
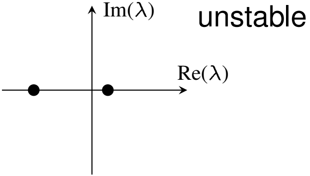
The Damped Pendulum: Dynamical Approach

- Find eigenvalues of the operator $\mathcal{L}$.
  Our example:

$$(0,0):\lambda_\pm = -\frac{\alpha}{2} \pm \left[\frac{\alpha^2}{4} - A\right]^{1/2} \quad ; \quad (\pi,0):\lambda_\pm = -\frac{\alpha}{2} \pm \left[\frac{\alpha^2}{4} + A\right]^{1/2}$$

Dynamical spectrum:





- If any real parts are positive, the stationary state is unstable
- As $A \to 0$, one real root equals zero, and the stable solution becomes unstable

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The Damped Pendulum: Noise Impacts

Our example: Calculate $\langle(\Delta x)^2\rangle$

$$\frac{\mathrm{d}\Delta x}{\mathrm{d}t} = \Delta y; \quad \frac{\mathrm{d}\Delta y}{\mathrm{d}t} = -A\Delta x - \alpha\Delta y + R; \quad \langle R(t)R(t')\rangle = \sigma^2\delta(t - t')$$

1. Find the left and right eigenvectors

$$\mathbf{w}_\pm = \frac{1}{\lambda_\pm - \lambda_\mp}\left[-\lambda_\mp, 1\right]; \quad \mathbf{v}_\pm = [1, \lambda_\pm]^T$$

normalized so that $\mathbf{w}_\pm\mathbf{v}_\pm = 1$; noting: $\mathbf{w}_\pm\mathbf{v}_\mp = 0$

2. Write the auto-correlation function of Δx

$$\mathbf{v} = a_+\mathbf{v}_+ + a_-\mathbf{v}_- \Rightarrow \Delta x = a_+ + a_-,$$

$$\langle(\Delta x)^2\rangle = \langle a_+{}^2\rangle + 2\langle a_+a_-\rangle + \langle a_-{}^2\rangle$$

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The Damped Pendulum: Noise Impacts

Our example: Calculate $\langle(\Delta x)^2\rangle$

$$\frac{d\Delta x}{dt} = \Delta y; \quad \frac{d\Delta y}{dt} = -A\Delta x - \alpha\Delta y + R; \quad \langle R(t)R(t')\rangle = \sigma^2\delta(t-t')$$

1. Find the left and right eigenvectors
2. Write the auto-correlation function of Δx
3. Find the solution to the Langevin equations:

$$\frac{da_\pm}{dt} = \lambda_\pm a_\pm + w_\pm[0, R]^T = \lambda_\pm a_\pm + \frac{R}{\lambda_\pm - \lambda_\mp},$$

$$\Rightarrow a_\pm = \frac{1}{\lambda_\pm - \lambda_\mp}\int_{-\infty}^{t} R\exp\left[\lambda_\pm(t-t')\right]$$

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The Damped Pendulum: Noise Impacts

Our example: Calculate $\langle(\Delta x)^2\rangle$

$$\frac{d\Delta x}{dt} = \Delta y; \quad \frac{d\Delta y}{dt} = -A\Delta x - \alpha\Delta y + R; \quad \langle R(t)R(t')\rangle = \sigma^2\delta(t - t')$$

1. Find the left and right eigenvectors
2. Write the auto-correlation function of Δx
3. Find the solution to the Langevin equations
4. Calculate expectations

$$\langle {a_\pm}^2\rangle = \frac{\sigma^2}{(\lambda_+ - \lambda_-)^2}\frac{1}{2\lambda_\pm}; \quad \langle a_+ a_-\rangle = -\frac{\sigma^2}{(\lambda_+ - \lambda_-)^2}\frac{1}{\lambda_+ - \lambda_-}$$

5. Combining yields

$$\langle(\Delta x)^2\rangle = \frac{\sigma^2}{2A}$$

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Origins of the Dynamical Approach

These dynamical ideas are very old!

- The pendulum clock:

  (Galileo – 1632; Huygens – 1673; Euler – 1736)

- Stability of the solar system:
  - Two body problem: Newton – 1686
  - Three body problem:

    In general, not solvable, but . . .

    stable fixed points found by Lagrange – 1772 (observed 1906)

- Application to continuous systems . . .

  (described by partial differential equations)

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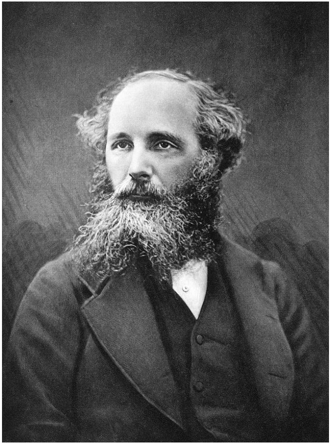
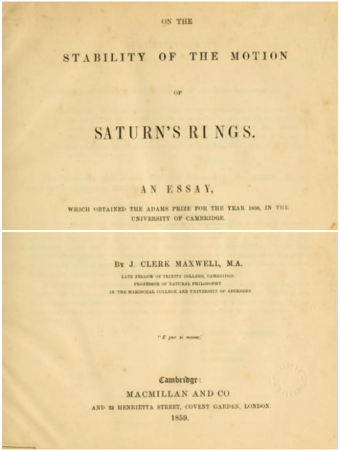
Origins of the Dynamical Approach

Before Maxwell's Equations (1861) . . .

Before the Maxwell-Boltzmann distribution (1865) . . .

Maxwell explained the stability of Saturn's rings (1859)




ON THE

STABILITY OF THE MOTION

OF

SATURN'S RINGS.

AN ESSAY,

WHICH OBTAINED THE ADAMS PRIZE FOR THE YEAR 1856, IN THE UNIVERSITY OF CAMBRIDGE.

BY J. CLERK MAXWELL, M.A.

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.
PROFESSOR OF NATURAL PHILOSOPHY
IN THE MARISCHAL COLLEGE AND UNIVERSITY OF ABERDEEN.

"E pur si muove."

Cambridge:
MACMILLAN AND CO.
AND 23 HENRIETTA STREET, COVENT GARDEN, LONDON.
1859.

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Numerous Applications

- Control of satellite orbits
- Electronic system design
- Plasma systems (tokamaks, ionosphere, ...)
- Mechanical, chemical, and fluid systems
- Biological systems (heart, animal populations)
- Economic systems
- Lasers/other optical resonators
  - but mostly in highly simplified, almost analytical approximations!

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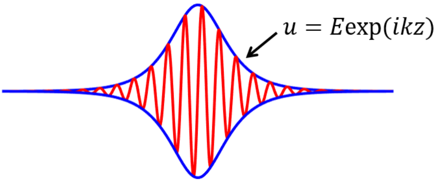
Nonlinear Schrödinger Equation

$$i\frac{\partial u}{\partial t} - \frac{\beta_2}{2}\frac{\partial^2 u}{\partial x^2} + \gamma|u|^2 u = 0$$

- First used to describe optical beams

  [Chiao et al., Phys. Rev. Lett. **14**, 1056 (1965)]

- Appears as a “lowest-order" envelope equation
  - in optics (lasers, optical fibers, beams, resonators)
  - in plasma, fluid systems
  - Bose-Einstein condensates



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Nonlinear Schrödinger Equation

Why lowest order?

- Neglects higher-order dispersion

$$\frac{\mathrm{d}^3 u}{\mathrm{d}z^3}, \quad \frac{\mathrm{d}^4 u}{\mathrm{d}z^4}, \quad \ldots$$

- Neglects higher-order in-band nonlinearity

$$u\frac{\mathrm{d}|u|^2}{\mathrm{d}z}, \quad |u|^4 u, \quad \ldots$$

Requires: Narrow bandwidth; weak nonlinearity

Qualitatively describes many systems.

Quantitatively describes none!

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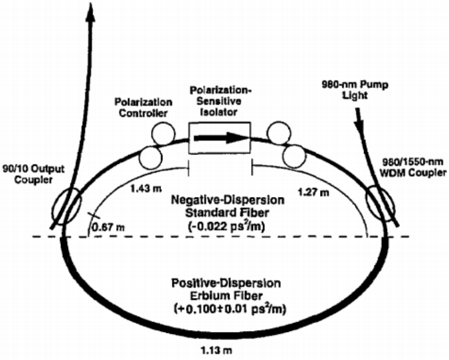
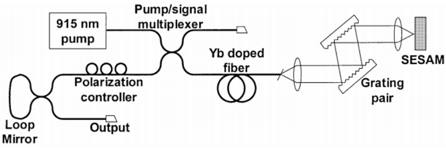
Frequency Comb Systems

Fiber Lasers

Fast Saturable Absorber[1]



Slow Saturable Absorber[2]



[1]K. Tamura et al., Opt. Lett. **18**, 1080 (1993).

[2]L. A. Gomes et al., IEEE J Sel. Top. Quantum Electron. **10**, 129 (2004).

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Frequency Comb Systems

Passively modelocked lasers (slow saturable gain):

$$\frac{\partial u}{\partial T} = \left[\frac{g\left(|u|\right)}{2}\left(1 + \frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2}\right) - \frac{l}{2} - i\frac{\beta''}{2}\frac{\partial^2}{\partial t^2} + i\gamma|u|^2 - f_{\text{sa}}(|u|)\right]u,$$

$$g\left(|u|\right) = g_0\left[1 + w_0/(P_{\text{sat}}T_R)\right]^{-1}$$

cubic-quintic model (fast saturable absorber)[1]

$$f_{\text{sa}}\left(|u|\right) = \delta|u|^2 - \sigma|u|^4$$

SESAM model (slow saturable absorber)[2]

$$f_{\text{sa}}\left(|u|\right) = -\frac{\rho}{2}n(t,T), \quad \frac{\partial n(t,T)}{\partial T} = \frac{1-n}{T_A} - \frac{\left|u(t,T)\right|}{w_A}n$$

[1] S. Wang et al., J. Opt. Soc. Am. B **31**, 2914 (2014).
[2] S. Wang et al., Opt. Lett. **42**, 2362 (2017).

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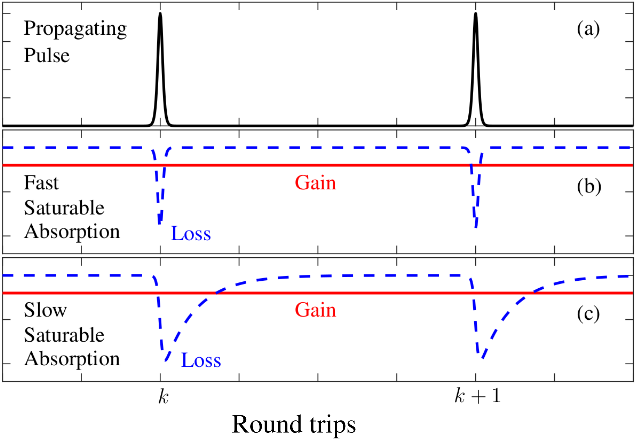
Saturable Absorption



The loss is saturated by the incoming pulse and then recovers

- almost instantaneously with fast absorbers
- slowly compared to the pulse duration with slow absorbers

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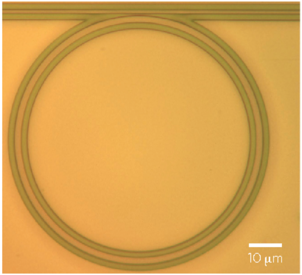
Frequency Comb Systems

Microresonators

The Lugiato-Lefever Equation:[3]

$$\frac{\partial \psi}{\partial t} - i\frac{\partial^2 \psi}{\partial x^2} - i|\psi|^2\psi + (1 + i\alpha)\,\psi - F = 0$$



System Parameters:

- α : frequency detuning (-5 to 10)
- F : pump amplitude (0 to 4)
- L : microresonator length (25 to 200)

[3] Z. Qi et al., Conf. Lasers Elect.-Opt. (2018), paper SF2A.6.

References

- Haus modelocking equation:
  - S. Wang et al., J. Opt. Soc. Am. B **31**, 2914 (2014)
  - Martinez, Fork, and Gordon, J. Opt. Soc. Am. B **2**, 753 (1985)
- Cubic-quintic modelocking equation:
  - Moores, Opt. Comm. **96**, 65 (1993)
  - Soto-Crespo et al., J. Opt. Soc. Am. B **13**, 1439 (1996)
  - Articles in *Dissipative Solitons*, Akhmediev and Ankiewicz ed. (2005)
- SESAM equation:
  - Kärtner et al., IEEE J. Sel. Quantum Electron. **2**, 540 (1996)
- Lugiato-Lefever equation:
  - Lugiato and Lefever, Phys. Rev. Lett. **58**, 2209 (1987)
  - Haelterman et al., Opt. Comm. **91**, 401 (1992)
  - Matsko et al., Opt. Lett. **36**, 2845 (2011)
  - Coen et al., Opt. Lett. **38**, 1790 (2013)
  - Chembo and Menyuk, Phys. Rev. A **87**, 053852 (2013)

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Dynamical Systems Methods

Standard, "brute-force" approach

- Solve the evolution equations for many roundtrips
- Use a noisy initial condition
- Convergence $\Longleftrightarrow$ existence + stability
- Change parameters; repeat

Advantages:

- Easy to program
- Intuitive (mimics experiments)

Disadvantages:

- Computationally slow
- Ambiguous near a stability boundary
- Limited insight into sources of instability

This approach is better for analysis than synthesis!

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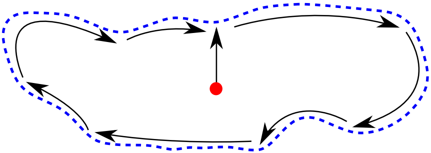
Our Approach

- Solve the evolution equations **once** to find a stationary solution
  - in a highly stable case
- Determine the stationary solution as parameters vary by solving a **root-finding problem**
- In parallel, find the eigenvalues of the linearized evolution equation
  - **The dynamical spectrum**
  - A stable solution has no eigenvalues with positive real parts
- Find parameters where one or more eigenvalues hit the imaginary axis
- Track the stability boundary



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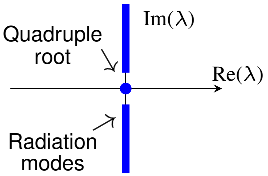
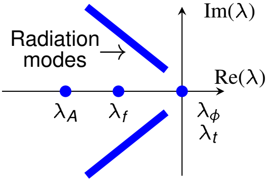
Atlas of Dynamical Spectra

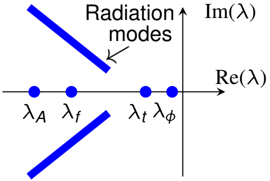
Classical NLS Spectrum



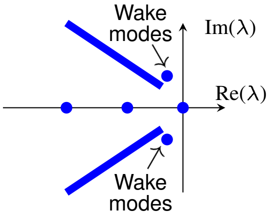
Soliton Laser



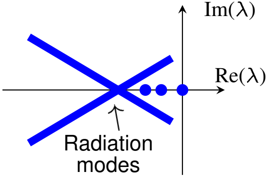
With locking



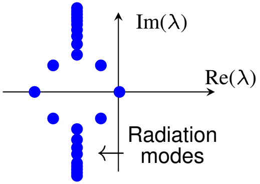
SESAM laser



Normal dispersion lasers



Microresonator



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Dynamical Systems Methods

Advantages

- $10^3 - 10^5$ times faster than brute-force solutions
- Unambiguous determination of stable operating parameter regimes
- Allows rapid mapping and optimization of solution properties
  - bandwidth, power, noise. . .
- Yields insight into the sources of instability

Disadvantages

- More difficult to program
- **The concepts are unfamiliar to many optical experimentalists**

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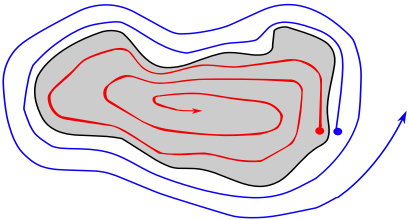
Two important caveats

- Accessibility vs. stability
  - Dynamical methods do not tell you how to access stable solutions
    - ★ Example: single solitons are hard to access in microresonators



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Dynamical Systems Methods

Two important caveats

- Accessibility vs. stability
  - Dynamical methods do not tell you how to access stable solutions
    - Example: single solitons are hard to access in microresonators
- Unstable system evolution
  - Dynamical methods do not tell you how an unstable solution evolves
    - Chaos, another stable solution, breathers are all possible

Dynamical and evolutionary methods are complementary!

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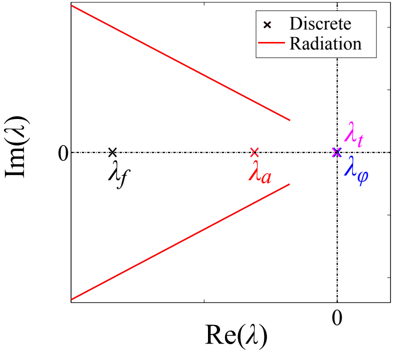
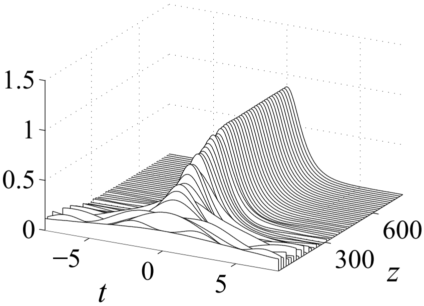
Linear Stability (Cubic-Quintic Model)

- The eigenmodes include:
  radiation modes and discrete eigenmodes
  - for soliton lasers, there are four discrete eigenmodes[1]
- The system is linearly stable if all $\mathrm{Re}\,(\lambda) \leq 0$





[1] Haus and Lai, J. Opt. Soc. Am. B **7**, 386 (1992).

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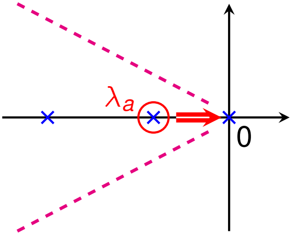
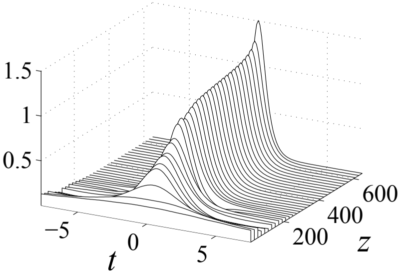
Instability of the Amplitude Eigenmode

When $\mathrm{Re}(\lambda_a) \to 0$:

- A saddle-node bifurcation occurs
- The corresponding pulse solution ceases to exist
- A small signal blows up or evolves to different solution





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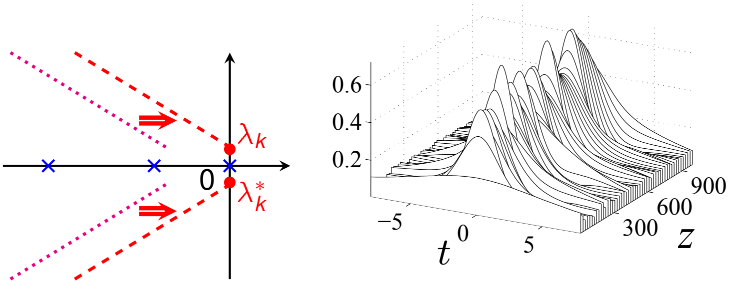
Instability of Radiation Modes

When $\mathrm{Re}\,(\lambda_k) \to 0$

- The radiation modes become unstable (Hopf bifurcation)
- We always observe that the pulse envelope fluctuates



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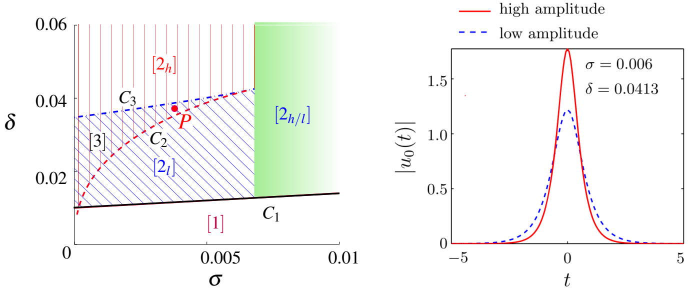
Stability Boundaries



- Two stable solutions coexist in a certain parameter regime:[1]
  - Low-amplitude solution (LAS)
  - High-amplitude solution (HAS)
- The boundaries of the stability regions are labeled C_1, C_2, and C_3

C_1	LAS	Hopf (essential) instability
C_2	HAS	saddle-node instability
C_3	LAS	saddle-node instability

[1] S. Wang et al., J. Opt. Soc. Am. B **31**, 2914 (2014).

Figure 2. The stability regions of the GHME with a cubic-quintic saturable absorber $f_{\mathrm{sa,cq}}(|u|)$. The stability boundaries are marked by three curves, C_1, C_2, and C_3. This figure reproduces Fig. 16 of ref. [22].

When $\sigma \neq 0$, the solution in Eq. (11) does not hold any more, and we have found two stable numerical pulse solutions of the GHME: a low-amplitude solution (LAS) and a high-amplitude solution (HAS). The LAS is stable in the blue-hatched region that is marked with $[2_l]$ as shown in Fig. 2, and it becomes unstable ... modes become unstable via a Hopf bifurcation or an essential instability [30]. The amplitude eigenmode becomes unstable when we cross C_3 from region $[2_l]$, which corresponds to a saddle-node instability. Meanwhile, there ... to as the high-amplitude solution (HAS). The HAS is stable in the red-hatched region which is marked with $[2_h]$, and its stability region is lower-bounded by C_2, below which the amplitude eigenmode becomes unstable via a saddle-node bifurcation. The LAS and the HAS coexist

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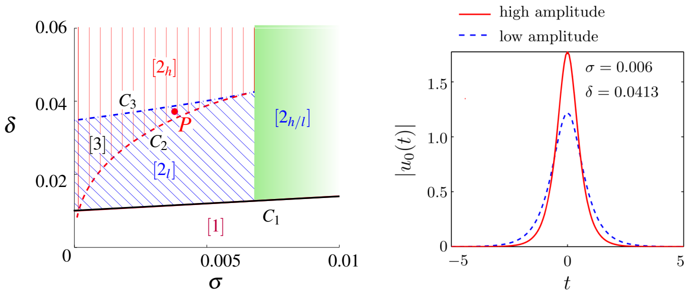
Stability Boundaries



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When $\eta \neq 0$, the solution in Eq. (11) does not hold any more, and we have found two stable numerical pulse solutions of the GHME: a low-amplitude solution (LAS) and a high-amplitude solution (HAS). The LAS is stable in the blue-hatched region that is marked with $[2_l]$ as shown in Fig. 2, and it becomes unstable in region [1] (below the curve C_1), where the continuous modes become unstable via a Hopf bifurcation or an essential instability [30]. The amplitude eigenmode becomes unstable when we cross C_3 from region $[2_l]$, which corresponds to a saddle-node instability. Meanwhile, there also exists a second stationary solution [22], which we refer to as the high-amplitude solution (HAS). The HAS is stable in the red-hatched region which is marked with $[2_h]$, and its stability region is lower-bounded by C_2, below which the amplitude

[1]	No stable pulse solution	$[2_l]$	low amplitude pulse solution
$[2_{h/l}]$	A single stable pulse solution	$[2_h]$	high amplitude pulse solution
[3]	Two stable pulse solutions		

Hypothesis verified experimentally:[1]

[1] C. Bao et al., Phys. Rev. Lett. **115**, 253903 (2015).

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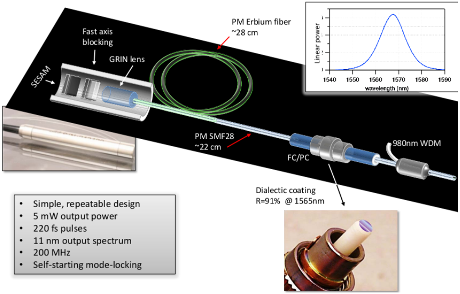
The SESAM Modelocked Fiber Laser

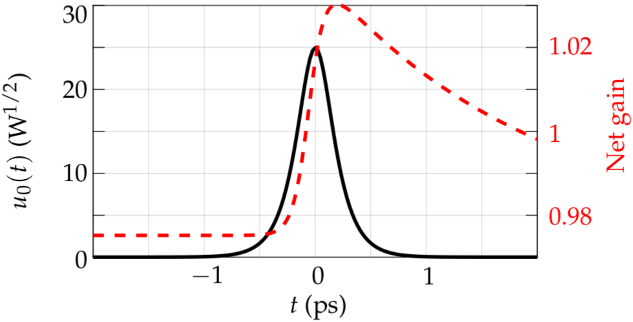
- The system is built using
  - telecom grade polarization-maintaining (PM) components
  - highly-doped erbium-doped fiber
  - highly non-linear PM fibers
  - a semiconductor saturable absorber mirror (SESAM)
- Output: highly stable 200 MHz combs with $P_{\text{av}} = 5$ mW



[1] Sinclair et al., Opt. Express **22**, 6996 (2014).

Balance of Energy

- The SESAM and the linear gain open a gain window that allows the pulse to grow
- A soliton wake instability will occur when g_0 becomes sufficiently large or β'' becomes sufficiently small



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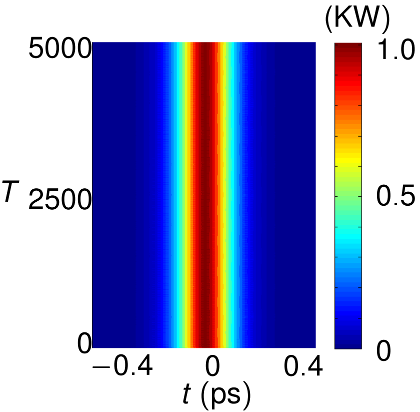
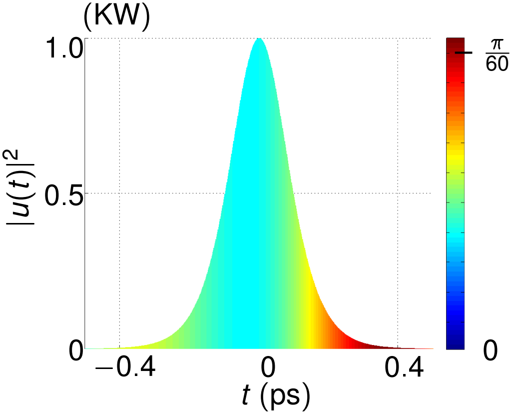
Stable Operation

The stable pulse is close in shape to a sech pulse
(soliton solution of the nonlinear Schrödinger equation)



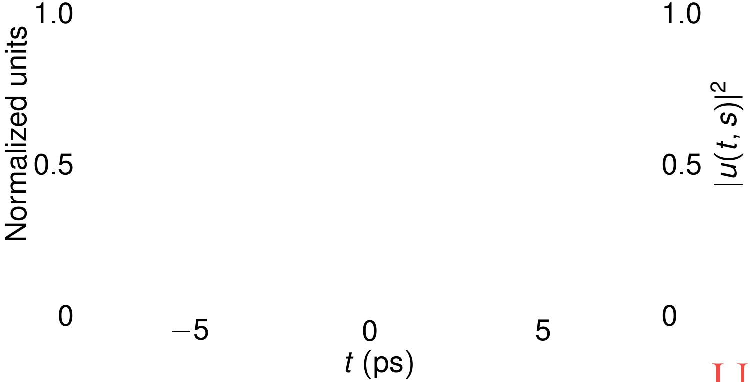
Color indicates the pulse power



Color indicates the phase of the pulse

Soliton Wake Instability

Quasi-periodicity is observed in the evolution



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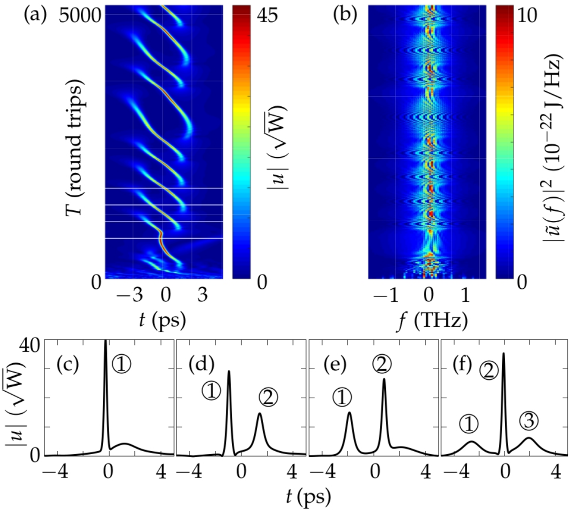
Soliton Wake Instability

Quasi-perio Letter is observed in the evolution



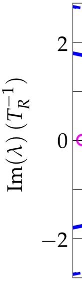


Fig. 4. The
saturated g
$\lambda_{w\pm} = -7$
9.09×10^{-3}
dicate how t
The eigenva

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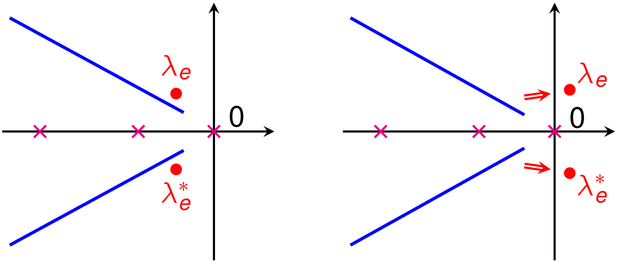
Dynamical Spectrum

- As g_0 increases:
  - a pair of eigenvalues λ_e, λ_e^* emerge from the radiation modes (edge bifurcation)
- Re$[\lambda_e]$, Re$[\lambda_e^*]$ become positive, leading to instability (Hopf bifurcation)[1]



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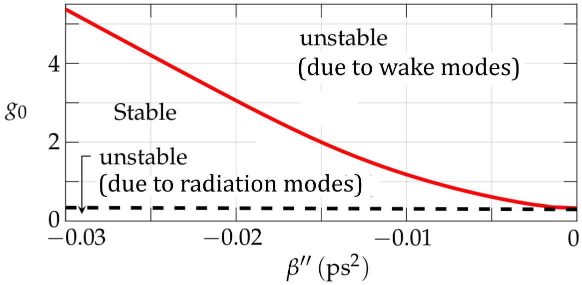
[1] S. Wang et al., Opt. Lett. **42**, 2362 (2017).

Stable Region

- Continuous modes become unstable when the gain is too low
- The wake mode instability occurs when
  - the unsaturated gain becomes large
  - the group delay dispersion becomes small



Once the stable region is known, optimization is possible!

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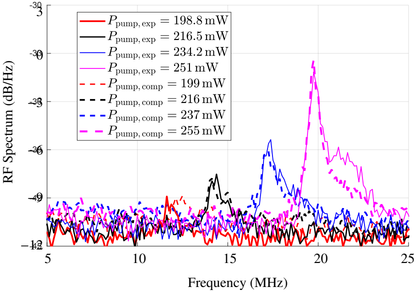
The Wake Mode Sidebands

Figure 3. An illustration of unwrapping the optical field $u(t,T)$ to $u(T)$, where T_w is the computational time window and T_R is the roundtrip time.

We show in Fig. 5 the obtained display trace of the spectrum analyzer in experiments. We show the RF spectrum of the photocurrent $I(t,T)$ where the sidebands' profiles are apparent. As the pump power increases, in addition to the shifts of the offset frequency, the power of the wake mode sidebands also increases. We have shown in prior computation a qualitative agreement [17]. The laser becomes unstable when the pump power $P_{\text{pump}} > 255$ mW. When $P_{\text{pump}} = 255$ mW, the sidebands is centered at 19.5 MHz, and the magnitude of the sidebands is about 13 dB above the background noise.

Even when stable, the wake modes generate sidebands[1]

Lumped



Averaged

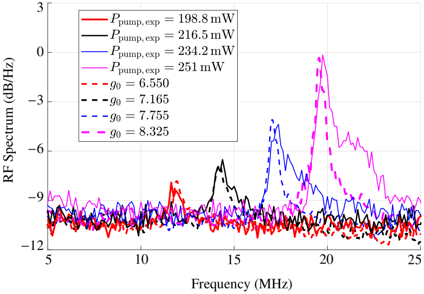


Figure 4. The display of the spectrum analyzer as the pump power increases and our computational simulations. The agreement is excellent.

As the pump power increases:

the magnitude and the offset frequency of sidebands grow

3. Optimizing the Laser Cavity

The lumped model accurately captures the behavior of each cavity component and thus provides good reference to further optimization study. However, the computational results we have shown in Fig. 5 is obtained using evolutionary approach, in which the computation time for each

[1] S. Wang, PhD Dissertation, UMBC (2018).

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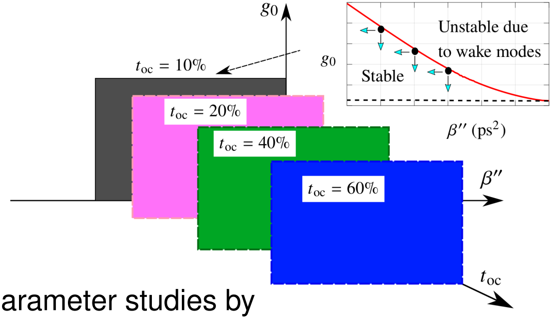
Parameter Studies

Our algorithms make 3D parameter space optimization possible[1]



We perform parameter studies by

- finding the stability boundary
  - when the output coupling ratio t_{oc} increases
  - the unsaturated gain decreases
  - the group delay dispersion decreases

[1] S. Wang, PhD Dissertation, UMBC (2018).

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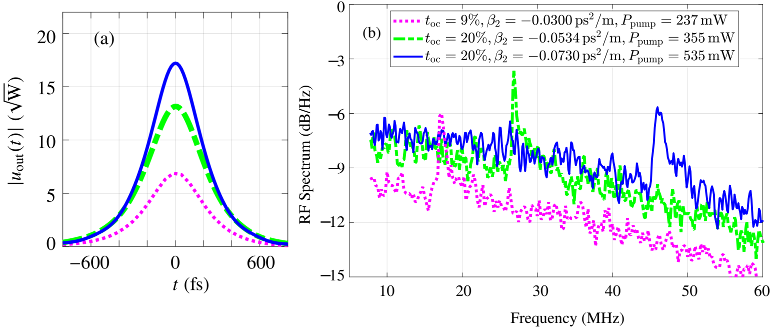
Optimal Pulse



To obtain higher output power and smaller wake mode sidebands:[1,2]

- increase the output coupling ratio

  and then

- maximize the cavity gain
- decrease the group delay dispersion

[1] S. Wang, PhD Dissertation, UMBC (2018).
[2] S. Wang et. al., paper SF1C.2, CLEO 2017.

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Evaluating Noise Levels

- To characterize the laser noise levels
  - The Haus-Mecozzi model is widely used
    - ★ Qualitatively useful for soliton lasers, but inaccurate
    - ★ Not reliable for non-soliton lasers
      (e.g., lasers that operate with normal dispersion)
  - Monte Carlo simulations can be prohibitively expensive
  - The dynamical method is far more computationally rapid
- We develop the dynamical method and compare the computational efficiency to Monte Carlo simulations

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Evaluating Noise Levels

Our approach

- Characterize the noise input
  - amplifier noise (modelocked lasers)
  - laser pump noise; thermal noise (microresonators)
- Calculate the noise contribution to each (right) eigenvector
  - Use the inner product with left eigenvectors
- Calculate the autocorrelation function for statistical quantities of interest
  - SESAM laser:
    central frequency (f_c), pulse energy (w), RF phase error (ψ)
- Calculate their power spectra densities
  (Fourier transform of autocorrelation functions)
  - $S_w(f)$, $S_{f_c}(f)$, $S_\psi(f)$

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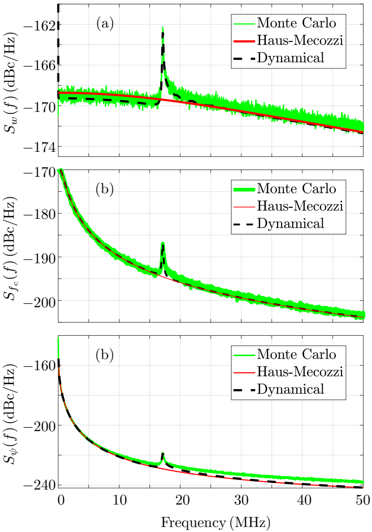
ESAM fiber laser, where $\sigma_{\Delta w}$, $\sigma_{\Delta f_c}$, and $\sigma_{\delta t_c}$ are time-dependent variances of the
rgy w, central frequency f_c, and the central time t_c (or equivilently, the roundtrip
he results from Haus-Mecozzi predictions are derived from Eqs. (6) and (13) using
utational stationary pulse solution.

Evaluating Noise Levels



he noise spectra of (a) the energy jitter, (b) the frequency jitter, and (c) the timing
ter that we obtain from the Monte Carlo approach, the Haus-Mecozzi formulae,

We evaluate the noise level of the SESAM fiber comb laser[1]

- We have achieved excellent agreement between the dynamical method and the Monte Carlo method
- The Haus-Mecozzi equation misses the wake mode sidebands

[1] S. Wang et al., J. Opt. Soc. Am. B **35**, 2521 (2018).

Computational Efficiency

Monte Carlo method vs. Dynamical method[1]

Experiment	# of cores	Time cost	Memory usage	Storage
MC runs	256	20 min/core	314 MB/core	1.7 MB/core
Dynamical	1	$<$ 4 min	900 MB	144 MB

We integrate the system for 2×10^5 roundtrips in each run of the Monte Carlo simulations.

The computational efficiency is improved by a factor $> 10^3$!

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[1] S. Wang et al., J. Opt. Soc. Am. B **35**, 2521 (2018).

Cnoidal Waves in Microresonators

- Solitons are a special case of a broader class of periodic wave functions
  - Referred to as:

    Turing rolls, **cnoidal waves**
  - "Cnoidal waves" is the most common nomenclature in the nonlinear waves / dynamical system community

- Cnoidal wave attributes:

  On the one hand, they. . .
  - have clean spectra with evenly spaced comb lines (like solitons)
  - have analytical solutions that exist in the no-loss limit (like solitons)[1]
  - can have a broad bandwidth (like solitons)

[1] Z. Qi et al., J. Opt. Soc. Am. B **34**, 785 (2017).

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Cnoidal Waves in Microresonators

- Solitons are a special case of a broader class of periodic wave functions
  - Referred to as:

    Turing rolls, **cnoidal waves**
  - "Cnoidal waves" is the most common nomenclature in the nonlinear waves / dynamical system community

- Cnoidal wave attributes:

  On the other hand, they...
  - can be easily and deterministically accessed (unlike solitons)
  - use the pump more efficiently and produce higher power comb lines

BUT

with lines spaced farther apart

They have often been observed, but little remarked upon!

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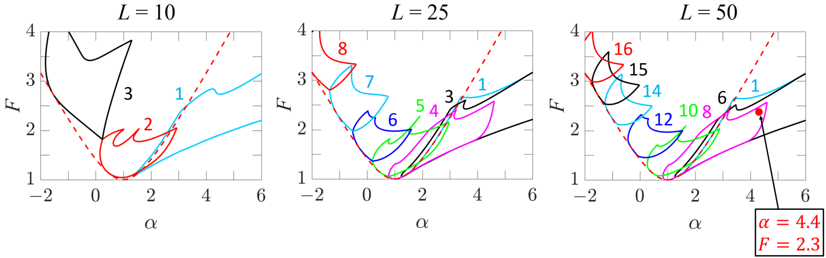
Stable Regions

Stable regions for different periodicities

F = normalized pump power
α = normalized detuning



- Continuous waves are stable below the red-dashed line
- Below $\alpha = 41/30 \simeq 1.37$, cnoidal waves can be easily accessed by raising the pump power
- Cnoidal waves are stable in a U-shaped region in α-F space
- Moving along this region, different values of N_{per} can be deterministically accessed

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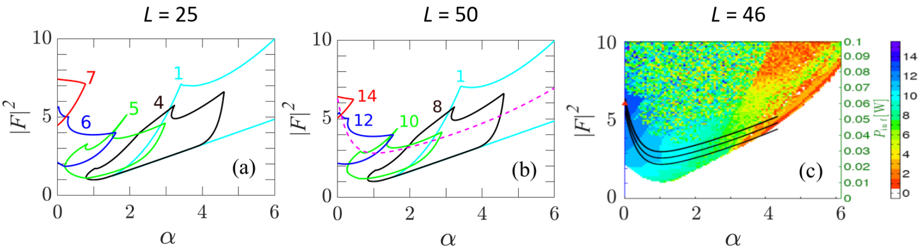
Stable Regions

Cnoidal Wave Solutions:

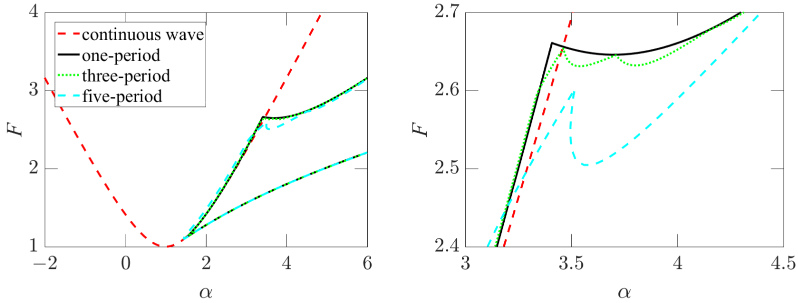


$\lvert F\rvert^2$:		normalized input power
α :		normalized detuning
L :		normalized circumference

[1] J. Jaramillo-Villegas et al., *Opt. Express* **23**, 9618 (2015)

Stable Regions ($L = 50$)

Why are single solitons hard to access



- Substantial overlap with other cnoidal waves
- Almost complete overlap with continuous waves

By contrast the periodicity-8 cnoidal wave can be deterministically accessed!

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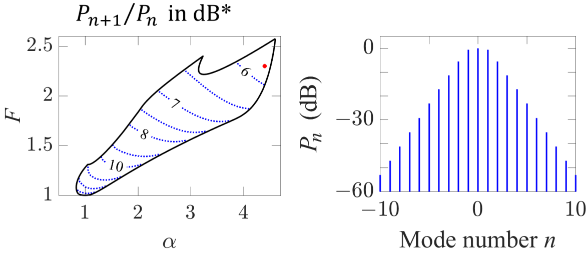
Periodicity-8 Cnoidal Waves ($L = 50$)

Controllability and bandwidth



- With FSR = 125 GHz; 30 dB down bandwidth = 24 THz
- Uses the pump more efficiently than a single soliton

Large bandwidth can be obtained!

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* large n limit ($n \gtrsim 4$)

Work in Progress

- With A. Weiner and M. Qi (Purdue): Investigations of cnoidal waves and molecules
- With A. Coillet and Y. Chembo (CNRS): Secondary and higher-order combs
- With F. Li and P. K. A. Wai (HKPU): Dual-frequency Kerr combs
- With O. Gat (Hebrew U.): Inclusion of temperature effects
- With J. Zweck (U. T. Dallas): Periodically-stationary systems

Conclusions

- Powerful dynamical methods can be combined with modern computer algorithms to:
  - rapidly determine where in the parameter space stable solutions lie
  - yield important insights into the sources of instability
  - rapidly determine the noise performance
  - optimize the system performance

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Conclusions

- We have applied these methods to:
  - The laser models with slow saturable gain
    - Identified parameters ranges where two stable solutions exist
    - Compare the cubic-quintic model to other models
  - SESAM laser
    - Characterized the wake mode instability
    - Determined where stable solutions exist
    - Explained the appearance of sidebands
    - Characterized the noise performance
    - Optimized the system parameters
  - Microresonators
    - Determined where cnoidal waves are stable
    - Explained why single solitons are hard to access
    - Found broadband, easily accessible cnoidal wave solutions

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Summary

These methods are an important complement to widely used evolutionary methods and should be commonly used!

Our software is available at:

`http://www.umbc.edu/photonics/software.html`

See: "Dynamical method to evaluate noise. . ."
"Boundary tracking algorithms"

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