Adaptive Partitioned Methods for the Time-Accurate Approximation of the Evolutionary Stokes-Darcy System

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Abstract

This paper develops, analyzes and tests a time-accurate partitioned method for the Stokes-Darcy equations. The method combines a time filter and Backward Euler scheme, is second order accurate and provide, at no extra complexity, an estimated the temporal error. This approach post-processes the solutions of Backward Euler scheme by adding three lines to original codes to increase the time accuracy from first order to second order. We prove long time stability and error estimates of Backward Euler plus time filter with constant time stepsize. Moreover, we extend the approach to variable time stepsize and construct adaptive algorithms. Numerical tests show convergence of our method and support the theoretical analysis.

Keywords: Time filter; Backward Euler; Stokes-Darcy; Variable time stepsize; Adaptive algorithms.

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1 Introduction

The coupling of a fluid flowing between a porous media and a free flow region is a typical multi-physics and multi-domain problem, which plays an important role in many industrial and engineering applications and in transport between ground water and surface water. The Stokes-Darcy model is a fundamental model of this proem and it has a close relationship with other multi-physics model[1]. Numerical methods for the coupled model have been extensively studied and tested, including finite element methods [2], spectral methods [3], discontinuous Galerkin methods [4], discontinuous finite volume methods [5], mortar element methods [6], boundary integral methods [7], hybrid discontinuous Galerkin methods [8], least square methods [9], optimization based methods [10], weak Galerkin methods [11], various domain decomposition methods [12, 13, 14], two grid methods [15], multigrid methods [16], and time partitioned methods [17].

Most recently research has focused on partitioned methods for the non-stationary Stokes-Darcy model. See [18, 19, 20, 21] for a first-order partitioned methods, [22, 17] for second order partitioned

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method and [23] for a third-order partitioned method. Current directions include methods using different timesteps in different subdomains [21, 17] and higher order methods [22, 17, 23]. The aim of this research is to develop a fast partitioned method that provides time accurate approximations by time adaptive partitioned methods using time filters. Time filters are effective tools to offset the weakness of lower-order partitioned methods. Partitioning can be accomplished by using an IMEX method with explicit discretization of interface coupling terms. If this induces oscillations, time filters can also be used to damp non-physical oscillations here, as in GFD, see Section 1.1 for a summary of this work. Recently it was noted in [24] that time filters also can increase the time accuracy of simple, lower accuracy methods. This yields, at low cost, two approximations of different accuracy. Thus, it also gives a low cost error estimator for adapting the timestep to ensure time accuracy. This report develops, analyzes and tests adaptive algorithms, based on this idea, for the coupled, evolutionary Stokes-Darcy problem. The method herein is based on finite element discretization in space. The subdomain/subphysics terms are discretized in time by the usual (first order) fully implicit Backward Euler method. The coupling terms are discretized by the explicit second order's extrapolation method. These terms are skew symmetric (express conservation of material flowing from one subdomain into the other), must be treated explicitly to produce a partitioned method and represent the critical physical effect. For all these reasons we discretize by a second order extrapolation formula (rather than forward Euler). Adding 3 lines of code (time filtering the flow variables) increases the accuracy to $O(\Delta t^2)$ and gives (as noted above) an error estimator to adapt the time step.

The paper is organized as follows. Section 2 gives the coupled Stokes-Darcy model and the associated weak formulation. The BETF algorithm and the long time stability are given in Section 3. Section 4 is devoted to the error analysis of the fully discretized scheme. In Section 5, we introduce BE and BETF algorithm for variable time stepsize and construct adaptive algorithm with performing stepsize selections to control time accuracy and computational efficiency. We presented the numerical tests to illustrate the time accuracy of our numerical methods in Section 6. Final conclusions are given in Section 7.

1.1 Previous work on time filters

The first time filter called RA time filter was constructed by Robert [25] and analyzed by Asselin [26]. The combination of RA filter with leapfrog is used to control the leapfrog method's computational mode [27]. Williams [28, 29] made an important modification to the RA filter, by proposing RAW time filter which increases the numerical accuracy for amplitude errors from the first order to the third-order accuracy. Li and Trenchea [30] proposed a higher-order Robert-Asselin (hoRA) type time filter which is non-intrusive, easily implementable and achieves third-order accuracy. The approach of using time filters, as herein, is quite recent but has already been shown to increase time accuracy in other flow problems. These include the fully coupled and nonlinear Navier-Stokes equations [31], slightly compressible flow problem [32] and shows promise for variable order, time adaptive method [33].

2 The Stokes-Darcy model and weak formulation

We consider the time-dependent Stokes-Darcy model consisting of Stokes equations and Darcy equations. The Stokes equations which describe the motion of free flow are given by: find the fluid velocity u: $\Omega_f \times [0,T] \to R^d$, the pressure p: $\Omega_f \times [0,T] \to R$ satisfying

$$\frac{\partial u}{\partial t} - \nu \triangle u + \nabla p = f_1 \quad in\ \Omega_f, \tag{2.1}$$

$$\nabla \cdot u = 0 \quad in\ \Omega_f. \tag{2.2}$$

In the porous medium region Ω_p, the Darcy equations which describe the behavior of the porous media flow is given by: find hydraulic head ϕ: $\Omega_p \times [0,T] \to R^d$ satisfying

$$S_0 \frac{\partial \phi}{\partial t} + \nabla \cdot u_p = f_2 \quad in\ \Omega_p, \tag{2.3}$$

$$u_p = -K \nabla \phi \quad in\ \Omega_p. \tag{2.4}$$

Combining the continuity equation (2.3) with Darcy's law (2.4), we can get the following equation

$$S_0 \frac{\partial \phi}{\partial t} - \nabla \cdot (K \nabla \phi) = f_2 \quad in\ \Omega_p, \tag{2.5}$$

Here, in Figure 1, $\Omega \in R^d (d = 2\ or\ 3)$ is a bounded domain and $\overline{\Omega} = \overline{\Omega_f} \cup \overline{\Omega_p}$. Ω_f and Ω_p are the fluid region and the porous medium region, respectively. And $\Gamma = \partial\Omega_f \cap \partial\Omega_p$ is the interface between the fluid and the porous media regions. Both Ω_f and Ω_p have Lipschitz continuous boundaries. Define $\Gamma_i = \partial\Omega_i \backslash \Gamma$ for $i = f, p$. Moreover, we denote by n_f and n_p the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, and τ_f the unit tangential vectors on the interface Γ. It is clear that $n_p = -n_f$ on Γ. Here, u and u_p denote the fluid velocity and the specific discharge rate in the porous medium, p

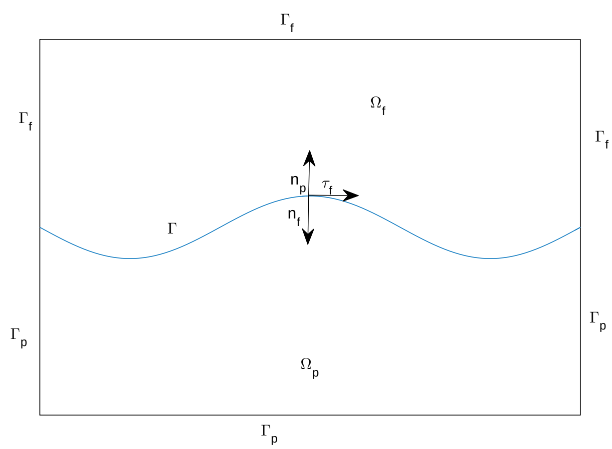

Figure 1: A global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p separated by the interface Γ.

denotes the kinematic pressure, and f_1 and f_2 denote a general body force in Stokes equations and a source term in Darcy equations, ϕ denotes the hydraulic head, K is the hydraulic conductivity tensor, and S_0 is the soil compressibility. For simplicity, we assume that $K = \{K_{ii}\}_{d\times d}$ is a symmetric and positive definite matrix with the smallest eigenvalue $K_{min} > 0$. It is important to note that in many applications these are not $O(1)$ parameters.

We impose homogeneous Dirichlet boundary conditions and the initial condition:

$$u = 0 \quad on\ \Gamma_f, \tag{2.6}$$
$$\phi = 0 \quad on\ \Gamma_p, \tag{2.7}$$
$$u(x,0) = u^0 \quad in\ \Omega_f, \tag{2.8}$$
$$\phi(x,0) = \phi^0 \quad in\ \Omega_p. \tag{2.9}$$

The coupling conditions on the interface are the conservation of mass, the balance of normal forces and the Beavers-Joseph-Saffman condition:

$$u \cdot n_f + u_p \cdot n_p = 0 \qquad on\ \Gamma, \tag{2.10}$$
$$p - \nu n_f \cdot \frac{\partial u}{\partial n_f} = g\phi \qquad on\ \Gamma, \tag{2.11}$$
$$-\nu \tau_f \cdot \frac{\partial u}{\partial n_f} = \frac{\alpha \nu \sqrt{d}}{\sqrt{trace(\Pi)}} \tau_f \cdot u \quad on\ \Gamma. \tag{2.12}$$

Here, d is the space dimension, g is the gravitational acceleration, α is a positive parameter depending on the properties of the medium and must be experimentally determined, and the permeability $\Pi = \frac{K\nu}{g}$. Equation (2.12) is the Beavers-Joseph-Saffman condition.

We introduce the following spaces

$$H_f = \{v \in (H^1(\Omega_f)^d : v = 0\ on\ \partial\Omega_f \backslash \Gamma\},$$
$$H_p = \{\psi \in H^1(\Omega_p) : \psi = 0\ on\ \partial\Omega_p \backslash \Gamma\},$$
$$Q = L_0^2(\Omega_f).$$

For the domain D, $(\cdot,\cdot)_D$ refers to the scalar inner product in D for $D = \Omega_f$ or Ω_p . In particular, we denote the $H^1(\Omega_{f/p})$ norm by $\|\cdot\|_{H_f/H_p}$, the $L^2(\Gamma)$ norm by $\|\cdot\|_\Gamma$ and the $L^2(\Omega_{f/p})$ norm by $\|\cdot\|_{f/p}$, and define the corresponding norms and the notation hereafter:

$$\|u\|_f = \|u\|_{L^2(\Omega_f)}, \qquad \|u\|_{H_f} = \|\nabla u\|_{L^2(\Omega_f)},$$
$$\|\phi\|_p = \|\phi\|_{L^2(\Omega_p)}, \qquad \|\phi\|_{H_p} = \|\nabla \phi\|_{L^2(\Omega_p)}.$$

With these notations, the weak formulation of the coupled Stokes-Darcy problem is given as follows: find $(u, \phi) \in H_f \times H_p$ and $p \in Q$ such that, $\forall t \in (0, T]$,

$$(u_t, v)_{\Omega_f} + gS_0(\phi_t, \psi)_{\Omega_p} + a_f(u, v) + a_p(\phi, \psi) + c_\Gamma(v, \phi) - c_\Gamma(u, \psi) + b(v, p)$$
$$= (f_1, v)_{\Omega_f} + g(f_2, \psi)_{\Omega_p} \quad \forall v \in H_f, \psi \in H_p, \tag{2.13}$$
$$b(u, q) = 0 \quad \forall q \in Q, \tag{2.14}$$

where

$$a_f(u, v) = \nu(\nabla u, \nabla v)_{\Omega_f} + a_\Gamma(u, v),$$
$$a_p(\phi, \psi) = g(K\nabla\phi, \nabla\psi)_{\Omega_p},$$
$$c_\Gamma(v, \psi) = g\int_\Gamma \psi v \cdot n_f,$$
$$b(v, q) = -(p, \nabla \cdot v)_{\Omega_f}.$$
$$a_\Gamma(u, v) = \sum_{i=1}^{d-1} \int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{trace(\Pi)}} (u \cdot \tau_i)(v \cdot \tau_i).$$

For further investigation, we also recall the Poincaré, trace and Sobolev inequalities that are useful in the following analysis. There exist constants C_d, C_s, which depend only on the domain Ω_f, and $\widetilde{C}_d, \widetilde{C}_s$, which depend only on the domain Ω_p, such that, for all $v \in H_f$ and $\phi \in H_p$,

$$\|v\|_f \le C_d \|v\|_{H_f}, \qquad \|\phi\|_p \le \widetilde{C}_d \|\phi\|_{H_p}. \tag{2.15}$$

$$\|v\|_\Gamma \le C_s \|v\|_f^{\frac{1}{2}} \|\nabla v\|_f^{\frac{1}{2}}, \ \ \|\phi\|_\Gamma \le \widetilde{C}_s \|\phi\|_p^{\frac{1}{2}} \|\nabla \phi\|_p^{\frac{1}{2}}. \tag{2.16}$$

Lemma 2.1 *[17] There exist constants $C_1 = C_s^2 \widetilde{C}_s^2 \ge 0$ and $C_2 = C_d \widetilde{C}_d \ge 0$, such that for all $(v, \phi) \in H_f \times H_p$ and $\epsilon, \epsilon_1, \epsilon_2 > 0$, we have*

$$c_\Gamma(v, \phi) \le \epsilon\nu \|v\|_{H_f}^2 + \frac{g^2 C_1 C_2}{4\epsilon\nu} \|\phi\|_{H_p}^2, \tag{2.17}$$

$$c_\Gamma(v, \phi) \le \epsilon g \|K^{\frac{1}{2}} \nabla\phi\|_p^2 + \frac{g C_1 C_2}{4\epsilon K_{min}} \|u\|_{H_f}^2, \tag{2.18}$$

$$c_\Gamma(v, \phi) \le \epsilon_1 \nu \|v\|_{H_f}^2 + \epsilon_2 \|K^{\frac{1}{2}} \nabla\phi\|_p^2 + \frac{g^4 C_1^2 C_d^2}{64 \epsilon_1^2 \epsilon_2 \nu^2 K_{min}} \|\phi\|_p^2, \tag{2.19}$$

and

$$c_\Gamma(v, \phi) \le \epsilon_1 \|K^{\frac{1}{2}} \nabla\phi\|_p^2 + \epsilon_2 \nu \|v\|_{H_f}^2 + \frac{g^4 C_1^2 \widetilde{C}_d^2}{64 \epsilon_1^2 \epsilon_2 \nu K_{min}^2} \|v\|_f^2, \tag{2.20}$$

$$c_\Gamma(v, \phi) \le \epsilon\nu \|v\|_{H_f}^2 + \frac{g^2 C_1 \widetilde{C}_I C_d}{4\epsilon\nu h} \|\phi\|_p^2, \tag{2.21}$$

$$c_\Gamma(v, \phi) \le \epsilon g \|K^{\frac{1}{2}} \nabla\phi\|_p^2 + \frac{g^2 C_1 \widetilde{C}_d C_I}{4\epsilon g K_{min} h} \|u\|_f^2. \tag{2.22}$$

3 Numerical algorithms

In this section, we propose the decoupled scheme for the coupled Stokes-Darcy model. We choose a uniform partition of $[0, T]$ with $t_m = m\Delta t$, $m = 0, 1, ..., N$, where $\Delta t = \frac{T}{N}$, and (u^m, p^m, ϕ^m) denotes the discrete approximation in time by following schemes to $(u(t_m), p(t_m), \phi(t_m))$. These are presented below for constant time steps. Their variable Δt and adaptive versions are in Section 5.

3.1 Backward Euler plus time filter (BETF)

- Given (u^0, p^0, ϕ^0) and (u^1, p^1, ϕ^1). Find $(\hat{u}^{m+1}, \hat{p}^{m+1}) \in (H_f, Q)$ with $m = 0, 1, ..., N-2$, such that for any $v \in H_f$, and $q \in Q$,

$$(\frac{\hat{u}^{m+1} - u^m}{\Delta t}, v)_{\Omega_f} - a_f(\hat{u}^{m+1}, v) + b(v, \hat{p}^{m+1}) = (f_1^{m+1}, v)_{\Omega_f} - c_\Gamma(v, 2\phi^m - \phi^{m-1}), \tag{3.1}$$

$$b(\hat{u}^{m+1}, q) = 0. \tag{3.2}$$

- Find $\hat{\phi}^{m+1} \in H_p$, with $m = 1, ..., N-2$, such that for any $\psi \in H_p$,

$$g S_0 (\frac{\hat{\phi}^{m+1} - \phi^m}{\Delta t}, \psi)_{\Omega_p} + a_p(\hat{\phi}^{m+1}, \psi) = g(f_2^{m+1}, \psi)_{\Omega_p} + c_\Gamma(2u^m - u^{m-1}, \psi). \tag{3.3}$$

- Apply *time filter* to update the previous solutions $(\hat{u}^{m+1}, \hat{p}^{m+1}, \hat{\phi}^{m+1})$,

$$u^{m+1} = \hat{u}^{m+1} - \frac{1}{3}(\hat{u}^{m+1} - 2u^m + u^{m-1}), \tag{3.4}$$

$$p^{m+1} = \hat{p}^{m+1} - \frac{1}{3}(\hat{p}^{m+1} - 2p^m + p^{m-1}), \tag{3.5}$$

$$\phi^{m+1} = \hat{\phi}^{m+1} - \frac{1}{3}(\hat{\phi}^{m+1} - 2\phi^m + \phi^{m-1}). \tag{3.6}$$

3.2 Equivalent Backward Euler plus time filter

To analyze the algorithm we will eliminate the intermediate variables and reduce BETF to an equivalent 2 step method. This reduction is a repetition of the NSE case in [31] so we omit the routine algebraic details, yielding the following.

- Given (u^0, p^0, ϕ^0) and (u^1, p^1, ϕ^1), find $(u^{m+1}, p^{m+1}) \in (H_f, Q)$, with $m = 1, ..., N-1$, such that for any $v \in H_f$, and $q \in Q$,

$$\begin{aligned}&(\frac{3u^{m+1} - 4u^m + u^{m-1}}{2\Delta t}, v)_{\Omega_f} - a_f(\frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}, v)\\ &+ b(v, \frac{3}{2}p^{m+1} - p^m + \frac{1}{2}p^{m-1}) = (f_1^{m+1}, v)_{\Omega_f} - c_\Gamma(v, 2\phi^m - \phi^{m-1}),\end{aligned} \tag{3.7}$$

$$b(\frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}, q) = 0. \tag{3.8}$$

- Given ϕ^0 and ϕ^1, find $\phi^{m+1} \in H_p$, with $m = 1, ..., N-1$, such that for any $\psi \in H_p$,

$$\begin{aligned}&gS_0(\frac{3\phi^{m+1} - 4\phi^m + \phi^{m-1}}{2\Delta t}, \psi)_{\Omega_p} + a_p(\frac{3}{2}\phi^{m+1} - \phi^m + \frac{1}{2}\phi^{m-1}, \psi)\\ &= g(f_2^{m+1}, \psi)_{\Omega_p} + c_\Gamma(2u^m - u^{m-1}, \psi).\end{aligned} \tag{3.9}$$

Define the following difference operators:

$$A(u^{m+1}) = \frac{3}{2}u^{m+1} - 2u^m + \frac{1}{2}u^{m-1}, B(u^{m+1}) = \frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}.$$

There are some important identities we need to use in the later section.

$$\begin{aligned}&(A(u^{m+1}), u^{m+1})_{\Omega_f} = \frac{\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2}{4}\\ &- \frac{\|u^m\|_f^2 + \|2u^m - u^{m-1}\|_f^2}{4} + \frac{\|u^{m+1} - 2u^m + u^{m-1}\|_f^2}{4},\\ &(B(u^{m+1}), u^{m+1})_{\Omega_f} = \frac{3\|u^{m+1}\|_f^2 + \|u^m\|_f^2}{4} - \frac{3\|u^m\|_f^2 + \|u^{m-1}\|_f^2}{4}\\ &+ \frac{\|u^{m+1} - u^m\|_f^2}{2} + \frac{\|u^{m+1} + u^{m-1}\|_f^2}{4}.\end{aligned} \tag{3.10}$$

3.3 Time stability of the decoupled scheme

In this section, we prove the long time stability of BETF decoupled scheme for constant time step. Assume u^1 and u^0 is divergence free, i.e., $b(u^i, q) = 0, \forall q \in Q, i = 1, 0$. From (3.8), it is easy to know $b(u^i, q) = 0, \forall q \in Q, i \geq 0$. Long time stability holds under a condition (below) relating Δt to problem parameters. We make this condition explicit as parameters can vary from large to very small in different applications.

Theorem 3.1 *Define Energy*

$$\begin{aligned} E^{m+1} =& \frac{1}{4}(\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2) + \frac{\Delta t}{4}\big(3a_f(u^{m+1}, u^{m+1}) + a_f(u^m, u^m)\big) \\ &+ \frac{gS_0}{4}(\|\phi^{m+1}\|_p^2 + \|2\phi^{m+1} - \phi^m\|_p^2) + \frac{\Delta t}{4}\big(3a_p(\phi^{m+1}, \phi^{m+1}) + a_p(\phi^m, \phi^m)\big). \end{aligned} \tag{3.11}$$

Suppose Δt satisfies the time step condition $\Delta t \leq min\{\frac{\nu K_{min}}{288 C_1^2 \widetilde{C}_d^2 g^2}, \frac{\nu^2 K_{min} S_0}{288 C_1^2 C_d^2 g^2}\}$, then the decoupled scheme is stable uniformly in time and there holds

$$\begin{aligned} & E^N + \frac{\nu \Delta t}{8}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu \Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\ & + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\ \leq & E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\ & + \frac{3\nu\Delta t}{8}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{3g\Delta t}{8}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2). \end{aligned} \tag{3.12}$$

If there is no restriction on Δt, the decoupled scheme is stable in finite time and there holds

$$\begin{aligned} & E^N + \frac{1}{4}\sum_{m=1}^{N-1}\Big(\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + gS_0\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2\Big) \\ & + \frac{13\nu\Delta t}{36}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{9}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\ & + \frac{13g\Delta t}{36}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{9}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\ \leq & C(T)\Big\{E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\ & + \frac{\nu\Delta t}{2}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{g\Delta t}{2}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2)\Big\}. \end{aligned} \tag{3.13}$$

with $C(T) = exp\big(\sum_{m=0}^{N} max\{\frac{CC_1^2 C_d^2 g^2 \Delta t}{\nu^2 K_{min} S_0}, \frac{CC_1^2 \tilde{C}_d^2 g^2 \Delta t}{\nu K_{min}^2}\}\big)$.

Proof. In (3.7)-(3.9), we set $v = \Delta t u^{m+1}$, $q = \Delta t p^{m+1}$ and $\psi = \Delta t \phi^{m+1}$, and add them together

$$\begin{aligned}
&(A(u^{m+1}), u^{m+1})_{\Omega_f} + gS_0(A(\phi^{m+1}), \phi^{m+1})_{\Omega_p} \\
&+ \Delta t a_f(B(u^{m+1}), u^{m+1}) + \Delta t a_p(B(\phi^{m+1}), \phi^{m+1}) \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1}))_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m-1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned} \tag{3.14}$$

From (3.10) and (3.14), we get

$$\begin{aligned}
&\frac{1}{4}(\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2) + \frac{\Delta t}{4}\big(3a_f(u^{m+1}, u^{m+1}) + a_f(u^m, u^m)\big) \\
&+ \frac{gS_0}{4}(\|\phi^{m+1}\|_p^2 + \|2\phi^{m+1} - \phi^m\|_p^2) + \frac{\Delta t}{4}\big(3a_p(\phi^{m+1}, \phi^{m+1}) + a_p(\phi^m, \phi^m)\big) \\
&- \frac{1}{4}(\|u^m\|_f^2 + \|2u^m - u^{m-1}\|_f^2) - \frac{\Delta t}{4}\big(3a_f(u^m, u^m) + a_f(u^{m-1}, u^{m-1})\big) \\
&- \frac{gS_0}{4}(\|\phi^m\|_p^2 + \|2\phi^m - \phi^{m-1}\|_p^2) - \frac{\Delta t}{4}\big(3a_p(\phi^m, \phi^m) + a_p(\phi^{m-1}, \phi^{m-1})\big) \\
&+ \frac{1}{4}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + \frac{gS_0}{4}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{\nu\Delta t}{2}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{4}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{2}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m+1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned} \tag{3.15}$$

Then we can rearrange the equality

$$\begin{aligned}
&E^{m+1} - E^m + \frac{1}{4}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + \frac{gS_0}{4}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{\nu\Delta t}{2}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{4}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{2}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m+1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned} \tag{3.16}$$

Note that $u^{m+1} = \frac{1}{2}(u^{m+1} - u^m + u^{m+1} + u^{m-1} + u^m - u^{m-1})$ and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, by using Young's and Hölder's inequalities, we have

$$\begin{aligned}
&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
\leq&\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 + \frac{\nu\Delta t}{12} \parallel \nabla u^{m+1} \parallel_f^2 + \frac{g\Delta t}{12} \parallel K^{\frac{1}{2}}\nabla\phi^{m+1} \parallel_p^2 \\
\leq&\frac{\nu\Delta t}{16}\Big(\|\nabla(u^{m+1} - u^m)\|_f^2 + \|\nabla(u^{m+1} + u^{m-1})\|_f^2 + \|\nabla(u^m - u^{m-1})\|_f^2\Big) \\
&+ \frac{g\Delta t}{16}\Big(\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 + \|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2\Big) \\
&+ \frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 .
\end{aligned} \tag{3.17}$$

Take $\epsilon_1 = \frac{1}{12}$, $\epsilon_2 = \frac{g}{32}$ in (2.19) and $\epsilon_1 = \frac{g}{12}$, $\epsilon_2 = \frac{1}{32}$ in (2.20), for the interface terms on the right hand side of (3.16),

$$\begin{aligned}
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m-1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}) \\
=&- \Delta t c_\Gamma(u^{m+1}, \phi^{m+1} - 2\phi^m + \phi^{m-1}) + \Delta t c_\Gamma(u^{m+1} - 2u^m + u^{m-1}, \phi^{m+1}) \\
\le&\frac{\nu\Delta t}{12}\|\nabla u^{m+1})\|_f^2 + \frac{g\Delta t}{32}\|K^{\frac{1}{2}}(\phi^{m+1} - 2\phi^m + \phi^{m-1})\|_p^2 \\
&+ \frac{g\Delta t}{12}\|K^{\frac{1}{2}}\nabla\phi^{m+1}\|_p^2 + \frac{\nu\Delta t}{32}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
&+ \frac{72C_1^2C_d^2g^3\Delta t}{\nu^2K_{min}}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 + \frac{72C_1^2\tilde{C}_dg^2\Delta t}{\nu K_{min}^2}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
\le&\frac{\nu\Delta t}{16}\Big(2\|\nabla(u^{m+1} - u^m)\|_f^2 + \|\nabla(u^{m+1} + u^{m-1})\|_f^2 + 2\|\nabla(u^m - u^{m-1})\|_f^2\Big) \\
&+ \frac{g\Delta t}{16}\Big(2\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 + 2\|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2\Big) \\
&+ \frac{72C_1^2C_d^2g^3\Delta t}{\nu^2K_{min}}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 + \frac{72C_1^2\tilde{C}_dg^2\Delta t}{\nu K_{min}^2}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2.
\end{aligned} \tag{3.18}$$

Inserting (3.18) and (3.17) to (3.16), we arrive at

$$\begin{aligned}
&E^{m+1} - E^m + (\frac{1}{4} - \frac{72C_1^2\tilde{C}_dg^2\Delta t}{\nu K_{min}^2})\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
&+ (\frac{gS_0}{4} - \frac{72C_1^2C_d^2g^3\Delta t}{\nu^2K_{min}})\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{5\nu\Delta t}{16}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\le&\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{3\nu\Delta t}{16}\|\nabla(u^m - u^{m-1})\|_f^2 + \frac{3g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2.
\end{aligned} \tag{3.19}$$

If Δt satisfies the time step condition $\Delta t \le min\{\frac{\nu K_{min}}{288C_1^2\widetilde{C}_d^2g^2}, \frac{\nu^2K_{min}S_0}{288C_1^2C_d^2g^2}\}$, summing up (3.19) from $m = 1$ to $N - 1$ leads to

$$\begin{aligned}
&E^N + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\le&E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{3\nu\Delta t}{8}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{3g\Delta t}{8}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2).
\end{aligned} \tag{3.20}$$

Thus we complete the proof of the uniform in time stability. Next we are going to prove the unconditional, finite time stability of BETF scheme. Using $2u^m - u^{m-1} = -\frac{1}{2}(u^{m+1} - u^m) + \frac{3}{2}(u^m - u^{m-1}) +$

$\frac{1}{2}(u^{m+1}+u^{m-1})$, $u^{m+1}=\frac{1}{2}(u^{m+1}-u^m+u^{m+1}+u^{m-1}+u^m-u^{m-1})$ and $(a+b+c)^2\le 3(a^2+b^2+c^2)$ and taking $\epsilon_1=\frac{1}{12}$, $\epsilon_2=\frac{g}{54}$ in (2.19) and $\epsilon_1=\frac{g}{12}$, $\epsilon_2=\frac{1}{54}$ in (2.20), for the interface term on the right hand side of (3.16),

$$
\begin{aligned}
&-\Delta t c_\Gamma(u^{m+1},2\phi^m-\phi^{m-1})+\Delta t c_\Gamma(2u^m-u^{m-1},\phi^{m+1})\\
\le&\frac{\nu\Delta t}{12}\|\nabla u^{m+1})\|_f^2+\frac{g\Delta t}{54}\|K^{\frac{1}{2}}(2\phi^m-\phi^{m-1})\|_p^2+\frac{243C_1^2C_d^2g^3\Delta t}{2\nu^2K_{min}}\|2\phi^m-\phi^{m-1}\|_p^2\\
&+\frac{g\Delta t}{12}\|K^{\frac{1}{2}}\nabla\phi^{m+1}\|_p^2+\frac{\nu\Delta t}{54}\|2u^m-u^{m-1}\|_f^2+\frac{243C_1^2\tilde{C}_dg^2\Delta t}{2\nu K_{min}^2}\|2u^m-u^{m-1}\|_f^2\\
\le&\frac{\nu\Delta t}{144}\Big(11\|\nabla(u^{m+1}-u^m)\|_f^2+11\|\nabla(u^{m+1}+u^{m-1})\|_f^2+27\|\nabla(u^m-u^{m-1})\|_f^2\Big)\\
&+\frac{g\Delta t}{144}\Big(11\|K^{\frac{1}{2}}\nabla(\phi^{m+1}-\phi^m)\|_p^2+11\|K^{\frac{1}{2}}\nabla(\phi^{m+1}+\phi^{m-1})\|_p^2+27\|K^{\frac{1}{2}}\nabla(\phi^m-\phi^{m-1})\|_p^2\Big)\\
&+\frac{243C_1^2C_d^2g^3\Delta t}{2\nu^2K_{min}}\|2\phi^m-\phi^{m-1}\|_p^2+\frac{243C_1^2\tilde{C}_dg^2\Delta t}{2\nu K_{min}^2}\|2u^m-u^{m-1}\|_f^2.
\end{aligned}
\tag{3.21}
$$

Inserting (3.21) and (3.17) to (3.16), we arrive at

$$
\begin{aligned}
&E^{m+1}-E^m+\frac{1}{4}\|u^{m+1}-2u^m+u^{m-1}\|_f^2\\
&+\frac{gS_0}{4}\|\phi^{m+1}-2\phi^m+\phi^{m-1}\|_p^2\\
&+\frac{13\nu\Delta t}{36}\|\nabla(u^{m+1}-u^m)\|_f^2+\frac{\nu\Delta t}{9}\|\nabla(u^{m+1}+u^{m-1})\|_f^2\\
&+\frac{13g\Delta t}{36}\|K^{\frac{1}{2}}\nabla(\phi^{m+1}-\phi^m)\|_p^2+\frac{g\Delta t}{9}\|K^{\frac{1}{2}}\nabla(\phi^{m+1}+\phi^{m-1})\|_p^2\\
\le&\frac{3C_d^2\Delta t}{\nu}\parallel f_1^{m+1}\parallel_f^2+\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}}\parallel f_2^{m+1}\parallel_p^2\\
&+\frac{\nu\Delta t}{4}\|\nabla(u^m-u^{m-1})\|_f^2+\frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^m-\phi^{m-1})\|_p^2\\
&+\frac{243C_1^2C_d^2g^3\Delta t}{2\nu^2K_{min}}\|2\phi^m-\phi^{m-1}\|_p^2+\frac{243C_1^2\tilde{C}_d^2g^2\Delta t}{2\nu K_{min}^2}\|2u^m-u^{m-1}\|_f^2.
\end{aligned}
\tag{3.22}
$$

Summing up (3.19) from $m=1$ to $N-1$ leads to

$$
\begin{aligned}
&E^N+\frac{1}{4}\sum_{m=1}^{N-1}\Big(\|u^{m+1}-2u^m+u^{m-1}\|_f^2+gS_0\|\phi^{m+1}-2\phi^m+\phi^{m-1}\|_p^2\Big)\\
&+\frac{13\nu\Delta t}{36}\sum_{m=1}^{N-1}\|\nabla(u^{m+1}-u^m)\|_f^2+\frac{\nu\Delta t}{9}\sum_{m=1}^{N-1}\|\nabla(u^{m+1}+u^{m-1})\|_f^2\\
&+\frac{13g\Delta t}{36}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1}-\phi^m)\|_p^2+\frac{g\Delta t}{9}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1}+\phi^{m-1})\|_p^2\\
\le& E^1+\sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu}\parallel f_1^{m+1}\parallel_f^2+\sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}}\parallel f_2^{m+1}\parallel_p^2\\
&+\frac{\nu\Delta t}{2}(\|\nabla u^1\|_f^2+\|\nabla u^0\|_f^2)+\frac{g\Delta t}{2}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2+\|K^{\frac{1}{2}}\nabla\phi^0\|_p^2)\\
&+\frac{CC_1^2C_d^2g^2\Delta t}{\nu^2K_{min}S_0}\sum_{m=0}^{N-1}\frac{gS_0}{4}\|\phi^m\|_p^2+\frac{CC_1^2\tilde{C}_d^2g^2\Delta t}{\nu K_{min}^2}\sum_{m=0}^{N-1}\frac{1}{4}\|u^m\|_f^2.
\end{aligned}
\tag{3.23}
$$

Applying the discrete Gronwall inequality, we get (3.13). □

4 Error analysis

This section gives an analysis of the error(constant time stepsize) of the fully discrete BETF algorithm, where spatial discretization is performed using finite element methods (FEMs). To discretize the Stokes-Darcy problem in space by finite element method, let h_i be a positive parameter and $\mathcal{T}_{h_i}$ be quasi-uniform partition of triangular or quadrilateral elements of Ω_i, $i=f,p$. We assume $\mathcal{T}_{h_f}$ and $\mathcal{T}_{h_p}$ match at the interface and the interface is straight in the analysis. We select continuous piecewise polynomials of degrees k, k, and $k-1$ for the finite element spaces $H_f^h\subset H_f, H_p^h\subset H_p, Q^h\subset Q$ which are conforming finite element spaces. We assume the fluid velocity space H_f^h and the pressure space Q^h satisfy the discrete inf-sup condition: there exists a positive constant γ, independent of h, such that, $\forall q\in Q^h, \exists\ v\in H_f^h, v\neq 0$,

$$
b(v,q)\ge\gamma\|v\|_{H_f}\|q\|_f. \tag{4.1}
$$

Furthermore, we will use the inverse inequalities: there exist constants $C_I,\widetilde{C}_I$, which depend on the angles in the finite element mesh, such that, for all $v\in H_f^h$ and $\phi\in H_p^h$,

$$
\|\nabla v\|_f\le C_Ih^{-1}\|v\|_f,\quad \|\nabla\phi\|_p\le\widetilde{C}_Ih^{-1}\|\phi\|_p. \tag{4.2}
$$

The fully discrete approximation of BETF algorithm is: Given (u_h^0,p_h^0,ϕ_h^0) and (u_h^1,p_h^1,ϕ_h^1), find $(u_h^{m+1},p_h^{m+1},\phi_h^{m+1})\in(H_f^h,Q^h,H_p^h)$, with $m=1,....,N-1$, such that for any $v_h\in H_f^h$, $\psi_h\in H_p^h$ and $q_h\in Q^h$,

$$
\begin{aligned}
&(\frac{3u_h^{m+1}-4u_h^m+u_h^{m-1}}{2\Delta t},v_h)_{\Omega_f}-a_f(\frac{3}{2}u_h^{m+1}-u_h^m+\frac{1}{2}u_h^{m-1},v_h)\\
&+b(v_h,\frac{3}{2}p_h^{m+1}-p_h^m+\frac{1}{2}p_h^{m-1})=(f_1^{m+1},v_h)_{\Omega_f}-c_\Gamma(v_h,2\phi_h^m-\phi_h^{m-1}),
\end{aligned}
\tag{4.3}
$$

$$b(\frac{3}{2}u_h^{m+1}-u_h^m+\frac{1}{2}u_h^{m-1},q_h)=0, \tag{4.4}$$

$$\begin{aligned}&gS_0(\frac{3\phi_h^{m+1}-4\phi_h^m+\phi_h^{m-1}}{2\Delta t},\psi_h)_{\Omega_p}+a_p(\frac{3}{2}\phi_h^{m+1}-\phi_h^m+\frac{1}{2}\phi_h^{m-1},\psi_h)\\&=g(f_2^{m+1},\psi_h)_{\Omega_p}+c_\Gamma(2u_h^m-u_h^{m-1},\psi_h).\end{aligned} \tag{4.5}$$

We assume that the solution of Stokes-Darcy problem satisfies the following regularity: $(u(t),\phi(t))\in(H^{k+1}(\Omega_f)^d,H^{k+1}(\Omega_p))$ and $p(t)\in H^k(\Omega_f)$, define the linear projection operator $P:(u(t),\phi(t),p(t))\in(H_f,H_p,Q)\to(\tilde{u}(t),\tilde{\phi}(t),\tilde{p}(t))\in(H_f^h,H_p^h,Q^h),\forall t\in[0,T]$ by:

$$\begin{aligned}&a_f(u(t),v)+c_\Gamma(v,\phi(t))+b(v,p(t))+a_p(\phi(t),\psi)-c_\Gamma(u(t),\psi)\\&=a_f(\tilde{u}(t),v)+c_\Gamma(v,\tilde{\phi}(t))+b(v,\tilde{p}(t))\\&+a_p(\tilde{\phi}(t),\psi)-c_\Gamma(\tilde{u}(t),\psi)\quad\forall v(t)\in H_f^h,\psi(t)\in H_p^h,\end{aligned} \tag{4.6}$$

$$b(\tilde{u}(t),q)=0\quad\forall q(t)\in Q^h, \tag{4.7}$$

then we have the following error estimates[20, 21]:

$$\|\tilde{u}(t)-u(t)\|_f+h\|\tilde{u}(t)-u(t)\|_{H_f}\le Ch^{k+1}\|u(t)\|_{H^{k+1}(\Omega_f)^d}, \tag{4.8}$$

$$\|\tilde{\phi}(t)-\phi(t)\|_p+h\|\tilde{\phi}(t)-\phi(t)\|_{H_p}\le Ch^{k+1}\|\phi(t)\|_{H^{k+1}(\Omega_p)}, \tag{4.9}$$

$$\|\tilde{p}(t)-p(t)\|_f\le Ch^k\|p(t)\|_{H^k(\Omega_f)}. \tag{4.10}$$

For $\forall(v_h,\psi_h,q_h)\in(H_f^h,H_p^h,Q^h)$, the true solution $(u(t_{m+1}),p(t_{m+1}),\phi(t_{m+1}))$ satisfies:

$$\begin{aligned}&(\frac{A(u(t_{m+1}))}{\Delta t},v_h)_{\Omega_f}+a_f(B(u(t_{m+1})),v_h)+b(v_h,B(p(t_{m+1})))\\&=(\xi_f^{m+1},v_h)_{\Omega_f}+(f_1^{m+1},v_h)_{\Omega_f}-g\int_\Gamma\phi(t_{m+1})v_h\cdot n_f\\&+a_f(B(u(t_{m+1}))-u(t_{m+1}),v_h)+b(v_h,B(p(t_{m+1}))-p(t_{m+1})),\end{aligned} \tag{4.11}$$

$$-b(u(t_{m+1}),q_h)=0, \tag{4.12}$$

$$\begin{aligned}&gS_0(\frac{A(\phi(t_{m+1}))}{\Delta t},\psi_h)_{\Omega_p}+a_p(B(\phi(t_{m+1})),\psi_h)=gS_0(\xi_p^{m+1},\psi_h)_{\Omega_p}\\&+g(f_2^{m+1},\psi_h)_{\Omega_p}+g\int_\Gamma\psi_h u(t_{m+1})\cdot n_f+a_p(B(\phi(t_{m+1}))-\phi(t_{m+1}),\psi_h),\end{aligned} \tag{4.13}$$

where, ξ_f^{m+1},ξ_p^{m+1} are defined by

$$\xi_f^{m+1}:=\tfrac{A(u(t_{m+1}))}{\Delta t}-u_t(t_{m+1}),\ \xi_p^{m+1}:=\tfrac{B(\phi(t_{m+1}))}{2\Delta t}-\phi_t(t_{m+1}). \tag{4.14}$$

To derive the error estimates of BETF algorithm, we first give this method's consistency error.

Lemma 4.1 *The following inequalities hold:*

$$\|\tfrac{A(u(t_{m+1}))}{\Delta t}\|_f^2 \le \tfrac{9}{2\triangle t}\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_f^2 dt, \tag{4.15}$$

$$\|\xi_f^{m+1}\|_f^2 \le C\triangle t^3 \int_{t_m}^{t_{m+2}} \|u_{ttt}(t)\|_f^2 dt, \tag{4.16}$$

$$\|\widetilde{u}(t_{m+2}) - 2\widetilde{u}(t_{m+1}) + \widetilde{u}(t_m)\|_{H_f}^2 \le C\triangle t^3 \int_{t_m}^{t_{m+2}} \|u_{tt}\|_{H_f}^2 dt, \tag{4.17}$$

$$\|B(u(t_{m+1})) - u(t_{m+1})\|_{H_f}^2 \le C\triangle t^3 \int_{t_{m-1}}^{t_{m+1}} \|u_{tt}\|_{H_f}^2 dt, \tag{4.18}$$

$$\|\tfrac{A(\phi(t_{m+1}))}{\Delta t}\|_p^2 \le \tfrac{9}{2\triangle t}\int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_p^2 dt, \tag{4.19}$$

$$\|\xi_p^{m+2}\|_p^2 \le C\triangle t^3 \int_{t_m}^{t_{m+2}} \|\phi_{ttt}(t)\|_p^2 dt, \tag{4.20}$$

$$\|\widetilde{\phi}(t_{m+2}) - 2\widetilde{\phi}(t_{m+1}) - \widetilde{\phi}(t_m)\|_{H_p}^2 \le C\triangle t^3 \int_{t_m}^{t_{m+2}} \|\phi_{tt}\|_{H_p}^2 dt, \tag{4.21}$$

$$\|B(\phi(t_{m+1})) - \phi(t_{m+1})\|_{H_p}^2 \le C\triangle t^3 \int_{t_{m-1}}^{t_{m+1}} \|\phi_{tt}\|_{H_p}^2 dt. \tag{4.22}$$

Proof. The proof is similar to the Lemma 2 in [17]. □

Theorem 4.1 *For any $0 < t_N = T < \infty$, assume the solution satisfies the following regularity condition*

$$\begin{aligned} &u \in H^1(0,T; H^{k+1}(\Omega_f)) \cap H^2(0,T; H^1(\Omega_f)) \cap H^3(0,T; L^2(\Omega_f)),\\ &\phi \in H^1(0,T; H^{k+1}(\Omega_p)) \cap H^2(0,T; H^1(\Omega_p)) \cap H^3(0,T; L^2(\Omega_p)),\\ &p_{tt} \in L^2(0,T, L_0^2(\Omega_f)), \end{aligned} \tag{4.23}$$

and Δt satisfies $\Delta t \le min\{\frac{\nu K_{min}}{CC_1^2\widetilde{C}_d^2 g^2}, \frac{\nu^2 K_{min} S_0}{CC_1^2 C_d^2 g^2}\}$, then there exists a constant C independent of h and Δt, such that

$$\begin{aligned} &\|u(t_N) - u_h^N\|_f^2 + \|\phi(t_N) - \phi_h^N\|_p^2\\ &+ \sum_{m=1}^{N-1} \big(\frac{11}{8}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \frac{11}{8} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2\big)\\ &+ \sum_{m=1}^{N-1} \big(\frac{3}{4} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\big)\\ \le& C\Delta t^4\Big(\int_0^T \|u_{ttt}\|_f^2 dt + \int_0^T \|\phi_{ttt}\|_p^2 dt\Big)\\ &+ C\Delta t^4\Big(\int_0^T \|\phi_{tt}\|_{H_p}^2 dt + \int_0^T \|u_{tt}\|_{H_f}^2 dt + \int_0^T \|p_{tt}\|_f^2 dt\Big)\\ &+ Ch^{2k+2}\Big(\int_0^T \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_0^T \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big)\\ &+ C\Delta t(\|\nabla\eta_u^1\|_f^2 + \|\nabla\eta_u^0\|_f^2) + C\Delta t(\|K^{\frac{1}{2}}\nabla(\eta_\phi^1)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^0)\|_p^2)\\ &+ (\|\eta_u^1\|^2 + \|2\eta_u^1 - \eta_u^0\|^2) + gS_0(\|\eta_\phi^1\|^2 + \|2\eta_\phi^1 - \eta_\phi^0\|^2). \end{aligned} \tag{4.24}$$

Here and afterwards, we denote by C a generic positive constant which depends on the physical parameters ($\nu,\ g,\ S_0,\ K_{min}$), and it may has different values at different occasions.

Proof. Subtracting (4.11)-(4.13) from (4.3)-(4.5), we have the following error equations:

$$\begin{aligned}&(\frac{A(u(t_{m+1})-u_h^{m+1})}{\Delta t},v_h)_{\Omega_f}-a_f(B(u(t_{m+1})-u_h^{m+1}),v_h)+b(v_h,B(p(t_{m+1})-p_h^{m+1}))\\&=(\xi_f^{m+1},v_h)_{\Omega_f}-g\int_\Gamma(\phi(t_{m+1})-2\phi_h^m+\phi_h^{m-1})v_h\cdot n_f\\&+a_f(B(u(t_{m+1}))-u(t_{m+1}),v_h)+b(v_h,B(p(t_{m+1}))-p(t_{m+1})),\end{aligned}\tag{4.25}$$

$$b(u(t_{m+1})-B(u_h^{m+1}),q_h)=0,\tag{4.26}$$

$$\begin{aligned}&gS_0(\frac{A(\phi(t_{m+1})-\phi_h^{m+1})}{\Delta t},\psi_h)_{\Omega_p}+a_p(B(\phi(t_{m+1})-\phi_h^{m+1}),\psi_h)\\&=gS_0(\xi_p^{m+1},\psi_h)_{\Omega_p}+g\int_\Gamma\psi_h(u(t_{m+1})-2u_h^m+u_h^{m-1})\cdot n_f+a_p(B(\phi(t_{m+1}))-\phi^{m+1},\psi_h).\end{aligned}\tag{4.27}$$

Let $\tilde{u}$, $\tilde{p}$ and $\tilde{\phi}$ be the projection of u, p and ϕ in H_f^h, Q^h and H_p^h. Denote the error as follows:

$$u(t_{m+1})-u_h^{m+1}=u(t_{m+1})-\tilde{u}(t_{m+1})+\tilde{u}(t_{m+1})-u_h^{m+1}=\epsilon_u^{m+1}+\eta_u^{m+1},\tag{4.28}$$

$$\phi(t_{m+1})-\phi_h^{m+1}=\phi(t_{m+1})-\tilde{\phi}(t_{m+1})+\tilde{\phi}(t_{m+1})-\phi_h^{m+1}=\epsilon_\phi^{m+1}+\eta_\phi^{m+1},\tag{4.29}$$

$$p(t_{m+1})-p_h^{m+1}=p(t_{m+1})-\tilde{p}(t_{m+1})+\tilde{p}(t_{m+1})-p_h^{m+1}=\epsilon_p^{m+1}+\eta_p^{m+1}.\tag{4.30}$$

Then we can rewrite (4.25)-(4.27):

$$\begin{aligned}&(\frac{A(\eta_u^{m+1})}{\Delta t},v_h)_{\Omega_f}+a_f(B(\eta_u^{m+1}),v_h)+b(v_h,B(\eta_p^{m+1}))=(\xi_f^{m+1},v_h)_{\Omega_f}\\&-g\int_\Gamma(\phi(t_{m+1})-B(\phi(t_{m+1}))+B(\tilde{\phi}(t_{m+1}))-(2\tilde{\phi}(t_m)-\tilde{\phi}(t_{m-1})))v_h\cdot n_f\\&-g\int_\Gamma(2\eta_\phi^m-\eta_\phi^{m-1})v_h\cdot n_f-a_f(B(\epsilon_u^{m+1}),v_h)-b(v_h,B(\epsilon_p^{m+1}))-g\int_\Gamma B(\epsilon_\phi^{m+1})v_h\cdot n_f\\&+a_f(B(u(t_{m+1}))-u(t_{m+1}),v_h)+b(v_h,B(p(t_{m+1}))-p(t_{m+1}))-(\frac{A(\epsilon_u^{m+1})}{\Delta t},v_h)_{\Omega_f},\end{aligned}\tag{4.31}$$

$$b(u(t_{m+1})-B(u(t_{m+1}))+B(\eta_u^{m+1}),q_h)=-b(B(\epsilon_u^{m+1}),q_h),\tag{4.32}$$

$$\begin{aligned}&gS_0(\frac{A(\eta_\phi^{m+1})}{\Delta t},\psi_h)_{\Omega_p}+a_p(B(\eta_\phi^{m+1}),\psi_h)=gS_0(\xi_p^{m+1},\psi_h)_{\Omega_p}+g\int_\Gamma\psi_h(2\eta_u^m-\eta_u^{m-1})\cdot n_f\\&+g\int_\Gamma\psi_h(u(t_{m+1})-B(u(t_{m+1}))+B(\tilde{u}(t_{m+1}))-(2\tilde{u}(t_m)-\tilde{u}(t_{m-1})))\cdot n_f\\&-gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t},\psi_h)_{\Omega_p}-a_p(B(\epsilon_\phi^{m+1}),\psi_h)\\&+g\int_\Gamma\psi_hB(\epsilon_u^{m+1})\cdot n_f+a_p(B(\phi(t_{m+1}))-\phi(t_{m+1}),\psi_h).\end{aligned}\tag{4.33}$$

Setting $v_h = \eta_u^{m+1}$, $q_h = \eta_p^{m+1}$ in (4.31) and (4.32) yields

$$\begin{aligned}
&(\frac{A(\eta_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} + a_f(B(\eta_u^{m+1}), \eta_u^{m+1}) + b(\eta_u^{m+1}, B(\eta_p^{m+1})) = (\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} \\
&- g\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f \\
&- g\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\eta_u^{m+1}\cdot n_f - (\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} - a_f(B(\epsilon_u^{m+1}), \eta_u^{m+1}) \\
&- b(\eta_u^{m+1}, B(\epsilon_p^{m+1})) - g\int_\Gamma B(\epsilon_\phi^{m+1})\eta_u^{m+1}\cdot n_f + a_f(B(u(t_{m+1})) - u(t_{m+1}), \eta_u^{m+1}) \\
&+ b(\eta_u^{m+1}, B(p(t_{m+1})) - p(t_{m+1})),
\end{aligned} \tag{4.34}$$

$$b(B(\eta_u^{m+1}), \eta_p^{m+1}) = -b(u(t_{m+1}) - B(u(t_{m+1})), \eta_p^{m+1}) - b(B(\epsilon_u^{m+1}), \eta_p^{m+1}). \tag{4.35}$$

Choosing $\psi_h = \eta_\phi^{m+1}$ in (4.33) yields

$$\begin{aligned}
&gS_0(\frac{A(\eta_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p} + a_p(B(\eta_\phi^{m+1}), \eta_\phi^{m+1}) = gS_0(\xi_p^{m+1}, \eta_\phi^{m+1})_{\Omega_p} \\
&+ g\int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f - gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p} \\
&+ g\int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f \\
&- a_p(B(\epsilon_\phi^{m+1}), \eta_\phi^{m+1}) + g\int_\Gamma \eta_\phi^{m+1}B(\epsilon_u^{m+1})\cdot n_f + a_p(B(\phi(t_{m+1}) - \phi(t_{m+1}), \eta_\phi^{m+1}).
\end{aligned} \tag{4.36}$$

From (4.6), (4.7) and (4.32), we notice $b\big(B(\eta_u^{m+1}), q_h\big) = -b\big(u(t_{m+1}) - B(u(t_{m+1})), q_h\big) - b\big(B(\epsilon_u^{m+1}), q_h\big) = 0$, $-a_f\big(B(\epsilon_u^{m+1}), \eta_u^{m+1}\big) - b\big(\eta_u^{m+1}, B(\epsilon_p^{m+1})\big) + g\int_\Gamma B(\epsilon_\phi^{m+1})\eta_u^{m+1}\cdot n_f = 0$. Assuming $u_h^1 = \tilde{u}(t_1)$, $u_h^0 = \tilde{u}(t_0)$, and by (4.7) and $b\big(B(\eta_u^{m+1}), q_h\big) = 0$ we have $b(\eta_u^m, q_h) = 0\ \forall q_h \in Q^h$ for $m = 0, ...N$. From (3.10) and (3.14) and multiplying (4.34) by $4\Delta t$ gives

$$\begin{aligned}
&(\|\eta_u^{m+1}\|_f^2 + \|2\eta_u^{m+1} - \eta_u^m\|_f^2) + 3\Delta t a_f(\eta_u^{m+1}, \eta_u^{m+1}) + \Delta t a_f(\eta_u^m, \eta_u^m) \\
&- (\|\eta_u^m\|_f^2 + \|2\eta_u^m - \eta_u^{m-1}\|_f^2) - 3\Delta t a_f(\eta_u^m, \eta_u^m) - \Delta t a_f(\eta_u^{m-1}, \eta_u^{m-1}) \\
&+ \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + 2\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
&- 4\Delta t b(\eta_u^{m+1}, B(p(t_{m+1})) - p(t_{m+1})) \\
=&4\Delta t(\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} - 4g\Delta t\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f \\
&- 4g\Delta t\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f \\
&- 4\Delta t\Big((\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} - a_f(B(u(t_{m+1})) - u(t_{m+1}), \eta_u^{m+1})\Big).
\end{aligned} \tag{4.37}$$

Similarly, from (4.6), note that $a_p(B(\epsilon_\phi^{m+1}), \eta_\phi^{m+1}) + g\int_\Gamma \eta_\phi^{m+1}B(\epsilon_u^{m+1})\cdot n_f = 0$, from (3.10) and multiplying (4.36) by $4\Delta t$ yields

$$
\begin{aligned}
& gS_0(\|\eta_\phi^{m+1}\|_p^2 + \|2\eta_\phi^{m+1} - \eta_\phi^m\|_p^2) + 3\Delta t a_p(\eta_p^{m+1}, \eta_p^{m+1}) + \Delta t a_p(\eta_p^m, \eta_p^m) \\
& - gS_0(\|\eta_\phi^m\|_p^2 + \|2\eta_\phi^m - \eta_\phi^{m-1}\|_p^2) - 3\Delta t a_p(\eta_p^m, \eta_p^m) - \Delta t a_p(\eta_p^{m-1}, \eta_p^{m-1}) \\
& + gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + 2g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
& + g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 \\
= & 4gS_0\Delta t(\xi_p^{m+1}, \eta_\phi^{m+1})_{\Omega_p} + 4g\Delta t\int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f \\
& + 4g\Delta t\int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f \\
& - 4\Delta t\Big(gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p} - a_p(B(\phi^{m+1}) - \phi^{m+1}, \eta_\phi^{m+1})\Big).
\end{aligned}
\tag{4.38}
$$

Let $F_{m+1} = (\|\eta_u^{m+1}\|_f^2 + \|2\eta_u^{m+1} - \eta_u^m\|_f^2) + 3\Delta t a_f(\eta_u^{m+1}, \eta_u^{m+1}) + \Delta t a_f(\eta_u^m, \eta_u^m) + gS_0(\|\eta_\phi^{m+1}\|_p^2 + \|2\eta_\phi^{m+1} - \eta_\phi^m\|_p^2) + 3\Delta t a_p(\eta_p^{m+1}, \eta_p^{m+1}) + \Delta t a_p(\eta_p^m, \eta_p^m)$, adding the above equalities (4.37)-(4.38) together, we get

$$
\begin{aligned}
& F_{m+1} - F_m + \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + 2\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 \\
& + gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + 2g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
& + g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
= & 4\Delta t\Big((\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} - (\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} \\
& + gS_0(\xi_p^{m+1}, \eta_\phi^{m+1})_{\Omega_p} - gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p}\Big) \\
& - 4g\Delta t\Big(\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f - \int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f\Big) \\
& - 4g\Delta t\Big(\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f \\
& - \int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f\Big) \\
& + 4\Delta t\Big(a_f(B(u(t_{m+1})) - u(t_{m+1}), \eta_u^{m+1}) + b(\eta_u^{m+1}, B(p(t_{m+1}) - p(t_{m+1}))) \\
& + a_p(B(\phi(t_{m+1})) - \phi(t_{m+1}), \eta_\phi^{m+1})\Big).
\end{aligned}
\tag{4.39}
$$

Based on Lemma 4.1 and $\|\frac{A(\epsilon_u^{m+1})}{\Delta t}\|^2 \le \frac{9}{2\Delta t}\int_{t_{m-1}}^{t_{m+1}} \|u_t - \tilde{u}_t\|^2 dt$, the first term on the right hand side of (4.39) can be bounded by

$$
\begin{aligned}
&4\Delta t\Big((\xi_f^{m+1},\eta_u^{m+1})_{\Omega_f}-(\frac{A(\epsilon_u^{m+1})}{\Delta t},\eta_u^{m+1})_{\Omega_f}\\
&+gS_0(\xi_p^{m+1},\eta_\phi^{m+1})_{\Omega_p}-gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t},\eta_\phi^{m+1})_{\Omega_p}\Big)\\
\le& C\Delta t^4\int_{t_{m-1}}^{t_{m+1}}\|u_{ttt}\|_f^2dt+\frac{\nu\Delta t}{16}\|\nabla\eta_u^{m+1}\|_f^2+C\Delta t\|\frac{A(\epsilon_u^{m+1})}{\Delta t}\|_f^2\\
&+C\Delta t^4\int_{t_{m-1}}^{t_{m+1}}\|\phi_{ttt}\|_p^2dt+\frac{g\Delta t}{16}\|K^{\frac12}\nabla\eta_\phi^{m+1}\|_p^2+C\Delta t\|\frac{A(\epsilon_\phi^{m+1})}{\Delta t}\|_p^2\\
\le& C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}}\|u_{ttt}\|_f^2dt+\int_{t_{m-1}}^{t_{m+1}}\|\phi_{ttt}\|_p^2dt\Big)\\
&+Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}}\|u_t\|_{H^{k+1}(\Omega_f)}^2dt+\int_{t_{m-1}}^{t_{m+1}}\|\phi_t\|_{H^{k+1}(\Omega_p)}^2dt\Big)\\
&+\frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1}-\eta_u^m)\|_f^2+\|\nabla(\eta_u^m-\eta_u^{m-1})\|_f^2+\|\nabla(\eta_u^{m+1}+\eta_u^{m-1})\|_f^2\big)\\
&+\frac{g\Delta t}{16}\big(\|K^{\frac12}\nabla(\eta_\phi^{m+1}-\eta_\phi^m)\|_p^2+\|K^{\frac12}\nabla(\eta_\phi^m-\eta_\phi^{m-1})\|_p^2+\|K^{\frac12}\nabla(\eta_\phi^{m+1}+\eta_\phi^{m-1})\|_p^2\big).
\end{aligned}
\tag{4.40}
$$

Taking the same technique used in (3.18), we can bound the second term on the right hand side of (4.39)

$$
\begin{aligned}
&-4g\Delta t\Big(\int_\Gamma(2\eta_\phi^m-\eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f-\int_\Gamma\eta_\phi^{m+1}(2\eta_u^m-\eta_u^{m-1})\cdot n_f\Big)\\
\le&\frac{\nu\Delta t}{16}\Big(2\|\nabla(\eta_u^{m+1}-\eta_u^m)\|_f^2+\|\nabla(\eta_u^{m+1}+\eta_u^{m-1})\|_f^2+2\|\nabla(\eta_u^m-\eta_u^{m-1})\|_f^2\Big)\\
&+\frac{g\Delta t}{16}\Big(2\|K^{\frac12}\nabla(\eta_\phi^{m+1}-\eta_\phi^m)\|_p^2+\|K^{\frac12}\nabla(\eta_\phi^{m+1}+\eta_\phi^{m-1})\|_p^2+2\|K^{\frac12}\nabla(\eta_\phi^m-\eta_\phi^{m-1})\|_p^2\Big)\\
&+\frac{CC_1^2C_d^2g^3\Delta t}{\nu^2K_{min}}\|\eta_\phi^{m+1}-2\eta_\phi^m+\eta_\phi^{m-1}\|_p^2+\frac{CC_1^2\tilde{C}_dg^2\Delta t}{\nu K_{min}^2}\|\eta_u^{m+1}-2\eta_u^m+\eta_u^{m-1}\|_f^2.
\end{aligned}
\tag{4.41}
$$

Taking $\epsilon=\frac{\Delta t}{12}$ in (2.17) and (2.18) and using $u^{m+1}=\frac12(u^{m+1}-u^m+u^{m+1}+u^{m-1}+u^m-u^{m-1})$

and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the third term on the right hand side of (4.39) can be bounded by

$$\begin{aligned}
&-4g\Delta t\Big(\int_\Gamma(\phi(t_{m+1})-B(\phi(t_{m+1}))+B(\tilde{\phi}(t_{m+1}))-(2\tilde{\phi}(t_m)-\tilde{\phi}(t_{m+1})))\eta_u^{m+1}\cdot n_f \\
&-\int_\Gamma \eta_\phi^{m+1}(u(t_{m+1})-B(u(t_{m+1}))+B(\tilde{u}(t_{m+1}))-(2\tilde{u}(t_m)-\tilde{u}(t_{m-1})))\cdot n_f\Big) \\
=&-4g\Delta t\int_\Gamma\Big((-\frac{1}{2}\phi(t_{m+1})+\phi(t_m)-\frac{1}{2}\phi(t_{m-1}))+\frac{3}{2}(\tilde{\phi}(t_{m+1})-2\tilde{\phi}(t_m)+\tilde{\phi}(t_{m-1}))\Big)\eta_u^{m+1}\cdot n_f \\
&+4g\Delta t\int_\Gamma \eta_\phi^{m+1}\Big((-\frac{1}{2}u(t_{m+1})+u(t_m)-\frac{1}{2}u(t_{m-1}))+\frac{3}{2}(\tilde{u}(t_{m+1})-2\tilde{u}(t_m)+\tilde{u}(t_{m-1}))\Big)\cdot n_f \\
\leq&\frac{gC_3}{\nu}\Big(\|\nabla\phi(t_{m+1})-2\nabla\phi(t_m)+\nabla\phi(t_{m-1})\|_p^2+\|\nabla\tilde{\phi}(t_{m+1})-2\nabla\tilde{\phi}(t_m)+\nabla\tilde{\phi}(t_{m-1})\|_p^2\Big) \\
&+\frac{C_3}{K_{min}}\Big(\|\nabla u(t_{m+1})-2\nabla u(t_m)+\nabla u(t_{m-1})\|_f^2+\|\nabla\tilde{u}(t_{m+1})-2\nabla\tilde{u}(t_m)+\nabla\tilde{u}(t_{m-1})\|_f^2\Big) \\
&+\frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1}-\eta_u^m)\|_f^2+\|\nabla(\eta_u^m-\eta_u^{m-1})\|_f^2+\|\nabla(\eta_u^{m+1}+\eta_u^{m-1})\|_f^2\big) \\
&+\frac{g\Delta t}{16}\big(\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}-\eta_\phi^m)\|_p^2+\|K^{\frac{1}{2}}\nabla(\eta_\phi^m-\eta_\phi^{m-1})\|_p^2+\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}+\eta_\phi^{m-1})\|_p^2\big),
\end{aligned} \tag{4.42}$$

where, the parameter $C_3 = 216C_1C_2g\Delta t$. From the definition of the bilinear forms a_f, a_p and b, the rest of the right hand side of the (4.39) can be bounded by

$$\begin{aligned}
&-4\Delta t\Big(a_f(B(u(t_{m+1})-u(t_{m+1}),\eta_u(t_{m+1}))+b(\eta_u(t_{m+1}),B(p(t_{m+1}))-p(t_{m+1})) \\
&+4\Delta t a_p(B(\phi(t_{m+1}))-\phi(t_{m+1}),\eta_\phi(t_{m+1}))\Big) \\
\leq&C\Delta t\Big(\|\nabla(B(u(t_{m+1}))-u(t_{m+1}))\|_f^2+\|B(p(t_{m+1}))-p(t_{m+1})\|_f^2 \\
&+\|K^{\frac{1}{2}}\nabla(B(\phi(t_{m+1}))-\phi(t_{m+1}))\|_p^2\Big) \\
&+\frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1}-\eta_u^m)\|_f^2+\|\nabla(\eta_u^m-\eta_u^{m-1})\|_f^2+\|\nabla(\eta_u^{m+1}+\eta_u^{m-1})\|_f^2\big) \\
&+\frac{g\Delta t}{16}\big(\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}-\eta_\phi^m)\|_p^2+\|K^{\frac{1}{2}}\nabla(\eta_\phi^m-\eta_\phi^{m-1})\|_p^2+\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}+\eta_\phi^{m-1})\|_p^2\big).
\end{aligned} \tag{4.43}$$

Combining all the above estimates (4.39)-(4.43) gives

$$
\begin{aligned}
&F_{m+1} - F_m + \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + \frac{27}{16}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 \\
&+ gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + \frac{27}{16}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
&+ \frac{3}{4}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
\le& C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ \frac{CC_1^2C_d^2g^3\Delta t}{\nu^2K_{min}}\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + \frac{CC_1^2\tilde{C}_d^2g^2\Delta t}{\nu K_{min}^2}\|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 \\
&+ C\Delta t\Big(\|\nabla\phi(t_{m+1}) - 2\nabla\phi(t_m) + \nabla\phi(t_{m-1})\|_p^2 + \|\nabla\tilde{\phi}(t_{m+1}) - 2\nabla\tilde{\phi}(t_m) + \nabla\tilde{\phi}(t_{m-1}))\|_p^2\Big) \\
&+ C\Delta t\Big(\|\nabla u(t_{m+1}) - 2\nabla u(t_m) + \nabla u(t_{m-1})\|_f^2 + \|\nabla\tilde{u}(t_{m+1}) - 2\nabla\tilde{u}(t_m) + \nabla\tilde{u}(t_{m-1})\|_f^2\Big) \\
&+ C\Delta t\Big(\|\nabla(B(u(t_{m+1})) - u(t_{m+1}))\|_f^2 + \|B(p(t_{m+1})) - p(t_{m+1})\|_f^2 \\
&+ \|K^{\frac{1}{2}}\nabla(B(\phi(t_{m+1})) - \phi(t_{m+1}))\|_p^2\Big) + \frac{5\nu\Delta t}{16}\|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 + \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2.
\end{aligned} \tag{4.44}
$$

From (4.17), (4.18), (4.21) and (4.22), and assuming Δt satisfies $\Delta t \le min\{\frac{\nu K_{min}}{CC_1^2\tilde{C}_d^2g^2}, \frac{\nu^2K_{min}S_0}{CC_1^2C_d^2g^2}\}$, we have

$$
\begin{aligned}
&F_{m+1} - F_m + \frac{27}{16}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \frac{27}{16}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
&+ \frac{3}{4}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
\le& C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|\phi_{tt}\|_{H_p}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|u_{tt}\|_{H_f}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|p_{tt}\|_f^2 dt\Big) \\
&+ \frac{5\nu\Delta t}{16}\|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 + \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2.
\end{aligned} \tag{4.45}
$$

Summing up from $m=1$ to $m=N-1$ yields

$$
\begin{aligned}
&F_N+\sum_{m=1}^{N-1}\Big(\frac{11}{8}\nu\Delta t\|\nabla(\eta_u^{m+1}-\eta_u^{m})\|_f^2+\frac{11}{8}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}-\eta_\phi^{m})\|_p^2\Big)\\
&+\sum_{m=1}^{N-1}\Big(\frac{3}{4}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1}+\eta_\phi^{m-1})\|_p^2+\frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1}+\eta_u^{m-1})\|_f^2\Big)\\
\leq &F_1+C\Delta t^4\Big(\int_0^T\|u_{ttt}\|_f^2dt+\int_0^T\|\phi_{ttt}\|_p^2dt\Big)\\
&+C\Delta t^4\Big(\int_0^T\|\phi_{tt}\|_{H_p}^2dt+\int_0^T\|u_{tt}\|_{H_f}^2dt+\int_0^T\|p_{tt}\|_f^2dt\Big)\\
&+Ch^{2k+2}\Big(\int_0^T\|u_t\|_{H^{k+1}(\Omega_f)}^2dt+\int_0^T\|\phi_t\|_{H^{k+1}(\Omega_p)}^2dt\Big)\\
&+\frac{5}{8}\nu\Delta t(\|\nabla\eta_u^1\|_f^2+\|\nabla\eta_u^0\|_f^2)+\frac{5}{8}g\Delta t(\|K^{\frac{1}{2}}\nabla(\eta_\phi^1)\|_p^2+\|K^{\frac{1}{2}}\nabla(\eta_\phi^0)\|_p^2).
\end{aligned}
\tag{4.46}
$$

From (4.8)-(4.10) and using triangle inequality on (4.28)-(4.30) yields (4.24). □

Remark 4.1 *: The stability and error estimation require Δt satisfies that limitation conditions, which implies the time step size Δt generally depends on the physical parameters (ν, K_{min}, S_0), and in some pratical cases we have to take very small time step. The relationship between time step size and physical parameters is still an open question, and it depends on the numerical scheme and equations. The structure and stiffness matrices of the Stokes equation and of Darcy equations become more dissimilar when the physical parameters decrease. A sequential Robin-Robin (sRR) method as an alternating direction scheme is robust in convergence rate with independent of the physical quantities characterizing the filtration[34].*

5 Variable stepsize Backward Euler plus time filter and adaptive algorithms

By using Newton interpolation, the variable stepsize BDF methods of order p ($BDF-p$) can be written in [35]. Define the jth order backward divided difference by $\sigma^j u = u[t_{n+m}, t_{n+m-1}, ..., t_{n+m-j}]$, and the parameter in time filter by $\eta^{p+1}=\frac{\prod_{i=1}^{p}(t_{n+m}-t_{n+m-i})}{\sum_{j=1}^{p+1}(t_{n+m}-t_{n+m-j})^{-1}}$. Based on divided difference, let $k_m = t_{m+1}-t_m$, $\tau_m=\frac{k_{m+1}}{k_m}$. It is easy to state the variable stepsize Backward Euler plus time filter for Stokes-Darcy equations in terms of divided difference:

- Given (u_h^0, p_h^0, ϕ_h^0) and (u_h^1, p_h^1, ϕ_h^1), find $(u_h^{m+1}, p_h^{m+1}, \phi^{m+1}) \in (H_f^h, Q^h, H_p^h)$, with $m=0,1,....,N-1$, such that:

  BE for Stokes $\quad (\sigma^1\hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^*),$
  $b(\hat{u}^{m+1}, q_h)=0.$

  BE for Darcy $\quad gS_0(\sigma^1\hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^*, \psi_h).$

  Here, choosing $\phi_h^* = \phi_h^{m-1}$ and $u_h^* = u_h^{m-1}$ gives a standard variable time stepsize BE method for Stokes-Darcy equations and choosing $\phi_h^* = (1+\tau_{m-1})\phi_h^m - \tau_{m-1}\phi_h^{m-1}$, and $u_h^* = (1+$

$\tau_{m-1})u_h^m - \tau_{m-1}u_h^{m-1}$ is explored herein. Implementing this change involves changing one line of code redefining ϕ_h^* and u_h^*.

- Apply time filter to update the previous solution

$$u_h^{m+1} = \hat{u}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{u}_h^{m+1}, \tag{5.1}$$

$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{\phi}_h^{m+1}, \tag{5.2}$$

$$p_h^{m+1} = \hat{p}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{p}_h^{m+1}. \tag{5.3}$$

Using algebraic manipulation, the above three equality can be written in terms of stepsize ratio τ as follows:

$$u_h^{m+1} = \hat{u}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{u}_h^{m+1} - u_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}u_h^{m-1}\Big), \tag{5.4}$$

$$p_h^{m+1} = \hat{p}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{p}_h^{m+1} - p_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}p_h^{m-1}\Big), \tag{5.5}$$

$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{\phi}_h^{m+1} - \phi_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}\phi_h^{m-1}\Big). \tag{5.6}$$

The combination of BE, BEplustimefilter and general adaptive method lead to adaptive BE and adaptive BEplustimefilter algorithm. Since the time accuracy of BEplustimefilter is $O(k^2)$, we can use $\mathrm{Est}_u = |u_h^{m+1} - \hat{u}_h^{m+1}|$ and $\mathrm{Est}_\phi = |\phi_h^{m+1} - \hat{\phi}_h^{m+1}|$ as two estimate for the local error of velocity and hydraulic head in BE. In order to estimate the local error of velocity and hydraulic head for BEplustimefilter, it is easy to take $\mathrm{Est}_u = \eta^3\sigma^3\hat{u}_h^{m+1}$, $\mathrm{Est}_\phi = \eta^3\sigma^3\hat{\phi}_h^{m+1}$ as two estimate because σ^3 is the third order backward divided difference. We give a simple formula for stepsize selection which is the combination of some general adaptive methods. We denote γ and $\tilde{\gamma}$ two safety factors. The first safety factor γ is used to prevent the next step size becoming too big to decrease the chance that the next solution will be rejected. The effect of the second factor $\tilde{\gamma}$ is making the stepsize growing more slowly so that the recomputed solution is more likely to be accepted. We took $\gamma = 0.9$, and $\tilde{\gamma} = 0.6$.

Algorithm 5.1 (Adaptive BE) *Let* $m = 1$. *Given* ε, $\tilde{\gamma}$, γ, $\{u_h^{m-1}, u_h^m\}$, $\{p_h^{m-1}, p_h^m\}$ *and* $\{\phi_h^{m-1}, \phi_h^m\}$, *compute* $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

BE for Stokes

$$(\sigma^1 u_h^{m+1}, v_h)_{\Omega_f} - a_f(u_h^{m+1}, v_h) + b(v_h, p^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^m),$$
$$b(u_h^{m+1}, q_h) = 0.$$

BE for Darcy

$$gS_0(\sigma^1\phi_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\phi_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^m, \psi_h).$$

Choose

$$Est_u = \eta^2\sigma^2\hat{u}_h^{m+1}, \quad Est_\phi = \eta^2\sigma^2\hat{\phi}_h^{m+1},$$

if $\min\{|Est_u|, |Est_\phi|\} < \frac{\varepsilon}{4}$,

$$\tau_m = \min\left\{2, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{2}}\right\}, \tag{5.7}$$

$$k_{m+1} = \gamma \cdot \tau_m \cdot k_m; \tag{5.8}$$

if $\frac{\varepsilon}{4} \leq \min\{|Est_u|, |Est_\phi|\} \leq \varepsilon$,

$$\tau_m = \min\left\{1, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{2}}\right\}, \tag{5.9}$$

$$k_{m+1} = \gamma \cdot \tau_m \cdot k_m. \tag{5.10}$$

If none satisfy the tolerance, set

$$k_m = \tilde{\gamma} \cdot \tau_{m-1} \cdot k_{m-1},$$

and recompute the above steps.

Algorithm 5.2 (Adaptive BETF) *Let* $m = 2$*. Given* ε, $\tilde{\gamma}$, γ, $\{u_h^{m-2}, u_h^{m-1}, u_h^m\}$, $\{p_h^{m-2}, p_h^{m-1}, p_h^m\}$, $\{\phi_h^{m-2}, \phi_h^{m-1}, \phi_h^m\}$*, compute* $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

Modified BE for Stokes

$$\begin{aligned}&(\sigma^1 \hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1})\\ &= (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, (1+\tau_{m-1})\phi_h^m - \tau_{m-1}\phi_h^{m-1}),\\ &b(\hat{u}^{m+1}, q_h) = 0.\end{aligned}$$

BE for Darcy

$$\begin{aligned}&gS_0(\sigma^1 \hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h)\\ &= g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma((1+\tau_{m-1})u^m - \tau_{m-1}u^{m-1}, \psi_h).\end{aligned}$$

Time filter for $\hat{u}_h^{m+1}, \hat{p}_h^{m+1}, \hat{\phi}_h^{m+1}$

$$\begin{aligned}u_h^{m+1} &= \hat{u}_h^{m+1} - \eta^2\sigma^2\hat{u}_h^{m+1},\\ \phi_h^{m+1} &= \hat{\phi}_h^{m+1} - \eta^2\sigma^2\hat{\phi}_h^{m+1},\\ p_h^{m+1} &= \hat{p}_h^{m+1} - \eta^2\sigma^2\hat{p}_h^{m+1}.\end{aligned}$$

Choose

$$Est_u = \eta^3\sigma^3\hat{u}_h^{m+1}, \quad Est_\phi = \eta^3\sigma^3\hat{\phi}_h^{m+1},$$

if $\min\{|Est_u|, |Est_\phi|\} < \frac{\varepsilon}{4}$,

$$\tau_m = \min\left\{2, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{3}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{3}}\right\}, \tag{5.11}$$

$$k_{m+1} = \gamma \cdot \tau_m \cdot k_m; \tag{5.12}$$

if $\frac{\varepsilon}{4} \leq \min\{|Est_u|, |Est_\phi|\} \leq \varepsilon$,

$$\tau_m = \min\left\{1, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{3}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{3}}\right\}, \tag{5.13}$$

$$k_{m+1} = \gamma \cdot \tau_m \cdot k_m. \tag{5.14}$$

If none satisfy the tolerance, set

$$k_m = \tilde{\gamma} \cdot \tau_{m-1} \cdot k_{m-1},$$

and recompute the above steps.

Extension to a Third Order Method

Recently [33] has shown how time filter can be extended from backward Euler to some higher order BDF methods. We also give a test of BDF2 discretization of subdomain terms with a third order extrapolation formula for the interface terms. Errors are estimated and accuracy is increased by an added time filter as presented next.

Algorithm 5.3 (BDF-2 plus time filter) *Let* $\sigma^3 x^{m+1} = \sum_{i=0}^{3} C^i x^{m+1-i}$. *Given* $\{u_h^{m-2}, u_h^{m-1}, u_h^m\}$, $\{p_h^{m-2}, p_h^{m-1}, p_h^m\}$ *and* $\{\phi_h^{m-2}, \phi_h^{m-1}, \phi_h^m\}$, *compute* $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

BDF-2 for Stokes

$$(\sigma^2 \hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^*),$$
$$b(\hat{u}^{m+1}, q_h) = 0.$$

BDF-2 for Darcy

$$gS_0(\sigma^2 \hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^*, \psi_h).$$

Time filter for $\hat{u}_h^{m+1}, \hat{p}_h^{m+1}, \hat{\phi}_h^{m+1}$

$$u_h^{m+1} = \hat{u}_h^{m+1} - \eta^3 \sigma^3 \hat{u}_h^{m+1},$$
$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \eta^3 \sigma^3 \hat{\phi}_h^{m+1},$$
$$p_h^{m+1} = \hat{p}_h^{m+1} - \eta^3 \sigma^3 \hat{p}_h^{m+1}.$$

where, $u_h^* = -\sum_{i=0}^{2} \frac{C^{i+1}}{C^0} u_h^{m-i}$, $\phi_h^* = -\sum_{i=0}^{2} \frac{C^{i+1}}{C^0} \phi_h^{m-i}$. We approximate u_h^{m+1} and ϕ_h^{m+1} by u_h^* and ϕ_h^* in the interface coupling terms due to $u_h^{m+1} = u_h^* + \frac{\sigma^{p+1} u_h^{m+1}}{C^0} = u_h^* + O(k^{p+1})$.

6 Numerical Experiments

In this section, three time filters are performed with constant time stepsize and variable time stepsize to show the validity and accuracy of the decoupled scheme. Furthermore, we implemented the codes using the software package FreeFEM++[36].

Example 1: Consider the computational domain $\Omega_f = (0,1) \times (1,2), \Omega_p = (0,1) \times (0,1)$, and interface $\Gamma = (0,1) \times \{1\}$. We take the exact solution:

$$\phi(x,y,t) = [2 - \pi sin(\pi x)][1 - y - cos(\pi y)]cos(t),$$
$$u(x,y,t) = \Big([x^2(y-1)^2 + y]cos(t), [-\frac{2}{3}x(y-1)^3 + 2 - \pi sin(\pi x)]cos(t)\Big),$$
$$p(x,y,t) = [2 - \pi sin(\pi x)]sin(\frac{\pi}{2}y)cos(t).$$

Table 1: The convergence performance for BE method at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	1.7317e-3		7.4639e-2		1.8125e-2	
1/16	8.7666e-4	0.98	3.7162e-2	1.01	9.2799e-3	0.97
1/32	4.4107e-4	0.99	1.8558e-2	1.00	4.6920e-3	0.98
1/48	2.9466e-4	0.99	1.2368e-2	1.00	3.1393e-3	0.99
1/64	2.2123e-4	1.00	9.2751e-3	1.00	2.3587e-3	0.99

Table 2: The convergence performance for BETF method at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	9.1743e-3		2.2842e-2		8.2014e-3	
1/16	2.1750e-3	2.08	6.0040e-3	1.93	1.9380e-3	2.08
1/32	5.1999e-4	2.06	1.5002e-3	2.00	4.6396e-4	2.06
1/48	2.2759e-4	2.04	6.6679e-4	2.00	2.0314e-4	2.04
1/64	1.2703e-4	2.03	3.7883e-4	1.97	1.1341e-4	2.03

Table 3: The convergence performance for BDF2 at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	2.0238e-4		1.1384e-2		1.5852e-3	
1/16	4.7980e-5	2.08	2.7673e-3	2.04	3.6286e-4	2.13
1/32	1.1647e-5	2.04	6.805e-4	2.02	8.6495e-5	2.07
1/48	5.1234e-6	2.03	3.0068e-4	2.01	3.7808e-5	2.04
1/64	2.8670e-6	2.02	1.6863e-4	2.01	2.1088e-5	2.03

Here, we set the parameters $\nu = 1, g = 1, z = 0, S_0 = 1, \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}} = 1$, and $K = kI$, where $k = 1$, and the initial conditions, boundary conditions, source terms and the initial numerical solutions follow from the exact solution. We consider uniform triangular meshes with fixed mesh size h by the following way: first dividing two subdomains Ω_f and Ω_p into many identical squares and then dividing every square into two triangles. For BE and BETF, we use the well-known Taylor-Hood elements $(P2-P1)$ for fluid velocity and pressure and the continuous piecewise quadratic functions $(P2)$ for hydraulic head. For BDF2 and BDF2 plus time filter, we use $(P3-P2)$ for fluid velocity and pressure and $(P3)$ for hydraulic head.

Table 4: The convergence performance for BDF2 plus time filter at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	5.8394e-4		1.6579e-3		4.8543e-4	
1/16	7.9257e-5	2.88	1.9993e-4	3.05	6.5981e-5	2.88
1/32	1.0271e-5	2.95	2.5728e-5	2.96	8.5509e-6	2.95
1/48	3.0774e-6	2.97	7.7075e-6	2.97	2.5621e-6	2.97
1/64	1.3053e-6	2.98	3.2752e-6	2.97	1.0868e-6	2.98

Table 5: The convergence performance of Global error for adaptive BE method with $h = 1/120$.

$\overline{\Delta t}$	$\left(\sum_{i=2}^{N} k_i \frac{\|u(t_i)-u_h^i\|_f^2}{\|u(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|p(t_i)-p_h^i\|_f^2}{\|p(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|\phi(t_i)-\phi_h^i\|_p^2}{\|\phi(t_i)\|_p^2}\right)^{\frac{1}{2}}$	rate
$\frac{1}{34}$	2.3617e-4		1.1944e-2		2.3434e-3	
$\frac{1}{114}$	7.4300e-5	0.96	3.7714e-3	0.95	7.4303e-4	0.95
$\frac{1}{362}$	2.3382e-5	1.00	1.1975e-3	0.99	2.3383e-4	1.00
$\frac{1}{1149}$	7.2449e-6	1.01	3.7923e-4	1.00	7.3430e-5	1.00
$\frac{1}{1636}$	4.9781e-6	1.06	2.6662e-4	1.00	5.0903e-5	1.04

Table 6: The convergence performance of Global error for adaptive BETF method with $h = 1/120$.

$\overline{\Delta t}$	$\left(\sum_{i=2}^{N} k_i \frac{\|u(t_i)-u_h^i\|_f^2}{\|u(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|p(t_i)-p_h^i\|_f^2}{\|p(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|\phi(t_i)-\phi_h^i\|_p^2}{\|\phi(t_i)\|_p^2}\right)^{\frac{1}{2}}$	rate
$\frac{1}{13}$	3.0781e-3		1.7269e-2		2.7196e-3	
$\frac{1}{41}$	3.3127e-4	1.94	2.2055e-3	1.79	2.9245e-4	1.94
$\frac{1}{119}$	4.0027e-5	1.98	5.4664e-4	1.31	3.5363e-5	1.98
$\frac{1}{373}$	3.6975e-6	2.08	1.4386e-4	1.17	3.2948e-6	2.08
$\frac{1}{553}$	1.6668e-6	2.02	9.1707e-5	1.14	1.5253e-6	1.96

In Tables 1-4, we use BE, BETF, BDF2 and BDF2 plus time filter with constant time stepsize to run simulations at final time $T = 1.0$ and set the mesh size $h = \frac{1}{120}$. The results show that the numerical convergence rate of BETF and BDF2 plus time filter are approximately second order and third order in time respectively for u, p and ϕ. In Figure 2-4, we present the log-log plots of the relative error for velocity, pressure and hydraulic head which clearly shows applying time filter leads to higher order.

In Tables 5-6, we allow the time stepsize to be variable, and final time and the same mesh from the constant stepsize test were used. Various tolerance were tested from 1e-1 to 1e-7. Since the time stepsize is variable, $\overline{\Delta t}$ in Table 5-6 is the average time stepsize. It can be seen the convergence order of adaptive BETF is also higher than adaptive BE. Figure 5-6 displays the log-log plots of the global error for velocity, pressure and hydraulic head. *It is observed that adaptive BETF reduced the total amount of work (number of steps taken), and it requires less time steps than adaptive BE for the smaller tolerance.*

In Tables 7-8, we choose the same finite elements, mesh size and time step size for BETF and BDF2 to compare their computational cost and accuracy. We use the Taylor-Hood elements ($P2 - P1$) for fluid velocity and pressure and the continuous piecewise quadratic functions ($P2$) for hydraulic head. From the results of these two tables, we see that BETF and BDF2 are second order accuracy in time, as predicted. Further, although BDF2 is slightly more accurate than BETF, BETF required less CPU time and is more easier to implement its code than BDF2.

Table 7: Comparison of computation cost and accuracy between BETF and BDF2 with $\Delta t = h = 1/30$.

	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	CPU
BETF	5.9328e-4	6.9384e-4	1.9514e-3	5.2930e-4	1.0176e-3	12.89
BDF2	1.4473e-5	3.6636e-4	1.4336e-3	1.0305e-4	8.6190e-4	13.66

Table 8: Comparison of computation cost and accuracy between BETF and BDF2 with $\Delta t = h = 1/60$.

	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	CPU
BETF	1.4475e-4	1.6850e-4	4.8234e-4	1.2922e-4	2.4954e-4	162.33
BDF2	3.3383e-6	8.8016e-5	3.5121e-4	2.4295e-5	2.1179e-4	174.86

7 Conclusions

We presented an BETF algorithm, construct an adaptive BETF algorithm and another time filter for second order BDF method for Stokes-Darcy equation. Another second order method is derived by the combination of Backward Euler plus a time filter that is easily completed by adding three lines to the previous code and slightly modifying the matrix of right hand side based on the BE method. Both theoretical analysis and tests both indicate that adding the filter step to Backward Euler have the advantage of improving time accuracy and convergence order. The numerical tests performed verify that time adaptivity guarantees accuracy while decreasing storage required and overall complexity.

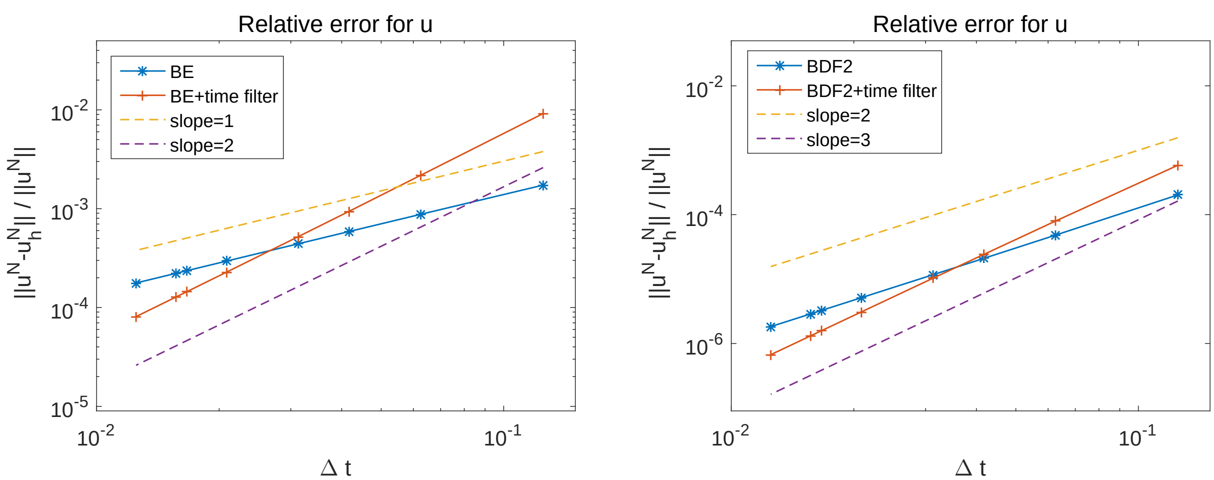

Figure 2: Relative errors of velocity u for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize

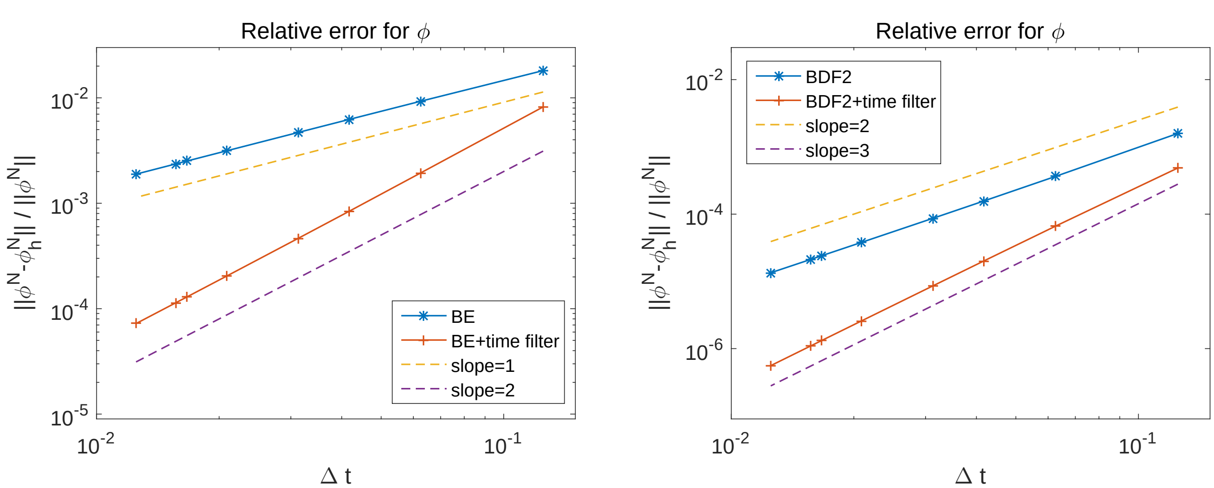

Figure 3: Relative errors of hydraulic head ϕ for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize.

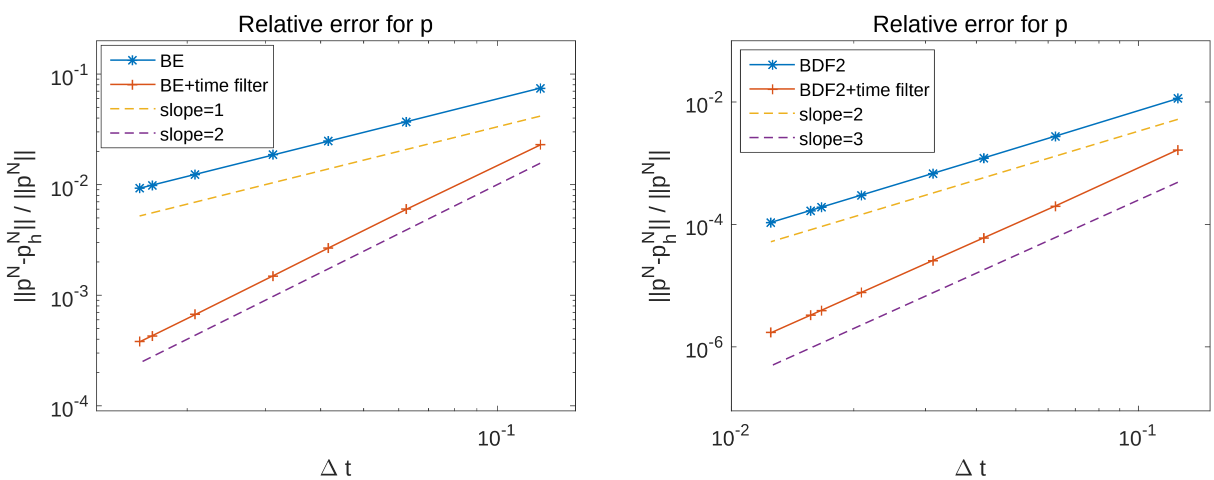

Figure 4: Relative errors of pressure p for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize.

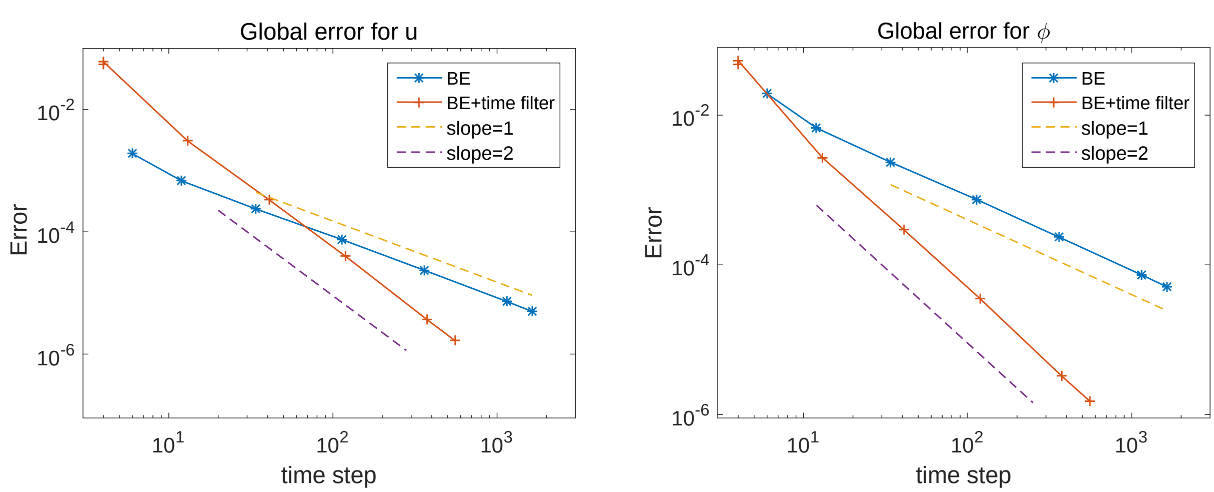

Figure 5: Global errors of velocity u (left) and hydraulic head ϕ (right) for adaptive BE and BE plus time filter.

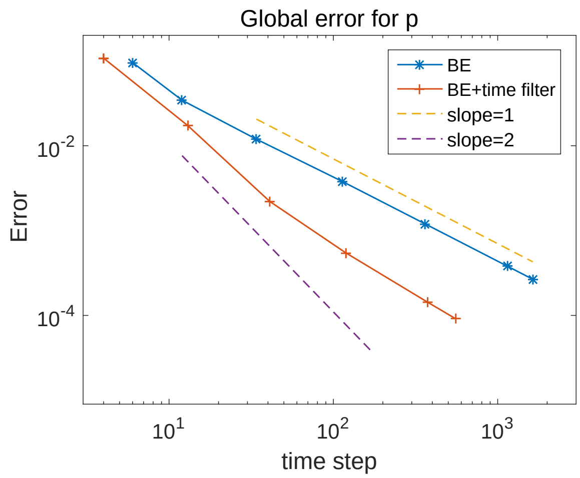

Figure 6: Global errors of pressure p for adaptive BE and BE plus time filter.

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