Mathematische Zeitschrift
https://doi.org/10.1007/s00209-019-02435-1

Mathematische Zeitschrift

Square Sierpiński carpets and Lattès maps

Mario Bonk[1] · Sergei Merenkov[2]

Received: 14 May 2018 / Accepted: 5 November 2019

Abstract

We prove that every quasisymmetric homeomorphism of a standard square Sierpiński carpet S_p, $p \geq 3$ odd, is an isometry. This strengthens and completes earlier work by the authors (Bonk and Merenkov in Ann Math (2) 177:591–643, 2013, Theorem 1.2). We also show that a similar conclusion holds for quasisymmetries of the double of S_p across the outer peripheral circle. Finally, as an application of the techniques developed in this paper, we prove that no standard square carpet S_p is quasisymmetrically equivalent to the Julia set of a postcritically-finite rational map.

1 Introduction

The standard square Sierpiński carpet S_p is constructed as follows. We fix an odd integer $p \geq 3$. We start with the closed unit square $Q = [0, 1]^2$ in the plane $\mathbb{R}^2$ and subdivide it into $p \times p$ subsquares of sidelength $1/p$. Next, we remove the interior of the middle subsquare of this subdivision. Note that this middle subsquare is well defined since p is odd. After this we repeat these two operations (i.e., subdividing and removing the middle subsquare) indefinitely on the remaining subsquares. We equip the residual set of this construction with the Euclidean metric and call it the *standard square Sierpiński p-carpet* and denote it by S_p. The sets S_p are all homeomorphic to each other. In general, we call a metrizable topological space Z a *Sierpiński carpet* if Z is homeomorphic to S_3 (Fig. 1).

The boundary of Q and the boundaries of all the squares that were removed from Q in the construction of S_p are the so-called *peripheral circles* of S_p. A Jordan curve $J \subseteq S_p$ is a peripheral circle if and only if its removal from S_p does not separate S_p. The boundary ∂Q of Q is called the *outer* peripheral circle of S_p. We denote it by O.

Mario Bonk was supported by NSF Grants DMS-1506099 and DMS-1808856. Sergei Merenkov was supported by NSF Grant DMS-1800180.

✉ Sergei Merenkov
smerenkov@ccny.cuny.edu

Mario Bonk
mbonk@math.ucla.edu

1 Department of Mathematics, University of California, Los Angeles, CA 90095, USA

2 Department of Mathematics, City College of New York and CUNY Graduate Center, New York, NY 10031, USA

Published online: 09 December 2019

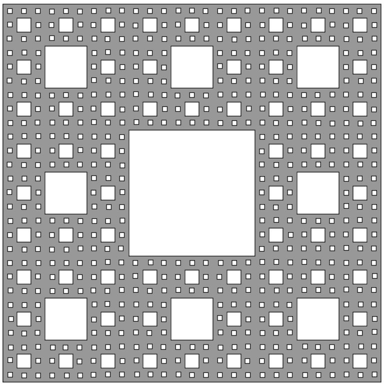

Fig. 1 The standard square Sierpiński 3-carpet S_3

A homeomorphism $f: X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is said to be *quasisymmetric* or a *quasisymmetry*, if there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for all distinct points $x, y, z \in X$. We also say that a map $f: X \to Y$ is a *quasisymmetric embedding* if the map $f: X \to f(X)$ is a quasisymmetry, where $f(X)$ is endowed with the restriction of the metric d_Y. Finally, if we want to emphasize a distortion function η, we say that f is η-*quasisymmetric*.

The class of quasisymmetries contains all bi-Lipschitz maps. The composition of two quasisymmetries (when defined) and the inverse of a quasisymmetry are quasisymmetric. So if we call two metric spaces X and Y *quasisymmetrically equivalent* if there exists a quasisymmetry $f: X \to Y$, then we have a notion of equivalence for metric spaces.

The question of when two metric spaces are quasisymmetrically equivalent has drawn much attention in recent years. This is motivated by questions in geometric group theory, for example, such as Cannon's conjecture or the Kapovich–Kleiner conjecture which can be reduced to quasisymmetric equivalence problems (see [2] for a survey of this topic).

The main result of this paper is the following statement.

Theorem 1.1 *Every quasisymmetry* $\xi: S_p \to S_p$, $p \geq 3$ *odd, is an isometry.*

This improves results in [5]. There it was shown that every quasisymmetry of S_3 is an isometry [5, Theorem 1.1] and that the group of all quasisymmetries of S_p, $p \geq 5$ odd, is a finite dihedral group [5, Theorem 1.2].

The methods of [5] do not seem to give the more general conclusion of Theorem 1.1 (see the discussion in [5, Remark 8.3]). In the present paper we do rely on the results in [5], but for the proof of Theorem 1.1 we combine this with new ideas that were developed in [4] for the study of quasisymmetries of Sierpiński carpets that arise as Julia sets of postcritically-finite rational maps. Our methods also allow us to prove other related rigidity results for quasisymmetries. For their formulation we require some more definitions.

We consider the double P of the unit square Q, i.e., P is obtained from two identical copies of Q glued together by identifying corresponding points on their boundaries. We refer to P as a *pillow* and endow it with the unique path metric whose restriction to each of the two copies of Q in P coincides with the Euclidean metric. We can identify one of the isometric copies of Q with Q itself and call it the *front* of P. Then $Q \subseteq P$. The other isometric copy Q' of Q in P is called the *back* of P.

We consider S_p as a subset of the front Q of P. The back Q' of P carries another isometric copy S'_p of S_p. We use the notation $D_p = S_p \cup S'_p$ for the union of these sets and equip it with the restriction of the path metric on P. Then D_p is a Sierpiński carpet (this easily follows from a topological characterization of Sierpiński carpets due to Whyburn [11]). It consists of two copies of S_p glued together along the outer peripheral circle.

Our methods give the following rigidity result for D_p.

Theorem 1.2 *Every quasisymmetry $\xi\colon D_p \to D_p$, $p \geq 3$ odd, is an isometry.*

The geometry of S_p distinguishes its outer peripheral circle O. This is supported by the fact that for the investigations in [5] and also for our proof of Theorem 1.1 the starting point is the non-trivial fact that every quasisymmetry $\xi\colon S_p \to S_p$ has to preserve the outer peripheral circle O as a set, i.e., $\xi(O) = O$. In contrast, the Sierpiński carpet D_p does not carry such a distinguished peripheral circle; this makes the rigidity result given by Theorem 1.2 somewhat more surprising.

To formulate our last result, we have to briefly review some standard facts from complex dynamics (see [1] for general background). Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a map on the Riemann sphere $\widehat{\mathbb{C}}$. For $n \in \mathbb{N}$, we denote by

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ factors}}$$

the n-th iterate of f. It is convenient to set $f^0 = \mathrm{id}_{\widehat{\mathbb{C}}}$, where $\mathrm{id}_{\widehat{\mathbb{C}}}$ is the identity map on $\widehat{\mathbb{C}}$.

Now suppose that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree ≥ 2. Then the *Fatou set* of f, denoted by $\mathcal{F}(f)$, is the set of all points in $\widehat{\mathbb{C}}$ that have neighborhoods where the sequence $\{f^n\}_{n\in\mathbb{N}}$ of iterates of f is a normal family. The complement of $\mathcal{F}(f)$ in $\widehat{\mathbb{C}}$ is called the *Julia set* of f and denoted by $\mathcal{J}(f)$. It is a standard fact that $\mathcal{J}(f)$ is a non-empty compact set that is completely invariant under f, i.e., $f^{-1}(\mathcal{J}(f)) = \mathcal{J}(f) = f(\mathcal{J}(f))$.

The *critical set* of f consists of all points in $\widehat{\mathbb{C}}$ near which f is not a local homeomorphism. This is a finite subset of $\widehat{\mathbb{C}}$. The *postcritical set*

$$\bigcup_{n\in\mathbb{N}} \{f^n(c) : c \in \widehat{\mathbb{C}} \text{ critical point of } f\}$$

of f consists of all forward iterates of critical points. A rational map f is said to be *postcritically-finite* if its postcritical set is finite.

In [4] it was shown that every quasisymmetry between two Sierpiński carpets that arise as Julia sets of postcritically-finite rational maps is a Möbius transformation (i.e., a fractional linear or conjugate fractional linear map on the Riemann sphere $\widehat{\mathbb{C}}$). It is a natural question whether any of the carpets S_p or D_p can be quasisymmetrically equivalent to such a Julia set. The following statement shows that this is never the case.

Theorem 1.3 *No Sierpiński carpet S_p or D_p, $p \geq 3$ odd, is quasisymmetrically equivalent to the Julia set $\mathcal{J}(g)$ of a postcritically-finite rational map g.*

Even though there is only one topological type of Sierpiński carpets [11], Theorem 1.3 shows that standard square carpets and Julia sets of postcritically-finite rational maps are in different quasisymmetric equivalence classes.

By the authors' earlier work [5] the carpets S_p and S_q for different odd integers p and q are never quasisymmetrically equivalent. In [10], the second author proved that a Sierpiński carpet that arises as the boundary at infinity of a torsion-free hyperbolic group cannot be quasisymmetrically equivalent to a standard carpet S_p or the Julia set of a rational map. Moreover, in [4] it was shown that no Sierpiński carpet Julia set of a postcritically-finite rational map is quasisymmetrically equivalent to the limit set of a Kleinian group.

To summarize, these results tell us that there are at least three quasisymmetrically distinct classes or "universes" of Sierpiński carpets: standard square carpets, boundaries at infinity of hyperbolic groups (or limit sets of Kleinian groups), and Julia sets of postcritically-finite rational maps. Moreover, even within these universes one often encounters infinitely many quasisymmetric equivalence classes.

Before we go into the details, we will discuss some of the ideas that are used in the proofs of the main results. Our main observation is that a quasisymmetry $\xi\colon D_p \to D_p$ as in Theorem 1.2 is related to the dynamics of a Lattès map T (depending on p) that is defined on the pillow P and leaves the Sierpiński carpet D_p forward-invariant. More precisely, we have a relation of the form

$$T^m \circ \xi = T^\ell \circ \xi \circ T^k \tag{1.1}$$

with (arbitrarily large) $k, \ell, m \in \mathbb{N}$ (see Proposition 5.1). Once (1.1) is established, the proofs of Theorems 1.1 and 1.2 are completed by carefully analyzing the implications for the mapping behavior of ξ in combination with known results from [5]. For the proof of Theorem 1.3 one derives similar dynamical relations for a quasisymmetry ξ of D_p or S_p onto the Julia set $\mathcal{J}(g)$ of a postcritically-finite rational map g (see (7.5) and (7.8)) which ultimately lead to a contradiction.

In order to establish (1.1), we rely on a dynamical "blow down-blow up" procedure very similar to the one used in [4]. This is combined with a uniformization result for Sierpiński carpets proved by the first author [3] and rigidity results for Schottky maps established by the second author [8,9].

The paper is organized as follows. In Sect. 2 we introduce the Lattès map T mentioned above and some geometric facts related to the dynamics of T. Section 3 is devoted to the resolution of some technicalities that are ultimately caused by the lack of backward invariance of D_p under T. This relies on the concept of an *admissible map* that is introduced and studied in this section. In Sect. 4 we review the necessary background from the theory of Schottky maps and the required rigidity results (in particular, Theorems 4.2 and 4.3). In Sect. 5 we prove Proposition 5.1 that provides the crucial relation (1.1). The proof of Theorems 1.1, 1.2 and 1.3 are then given in the two subsequent sections.

Our arguments heavily rely on previous results obtained in [3–5,8,9]. A detailed knowledge of these works is not necessary for the reader of the present paper, because we will review all the relevant facts. It may be helpful for the reader though to take a more careful look at [4, Section 8], because our arguments in Sect. 5 and part of the proof of Theorem 1.3 (leading to (7.5) and (7.8)) are very similar to the reasoning there.

2 The Lattès map *T*

Throughout this paper $p \geq 3$ is a fixed odd integer. Our pillow P as defined in the introduction is equipped with a path metric that agrees with the Euclidean metric on the front Q and on the back Q' of P. In the following, all metric notions related to P will be based on this metric. The pillow P is an (abstract) polyhedral surface and so it carries a natural conformal structure making it conformally equivalent to the Riemann sphere. On the subsquare $[0, 1/p]^2$ of the front $Q = [0, 1]^2$ of P, we consider the map $z \in [0, 1/p]^2 \mapsto pz \in Q$. By Schwarz reflection this naturally extends to a map $T\colon P \to P$. Note that this extension of T to all of P using Schwarz reflection is possible, because in the obvious subdivision of P into $2p^2$ subsquares of equal size, each corner of every subsquare is common to an even number of subsquares in the subdivision. Of course, T depends on p, but we suppress this from our notation.

With the conformal structure on P, the map T is holomorphic. By the uniformization theorem there is a conformal map of P onto $\widehat{\mathbb{C}}$. Under such a conformal identification $P \cong \widehat{\mathbb{C}}$, the map T is a rational map on $\widehat{\mathbb{C}}$, a so-called *Lattès map* (see [6, Chapter 3] for a detailed discussion of Lattès maps from this point of view). Note that $T(D_p) = D_p$, i.e., D_p is forward invariant under T, but clearly not backward invariant.

Let $n \in \mathbb{N}_0$. Then each of the two faces Q and Q' of the pillow P is in a natural way subdivided into p^{2n} squares of side length p^{-n}. We call a square obtained in this way from the subdivision of Q or Q' a *tile of level n* or simply an *n-tile*. So there are $2p^{2n}$ tiles of level n. Similarly, we call the sides of these n-tiles the *n-edges* and their corners the *n-vertices* (this terminology is motivated by the language in [6, Section 5.3]).

On each n-tile X^n the iterate T^n behaves like a similarity map and sends X^n homeomorphically to either Q or Q'. Here and elsewhere we use the convention that T^0 denotes the identity map on P. We assign the color white or black to the n-tile X^n as follows: if $T^n(X^n) = Q$, then we assign to X^n the color white, and if $T^n(X^n) = Q'$ the color black. Colors on n-tiles alternate so that two n-tiles sharing a side have different colors. Therefore, the n-tiles form a *checkerboard tiling* of P (as defined in [6, Section 5.3]).

More generally, if $k, n \in \mathbb{N}_0$, and X^{n+k} is an $(n+k)$-tile, then T^n is a homeomorphism of X^{n+k} onto the k-tile $X^k := T^n(X^{n+k})$. Moreover, T^n is color-preserving in the sense that X^{n+k} and X^k have the same color.

In general, an *inverse branch* T^{-n} for $n \in \mathbb{N}_0$ is a right inverse of T^n defined on some subset of P. In this paper, we will consider very specific inverse branches defined on Q. To define them, let $c := (0, 0) \in Q$ be the lower left corner of Q. Then $Z^n = [0, 1/p^n]^2$ is the unique n-tile Z^n with $c \in Z^n \subseteq Q$ and T^n sends Z^n homeomorphically onto Q. We define $T^{-n} := (T^n|Z^n)^{-1}$ and so $T^{-n}\colon Q \to Z^n$ is the unique map such that $T^n \circ T^{-n}$ is the identity on Q.

If $k, n \in \mathbb{N}_0$, then with these definitions we have $T^{-(n+k)} = T^{-n} \circ T^{-k}$ and, if $n > k$ in addition, $T^{n-k} \circ T^{-n} = T^{-k}$. This latter consistency condition for inverse branches will be important in Sect. 5 (see (5.2)).

For some n-tiles X^n the interior $\operatorname{int}(X^n)$ is disjoint from D_p, because $\operatorname{int}(X^n)$ falls into one of the sets that were removed from Q or Q' in the construction of S_p and S'_p. We call an n-tile X^n *good* if $\operatorname{int}(X^n) \cap D_p \neq \emptyset$. There are precisely $2(p^2 - 1)^n$ good n-tiles. It follows from the self-similar construction of S_p that if X^n is a good white or black n-tile, then $D_p \cap X^n$ is a scaled copy of S_p. Moreover, then T^n is a homeomorphism of $D_p \cap X^n$ onto S_p or S'_p, respectively.

The inverse branches T^{-n} defined above preserve the color of a tile. Moreover, T^{-n} induces a bijection between the good subtiles of Q and the good subtiles of $Z^n = T^{-n}(Q)$. So in particular, if $k \in \mathbb{N}_0$ and $X^k \subseteq Q$ is a k-tile, then $X^{n+k} := T^{-n}(X^k)$ is an $(n+k)$-tile with the same color as X^k. Moreover, X^k is a good tile if and only if X^{n+k} is.

As before, we denote by O the boundary of Q and consider it as subset of the pillow P. Then $O = Q \cap Q' \subseteq P$. For each side e of O, i.e., for each 0-edge e, we have

$$T(e) = e. \tag{2.1}$$

This is clearly true for the two sides of Q that contain the origin $c = (0, 0)$ (i.e., the lower left corner of $Q \subseteq P$). It is also true for the two other sides of Q since p is odd. The identity (2.1) implies that

$$T(O) = O. \tag{2.2}$$

The *middle peripheral circle* M of $S_p \subseteq P$ is the boundary of the subsquare of Q that is removed in the first stage of the construction of S_p; this is the only peripheral circle of S_p other than the outer peripheral circle O that is invariant under the isometries of the square Q. Similarly, we denote by $M' \subseteq P$ the corresponding peripheral circle of the back copy S'_p. Then we have

$$T(M) = T(M') = O. \tag{2.3}$$

We will now establish a geometric fact about quasisymmetries and tiles that will be used later (see Lemma 2.2). First, we prove an auxiliary result. In both of the following lemmas and their proofs $p \in \mathbb{N}$, $p \geq 3$ odd, is fixed.

All metric notions refer to the piecewise Euclidean metric on P discussed above. We use $\operatorname{dist}(x, y)$ to indicate the distance of two points $x, y \in P$ with respect to this metric. We denote by $B(a, r) = \{x \in P : \operatorname{dist}(a, x) < r\}$ the open ball of radius $r > 0$ centered at $a \in P$. If $A, B \subseteq P$, we let

$$\operatorname{diam}(A) = \sup\{\operatorname{dist}(x, y) : x, y \in A\}$$

be the diameter of A, and

$$\operatorname{dist}(A, B) = \inf\{\operatorname{dist}(x, y) : x \in A,\ y \in B\}$$

be the distance of A and B. If $x \in P$, we set $\operatorname{dist}(x, A) = \operatorname{dist}(\{x\}, A)$.

Lemma 2.1 *Let $m, \ell \in \mathbb{N}_0$, $\ell \geq 1$, $v \in P$ be an m-vertex, K be the union of all m-edges that meet v, and Ω be the interior of the union of all $(m + \ell)$-tiles that meet K. Then Ω is a simply connected region that contains the open $p^{-(m+\ell)}$-neighborhood of K, but does not contain any ball of radius $r > \sqrt{2} \cdot p^{-(m+\ell)}$.*

Proof Note that unless v is a corner of P, the set K forms a "cross" (possibly "folded" if $v \in \partial Q = \partial Q'$). If v is a corner of P, then K consists of two line segments of length p^{-m} meeting perpendicularly at the common endpoint v.

Obviously, K is contained in Ω. Moreover, Ω is connected, because two arbitrary points $x, y \in \Omega$ can be joined by a path in Ω as follows. There exist $(m + \ell)$-tiles X and Y with $x \in X$, $y \in Y$, $X \cap K \neq \emptyset$, and $Y \cap K \neq \emptyset$. Then one runs from x to a point in $x' \in X \cap K$ along a path in $X \cap \Omega$, from x' along a path in $K \subseteq \Omega$ to a point in $y' \in Y \cap K$, and finally from y' to y along a path in $Y \cap \Omega$. This shows that Ω is a region.

The region Ω is simply connected, i.e., a contractible space, because Ω can be retracted to $K \subseteq \Omega$ and K is contractible.

Let $x \in K$ be arbitrary. Then there exists an $(m + \ell)$-edge $e \subseteq K$ such that $x \in e$. There are at most six $(m + \ell)$-tiles that have one of the endpoints of e as a corner. The union of these tiles is a set whose interior is contained in Ω and contains the ball $B(x, p^{-(m+\ell)})$. Hence $B(x, p^{-(m+\ell)}) \subseteq \Omega$ which implies that Ω contains the open $p^{-(m+\ell)}$-neighborhood of K.

Finally, every point $x \in \Omega$ is contained in an $(m + \ell)$-tile X that meets K. Every such tile X contains a corner $y \notin \Omega$. For the distance of x and y we have $\operatorname{dist}(x, y) \leq \sqrt{2} \cdot p^{-(m+\ell)}$. This implies that Ω cannot contain any ball of radius $r > \sqrt{2} \cdot p^{-(m+\ell)}$. □

Lemma 2.2 *Let $\xi : P \to P$ be a quasisymmetry with $\xi(D_p) \subseteq D_p$. Then there exist numbers $r_0, N \in \mathbb{N}_0$ and $C \geq 1$ with the following properties: if $n \in \mathbb{N}_0$ with $n \geq N$ and $X \subseteq P$ is a good n-tile, then there exist a good $(n + r_0)$-tile $Y \subseteq X$ and a good m-tile Z for some $m \in \mathbb{N}_0$ such that $\xi(Y) \subseteq Z$ and*

$$\frac{1}{C} p^{-m} \leq \operatorname{diam}(\xi(Y)) \leq C p^{-m}. \tag{2.4}$$

If A and B are two quantities, then we write $A \asymp B$ if there exists a constant $C \geq 1$ only depending on some ambient parameters such that $A/C \leq B \leq CA$. Similarly, we write $A \lesssim B$ or $B \gtrsim A$ if $A \leq CB$.

Then (2.4) can be written as $\operatorname{diam}(\xi(Y)) \asymp p^{-m} \asymp \operatorname{diam}(Z)$, where the implicit multiplicative constants are independent of the initial choice of the tile X. So Lemma 2.2 says that $\xi(Y)$ lies in a good m-tile Z of comparable size with constants of comparability independent of X. In general, one cannot guarantee that the set $\xi(X)$ itself lies in a good tile of comparable size.

Proof Let X be a good n-tile, where $n \in \mathbb{N}_0$. Since ξ is a quasisymmetry, the image $\xi(X)$ is a "quasi-ball". So if x_1 is the center of the square X, then $\xi(x_1)$ has a distance to the Jordan curve $J := \xi(\partial X)$ that is comparable to $\operatorname{diam}(J)$. Similarly, there exists a point $x_2 \in P \backslash X$ (for example, for x_2 we can take the center of the face of P on the opposite side of X) such that $\operatorname{dist}(\xi(x_2), J) \gtrsim \operatorname{diam}(J)$, i.e., we have $\operatorname{dist}(\xi(x_2), J) \geq \operatorname{diam}(J)/C$ for some constant $C \geq 1$ that depends only on ξ. Let $y_i = \xi(x_i)$ for $i = 1, 2$. Then y_1 and y_2 lie in different components of $P \backslash J$. Moreover, there exists a constant $\delta > 0$ independent of n and X such that $\operatorname{dist}(y_i, J) > \delta \operatorname{diam}(J)$. This shows that each of the two complementary components of J in P contains a ball of radius $r := \delta \operatorname{diam}(J)$.

Uniform continuity of ξ implies that there exists $N \in \mathbb{N}_0$ that depends only on ξ such that if $n \geq N$, then $\operatorname{diam}(J) < 1/3$. In this case, we can choose the largest number $m \in \mathbb{N}_0$ such that $\operatorname{diam}(J) < \frac{1}{3} p^{-m}$. Then $\frac{1}{3} p^{-(m+1)} \leq \operatorname{diam}(J) < \frac{1}{3} p^{-m}$, and so $\operatorname{diam}(J) \asymp p^{-m}$. We can choose $\ell \in \mathbb{N}$ only depending on δ (and independent of X) such that $r = \delta \operatorname{diam}(J) > \sqrt{2} \cdot p^{-(m+\ell)}$. By choice of δ, each of the two complementary components of J contains a ball of radius $r > \sqrt{2} \cdot p^{-(m+\ell)}$.

Claim. Let $E \subseteq P$ denote the union of all m-edges. Then there exists a point $a \in J$ such that $\operatorname{dist}(a, E) \geq \epsilon := p^{-(m+\ell)}$.

In order to prove the claim, we argue by contradiction and assume that there is no such point. Then J is contained in the open ϵ-neighborhood of E. In particular, there exists an m-edge e such that $\operatorname{dist}(e, J) < \epsilon$.

If e_1 and e_2 are two disjoint m-edges, then the connected set J cannot be ϵ-close to both of them. Indeed, if this were the case, then it follows from $\operatorname{dist}(e_1, e_2) \geq p^{-m}$, $\epsilon \leq p^{-(m+1)} \leq \frac{1}{3} p^{-m}$ and $\operatorname{diam}(J) < \frac{1}{3} p^{-m}$ that

$$\tfrac{1}{3} p^{-m} > \operatorname{diam}(J) \geq \operatorname{dist}(e_1, e_2) - 2\epsilon \geq \tfrac{1}{3} p^{-m}.$$

This is a contradiction.

Since J cannot be ϵ-close to two disjoint m-edges, one of the endpoints v of e, which is an m-vertex, has the following property: if K is the set of all m-edges that meet v, then J is contained in the open ϵ-neighborhood of K. In particular, the Jordan curve J is contained in the simply connected region Ω as defined in Lemma 2.1 for the m-vertex v and our choice of ℓ.

Then one of the two complementary components U of J is also contained in Ω, because Ω is simply connected. This is a contradiction, because U contains a ball of radius $r =$

$\delta \operatorname{diam}(J) > \sqrt{2} \cdot p^{-(m+\ell)}$ by what we have seen above, while $\Omega \supseteq U$ contains no such ball by Lemma 2.1. The Claim follows.

Since ξ is a quasisymmetry, we can choose $r_0 \in \mathbb{N}_0$ sufficiently large independent of X with the following property: if Y is any $(n + r_0)$-tile with $Y \subseteq X$ and $Y \cap \partial X \neq \emptyset$, then

$$\operatorname{diam}(\xi(Y)) \leq p^{-\ell} \operatorname{diam}(\xi(\partial X)) = p^{-\ell} \operatorname{diam}(J) < \tfrac{1}{3} p^{-(m+\ell)}.$$

Note that these tiles Y are lined up along the boundary of X and cover ∂X. Each such tile Y is a good tile, because X is a good tile.

Therefore, we can choose such a tile Y so that $\xi(Y)$ contains a point $a \in J$ with $\operatorname{dist}(a, E) \geq p^{-(m+\ell)}$ as provided by the Claim. Then

$$\operatorname{dist}(\xi(Y), E) \geq \operatorname{dist}(a, E) - \operatorname{diam}(\xi(Y)) \geq p^{-(m+\ell)} - \tfrac{1}{3} p^{-(m+\ell)} > 0,$$

and so $\xi(Y)$ does not meet the union E of all m-edges. Since $\xi(Y)$ is a connected set, it must be contained in the interior of an m-tile, because these interiors are precisely the complementary components of E. In particular, there exists an m-tile Z such that $\xi(Y) \subseteq Z$. Since Y is a good tile, there exists a point $b \in \operatorname{int}(Y) \cap D_p$. Then

$$\xi(b) \in \xi(\operatorname{int}(Y)) \cap \xi(D_p) \subseteq \operatorname{int}(Z) \cap D_p.$$

This implies that Z is a good tile.

Since r_0 is fixed and independent of X, the fact that ξ is a quasisymmetry implies that

$$\operatorname{diam}(\xi(Y)) \asymp \operatorname{diam}(J) \asymp p^{-m}$$

with implicit multiplicative constants independent of X and Y. It follows that we can find a suitable constant $C \geq 1$ independent of X such that inequality (2.4) is always valid. The statement follows. □

3 Admissible maps

In order to prove Theorems 1.1 and 1.2 , we want to establish a relation between a given quasisymmetry $\xi : D_p \to D_p$ and our Lattès map T (see Proposition 5.1). This relation can be obtained by arguments similar to [4] relying on rigidity statements for Schottky maps. These Schottky maps are obtained after a quasisymmetric uniformization of D_p by a *round* Sierpiński carpet, i.e., a Sierpiński carpet in $\widehat{\mathbb{C}}$ all of whose peripheral circles are geometric circles. We will discuss the necessary results in Sect. 4.

Unfortunately, there are some technicalities that are essentially due to the lack of backward invariance of D_p under T (see [4, Lemma 6.1], where a related statement relied on backward invariance). To work around this problem, we introduce in this section the ad hoc notion of an *admissible map*. We will prove several statements about these maps that will allow us to apply the results on Schottky maps. We now present the details.

Let S^2 be a topological 2-sphere. We think of it as equipped with an orientation and a metric d. Subsets of S^2 will carry the restriction of d, and so it makes sense to speak of quasisymmetries between such sets. In our applications, S^2 will be the pillow P equipped with the piecewise Euclidean metric described earlier or the Riemann sphere $\widehat{\mathbb{C}}$ equipped with the chordal metric.

Let $Z \subseteq S^2$ be a set and $f : U \to S^2$ be a map defined on a set $U \subseteq S^2$. We say that $x \in Z$ is a *good point* for f and Z if the following condition is true: there exists an (open) Jordan region $V \subseteq S^2$ with $x \in V$ such that f is defined on V, the set $W = f(V)$ is also a Jordan region, and $f|V : V \to W$ is an orientation-preserving quasisymmetric homeomorphism with $f(V \cap Z) = W \cap Z$. In particular, f is then a homeomorphism of $V \cap Z$ onto $W \cap Z$.

Let $Z \subseteq S^2$ be a Sierpiński carpet, and $f : S^2 \to S^2$ be a branched covering map (for the definition of a branched covering map and more background on this topic see [6, Chapter 2]). We say that f is *admissible* for the given Sierpiński carpet Z if $f(Z) \subseteq Z$ and if there exists a set $E \subseteq Z$ that is contained in a union of a finite set and finitely many peripheral circles of Z such that each point $x \in Z \backslash E$ is a good point for f and Z. We call E an *exceptional set* for f and Z. Note that E is not necessarily the complement in Z of all good points, but it contains this complement.

Lemma 3.1 *Let $Z \subseteq \widehat{\mathbb{C}}$ be a Sierpiński carpet, and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiregular map with $f^{-1}(Z) = Z$. Then f is an admissible map for Z.*

For the definition of a quasiregular map and some related facts in a similar context see [4, Section 2]. The lemma implies that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map and its Julia set $\mathcal{J}(f)$ is a Sierpiński carpet, then f is admissible for $\mathcal{J}(f)$.

Proof The statement follows from [4, Lemma 6.1] and its proof. The considerations there imply that each point in Z distinct from the finitely many critical points of f is a good point for f and Z. In particular, f is an admissible map for Z. □

Lemma 3.2 *The Lattès map $T : P \to P$ is admissible for D_p.*

Proof We know that T is a branched covering map and that $T(D_p) \subseteq D_p$. So we have to find an exceptional set for T and the Sierpiński carpet D_p.

Recall that M denotes the middle peripheral circle of S_p, and M' the corresponding peripheral circle in the back copy S'_p. Let F be the finite set consisting of all 1-vertices, i.e., the corners of all squares that arise in the natural subdivision of Q and Q' into squares of side length $1/p$. Then F contains all critical points of T (and actually four non-critical points of T, namely the four corners of P).

We claim that $E := F \cup M \cup M'$ is an exceptional set for T and D_p. To see this, let $x \in D_p \backslash E$ be arbitrary. We want to show that x is a good point for T and D_p. There exists a good 1-tile X with $x \in X$. We will assume that X is white (if X is black, the argument is completely analogous). We now consider two cases.

Case 1: $x \in \operatorname{int}(X)$. Since X is white, $T|X$ is a homeomorphism from X to Q. Actually, $T|X$ is a quasisymmetry, because on X the map behaves like a similarity scaling distances by the factor p. Then $U = \operatorname{int}(X)$ and $V = \operatorname{int}(Q)$ are Jordan regions and T is quasisymmetry from U onto V. Since X is a good 1-tile, we also have $T(X \cap D_p) = Q \cap D_p$ which implies that $T(U \cap D_p) = V \cap D_p$. Hence x is a good point for T and D_p.

Case 2: $x \in \partial X$. Since x does not lie in $E \supseteq F$, this point belongs to the boundary of X, but is not a corner of the square X. Hence there exists a unique side $e \subseteq \partial X$ of X with $x \in e$. Moreover, since $x \notin E \supseteq M \cup M'$, the side e is not contained in $M \cup M'$. Hence there exists a unique good 1-tile $Y \neq X$ that shares the side e with X. Since X is white, Y is black. Let $\operatorname{int}(e)$ be the set of interior points of the closed arc e, i.e., e with its two endpoints removed. Then $x \in \operatorname{int}(e)$. Moreover,

$$U' := \operatorname{int}(X) \cup \operatorname{int}(e) \cup \operatorname{int}(Y)$$

is a simply connected region with $x \in U'$ that is mapped by T homeomorphically onto the simply connected region

$$V' = \operatorname{int}(Q) \cup \operatorname{int}(\widetilde{e}) \cup \operatorname{int}(Q').$$

Here $\widetilde{e} := T(e)$ is a common side of Q and Q'. We have $T(U' \cap D_p) = V' \cap D_p$, because X and Y are good 1-tiles. Moreover, $T|U'$ scales lengths of paths in U' by the factor p, i.e.,

$$\text{length}(T \circ \gamma) = p \cdot \text{length}(\gamma),$$

whenever γ is a path in U'. The metric on P is a geodesic metric. So these considerations imply that if $r > 0$ is sufficiently small, then the open ball $U := B(x, r)$ is a Jordan region contained in U' and T is a quasisymmetry of U onto the Jordan region $V := B(T(x), pr)$ such that $T(U \cap D_p) = V \cap D_p$. Hence x is a good point for T and D_p.

Since Cases 1 and 2 exhaust all possibilities, every point $x \in D_p \backslash E$ is a good point for T and D_p. The statement follows. □

Lemma 3.3 *Let $f : S^2 \to S^2$ be a branched covering map that is an admissible map for the Sierpiński carpet $Z \subseteq S^2$, and let $J \subseteq Z$ be a peripheral circle of Z. Then $f^{-1}(J) \cap Z$ is contained in a union of finitely many peripheral circles of Z.*

This implies that if $E \subseteq Z$ is an exceptional set for f and Z, then the set $f^{-1}(E) \cap Z$ is contained in a union of a finite set and finitely many peripheral circles of Z.

Proof Let $A \subseteq Z$ be the union of all peripheral circles of Z. Then A consist precisely of those points in Z that are accessible by a (half-open) path contained in the complement of Z. This characterization of the points in A together with the definition of a good point implies that if $x \in Z$ is a good point for f and Z, then $x \in A$ if and only if $f(x) \in A$.

We also need the following topological fact: if K is a non-degenerate continuum (i.e., a compact connected set consisting of more than one point) and if K meets a point in $Z \backslash A$ or two distinct peripheral circles of Z, then $K \cap (Z \backslash A)$ is an uncountable set. To see this, we collapse the closure of each complementary component of Z to a point. Then by Moore's theorem (see [6, Theorem 13.8]) the quotient space obtained in this way is also a topological 2-sphere. The image K' of K under the quotient map is also a compact and connected set. The assumptions on K imply that K' contains more than one point, and is hence a non-degenerate continuum. Therefore, K' is an uncountable set. In particular, K' contains uncountably many points distinct from the countably many points obtained by collapsing the complementary components of Z. It follows that $K \cap (Z \backslash A)$ is uncountable, as desired.

Now let K be a connected component of $f^{-1}(J)$. Then $f(K) = J$ (this follows from a general fact for open and continuous maps—see [6, Lemma 13.13]; since J is a Jordan curve, one can also give a simple direct argument based on path lifting). Since f is finite-to-one, it follows that there are only finitely many such components K of $f^{-1}(J)$. Each of these components K is a non-degenerate continuum.

Let $x \in Z \backslash A$ be a good point of f and Z. Then $f(x) \in Z \backslash A \subseteq Z \backslash J$ by what we have seen in the beginning of the proof. In particular, $x \notin K \subseteq f^{-1}(J)$. Since every point in $Z \backslash A$ is a good point with finitely many exceptions, the set $K \cap (Z \backslash A)$ is finite. But then actually $K \cap (Z \backslash A) = \emptyset$, because otherwise $K \cap (Z \backslash A)$ would be uncountable. So $K \cap Z \subseteq A$. This implies that $K \cap Z$ is contained in a single peripheral circle of Z (or is empty), because if $K \cap Z$ met two distinct peripheral circles, then $K \cap (Z \backslash A)$ would again be an uncountable set.

We have seen that the intersection of each of the finitely many components of $f^{-1}(J)$ with Z lies in a single peripheral circle of Z. The statement follows. □

Lemma 3.4 *Let $f, g : S^2 \to S^2$ be two branched covering maps that are admissible maps for the Sierpiński carpet $Z \subseteq S^2$. Then $f \circ g$ is also admissible for Z.*

Proof As a composition of two branched covering maps, $h := f \circ g$ is also a branched covering map on S^2. Moreover, we have $h(Z) \subseteq Z$.

Let E be an exceptional set for f (and Z, the relevant Sierpiński carpet for all maps in this proof), and E' be an exceptional set for g. Then by the remark after Lemma 3.3 we know that $f^{-1}(E) \cap Z$ is contained in a union of a finite set and finitely many peripheral circles of Z. The same is then true for $(E' \cup f^{-1}(E)) \cap Z$. So to finish the proof, it is enough to show that each point $x \in Z \backslash (E' \cup f^{-1}(E))$ is a good point for h.

By our assumptions $x \in Z \backslash E'$ is a good point for g, and $y := g(x) \in Z \backslash E$ is a good point for f. By possibly shrinking the regions in the definition of a good point if necessary, we can find Jordan regions $U, V, W \subseteq S^2$ with the following properties: $x \in U$ and $y \in V$, the map g is a quasisymmetry from U onto V, the map f is a quasisymmetry from V onto W, and we have $g(U \cap Z) = V \cap Z$ and $f(V \cap Z) = W \cap Z$. Then $h = f \circ g$ is a quasisymmetry from U onto W and $h(U \cap Z) = W \cap Z$. This show that x is a good point for h, as desired. □

Lemma 3.5 *Let $k, n \in \mathbb{N}_0$ and $\xi : P \to P$ be a quasisymmetry with $\xi(D_p) = D_p$. Then the map $f := \xi^{-1} \circ T^n \circ \xi \circ T^k$ is admissible for D_p.*

Note that if the homeomorphism ξ reverses orientation, then it is not a branched covering map according to the definition given in [6, Section 2.1]. Conjugation by ξ still preserves the class of branched covering maps.

Proof It is clear that f is a branched covering map with $f(D_p) \subseteq D_p$. Moreover, it follows from Lemma 3.2 and repeated application of Lemma 3.4 that the maps T^n and T^k are admissible for D_p. It is also clear that conjugation of T^n by ξ leads to a branched covering map $\xi^{-1} \circ T^n \circ \xi$ that is admissible for D_p, because ξ induces a bijection on the peripheral circles of D_p. The statement now follows from another application of Lemma 3.4. □

4 Schottky maps

A *relative Schottky set* S in a region $D \subseteq \widehat{\mathbb{C}}$ is a subset of D whose complement in D is a union of open geometric disks whose closures are contained in D and are pairwise disjoint. The boundaries of these disks are called the *peripheral circles* of S. A relative Schottky set in $D = \widehat{\mathbb{C}}$ is called a *Schottky set*.

Let S be a relative Schottky set and $U \subseteq \widehat{\mathbb{C}}$ be an open set. A map $f : U \cap S \to \widehat{\mathbb{C}}$ is called *conformal* at a point $z_0 \in U \cap S$ if the *derivative* of f at z_0,

$$f'(z_0) = \lim_{z \in U \cap S,\, z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists and is non-zero. If $z_0 = \infty$ or $f(z_0) = \infty$, one has to interpret this in suitable charts on $\widehat{\mathbb{C}}$. In order to avoid this technicality, in the following we will only consider relative Schottky sets S that do not contain ∞ and so $S \subseteq \mathbb{C}$.

Let $S, \widetilde{S} \subseteq \mathbb{C}$ be two relative Schottky sets, $U \subseteq \widehat{\mathbb{C}}$ be an open set, and $f : U \cap S \to \widetilde{S}$ be a local homeomorphism. Such a map f is called a *Schottky map* if it is conformal at every point of $U \cap S$ and its derivative is a continuous function on $U \cap S$.

Under some mild additional assumptions quasisymmetries on relative Schottky sets are Schottky maps. More precisely, the following statement is true.

Theorem 4.1 *Let $S \subseteq \mathbb{C}$ be a relative Schottky set of measure zero in a region $D \subseteq \widehat{\mathbb{C}}$. Suppose $U \subseteq \widehat{\mathbb{C}}$ is open and $f : U \to \widehat{\mathbb{C}}$ is a continuous map with $f(U \cap S) \subseteq S$ such that* *each point $x \in U \cap S$ is a good point for f and S. Then $f|U \cap S\colon U \cap S \to S$ is a Schottky map.*

Proof A special case of this statement immediately follows from [8, Theorem 1.2]. Namely, if $U \subseteq \mathbb{C}$ is a Jordan region with partial $\partial U \subseteq S$ and f is an orientation-preserving quasisymmetry from U onto $f(U)$ with $f(U \cap S) = f(U) \cap S$, then $f|U \cap S\colon U \cap S \to S$ is a Schottky map.

In the general case, it is enough to show that $f|U \cap S$ is a Schottky map locally near each point $x \in U \cap S$. We can reduce this to the special case, because x is a good point for f and S. The details of the argument are very similar to the proof of Lemma 6.1 in [4] and so we will only give an outline.

By our assumptions for each $x \in U \cap S$ we can find Jordan regions $V, W \subseteq \widehat{\mathbb{C}}$ with $x \in V \subseteq U$ such that $f|V$ is an orientation-preserving quasisymmetry of V onto W with $f(V \cap S) = W \cap S$. We would be done if $\partial V \subseteq S$.

Now, if x does not lie on a peripheral circle, then one can shrink V suitably so that $\partial V \subseteq S$ (see the proof of Lemma 6.1 in [4] for the details).

For the remaining case, suppose x lies on a peripheral circle of S. Then $x \in \partial B \subseteq S$, where B is one of the complementary disks of S in D. Then one doubles the Schottky set S by reflection in $C = \partial B$ to obtain a new Schottky set $\widetilde{S}$ that does not have C as a peripheral circle. By a Schwarz reflection procedure one modifies the map f in B to obtain a map $\widetilde{f}$ that agrees with f in the complement of B near x. One can then find Jordan regions $V, W \subseteq \widehat{\mathbb{C}}$ such that $x \in V$, $\partial V \subseteq \widetilde{S}$, and $\widetilde{f}$ is an orientation-preserving quasisymmetry from V onto W with $\widetilde{f}(V \cap \widetilde{S}) = W \cap \widetilde{S}$. This implies that $\widetilde{f}$ is a Schottky map $V \cap \widetilde{S} \to \widetilde{S}$. By construction f and $\widetilde{f}$ agree on $V \cap S = (V \cap \widetilde{S}) \backslash B$ and map this set into S. Hence $f|U \cap S$ is a Schottky map into S near $x \in V \cap S$, as desired. □

We require the following stabilization result.

Theorem 4.2 *Let $S \subseteq \mathbb{C}$ be a locally porous relative Schottky set, $a \in S$, $U \subseteq \widehat{\mathbb{C}}$ be an open neighborhood of a such that $U \cap S$ is connected, and $u\colon U \cap S \to S$ be a Schottky map with $u(a) = a$ that is not equal to the identity on $U \cap S$. For $n \in \mathbb{N}$ let $h_n\colon U \cap S \to S$ be a Schottky map such that for some open set $U_n \subseteq \widehat{\mathbb{C}}$ the map $h_n\colon U \cap S \to U_n \cap S$ is a homeomorphism.*

Suppose the sequence $\{h_n\}$ converges locally uniformly on $U \cap S$ to a homeomorphism $h\colon U \cap S \to \widetilde{U} \cap S$, where $\widetilde{U} \subseteq \widehat{\mathbb{C}}$ is an open set. Then there exists $N \in \mathbb{N}$ such that $h_n = h$ on $U \cap S$ for all $n \geq N$.

This is a version of [9, Theorem 5.2] formulated in a way that will be convenient for our applications. Note that our assumption on u implies $u'(a) \neq 1$ by [9, Theorem 4.1]. The existence of such a map u is a strong requirement on the geometry of the relative Schottky set S and intuitively says that S admits a non-trivial "self-similarity" u locally near a.

It does not make a difference whether one allows the open sets $U, U_n, \widetilde{U}$ to contain the point $\infty \in \widehat{\mathbb{C}}$ (as in our formulation) or requires them to be subsets of $\mathbb{C}$ (as in [9]), because $\infty \notin S$ and so we can always delete ∞ from these open sets.

We refer the reader to [9] for the definition of local porosity. It is easy to check that the condition of local porosity is satisfied by S_p and D_p and is invariant under quasisymmetric maps.

We also need the following uniqueness result [9, Corollary 4.2].

Theorem 4.3 *Let $S \subseteq \mathbb{C}$ be a locally porous relative Schottky set, and $U \subseteq \widehat{\mathbb{C}}$ be an open set such that $U \cap S$ is connected. Suppose $f, g\colon U \cap S \to S$ are Schottky maps, and consider*

$$A := \{x \in U \cap S\colon f(x) = g(x)\}.$$

If A has a limit point in $U \cap S$, then $A = U \cap S$ and so $f = g$.

We can apply these results in our context due to the following fact.

Lemma 4.4 *There exists a quasisymmetry $\beta\colon P \to \widehat{\mathbb{C}}$ such that $S := \beta(D_p)$ is a locally porous Schottky set contained in $\mathbb{C}$ and $U = \beta(\operatorname{int}(Q))$ is a bounded Jordan region in $\mathbb{C}$ such that $U \cap S$ is connected. Moreover, there exist a point $a \in U \cap S$ and a Schottky map $u\colon U \cap S \to S$ such that $u(a) = a$ and u is not the identity on $U \cap S$.*

Proof We use the following uniformization theorem proved in [3] (where the terminology is also explained): if $Z \subseteq \widehat{\mathbb{C}}$ is a Sierpiński carpet whose peripheral circles are uniformly relatively separated uniform quasicircles, then there exists a quasisymmetry $\beta\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\beta(Z) \subseteq \widehat{\mathbb{C}}$ is a *round* Sierpiński carpet, i.e., a Sierpiński carpet whose peripheral circles are geometric circles (see [3, Corollary 1.2]). Since P is bi-Lipschitz equivalent to $\widehat{\mathbb{C}}$ and our Sierpiński carpet D_p has peripheral circles that are uniform quasicircles and are uniformly relatively separated, we can apply this statement and obtain a quasisymmetry $\beta\colon P \to \widehat{\mathbb{C}}$ such that $S := \beta(D_p) \subseteq \widehat{\mathbb{C}}$ is a round Sierpiński carpet. By postcomposing β with a Möbius transformation if necessary, we may assume that β is orientation-preserving, $S \subseteq \mathbb{C}$ and that $U = \beta(\operatorname{int}(Q))$ is a bounded Jordan region in $\mathbb{C}$. Then S is a Schottky set. It is locally porous, because D_p is a locally porous subset of P and this property is preserved under quasisymmetries. In particular, S is a set of measure zero.

The set

$$U \cap S = \beta(\operatorname{int}(Q)) \cap \beta(D_p) = \beta(\operatorname{int}(Q) \cap D_p) = \beta(S_p \backslash O)$$

is connected as a continuous image of the connected set $S_p \backslash O$.

To find a point a and a map u with the desired properties, we consider $\sigma = (1/(p+1), 1/(p+1)) \in Q \subseteq P$. Then $\sigma \in S_p \backslash O$. Indeed, the identity

$$\frac{1}{p+1} = \sum_{k=0}^{\infty} \frac{p-1}{p^{2(k+1)}}$$

shows that the p-ary expansion of $1/(p+1)$ has only coefficients 0 and $p-1$, and thus σ belongs to the direct product $C_p \times C_p$, where C_p is a Cantor set constructed similarly to the standard Cantor set, but instead of subdividing $[0, 1]$ into three equal parts, we subdivide it into p equal parts, remove the interior of the middle part (which is well defined because p is assumed to be odd), and continue in the usual self-similar way. Now $C_p \times C_p$ is a subset of S_p, which implies that $\sigma \in S_p$. Clearly, σ does not belong to O, and so $\sigma \in S_p \backslash O$.

Actually, σ is contained in the interior of the 2-tile $X := [(p-1)/p^2, 1/p]^2$. This is a good 2-tile and T^2 is an orientation-preserving quasisymmetry from $\operatorname{int}(X)$ onto $\operatorname{int}(Q)$ with $T^2(\operatorname{int}(X) \cap D_p) = \operatorname{int}(Q) \cap D_p = S_p \backslash O$. The inverse map is an orientation-preserving quasisymmetry $v\colon \operatorname{int}(Q) \to \operatorname{int}(X)$ with $v(S_p \backslash O) = \operatorname{int}(X) \cap D_p$. We have $T^2(\sigma) = \sigma$ (essentially, this follows from $p^2/(p+1) \equiv 1/(p+1) \mod 2$), and so $v(\sigma) = \sigma$.

We now define $a := \beta(\sigma) \in \beta(S_p \backslash O) = U \cap S$, and consider the map $\widetilde{u} := \beta \circ v \circ \beta^{-1}$ defined on $U = \beta(\operatorname{int}(Q))$. Then $\widetilde{u}$ is an orientation-preserving quasisymmetry of U onto the open set $\widetilde{u}(U) = \beta(\operatorname{int}(X))$ with

$$\widetilde{u}(U \cap S) = \widetilde{u}(\beta(S_p \backslash O)) = \beta(\operatorname{int}(X) \cap D_p) = \beta(\operatorname{int}(X)) \cap S = \widetilde{u}(U) \cap S.$$

Theorem 4.1 implies that $u := \widetilde{u}|U \cap S$ is a Schottky map $u\colon U \cap S \to S$. Moreover, $u(a) = a$ and u is not the identity on S. □

Corollary 4.5 *Let $\beta\colon P \to \widehat{\mathbb{C}}$ be the quasisymmetry from Lemma 4.4 with $S = \beta(D_p)$, and $f, g\colon P \to P$ be admissible maps for D_p. Define*

$$\widetilde{f} := \beta \circ f \circ \beta^{-1}, \quad \widetilde{g} := \beta \circ g \circ \beta^{-1}.$$

Then there exists a region $U \subseteq \widehat{\mathbb{C}}$ such that $U \cap S$ is a connected set that is dense in S and $\widetilde{f}, \widetilde{g}\colon U \cap S \to S$ are Schottky maps.

So if we conjugate the admissible maps f and g for D_p by the uniformizing map β, then we obtain Schottky maps at least on the large part $U \cap S$ of S.

Proof Since f and g are admissible for D_p, the maps $\widetilde{f}$ and $\widetilde{g}$ are admissible for the Sierpiński carpet $S = \beta(D_p) \subseteq \mathbb{C}$. This implies that there exist a finite set $F \subseteq S$ and finitely many peripheral circles $J_1, \dots, J_N$ of S such that $E := F \cup J_1 \cup \dots \cup J_N$ is an exceptional set for $\widetilde{f}$ and for $\widetilde{g}$. Let $D_1, \dots, D_N$ be the closures of the complementary components of S (in $\widehat{\mathbb{C}}$) bounded by $J_1, \dots, J_N$, respectively. Since $S \subseteq \mathbb{C}$, we may assume that $\infty \in D_1$. Then

$$U := \widehat{\mathbb{C}} \backslash (F \cup D_1 \cup \dots \cup D_N)$$

is a region in $\mathbb{C}$. The set $U \cap S = S \backslash E$ is connected and dense in S (the quickest way to see this is again by an argument as in the proof of Lemma 3.3 based on Moore's theorem—we leave the details to the reader). Note that $\widetilde{f}(S), \widetilde{g}(S) \subseteq S$ and each point in $U \cap S = S \backslash E$ is a good point for the maps $\widetilde{f}$ and $\widetilde{g}$ and the set S. Theorem 4.1 implies that $\widetilde{f}$ and $\widetilde{g}$ are Schottky maps $U \cap S \to S$. □

Corollary 4.6 *Let $f, g\colon P \to P$ be admissible maps for D_p. If there exists a non-empty set $A \subseteq D_p$ that is relatively open in D_p such that $f = g$ on A, then $f = g$ on D_p.*

Proof If β is the map from Lemma 4.4 and U is as in Corollary 4.5, then $A' := U \cap \beta(A)$ is a non-empty and relatively open set in $U \cap S$, where $S = \beta(D_p)$. In particular, A' has a limit point in $U \cap S$, and the Schottky maps $\widetilde{f}, \widetilde{g}\colon U \cap S \to S$ as defined in Corollary 4.5 agree on A'. It follows from Theorem 4.3 that $\widetilde{f}$ and $\widetilde{g}$ agree on $U \cap S$ and hence on S, because $U \cap S$ is dense in S. Thus $f = g$ on $\beta^{-1}(S) = D_p$. □

5 Relation to Lattès maps

We now want to prove a crucial relation between an arbitrary quasisymmetry $\xi\colon D_p \to D_p$ and our Lattès map T (recall that the odd integer $p \geq 3$ is fixed).

Proposition 5.1 *Let $\xi\colon D_p \to D_p$ be a quasisymmetry. Then for each $N \in \mathbb{N}$ there exist $k, \ell, m \in \mathbb{N}$ with $k, \ell, m \geq N$ such that*

$$T^m \circ \xi = T^\ell \circ \xi \circ T^k \tag{5.1}$$

on D_p.

In other words, (5.1) holds with arbitrarily large k, ℓ, and m. The proof of this proposition will occupy the rest of this section. The main ideas for establishing the relation (5.1) are related to those for the proof of the similar relation (1.2) in [4].

Let $\xi\colon D_p \to D_p$ be the given quasisymmetry. Then it has a (non-unique) extension to a quasisymmetry $\xi\colon P \to P$. This follows from [3, Proposition 5.1] (see also [4, Theorem 1.11]); here it is important that P is bi-Lipschitz equivalent to $\widehat{\mathbb{C}}$ equipped with the chordal metric and that every quasiconformal map $F\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasisymmetry.

In order to prove (5.1), we may assume that this extension ξ is orientation-preserving, because otherwise we consider the homeomorphism $\widetilde{\xi}\colon P \to P$ given by $\widetilde{\xi} = R \circ \xi$, where $R\colon P \to P$ is the involution on the pillow P that interchanges corresponding points on the front and back of P. Since R is an orientation-reversing isometry on P with $R(D_p) = D_p$, the map $\widetilde{\xi}$ is also a quasisymmetry on P with $\widetilde{\xi}(D_p) = D_p$ and it will be orientation-preserving if ξ reverses orientation. Moreover, if we have a relation as in (5.1) for $\widetilde{\xi}$, then a corresponding relation for ξ with the same numbers k, ℓ, m immediately follows from the identity $R \circ T = T \circ R$. So in the following we may assume that the extension $\xi\colon P \to P$ of the quasisymmetry in Proposition 5.1 is orientation-preserving.

We consider the point $c = (0, 0) \in S_p \subseteq D_p$ (i.e., the lower left corner of D_p). We choose a nested sequence of tiles X_n, $n \in \mathbb{N}$, of strictly increasing levels $k_n \in \mathbb{N}_0$ with $X_n \subseteq Q$ and $c \in X_n$. Then each X_n is a good tile. There exists a unique branch T^{-k_n} on Q such that $T^{-k_n}(Q) = X_n$. These branches T^{-k_n} are *consistent* in the sense that

$$T^{-k_n} = T^{k_{n+1}-k_n} \circ T^{-k_{n+1}} \tag{5.2}$$

for $n \in \mathbb{N}$. This consistency relation is preserved if we replace the original sequence of tiles $\{X_n\}$ (and the corresponding sequence of branches $\{T^{-k_n}\}$) by a subsequence as we will do in the ensuing argument.

If k_1 is sufficiently large, as we may assume, Lemma 2.2 guarantees that for each $n \in \mathbb{N}$ we can find a good $(k_n + r_0)$-tile $Y_n \subseteq X_n$ and a good tile Z_n with $\xi(Y_n) \subseteq Z_n$ and $\operatorname{diam}(\xi(Y_n)) \asymp \operatorname{diam}(Z_n)$. Here $r_0 \in \mathbb{N}_0$ and the comparability constant are independent of n. For each $n \in \mathbb{N}$, let $Y_n' := T^{k_n}(Y_n)$. Then $Y_n' \subseteq Q$ is a good tile of level r_0 such that $T^{-k_n}(Y_n') = Y_n$. Since there are only finitely many r_0-tiles, we may assume, by passing to a subsequence of the original sequence $\{X_n\}$ if necessary, that the tiles Y_n' are all equal to the same good r_0-tile Y. We can find an orientation-preserving scaling map $\varphi\colon Q \to Y$ that maps Q onto Y.

Let $\ell_n \in \mathbb{N}_0$ be the level of Z_n. Then $T^{\ell_n}|Z_n$ is a scaling map on Z_n that sends Z_n to the front Q or the back Q' of P depending on whether Z_n is white or black. Since

$$\operatorname{diam}(\xi(Y_n)) \asymp \operatorname{diam}(Z_n),$$

we then have

$$\operatorname{diam}\left(T^{\ell_n}(\xi(Y_n))\right) \asymp \operatorname{diam}(T^{\ell_n}(Z_n)) \asymp 1,$$

and so $T^{\ell_n}(\xi(Y_n))$ has *uniformly large size*, i.e., there exists $\alpha > 0$ such that $\operatorname{diam}(T^{\ell_n}(\xi(Y_n)) \geq \alpha$ for all $n \in \mathbb{N}$.

Putting this all together, for each $n \in \mathbb{N}$ we obtain a map

$$h_n := T^{\ell_n} \circ \xi \circ T^{-k_n} \circ \varphi. \tag{5.3}$$

This is a quasisymmetric embedding of Q into P with uniformly large image $M_n := T^{\ell_n}(\xi(Y_n))$. Since the maps T^{ℓ_n}, T^{-k_n}, and φ just scale distances, the maps h_n are *uniformly* quasisymmetric embeddings, i.e., there exists a distortion function η such that $h_n\colon Q \to M_n$ is an η-quasisymmetry for each $n \in \mathbb{N}$. Note that each of the four maps on the right hand side of (5.3) has the property that a point in the source space of the map lies in D_p if and only if its image point lies in D_p. This implies that for $z \in Q$ we have

$$z \in D_p \text{ if and only if } h_n(z) \in D_p. \tag{5.4}$$

We now invoke the following subconvergence lemma which follows from [3, Lemma 3.3].

Lemma 5.2 *Let X and Y be compact metric spaces, and let $h_n\colon X \to Y$ be an η-quasisymmetric embedding for $n \in \mathbb{N}$. Suppose that there exists a constant $\alpha > 0$ such that for the diameter of the image set $h_n(X)$ we have $\operatorname{diam}(h_n(X)) \geq \alpha$ for each $n \in \mathbb{N}$. Then there exist an increasing sequence $\{i_n\}$ in $\mathbb{N}$ and a quasisymmetric embedding $h\colon X \to Y$ such that $h_{i_n} \to h$ uniformly on X as $n \to \infty$.*

In our situation Lemma 5.2 implies that by passing to a subsequence if necessary, we may assume that $h_n \to h$ uniformly on Q, where $h\colon Q \to P$ is a quasisymmetric embedding.

We claim that the relation (5.4) passes to the limit, i.e., for $z \in Q$ we have

$$z \in D_p \text{ if and only if } h(z) \in D_p. \tag{5.5}$$

Indeed, if $z \in Q \cap D_p$, then $h_n(z) \in D_p$ for each $n \in \mathbb{N}$ by (5.4). Moreover, $h_n(z) \to h(z)$ as $n \to \infty$ and so $h(z) \in D_p$, because D_p is a closed subset of P.

For the other implication, we argue by contradiction and assume that $z \in Q \backslash D_p$, but $h(z) \in D_p$. Since $O = \partial Q \subseteq D_p$, the point z lies in the interior of Q. Then we can find a small neighborhood W of z with $W \subseteq Q \backslash D_p$. Since $h_n \to h$ uniformly on W, a topological degree argument implies that for sufficiently large n there exists $z_n \in W$ such that $h_n(z_n) = h(z) \in D_p$. Then $z_n \in D_p$ by (5.4). This is a contradiction, because $z_n \in W \subseteq P \backslash D_p$. Relation (5.5) follows.

Let $n \in \mathbb{N}$. Then for one of the two open Jordan regions $\Omega_n \subseteq P$ bounded by $h_n(O)$ the map h_n is a quasisymmetry of $Q \backslash O = \operatorname{int}(Q)$ onto Ω_n. By (5.4) this implies that $h_n(\operatorname{int}(Q) \cap D_p) = \Omega_n \cap D_p$.

Similarly, there exists a Jordan region $\Omega \subseteq P$ such that h is a quasisymmetry of $\operatorname{int}(Q)$ onto Ω. By (5.5) we then have $h(\operatorname{int}(Q) \cap D_p) = \Omega \cap D_p$.

We are now in a situation that is very similar to what was established in Step III in the proof of Theorem 1.4 in [4]: we want to show that the sequence $\{h_n\}$ stabilizes and $h_n \equiv h$ for large n. As in [4], we will invoke rigidity statements for Schottky maps.

Let β be the map provided by Lemma 4.4, and, as in this lemma, let $U = \beta(\operatorname{int}(Q))$ and $S = \beta(D_p)$. Then $S \subseteq \mathbb{C}$ is a locally porous Schottky set. For each $n \in \mathbb{N}$ the map

$$\widetilde{h}_n := \beta \circ h_n \circ \beta^{-1}$$

is an orientation-preserving quasisymmetry of U onto $U_n := \beta(\Omega_n)$ such that

$$\widetilde{h}_n(U \cap S) = \beta(h_n(\operatorname{int}(Q) \cap D_p)) = \beta(\Omega_n \cap D_p) = U_n \cap S.$$

It follows from Theorem 4.1 that $\widetilde{h}_n\colon U \cap S \to S$ is a Schottky map, and a homeomorphism of $U \cap S$ onto $U_n \cap S$.

By the same reasoning the map $\widetilde{h} := \beta \circ h \circ \beta^{-1}$ is also a Schottky map $U \cap S \to S$, and a homeomorphism of $U \cap S$ onto $\widetilde{U} \cap S$, where $\widetilde{U} = \beta(\Omega)$. Moreover, we have $\widetilde{h}_n \to \widetilde{h}$ locally uniformly on $U \cap S$. Lemma 4.4 and Theorem 4.2 imply that there exists $N' \in \mathbb{N}$ such that $\widetilde{h}_n = \widetilde{h}_{n+1}$ on $U \cap S = \beta(S_p \backslash O)$, and so $h_n = h_{n+1}$ on $S_p \backslash O$ for all $n \geq N'$.

For such an n we then have

$$T^{\ell_{n+1}} \circ \xi \circ T^{-k_{n+1}} \circ \varphi = T^{\ell_n} \circ \xi \circ T^{-k_n} \circ \varphi \tag{5.6}$$

on $S_p \backslash O$, and hence on S_p by continuity. Since φ is a homeomorphism of S_p onto $Y \cap D_p$, this leads to

$$T^{\ell_{n+1}} \circ \xi \circ T^{-k_{n+1}} = T^{\ell_n} \circ \xi \circ T^{-k_n}$$

on $Y \cap D_p$. By the consistency relation (5.2) this gives

$$T^{\ell_{n+1}} \circ \xi \circ T^{-k_{n+1}} = T^{\ell_n} \circ \xi \circ T^{k_{n+1}-k_n} \circ T^{-k_{n+1}}$$

on $Y \cap D_p$, and so

$$\xi^{-1} \circ T^{\ell_{n+1}} \circ \xi = \xi^{-1} \circ T^{\ell_n} \circ \xi \circ T^{k_{n+1}-k_n} \tag{5.7}$$

on $T^{-k_{n+1}}(Y \cap D_p) = Y_{n+1} \cap D_p$.

Since Y_{n+1} is a good tile, the set $Y_{n+1} \cap D_p$ is a subset of D_p with non-empty relative interior. We can now apply Lemma 3.5 and Corollary 4.6 to conclude that (5.7) is true on the whole set D_p. Here we can cancel ξ^{-1}. Since $k_{n+1} > k_n$ by our initial choice of the tiles X_n, it follows that there exist numbers $k \in \mathbb{N}$ and $\ell, m \in \mathbb{N}_0$ such that

$$T^m \circ \xi = T^\ell \circ \xi \circ T^k \tag{5.8}$$

on D_p. We can make m and ℓ arbitrarily large by postcomposing both sides in this identity with iterates of T. Moreover, using this equation for a large enough multiple of the original ℓ in (5.8) and precomposing with iterates of T^k, we can also make k in (5.8) arbitrarily large. So for each $N \in \mathbb{N}$ we can find $k, \ell, m \geq N$ such that (5.8) holds. This shows that Proposition 5.1 is indeed true.

6 Proof of Theorem 1.1

We first state two relevant results from [5]. The following statement is part of [5, Lemma 8.1].

Theorem 6.1 *Every quasisymmetry $\xi : S_p \to S_p$, $p \geq 3$ odd, preserves the outer peripheral circle O setwise and so $\xi(O) = O$.*

We need the following special case of [5, Theorem 1.4].

Theorem 6.2 *Let $\xi : S_p \to S_p$, $p \geq 3$ odd, be an orientation-preserving quasisymmetry that fixes the four corners of Q. Then ξ is the identity on S_p.*

Here $\xi : S_p \to S_p$ is *orientation-preserving* or *orientation-reserving* if it has an extension to a homeomorphism on $\widehat{\mathbb{C}} \supseteq S_p$ with the same property. Every quasisymmetry $\xi : S_p \to S_p$ is either orientation-preserving or orientation-reversing.

We can now prove our first main result.

Proof of Theorem 1.1 Let $\xi : S_p \to S_p$ be a quasisymmetry. In order to show that ξ is an isometry, we can freely pre- or postcompose ξ with isometries of S_p without affecting our desired conclusion.

Without loss of generality we may assume that ξ is orientation-preserving; otherwise, we consider the composition of ξ with a reflection of S_p in, say, one of the diagonals of the square Q.

By Theorem 6.1 we know that $\xi(O) = O$ and so ξ restricts to an orientation-preserving homeomorphism on O. If ξ send each corner of Q to another corner of Q, then ξ has to preserve the cyclic order of these corners. This implies that on the set of corners ξ acts as a rotation by an integer multiple of $\pi/2$ around the center of Q. It follows that we can postcompose ξ with such a rotation of S_p so that the new quasisymmetry actually fixes the corners of Q. By Theorem 6.2 this map is then the identity on S_p and we conclude that ξ is an isometry as desired.

So we are reduced to the case where ξ sends a corner of Q to a non-corner point. Equivalently, the preimage of a corner of Q under ξ is not a corner point. Again, by using rotations of S_p, we may assume that the preimage $q = \xi^{-1}(c) \in O$ of the lower left corner $c = (0, 0) \in \mathbb{R}^2$ of S_p is not a corner.

If q does not lie in the open bottom side $(0, 1) \times \{0\}$ of Q, we can precompose ξ with two reflections R_1, R_2 in appropriate symmetry lines of Q (i.e., diagonals and lines through the centers of two opposite sides of Q) so that ξ is still orientation-preserving and

$$q = (R_1 \circ R_2 \circ \xi^{-1})(c) \in (0, 1) \times \{0\} \subseteq S_p.$$

Based on these considerations, we reduced to the case that $q = \xi^{-1}(c) \in (0, 1) \times \{0\} \subseteq S_p$. We want to derive a contradiction from this statement, which will establish the theorem.

Let R be the involution on P that interchanges corresponding points in the two copies of Q. Since ξ preserves the outer peripheral circle O of S_p, it induces a homeomorphism of D_p that agrees with ξ on the front copy of S_p and is given by $R \circ \xi \circ R$ on the back copy S'_p of S_p in D_p. We continue to denote this map on D_p by ξ. It is clear that $\xi\colon D_p \to D_p$ is a homeomorphism.

If D_p is, as before, endowed with the restriction of the path metric on the pillow P, then the map $\xi\colon D_p \to D_p$ is actually a quasisymmetry. To see this, first note that the original map ξ on S_p extends to a quasiconformal homeomorphism of the unit square Q (see, e.g., [3, Proposition 5.1]). By using this and the reflection R, we can find a quasiconformal homeomorphism on P that extends the homeomorphism $\xi\colon D_p \to D_p$. We call this new map on P also ξ. Now P is bi-Lipschitz equivalent to $\widehat{\mathbb{C}}$. If we conjugate $\xi\colon P \to P$ by such a bi-Lipschitz map, then we obtain a quasiconformal homeomorphism $\widetilde{\xi}$ on $\widehat{\mathbb{C}}$. Each quasiconformal map on $\widehat{\mathbb{C}}$ is a quasisymmetry. It follows that $\widetilde{\xi}$ is a quasisymmetry. Hence its bi-Lipschitz conjugate $\xi\colon P \to P$ is also a quasisymmetry, and so is the restriction $\xi\colon D_p \to D_p$.

Let $T\colon P \to P$ be the Lattès map defined in Sect. 2. Then it follows from Proposition 5.1 that we have a relation of the form

$$T^m \circ \xi = T^\ell \circ \xi \circ T^k \tag{6.1}$$

on D_p, where we may assume that $k, \ell, m \in \mathbb{N}$ are suitably large.

Let $I \subseteq (0, 1) \times \{0\} \subseteq S_p$ be a small open interval one of whose endpoints is q, such that $\xi(I)$ is completely contained in one of the sides of Q. Recall that $\xi(q) = c = (0, 0)$. Each side of Q is forward invariant under T (see (2.1)). For a large enough k we actually have $T^k(I) = [0, 1] \times \{0\}$, because, roughly speaking, T expands by the factor p. We may assume that (6.1) holds for this k. Since

$$(0, 0) = \xi(q) \in \xi\big((0, 1) \times \{0\}\big),$$

the set $A := \xi\big([0, 1] \times \{0\}\big)$ meets the interior of at least two sides of Q. It follows that

$$(T^\ell \circ \xi \circ T^k)(I) = (T^\ell \circ \xi)\big([0, 1] \times \{0\}\big) = T^\ell(A)$$

meets the interior of at least two sides of Q.

On the other hand, since $\xi(I)$ is contained in one side of Q and T leaves each side of Q invariant, we conclude from (6.1) that $T^\ell(A) = (T^m \circ \xi)(I)$ is contained in one side of Q. Therefore, $T^\ell(A)$ cannot meet the interior of two different sides of Q. This is a contradiction. □

7 Proofs of Theorems 1.2 and 1.3

For the proof of Theorem 1.2 we require auxiliary results about certain weak tangent spaces at points of D_p. We will discuss these statements first. For a more detailed treatment of similar weak tangents for S_p the reader may consult [5, Section 7].

Let (X, d) be a metric space. Then a *weak tangent* of X at a point $a \in X$ is the Gromov–Hausdorff limit of a sequence of pointed metric spaces $(X, a, \lambda_n d)$, where $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ (for the relevant definitions and general background see [7, Chapters 7 and 8]). We are interested in weak tangents of subsets of D_p equipped with the restriction of the piecewise Euclidean metric on the pillow P. In this case, it is convenient to restrict the scaling factors λ_n in the definition of weak tangents to powers of p, i.e., they have the form $\lambda_n = p^{k_n}$ with $k_n \in \mathbb{N}_0$ and $k_n \to \infty$ as $n \to \infty$.

For example, let $c = (0, 0) \in S_p$ be the lower left corner of Q. Then S_p has an essentially unique weak tangent at c isometric to the union

$$W := \bigcup_{n \in \mathbb{N}_0} p^n S_p \subseteq \mathbb{C} \tag{7.1}$$

with base point $0 \in \mathbb{C}$. In this union we consider S_p as a subset of $\mathbb{R}^2 \cong \mathbb{C}$ and use the notation $\lambda A = \{\lambda z : z \in A\}$ whenever $\lambda \in \mathbb{C}$ and $A \subseteq \mathbb{C}$. In particular, W is a subset of the first quadrant of $\mathbb{R}^2 \cong \mathbb{C}$.

Let $q = \frac{1}{2p}(p-1, p-1) \in S_p$. Then q is the lower left corner of the middle peripheral circle M of S_p. At q the set S_p has an essentially unique weak tangent, denoted by $\widetilde{W}$. It is obtained as the union of three copies of W. Up to isometry, we have

$$\widetilde{W} := iW \cup (-i)W \cup (-1)W, \tag{7.2}$$

with base point $0 \in \mathbb{C}$. Of course, W and $\widetilde{W}$ depend on p, but we suppress this from our notation.

Standard compactness arguments imply that a quasisymmetric map of D_p that takes a point $a \in D_p$ to another point $b \in D_p$ induces a quasisymmetric map between appropriate weak tangents at a and b, respectively. This observation along with the following lemma will help us to eliminate certain mapping possibilities.

Lemma 7.1 *There is no quasisymmetry from W onto $\widetilde{W}$ that fixes* 0.

This follows from [5, Proposition 7.3]. Note that in [5] the setup is slightly different, because weak tangents were considered as Hausdorff limits of sets in $\widehat{\mathbb{C}}$ under blow-ups by scaling maps. This is equivalent to our definition with the only difference that our weak tangents do not contain the point $\infty \in \widehat{\mathbb{C}}$ as in [5].

***Proof of Theorem* 1.2** This immediately reduces to Theorem 1.1 if we can show that $\xi(O) = O$, where as before O denotes common boundary of the two copies Q and Q' of the unit square that form the pillow P. In the following, we will rely on some mapping properties of ξ and the Lattès map T (as defined in Sect. 2) that we state first.

Since ξ is a quasisymmetry on D_p and hence a homeomorphism, it maps peripheral circles of D_p to peripheral circles. Note that O does not meet any peripheral circle of D_p. It follows from (2.2) and (2.3) that if C is a peripheral circle of D_p and $n \in \mathbb{N}_0$, then either $T^n(C)$ is also a peripheral circle or $T^n(C) = O$.

If $z \in D_p$ lies on a peripheral circle, then each preimage $z' \in D_p$ of z under any iterate of T also lies on a peripheral circle. To see this, suppose $z' \in D_p \cap T^{-n}(z)$ for some $n \in \mathbb{N}$. Then z' lies in a good tile X^n of level n. On X^n the iterate T^n behaves like a similarity map scaling distances by the factor p^n, and sends the set $D_p \cap X^n$ onto $S_p \subseteq D_p$ or onto the back copy $S'_p \subseteq D_p$ depending on whether X^n is white or black. This implies that if $z = T^n(z')$ lies on a peripheral circle of D_p, then the same is true for the point z'.

By Proposition 5.1 we have an identity of the form

$$T^m \circ \xi = T^n \circ \xi \circ T^k \tag{7.3}$$

on D_p, where $k, n, m \in \mathbb{N}$. Here we may assume that k, n, m are as large as we wish. Then

$$T^m(\xi(M)) = T^n(\xi(O)),$$

because $T^k(M) = O$ for any $k \in \mathbb{N}$ as follows from (2.3) and (2.2).

The set $\xi(O) \subseteq D_p$ does not meet any peripheral circle of D_p. This and the mapping property of T discussed above imply that the set $T^n(\xi(O)) = T^m(\xi(M))$ does not meet any peripheral circle of D_p either. But $\xi(M)$ is a peripheral circle, and so $T^m(\xi(M)) = O = T^n(\xi(O))$. It follows that $\xi(O) \subseteq D_p \cap T^{-n}(O)$. Note that $T^{-n}(O)$ forms a "grid" in P consisting of all n-edges.

If $p = 3$, then it is not hard to see that the only Jordan curve (such as $\xi(O)$) that is contained in $D_p \cap T^{-n}(O)$ and does not meet any peripheral circle is equal to O. In this case, we conclude that $\xi(O) = O$, as desired. To give an argument that is valid for any odd $p \geq 3$, more work is required.

Let $C := \xi(O) \subseteq D_p \cap T^{-n}(O)$. Then C can be considered as a polygonal loop consisting of n-edges. If we run through C according to some orientation, then two successive n-edges in C have an endpoint a in common, where they meet at a right angle or at the angle π. If they meet at a right angle, then we call a a *turn* of C.

Claim. Every turn a of C must have one of the four corners of O as a preimage under ξ.

To see this, we argue by contradiction and assume that there exists $b \in O$ that is not a corner of O such that $a = \xi(b)$ is a turn of C. Let I be a small open interval contained in O, one of whose endpoints is b. Here we may assume that I is contained in one side of O and that $\xi(I)$ is contained in one of the two n-edges $e \subseteq \xi(O)$ that meet at the turn a. Note that then $\xi(I) \subseteq e$ is an open interval, one of whose endpoints is $a = \xi(b)$.

We now use an identity

$$T^{m'} \circ \xi = T^{n'} \circ \xi \circ T^{k'} \tag{7.4}$$

as in (7.3) with suitably large $k', n', m' \in \mathbb{N}$ and apply it to I. Note that k', n', m' will in general be different from the original numbers k, n, m in (7.3).

First, we may assume that k' in (7.4) is so large that $T^{k'}(I)$ is equal to the whole side of O that contains b and I. Then $\xi(T^{k'}(I))$ is a neighborhood of a in C. In particular, if $n' \geq n$ is large enough, as we may assume, then there are two n'-edges e_1 and e_2 that meet at a right angle at the common endpoint a with $e_1 \cup e_2 \subseteq \xi(T^{k'}(I)) \subseteq \xi(O)$. Since the n'-edges e_1 and e_2 meet at a right angle, $T^{n'}(e_1 \cup e_2)$ consists of two sides of O. It follows that the set

$$T^{n'}(\xi(T^{k'}(I))) \supseteq T^{n'}(e_1 \cup e_2)$$

contains at least two sides of O. On the other hand, $\xi(I)$ is contained in the n-edge e. If $m' \geq n$ as we may assume, $T^{m'}(e)$ is equal to one of the sides of O as follows from (2.1). This implies that the set

$$T^{n'}(\xi(T^{k'}(I))) = T^{m'}(\xi(I)) \subseteq T^{m'}(e)$$

is contained in one of the sides of Q. This is a contradiction and the Claim follows.

There are now two cases to consider, depending on whether C does or does not have turns.

Case 1: C has no turns. Then $C = \xi(O) \neq O$ and C runs "parallel" to one of the sides of O in Q and in the back copy Q' of Q. In particular, the involution R (that interchanges corresponding points of Q and Q') is a quasisymmetry on D_p preserving C. Consider

$$g = \xi^{-1} \circ R \circ \xi.$$

Then g is a quasisymmetry on D_p with $g(O) = O$. Its fixed point set is the Jordan curve $\xi^{-1}(O) \neq \xi^{-1}(C) = O$.

It follows from Theorem 1.1 that every quasisymmetry on D_p that preserves O as a set is an isometry of D_p. In particular, g must be such an isometry. There are 16 such maps: eight isometries that preserve the front and back copies of S_p and eight obtained by composing these maps by the involution R that interchanges the front and back copies. Among these 16 maps there is exactly one, namely the involution R, whose fixed point set is a Jordan curve J. In this case $J = O$. In all other cases, the fixed point set is either empty, finite, a Cantor set, or all of D_p. On the other hand, g has the fixed point set given by the Jordan curve $\xi^{-1}(O) \neq \xi^{-1}(C) = O$. This is a contradiction, showing that Case 1 is impossible.

Case 2: C has at least one turn. We claim that such a turn must be a corner of O. To reach a contradiction, suppose C has a turn a other than a corner of O. Since $C = \xi(O) \subseteq D_p \cap T^{-n}(O)$ does not meet any peripheral circle of D_p, the point a is the common corner of four good n-tiles. The curve C is the common boundary of two complementary regions of the pillow P. Since a is a turn of C, one of these regions, which we denote by U, has angle $3\pi/2$ at a (the other region has the angle $\pi/2$ at a).

There are three good n-tiles, i.e., three copies of S_p scaled by the factor $1/p^n$ that are contained $\overline{U}$ and that share a as a corner. In fact, there are infinitely many such triples of copies of S_p that meet at a rescaled by the factor p^{-k} with $k \geq n$. This implies $\overline{U} \cap D_p$ has a unique weak tangent at a and that it is isometric to the set $\widetilde{W}$ in (7.2) with basepoint 0.

Since a is a turn of C, there exists a corner b of O with $\xi(b) = a$. Now $\xi(O) = C$, and so either S_p or the back copy S'_p of S_p is mapped to $\overline{U} \cap D_p$ by the quasisymmetry ξ. In both cases we get an induced basepoint-preserving quasisymmetry of the weak tangents of the source set at b and the image set $D_p \cap \overline{U}$ at a. Both S_p and S'_p have unique weak tangents at any corner b of O isometric to W in (7.1) with basepoint 0. This implies that there exists a quasisymmetry between the pointed metric spaces $(W, 0)$ and $(\widetilde{W}, 0)$. This is impossible by Lemma 7.1, and so we reach a contradiction.

We conclude that the Jordan curve $C = \xi(O) \subseteq T^{-n}(O)$ has at least one turn, and that every turn of C is a corner of O; but then necessarily $C = \xi(O) = O$, and we are done. □

***Proof of Theorem* 1.3** We argue by contradiction and assume that there exists a quasisymmetry $\xi \colon D_p \to \mathcal{J}(g)$, where $p \geq 3$ is odd, and $\mathcal{J}(g)$ is the Julia set of a postcritically-finite rational map g on $\widehat{\mathbb{C}}$. We will later consider the case when S_p is assumed to be quasisymmetrically equivalent to $\mathcal{J}(g)$.

The quasisymmetry ξ can be extended (non-uniquely) to a quasisymmetry, also called ξ, of the pillow P onto $\widehat{\mathbb{C}}$ (the justification for this is the same as in the beginning of the proof of Proposition 5.1). We may assume that ξ is orientation-preserving, because otherwise we can precompose ξ with an orientation-reversing isometry of P that leaves D_p invariant (such as the reflection R that interchanges corresponding points on the front and back of P).

As before, we denote by T the Lattès map as introduced in Sect. 2 (for given p). The main idea of the proof now is to establish an analog of Proposition 5.1 for the maps g, ξ, and T. Namely, we want to show that there exist $k \in \mathbb{N}$ and $\ell, m \in \mathbb{N}_0$ such that

$$g^m \circ \xi = g^\ell \circ \xi \circ T^k \tag{7.5}$$

on the set D_p.

Once (7.5) is established, we obtain a contradiction as follows. Since $\mathcal{J}(g)$ is homeomorphic to D_p, the set $\mathcal{J}(g)$ is a Sierpiński carpet. Let A be the set of all points in $\mathcal{J}(g)$ that lie on a peripheral circle of $\mathcal{J}(g)$. If J is a peripheral circle of $\mathcal{J}(g)$, then $g(J)$ is also a peripheral circle and $g^{-1}(J)$ consists of finitely many peripheral circles (see [4, Lemma 5.1]). This implies that A is completely invariant under g, and hence under all iterates of g, i.e., $g^n(A) = A = g^{-n}(A)$ for each $n \in \mathbb{N}_0$. Note also that the homeomorphism ξ sends the peripheral circles of D_p to the peripheral circles of $\mathcal{J}(g)$.

We now apply both sides of (7.5) to the middle peripheral circle M of $S_p \subseteq D_p$. Then the left hand side shows that $(g^m \circ \xi)(M) \subseteq A$. On the other hand, if we consider the right hand side of (7.5), we first note that $T^k(M) = O$. It follows that

$$(g^\ell \circ \xi \circ T^k)(M) = (g^\ell \circ \xi)(O)$$

is disjoint from A, because O does not meet any peripheral circle of D_p and so $\xi(O)$ and $(g^\ell \circ \xi)(O)$ are disjoint from A. This is a contradiction.

In order to establish (7.5), one uses ideas as in the proof of Proposition 5.1 combined with arguments for the proof of the similar relation (8.4) in [4, Section 8], where T plays the role of the rational map f and D_p the role of the Julia set $\mathcal{J}(f)$.

As in the proof in [4, Section 8], we want to implement a "blow down-blow up" argument applying "conformal elevators". Namely, one first uses inverse branches T^{-n} to blow down, and then iterates of the map $g_\xi := \xi^{-1} \circ g \circ \xi \colon P \to P$ to blow up. Note that D_p is completely invariant under the map g_ξ and its iterates.

As in the proof of Proposition 5.1, we choose a sequence T^{-k_n} of inverse branches mapping the front Q of the pillow to a good tile $X_n \subseteq Q$ of level $k_n \in \mathbb{N}_0$ containing the corner $c = (0, 0) \in Q$. Here we again assume that the sequence $\{k_n\}$ is strictly increasing. The branches T^{-k_n} are consistent as in (5.2). Each map T^{-k_n} is a scaling map, and in particular a quasisymmetry of Q onto $X_n \subseteq Q$. We also have $\operatorname{diam}(T^{-k_n}(Q)) \to 0$ as $n \to \infty$. All of this is similar to the choice of inverse branches in [4, Section 8, Step II], but easier.

Following the argument of [4, Section 8, Step II], we use the expansion property of g to find for each natural number $n \in \mathbb{N}$, a corresponding number $\ell_n \in \mathbb{N}_0$ such that the sets $(g_\xi^{\ell_n} \circ T^{-k_n})(Q)$ have uniformly large size independent of n. Now we argue as in [4, Section 8, Step III] and consider the maps $\widetilde{h}_n := g_\xi^{\ell_n} \circ T^{-k_n} \colon Q \to P$ for $n \in \mathbb{N}$. Under a conformal identification $P \cong \widehat{\mathbb{C}}$ these are actually uniformly quasiregular maps on the region $U := Q \backslash O$. By passing to a subsequence if necessary, we may assume that there exists a (non-constant) quasiregular map $\widetilde{h} \colon U \to P$ such that $\widetilde{h}_n \to \widetilde{h}$ locally uniformly on U as $n \to \infty$.

The map $\widetilde{h}$ is locally quasiconformal on U away from the branch points of $\widetilde{h}$. These branch points form a set with no limit points in U. This implies that we can find a point $q \in U \cap D_p$ and a small radius $r > 0$ such that $B(q, 2r) \subseteq U$ and $\widetilde{h}$ is quasiconformal on $B(q, 2r)$. It follows that on the smaller ball $B(q, r)$ the maps $\widetilde{h}_n$ are quasiconformal for large n. By discarding finitely many of the maps $\widetilde{h}_n$ if necessary, we may assume that they are quasiconformal for all $n \in \mathbb{N}$.

Since $q \in D_p$, we can find a good tile $Y \subseteq B(q, r)$ (as defined in Sect. 2). Since a quasiconformal map is a local quasisymmetry, we conclude that the maps $\widetilde{h}$ and $\widetilde{h}_n$ for $n \in \mathbb{N}$ are quasisymmetric embeddings of Y into P. We are now in a similar situation as in the proof of Proposition 5.1. We choose an orientation-preserving scaling map φ that sends Q onto Y. Note that then $\varphi(Q \cap D_p) = Y \cap D_p$. We now define

$$\begin{aligned} h &:= \widetilde{h} \circ \varphi, \\ h_n &:= \widetilde{h}_n \circ \varphi = g_\xi^{\ell_n} \circ T^{-k_n} \circ \varphi \end{aligned}$$

for $n \in \mathbb{N}$. These maps are quasisymmetries on Q with $h_n \to h$ uniformly on Q.

Note that we again have the relation (5.4), which follows from the mapping properties of φ and T^{-k_n}, in combination with the identities $\xi(D_p) = \mathcal{J}(g)$ and $g_\xi^{-1}(D_p) = g_\xi(D_p) = D_p$. As before, (5.4) implies (5.5).

Based on Theorem 4.2 and Lemma 4.4 we can again argue that the sequence h_n stabilizes and so $h_{n+1} = h_n$ on S_p for large n. This implies that there exists $n \in \mathbb{N}$ such that

$$g_\xi^{\ell_{n+1}} \circ T^{-k_{n+1}} = g_\xi^{\ell_n} \circ T^{-k_n} \tag{7.6}$$

on $Y \cap S_p$. Using the consistency relation for the inverse branches we see that

$$g_\xi^{\ell_{n+1}} = g_\xi^{\ell_n} \circ T^{k_{n+1}-k_n} \tag{7.7}$$

on $T^{-k_{n+1}}(Y \cap S_p) \subseteq D_p$. Note that here $k_{n+1} - k_n \in \mathbb{N}$, because $k_{n+1} > k_n$.

We want to argue that this identity remains valid on the whole set D_p. To see this, first note that by Lemma 3.1 for each $\ell \in \mathbb{N}_0$ the iterate g^ℓ of g is an admissible maps for the Sierpiński carpet $\mathcal{J}(g)$. Thus, $g_\xi^\ell = \xi^{-1} \circ g^\ell \circ \xi$ is an admissible map for D_p. Combined with Lemmas 3.2 and 3.4 this shows that both sides in (7.7) are admissible maps for D_p.

Corollary 4.6 then implies that (7.7) is valid on D_p. This is equivalent to a relation of the form (7.5) as required. This completes the proof when D_p is assumed to be quasisymmetrically equivalent to $\mathcal{J}(g)$.

Now suppose that there exists a quasisymmetry $\xi\colon S_p \to \mathcal{J}(g)$. Again we may assume that ξ has an extension to an orientation-preserving quasisymmetry $\xi\colon P \to \widehat{\mathbb{C}}$. Here we cannot expect an identity as in (7.5) to be valid on S_p. The main problem is that S_p is not forward-invariant under T.

In order to derive a contradiction, we have to slightly modify the above argument. We again implement a "blow down-blow-up" procedure as above, where D_p is replaced with S_p, up to the point where we conclude that the sequence $\{h_n\}$ stabilizes. We again obtain the relation (7.6) on $Y \cap S_p$, where $Y \subseteq Q$ is a suitable good tile. Instead of using the consistency relation (5.2) we now employ the identity

$$T^{-k_{n+1}} = T^{-(k_{n+1}-k_n)} \circ T^{-k_n}$$

for the unique branch $T^{-(k_{n+1}-k_n)}$ that maps Q to the good tile $Z \subseteq Q$ of level $k = k_{n+1} - k_n \in \mathbb{N}$ with $c \in Z$. This implies that there exist constants $\ell, \ell' \in \mathbb{N}_0$ such that

$$g_\xi^{\ell'} \circ T^{-k} = g_\xi^{\ell} \tag{7.8}$$

on the set $T^{-k_n}(Y \cap S_p) = Y' \cap S_p$, where $Y' := T^{-k_n}(Y) \subseteq Q$ is a good tile.

Let C and C' be the finite sets of critical points of g_ξ^ℓ and $g_\xi^{\ell'}$, respectively. If we define $W := Q \backslash (O \cup C \cup T^k(C'))$, then $W \cap S_p$ is connected and dense in S_p. Moreover, each point in $x \in W \cap S_p$ is a good point (as defined in Sect. 3) for each of the two maps in (7.8) and the Sierpiński carpet S_p. So if we conjugate these maps by β from Lemma 4.4, then we obtain Schottky maps from the locally porous relative Schottky set $\beta(W) \cap \beta(S_p)$ into $\beta(S_p)$. By Theorem 4.3 this implies that (7.8) holds on $W \cap S_p$. Since $W \cap S_p$ is dense in S_p, it follows that (7.8) is valid on S_p.

We want to see that this is impossible. Since the union of all peripheral circles of $\mathcal{J}(g)$ is completely invariant under g, the union of all peripheral circles of S_p is completely invariant under g_ξ. Now consider the point $a := (1, 1) \in S_p$. Then $T^{-k}(a) = (p^{-k}, p^{-k})$ does not lie on a peripheral circle of S_p and so the same is true for $b := (g_\xi^{\ell'} \circ T^{-k})(a)$. On the other hand, a lies on the peripheral circle O of S_p and so by (7.8),

$$b = (g_\xi^{\ell} \circ T^{-k})(a) = g_\xi^{\ell}(a) \in g_\xi^{\ell}(O)$$

lies on the peripheral circle $g_\xi^{\ell}(O)$ of S_p. This is a contradiction.

We conclude that neither D_p nor S_p can be quasisymmetrically equivalent to the Julia set of a postcritically-finite rational map. □

The essential point in the previous proof was the fact that while the union of all peripheral circles of $\mathcal{J}(g)$ is completely invariant under g, the union of all peripheral circles of S_p or D_p is not completely invariant under T.

References

1. Beardon, A.F.: Iteration of Rational Functions. Springer, New York (1991)
2. Bonk, M.: Quasiconformal geometry of fractals. In: Proceedings of the International Congress of Mathematicians (Madrid 2006), vol. II. European Mathematical Society, Zürich, pp. 1349–1373 (2006)
3. Bonk, M.: Uniformization of Sierpiński carpets in the plane. Invent. Math. **186**, 559–665 (2011)
4. Bonk, M., Lyubich, M., Merenkov, S.: Quasisymmetries of Sierpiński carpet Julia sets. Adv. Math. **301**, 383–422 (2016)
5. Bonk, M., Merenkov, S.: Quasisymmetric rigidity of square Sierpínski carpets. Ann. Math. (2) **177**, 591–643 (2013)
6. Bonk, M., Meyer, D.: Expanding Thurston Maps, Math. Survey and Monographs, vol. 225. American Mathematical Society, Providence (2017)
7. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry, Grad. Studies in Math., vol. 33. American Mathematical Society, Providence (2001)
8. Merenkov, S.: Planar relative Schottky sets and quasisymmetric maps. Proc. Lond. Math. Soc. (3) **104**, 455–485 (2012)
9. Merenkov, S.: Local rigidity of Schottky maps. Proc. Am. Math. Soc. **142**, 4321–4332 (2014)
10. Merenkov, S.: Local rigidity for hyperbolic groups with Sierpiński carpet boundaries. Compos. Math. **150**, 1928–1938 (2014)
11. Whyburn, G.T.: Topological characterization of the Sierpiński curve. Fund. Math. **45**, 320–324 (1958)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.