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The arithmetic fundamental lemma: An update

Dedicated to Professor Lo Yang on the Occasion of His 80th Birthday

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Abstract This is an expository article on the recent progress on the arithmetic fundamental lemma conjecture, based largely on Zhang (2019). Beside stating the local conjecture, we will present three global intersection problems along with some constructions of algebraic cycles.

Keywords arithmetic fundamental lemma, arithmetic Gan-Gross-Prasad conjecture, Gross-Zagier formula, relative trace formula

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1 Introduction

The Gross-Zagier theorem [6] relates the first central derivative of the base change (to an imaginary quadratic field $E = \mathbb{Q}[\sqrt{-D}]$) L-function of an elliptic curve A over $\mathbb{Q}$ to the Nerón-Tate height of the Heegner point. Through a parameterization of A by a modular curve $\phi : X_0(N) \to A$ and under the Heegner hypothesis, the theory of complex multiplication defines a special divisor (the Heegner divisor) on $X_0(N)$ and its image under ϕ yields rational point $\mathscr{P}_E \in A(E)$. Then their theorem is an identity

$$\frac{L'(f_E,1)}{(f,f)} = \frac{1}{\sqrt{|D|}}\frac{\langle \mathscr{P}_E, \mathscr{P}_E\rangle_{\mathrm{NT}}}{\deg(\phi)},$$

where f is the (normalized) cusp form of weight two associated to A, (f,f) is the Petersson inner product, and $\langle \mathscr{P}_E, \mathscr{P}_E\rangle_{\mathrm{NT}}$ is the Néron-Tate height pairing for A over E.

The arithmetic Gan-Gross-Prasad conjecture (see [4, Section 27] and [23, Subsection 3.2]) is a generalization of the Gross-Zagier theorem to high dimensional Shimura varieties (see variants by Rapoport et al. [16]). In the arithmetic Gan-Gross-Prasad conjecture, one considers the product Shimura variety attached to unitary or orthogonal groups and a special algebraic cycle, generalizing the above modular curve $X_0(N)$ and the Heegner divisor $\mathscr{P}_E$, respectively. This conjecture is inspired by the (usual) Gan-Gross-Prasad conjecture relating period integrals on classical groups to special values of Rankin-Selberg tensor product L-functions. In [8], Jacquet and Rallis proposed a relative trace formula (RTF) approach to this last conjecture in the case of unitary groups. Inspired by their approach, in [23] Zhang proposed a relative trace formula approach to the arithmetic Gan-Gross-Prasad conjecture for unitary Shimura varieties. This approach reduces the problem to certain local statements, notably the arithmetic fundamental lemma (AFL) conjecture formulated by Zhang [23], and the arithmetic transfer (AT) conjecture formulated by Rapoport et al. [15, 17].

In the AFL and AT conjectures, we consider the local counterpart of special cycles on Shimura varieties, i.e., cycles on Rapoport-Zink formal moduli spaces of p-divisible groups. The AFL and AT conjectures then predict a relation between the local intersection numbers and special values of the derivative of relative orbital integrals. The theorem of Rapoport-Zink on the uniformization of Shimura varieties relates the local cycles to the global ones, and this allows us to relate the intersection numbers of the global cycles to those of local ones. Similarly, the relative trace formula of Jacquet-Rallis on the general linear group allows us to understand special values of L-functions in terms of certain relative orbital integrals.

The goal of this article is to explain some geometric constructions in Zhang's recent proof [26] of the AFL conjecture over $\mathbb{Q}_p$ for p large (see Theorem 2.3). We will recall the special divisors introduced by Kudla and Rapoport [10] on the integral models $\widetilde{\mathcal{M}}_n$ of certain unitary Shimura varieties. Then we will introduce the main new construction, a class of "derived" complex multiplication (CM) cycles on the same $\widetilde{\mathcal{M}}_n$, which can be viewed as elements in the Chow group (of 1-cycles) of $\widetilde{\mathcal{M}}_n$. We will first introduce the analogous CM cycles on the moduli space of principally polarized abelian varieties $\mathcal{A}_g$ (see Subsection 3.6), and then the construction extends easily to the integral models $\widetilde{\mathcal{M}}_n$ (see Subsection 3.7).

Some related survey articles on the AFL conjecture and the arithmetic Gan-Gross-Prasad conjecture are [20, 24, 25].

It is a pleasure to dedicate this article to Professor Lo Yang on the occasion of his 80th birthday. Professor Yang's mathematical work has tremendously influenced my generation, and his effort devoted to the development of mathematics in China has benefitted many scholars in the past decades.

2 The arithmetic fundamental lemma conjecture

2.1 The statement of AFL

In this subsection we recall the statement of the AFL conjecture [17, 23].

Let F be a finite extension of $\mathbb{Q}_p$ for an *odd* prime p. Let O_F be the ring of integers in F, and denote by q the number of elements in the residue field of O_F. Let $\breve{F}$ be the completion of a maximal unramified extension of F.

Let F'/F be an *unramified* quadratic extension, and $\eta_{F'/F} : F^\times \to \{\pm 1\}$ the associated quadratic character $\eta_{F'/F}(a) = (-1)^{\mathrm{val}(a)}$.

Let $\mathcal{N}_n$ be the unitary Rapoport-Zink formal moduli space over $\mathrm{Spf}\, O_{\breve{F}}$. Over the residue field $\overline{k}$ of $O_{\breve{F}}$ there is a unique *Hermitian formal $O_{F'}$-module* $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ *of signature* $(n-1, 1)$ such that $\mathbb{X}_n$ is supersingular, up to O_F-linear quasi-isogeny compatible with the polarization. Then $\mathcal{N}_n$ represents the functor over $\mathrm{Spf}\, O_{\breve{F}}$ that associates to each $\mathrm{Spf}\, O_{\breve{F}}$-scheme S the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ over S, where the final entry is an O_F-linear quasi-isogeny of height zero defined over the special fiber,

$$\rho\colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\mathrm{Spec}\, \overline{k}} \overline{S},$$

such that $\rho^*((\lambda_{\mathbb{X}_n})_{\overline{S}}) = \lambda_{\overline{S}}$ (a *framing*). Here, $\overline{S} := S \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathrm{Spec}\, \overline{k}$.

The formal scheme $\mathcal{N}_n$ is smooth over $\mathrm{Spf}\, O_{\breve{F}}$ of relative dimension $n-1$. One can construct an F'/F-Hermitian space $\mathbb{V}_n$ attached to $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$:

$$\mathbb{V}_n := \mathrm{Hom}^\circ_{O_{F'}}(\mathbb{E}, \mathbb{X}_n),$$

where $\mathbb{E} = \mathbb{X}_1$ but with the conjugate action of $O_{F'}$ (so the Kottwitz signature is $(1,0)$ rather than $(0,1)$). Here and henceforth $\mathrm{Hom}^\circ := \mathrm{Hom} \bigotimes_{\mathbb{Z}} \mathbb{Q}$. Then the group

$$\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) \xrightarrow{\ \sim\ } \mathrm{U}(\mathbb{V}_n)(F),$$

acts naturally on $\mathcal{N}_n$ by changing the framing.

Let $\mathcal{N}_{n-1,n} = \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$. Then $\mathcal{N}_{n-1,n}$ admits an action by the product of two unitary groups $\mathrm{G}(F)$, where $\mathrm{G} := \mathrm{U}(\mathbb{V}_{n-1}) \times \mathrm{U}(\mathbb{V}_n)$. There is a natural closed embedding $\delta\colon \mathcal{N}_{n-1} \to \mathcal{N}_n$. Let

$$\Delta\colon \mathcal{N}_{n-1} \to \mathcal{N}_{n-1,n}$$

be the graph morphism of δ. We denote by $\Delta_{\mathcal{N}_{n-1}}$ the image of Δ. It is invariant under the action of the subgroup $\mathrm{H}(F)$, where

$$\mathrm{H} := \mathrm{U}(\mathbb{V}_{n-1}) \xrightarrow{\Delta} \mathrm{G} = \mathrm{U}(\mathbb{V}_{n-1}) \times \mathrm{U}(\mathbb{V}_n)$$

is embedded diagonally.

An element $g = (g_{n-1}, g_n) \in \mathrm{G}(F)$ is called *regular semisimple* (relative to $\mathrm{H} \times \mathrm{H}$) if the orbit $\mathrm{H}g\mathrm{H}$ is Zariski closed in G, and the stabilizer is trivial. A regular semisimple (for short, "rs") element is called *strongly regular semisimple* (for short, "srs") if it satisfies the additional condition that $g_{n-1}^{-1} \cdot g_n \in \mathrm{U}(\mathbb{V}_n)(F)$ is regular semisimple in the usual sense (i.e., its characteristic polynomial has distinct roots). For $g \in \mathrm{G}(F)_{\mathrm{rs}}$, we consider the intersection product on $\mathcal{N}_{n-1,n}$ of $\Delta_{\mathcal{N}_{n-1}}$ with its translate $g\Delta_{\mathcal{N}_{n-1}}$, defined through the derived tensor product of the structure sheaves,

$$\mathrm{Int}(g) := (\Delta_{\mathcal{N}_{n-1}}, g \cdot \Delta_{\mathcal{N}_{n-1}})_{\mathcal{N}_{n-1,n}} := \chi\Big(\mathcal{N}_{n-1,n}, \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \overset{\mathbb{L}}{\bigotimes} \mathcal{O}_{g\cdot\Delta_{\mathcal{N}_{n-1}}}\Big). \tag{2.1}$$

Here, χ denotes the Euler-Poincaré characteristic for the relative cohomology along the map $\pi : \mathcal{X} = \mathcal{N}_{n-1,n} \to \mathrm{Spf}\, O_{\breve{F}}$, i.e., for coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{X}$,

$$\chi\Big(\mathcal{X}, \mathcal{F} \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathcal{X}}}{\bigotimes}} \mathcal{G}\Big) := \sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \mathrm{length}_{O_{\breve{F}}} \mathrm{R}^i\pi_*(\mathrm{Tor}_j^{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})). \tag{2.2}$$

When g is regular semisimple, the (formal) schematic intersection $\Delta \cap g\Delta$ is a proper *scheme* over $\mathrm{Spf}\, O_{\breve{F}}$, and hence the Euler-Poincaré characteristic is a finite integer.

We now recall the relative orbital integrals. Consider the triple $(\mathrm{G}', \mathrm{H}'_1, \mathrm{H}'_2)$ where $\mathrm{G}' = \mathrm{R}_{F'/F}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n)$, and

$$\mathrm{H}'_1 = \mathrm{R}_{F'/F}\mathrm{GL}_{n-1}, \quad \mathrm{H}'_2 = \mathrm{GL}_{n-1} \times \mathrm{GL}_n.$$

Consider the quadratic character of $\mathrm{H}'_2(F)$:

$$\eta = \eta_{n-1,n} : (h_{n-1}, h_n) \in \mathrm{H}'_2(F) \mapsto \eta_{F'/F}^{n-2}(\det(h_{n-1}))\eta_{F'/F}^{n-1}(\det(h_n)).$$

Let $f' \in \mathscr{C}_c^\infty(\mathrm{G}'(F))$ and $s \in \mathbb{C}$. For a *regular semisimple* (relative to $H'_{1,2} := H'_1 \times H'_2$) element $\gamma \in \mathrm{G}'(F)_{\mathrm{rs}}$ we introduce the (weighted) orbital integral

$$\mathrm{Orb}(\gamma, f', s) = \int_{\mathrm{H}'_{1,2}(F)} f'(h_1^{-1}\gamma h_2)|\det(h_1)|^s \eta(h_2)\, dh_1\, dh_2. \tag{2.3}$$

Here the Haar measure is normalized such that $\mathrm{vol}(\mathrm{H}'_{1,2}(O_F)) = 1$. We set

$$\partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s).$$

Conjecture 2.1 (Arithmetic fundamental lemma (AFL) conjecture)**.** *Let* $\gamma \in \mathrm{G}'(F)_{\mathrm{srs}}$ *match an element* $g \in \mathrm{G}(F)_{\mathrm{srs}}$. *Then*

$$\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, \mathbf{1}_{\mathrm{G}'(O_F)}) = -2\, \mathrm{Int}(g) \cdot \log q.$$

Here the matching relation between orbits is defined in [17, 23] (see also [26, Section 2]), and $\omega(\gamma)$ is a certain transfer factor. In the original formulation [23], one only assumes that γ and g are regular semisimple. For global applications, the restriction to strongly regular semisimple elements is harmless.

We may interpret the orbital integrals in terms of "counting lattices" (see [19, Section 7]). See [17, Section 4] for some other equivalent formulations of the AFL conjecture, including a variant of the orbital integrals in terms of the symmetric space S_n defined by (2.9) below.

2.2 The status

Theorem 2.2. (i) *The AFL Conjecture* 2.1 *holds when* $n = 2, 3$.

(ii) *The AFL Conjecture* 2.1 *holds for* minuscule *elements* $g \in \mathrm{G}(F)$ *in the sense of* [19].

Part (i) was proved in [23]; a simplified proof when $p \geqslant 5$ is given by Mihatsch [13] by "reduction to Lie algebra". Part (ii) was proved by Rapoport et al. [19] when $p \geqslant \frac{n}{2}+1$ (a simplified proof is given by Li and Zhu [11]), and by He et al. [7] for general p.

Recently the author has proved the following theorem.

Theorem 2.3 (See [26])**.** *The AFL Conjecture* 2.1 *holds when* $F = \mathbb{Q}_p$ *with* $p \geqslant n$.

In the rest of the article, we will explain two geometric ingredients in proving this theorem:

- an equivalent version of the AFL conjecture, and this part is of local nature (see Subsection 2.3);
- some global intersection problems (see Section 3), where the local intersection problems are "embedded" into the global ones.

In both parts, an important point of view is to change the intersection of two high codimensional cycles to the intersection of a 1-cycle and a divisor. The advantage of this change is that, in the global setting, we can utilize the modularity of the generating series of the collection of special divisors (see Remarks 3.2 and 3.6).

2.3 An alternative formulation of AFL via special divisors

Now we introduce a variant of the AFL conjecture via special divisors [9].

Recall from [9], for every non-zero $u \in \mathbb{V}_n$, Kudla and Rapoport have defined a special divisor $\mathcal{Z}(u)$ in $\mathcal{N}_n$. This is the locus where the quasi-homomorphism $u\colon \mathbb{E} \to \mathbb{X}_n$ lifts to a homomorphism from the canonical lifting $\mathcal{E}$ of $\mathbb{E}$ to the universal object over $\mathcal{N}_n$. By [9, Proposition 3.5], $\mathcal{Z}(u)$ is a locally principal divisor (or empty) whenever $u \neq 0$. Then $\delta\colon \mathcal{N}_{n-1} \to \mathcal{N}_n$ induces an isomorphism by [9, Lemma 5.2]

$$\mathcal{N}_{n-1} \xrightarrow{\sim} \mathcal{Z}(u_0) \tag{2.4}$$

for a special vector u_0 with unit norm.

Relative to the (diagonal) action of $\mathrm{U}(\mathbb{V}_n)$ on $\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n$, we can define the notion of regular semisimplicity ("rs") and strongly regular semisimplicity ("srs"), for an element $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)$, similar to the action of $\mathrm{H} \times \mathrm{H}$ on $\mathrm{G} = \mathrm{U}(\mathbb{V}_{n-1}) \times \mathrm{U}(\mathbb{V}_n)$.

For a pair $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)_{\mathrm{rs}}$, we introduce our first variant of the intersection number (see (2.2))

$$\mathrm{Int}(g, u) := \chi\left(\mathcal{N}_n \times \mathcal{N}_n,\, \mathcal{O}_{\Gamma_g} \overset{\mathbb{L}}{\bigotimes} \mathcal{O}_{\Delta(\mathcal{Z}(u))}\right). \tag{2.5}$$

This is again a finite integer. This intersection number has appeared in the AFL conjecture in the context of Fourier-Jacobi cycles in the work of Liu [12].

For $g \in \mathrm{U}(\mathbb{V}_n)$, let $\Gamma_g \subset \mathcal{N}_n \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$ be the graph and define the (naive) fixed point locus $\mathcal{N}_n^g$ as the (formal) schematic intersection (i.e., fiber product of formal schemes)

$$\mathcal{N}_n^g := \Gamma_g \cap \Delta_{\mathcal{N}_n}.$$

We also form a "derived fixed point locus" ${}^{\mathbb{L}}\mathcal{N}_n^g$, i.e., the derived tensor product

$${}^{\mathbb{L}}\mathcal{N}_n^g := \mathcal{O}_{\Gamma_g} \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}}{\bigotimes}} \mathcal{O}_{\Delta_{\mathcal{N}_n}}, \tag{2.6}$$

viewed as an element in the Grothendieck group $K_0'(\mathcal{N}_n^g)$ of coherent sheaves on $\mathcal{N}_n^g$.

For a pair $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)_{\mathrm{rs}}$, we define another variant of the intersection number

$$\mathrm{Int}(g, u) := \chi\left(\mathcal{N}_n,\, {}^{\mathbb{L}}\mathcal{N}_n^g \overset{\mathbb{L}}{\bigotimes} \mathcal{O}_{\mathcal{Z}(u)}\right). \tag{2.7}$$

Similar to (2.1) and (2.5), this is a finite integer. As alluded earlier, this is now the intersection of a 1-cycle and a divisor, rather than two high codimensional cycles in (2.1) and (2.5).

Remark 2.4. By the projection formula for the closed immersion $\Delta : \mathcal{N}_n \to \mathcal{N}_n \times \mathcal{N}_n$, one can show that the two intersection numbers in (2.5) and (2.7) are equal. This is the reason we use the same notation.

Remark 2.5. By (2.4), one can show that, for $g \in \mathrm{U}(\mathbb{V}_n)(F)_{\mathrm{rs}}$ such that $(1, g) \in \mathrm{G}(F)_{\mathrm{rs}}$,

$$\mathrm{Int}(g, u_0) = \mathrm{Int}((1, g)),$$

where $\mathrm{Int}((1, g))$ is defined by (2.1).

Remark 2.6. One can define a $\mathrm{U}(\mathbb{V}_n)(F)$-invariant map

$$\mathrm{inv}\colon (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F) \to F'[T]_{\deg=n} \times F'^n \tag{2.8}$$

sending (g, u) to (a, b) where

$$a = \mathrm{char}(g) \in F'[T]_{\deg=n},$$

and

$$b = (b_i)_{i=0}^{n-1} \in F'^n \quad \text{with} \quad b_i = \langle g^i \circ u, u\rangle.$$

Then the intersection number $\mathrm{Int}(g, u)$ depends only on the invariants (a, b) of regular semisimple (g, u), and thus we may write $\mathrm{Int}(g, u) = \mathrm{Int}(a, b)$.

One can adjust the definition of the relative orbital integral (2.3) as follows. Consider the symmetric space

$$S_n := \{g \in \mathrm{Res}_{F'/F} \mathrm{GL}_n \mid g\bar{g} = 1_n\}. \tag{2.9}$$

Let $V'_n := F^n \times (F^n)^*$ and $\mathrm{H}' := \mathrm{GL}_n$. Consider the (diagonal) action of H' on the product $S_n \times V'_n$ by

$$h \cdot (\gamma, (u_1, u_2)) = (h^{-1}\gamma h, (h^{-1}u_1, u_2 h)).$$

For $(\gamma, u') \in (S_n \times V'_n)(F)_{\mathrm{rs}}$, $\Phi' \in \mathscr{C}_c^\infty((S_n \times V'_n)(F))$ and $s \in \mathbb{C}$, we define

$$\mathrm{Orb}((\gamma, u'), \Phi', s) := \int_{\mathrm{H}'(F)} \Phi'(h \cdot (\gamma, u'))|\det h|^s \eta_{F'/F}(\det(h))\, dh.$$

Here the Haar measure is normalized such that $\mathrm{vol}(\mathrm{H}'(O_F)) = 1$. We set

$$\partial\mathrm{Orb}((\gamma, u'), \Phi') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}((\gamma, u'), \Phi', s).$$

Conjecture 2.7 (Arithmetic fundamental lemma conjecture, the semi-Lie algebra version [26])**.** *Suppose that $(\gamma, u') \in (S_n \times V'_n)(F)_{\mathrm{srs}}$ matches an element $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)_{\mathrm{srs}}$. Then*

$$\omega(\gamma, u')\, \partial\mathrm{Orb}((\gamma, u'), \mathbf{1}_{(S_n\times V'_n)(O_F)}) = -\mathrm{Int}(g, u) \cdot \log q.$$

Here the matching relation between orbits is defined in [26, Section 2], and $\omega(\gamma, u')$ is a certain transfer factor.

Along the proof of Theorem 2.3, a key observation is the following "inductive" nature of the two statements combined:

Proposition 2.8 (See [26])**.** *Assume that $q \geqslant n$ where q denotes the cardinality of the residue field of O_F. Then*

(i) *Conjecture* 2.1 *for $\mathbb{V}_n$ is equivalent to Conjecture* 2.7 *for $\mathbb{V}_{n-1}$.*

(ii) *Conjecture* 2.1 *for $\mathbb{V}_n$ implies Conjecture* 2.7 *for $\mathbb{V}_n$ and $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)_{\mathrm{srs}}$ where the norm of u is a unit.*

Part (ii) essentially follows from Remark 2.5. Part (i) is much more subtle and was not noticed for quite a while (see [26]).

3 Some global intersection problems

The proof of Theorem 2.3 in [26] is through the study of some global intersection problems arising from the arithmetic Gan-Gross-Prasad conjecture (for $\mathrm{U}_n \times \mathrm{U}_{n+1}$ and $\mathrm{U}_n \times \mathrm{U}_n$), where the local intersection numbers are "embedded" into the global situation. In this expository article, we only aim to introduce three global intersection problems in their simplest cases, corresponding to (2.1), (2.5) and (2.7), respectively. We refer the reader to [12, 16, 26] for more details.

We will first define several moduli stacks of abelian schemes with certain additional structures, and Hecke correspondences on some of them. For simplicity, we will only spell out their points over a test scheme S (always assumed to be locally noetherian); the morphisms are usually self-evident. We will only consider the "base case" of these moduli stacks, i.e., without any "level-structure".

3.1 The moduli space of principally polarized abelian varieties $\mathcal{A}_g$

Let $g \in \mathbb{N}$. Let $\mathcal{A}_g = \mathcal{A}_{g,1}$ be the Siegel moduli space of principally polarized abelian varieties, i.e., for a scheme S, $\mathcal{A}_g(S)$ is the groupoid of (A, λ) where $A \to S$ is an abelian scheme of relative dimension g, and $\lambda : A \to A^\vee$ is a principal polarization.

The functor $\mathcal{A}_g$ is represented by a Deligne-Mumford stack, separated of finite type and smooth over $\operatorname{Spec} \mathbb{Z}$ with relative dimension $\frac{g(g+1)}{2}$ (see [14]). It admits a smooth (toroidal) compactification [2].

We consider the Hecke stack $\mathrm{Hk}_{\mathcal{A}_g}$, whose S-points are tuples $(A, \lambda, A', \lambda', \varphi)$, where $(A, \lambda), (A', \lambda') \in \mathcal{A}_g(S)$, and $\varphi : A \to A'$ is a quasi-isogeny (i.e., $\mathbb{Q}$-isogeny) such that $\varphi^*(\lambda') = \lambda$. We have two natural projection maps, sending the tuple to the "head" (A, λ) and the "tail" (A', λ'), respectively:

$$\begin{array}{ccc} & \mathrm{Hk}_{\mathcal{A}_g} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{A}_g & & \mathcal{A}_g. \end{array}$$

By the theory of Hilbert scheme, the morphism $(\pi_1, \pi_2) : \mathrm{Hk}_{\mathcal{A}_g} \to \mathcal{A}_g \times \mathcal{A}_g$ is representable by a relative scheme (of non-finite type). We may impose a bound to the "denominator" of the quasi-isogeny φ. For $d \in \mathbb{N}$, let $\mathrm{Hk}^d_{\mathcal{A}_g}$ be the (open-and-closed) subfunctor, where we further require that $d \cdot \varphi : A \to A'$ is an *isogeny* (not merely a quasi-isogeny). Then the two projection maps $\mathrm{Hk}^d_{\mathcal{A}_g} \to \mathcal{A}_g$ are proper, and if we restrict them to $\operatorname{Spec} \mathbb{Z}[1/d]$, the two projection maps are finite étale (see [2, Chapter VII.3]; in fact, Faltings and Chai considered the "Hecke stack" Isog_g which allows quasi-isogeny φ of more general type than ours).

3.2 Moduli space of Picard type $\mathcal{M}_n$

The moduli functor $\mathcal{A}_g$ defines a canonical integral model for a Shimura variety associated to the symplectic group. We now move to define an analog for unitary groups. Here, we follow the treatment of Kudla and Rapoport [10]. The more general situation is studied in [16, 18].

We fix an imaginary quadratic field $E = \mathbb{Q}[\sqrt{-D}]$ where $-D < 0$ is a fundamental discriminant. Assume that the prime 2 is unramified in E (i.e., D is odd). Let O_E be the ring of integers in E.

Let $n \in \mathbb{N}$. Let $\mathcal{M}_n$ be the following moduli space of "Picard type": for a scheme S over $\operatorname{Spec} O_E$, $\mathcal{M}_n(S)$ is the groupoid of (A, ι, λ), where $(A, \lambda) \in \mathcal{A}_n(S)$ and $\iota : O_E \to \operatorname{End}_S(A)$ is an action of O_E on A such that

- the Kottwitz condition of signature $(n-1, 1)$ holds, i.e., the characteristic polynomial

$$\operatorname{char}(\iota(a) \mid \operatorname{Lie} A) = (T - \varphi(a))^{n-1}(T - \varphi(\overline{a})) \in \mathcal{O}_S[T] \quad \text{for all} \quad a \in O_E,$$

where $\varphi : O_E \to \mathcal{O}_S$ denotes the structure morphism and $a \mapsto \overline{a}$ the Galois conjugation, and

- the Rosati involution induced by λ on O_E via ι is the Galois conjugation, i.e.,

$$\operatorname{Ros}_\lambda(\iota(a)) = \iota(\overline{a}) \quad \text{for all} \quad a \in O_E.$$

We also consider a "companion" of $\mathcal{M}_1$: let $\mathcal{M}_1^*$ be the functor defined similarly except that the Kottwitz signature is $(1,0)$ rather than $(0,1)$.

The functor $\mathcal{M}_n$ is represented by a Deligne-Mumford stack, separated of finite type over $\operatorname{Spec} O_E$. Its restriction to $\operatorname{Spec} O_E[1/D]$ is smooth of relative dimension $n-1$ (see [10]).

Remark 3.1. Strictly speaking, the moduli space $\mathcal{M}_n$ is not an integral model of a Shimura variety. Rather, its generic fiber is a disjoint union of (copies) Shimura varieties attached to n-dimensional Hermitian spaces V that admit *self-dual* lattices. We refer to [10, Sections 2 and 4] for this subtle point. In any case, each connected component of the orbifold $\mathcal{M}_n(\mathbb{C})$ is a quotient of the unit ball in $\mathbb{C}^{n-1}$ by an arithmetic group of the form $\operatorname{Aut}(L)$, where L is a self-dual Hermitian lattice.

Next, we consider the Hecke stack $\mathrm{Hk}_{\mathcal{M}_n}$, whose S-points are tuples $(A,\iota,\lambda,A',\iota',\lambda',\varphi)$, where (A,ι,λ), $(A',\iota',\lambda')\in\mathcal{M}_n(S)$, and $\varphi: A\to A'$ is a quasi-isogeny such that $\varphi^*(\lambda')=\lambda$ and $\varphi^*(\iota')=\iota$. We have two natural projection maps, sending the tuple to (A,ι,λ) and (A',ι',λ'), respectively:

$$\begin{array}{ccc} & \mathrm{Hk}_{\mathcal{M}_n} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{M}_n & & \mathcal{M}_n. \end{array}$$

For $d\in\mathbb{N}$, let $\mathrm{Hk}^d_{\mathcal{M}_n}$ be the (open-and-closed) subfunctor, where we further require that the quasi-isogeny $d\cdot\varphi: A\to A'$ is an *isogeny*. Then the two projection maps $\pi_1,\pi_2:\mathrm{Hk}^d_{\mathcal{M}_n}\to\mathcal{M}_n$ are proper, and if we restrict them to $\operatorname{Spec}\mathbb{Z}[1/d]$, both are finite étale.

Finally, when $d=1$, the diagonal morphism $\Delta:\mathcal{M}_n\to\mathcal{M}_n\times\mathcal{M}_n$ induces an isomorphism

$$\mathrm{Hk}^{d=1}_{\mathcal{M}_n}\xrightarrow{\sim}\Delta_{\mathcal{M}_n}.$$

3.3 The first intersection problem

Denote $\widetilde{\mathcal{M}}_n=\mathcal{M}_1^*\times\mathcal{M}_n$. There is a natural morphism defined by taking "products" in terms of their moduli interpretation

$$\begin{aligned} \delta_{n-1}:\widetilde{\mathcal{M}}_{n-1} &\mapsto \widetilde{\mathcal{M}}_n \\ (A_0,\iota_0,\lambda_0,A^\flat,\iota^\flat,\lambda^\flat) &\mapsto (A_0,\iota_0,\lambda_0,A^\flat\times A_0,\iota^\flat\times\iota_0,\lambda^\flat\times\lambda_0). \end{aligned}\tag{3.1}$$

Then the *arithmetic diagonal cycle* is the graph of the above morphism

$$\Delta_{n-1}:\widetilde{\mathcal{M}}_{n-1}\to\widetilde{\mathcal{M}}_{n-1,n}:=\widetilde{\mathcal{M}}_{n-1}\times\widetilde{\mathcal{M}}_n.\tag{3.2}$$

Here, the fiber product is taken over the base $\operatorname{Spec} O_E[1/D]$. Note that one can also replace the target in (3.2) by $\mathcal{M}_1^*\times\mathcal{M}_{n-1}\times\mathcal{M}_n$.

Note that the Hecke stack $\mathrm{Hk}^d_{\mathcal{M}_n}$ introduced earlier can be extended to $\widetilde{\mathcal{M}}_n$ by base change along $\mathcal{M}_1^*\to\operatorname{Spec} O_E[1/D]$:

$$\begin{array}{ccc} & \mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \widetilde{\mathcal{M}}_n & & \widetilde{\mathcal{M}}_n. \end{array}$$

For simplicity we denote

$$\mathrm{Hk}^{(d_1,d_2)}:=\mathrm{Hk}^{d_1}_{\widetilde{\mathcal{M}}_{n-1}}\times\mathrm{Hk}^{d_2}_{\widetilde{\mathcal{M}}_n}.$$

Then the first intersection problem is to take the fiber product,

$$\begin{array}{ccc} \Delta^{(d_1,d_2)}_{\widetilde{\mathcal{M}}_{n-1}} & \longrightarrow & \mathrm{Hk}^{(d_1,d_2)} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_{n-1}\times\widetilde{\mathcal{M}}_{n-1} & \xrightarrow{(\Delta_{n-1},\Delta_{n-1})} & \widetilde{\mathcal{M}}_{n-1,n}\times\widetilde{\mathcal{M}}_{n-1,n} \end{array}$$

and to study the class in the Chow group

$$(\Delta_{n-1}, \Delta_{n-1})^!([\mathrm{Hk}^{(d_1,d_2)}]) \in \mathrm{Ch}_0(\Delta^{(d_1,d_2)}_{\widetilde{\mathcal{M}}_{n-1}})_{\mathbb{Q}}.$$

Here, $(\Delta_{n-1}, \Delta_{n-1})^!$ is the Gysin homomorphism with respect to the morphism $(\Delta_{n-1}, \Delta_{n-1})$ (see [3,5]), which is a regular local immersion. This class in Chow group is closely related to the variants of the arithmetic Gan-Gross-Prasad conjecture formulated by Rapoport et al. [16].

We consider a special case where $d_1 = 1$ and $d_2 = d \in \mathbb{N}$. Then the intersection problem is essentially reduced to the following cartesian diagram:

$$\begin{array}{ccc} \Delta^d_{\widetilde{\mathcal{M}}_{n-1}} & \longrightarrow & \mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n} \\ \downarrow & & \downarrow{\scriptstyle (\pi_1,\pi_2)} \\ \widetilde{\mathcal{M}}_{n-1} & \xrightarrow{\Delta\circ\delta_{n-1}} & \widetilde{\mathcal{M}}_n \times \widetilde{\mathcal{M}}_n, \end{array} \tag{3.3}$$

and the class in the Chow group

$$(\Delta \circ \delta_{n-1})^!([\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n}]) \in \mathrm{Ch}_0(\Delta^d_{\widetilde{\mathcal{M}}_{n-1}})_{\mathbb{Q}}.$$

Since this will be a special case of the second intersection problem below, we postpone a more detailed analysis.

3.4 Special divisors

We recall the construction by Kudla and Rapoport [10] of (global) special divisors on the moduli stack $\widetilde{\mathcal{M}}_n = \mathcal{M}_1^* \times \mathcal{M}_n$. Let $\mathcal{Z}$ be the moduli functor over $\widetilde{\mathcal{M}}_n = \mathcal{M}_1^* \times \mathcal{M}_n$, whose S-points are tuples

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \phi),$$

where $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_1^*(S), (A, \iota, \lambda) \in \mathcal{M}_n(S)$ and $\phi \in \mathrm{Hom}_{O_E}(A_0, A)$.

Given a geometric point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \widetilde{\mathcal{M}}_n(\mathrm{Spec}\,\kappa)$ for any algebraically closed field κ, one can associate an $E/\mathbb{Q}$-Hermitian space as follows. Let

$$V(A_0, A) = \mathrm{Hom}^\circ_{O_E}(A_0, A), \tag{3.4}$$

which is an E-vector space, endowed with a Hermitian pairing by the formula

$$\langle x, y\rangle := \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}^\circ_{O_E}(A_0) \simeq E.$$

It is positive definite by [10, Lemma 2.7]. Call $\deg_E(x) := \langle x, x\rangle \in \mathbb{Q}$ the Hermitian degree of $x \in V(A_0, A)$. It induces a locally constant map (in Zariski topology)

$$\deg_E \colon \mathcal{Z} \to \mathbb{N}.$$

Let $\mathcal{Z}_m$ denote the preimage of $m \in \mathbb{N}$, an open-and-closed substack of $\mathcal{Z}$. The forgetful morphism

$$i_m \colon \mathcal{Z}_m \to \widetilde{\mathcal{M}}_n$$

is finite and unramified. Its image is a locally principal divisor on $\widetilde{\mathcal{M}}_n$.

It is easy to see that, when $m = 1$, the morphism δ_{n-1} induces an isomorphism

$$\mathcal{Z}_1 \xrightarrow{\sim} \widetilde{\mathcal{M}}_{n-1}.$$

Remark 3.2. One can define a generating series with coefficients in the Chow group

$$c_0 + \sum_{m\geqslant 1} \mathcal{Z}_m\, q^m \in \mathrm{Ch}^1(\widetilde{\mathcal{M}}_n)_{\mathbb{Q}}[\![q]\!],$$

where c_0 is a suitable multiple of the first Chern class of the Hodge bundle ω (see [1, Subsection 2.4]). Then a theorem of Bruinier et al. [1] asserts that this generating series is (the q-expansion of) a holomorphic form of weight n. In fact, in *loc. cit*, more is proved by upgrading special divisors $\mathcal{Z}_m$ to elements in the (Arakelov) arithmetic Chow group of the toroidal compactification of a regular integral model over the full ring of integers O_E.

3.5 The second intersection problem

With the collection of special divisors $\mathcal{Z}_m$, we come to the second intersection problem, i.e., to take the fiber product

$$\begin{CD} \Delta^d_{\mathcal{Z}_m} @>>> \mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n} \\ @VVV @VV{(\pi_1,\pi_2)}V \\ \mathcal{Z}_m @>{\Delta\circ i_m}>> \widetilde{\mathcal{M}}_n \times \widetilde{\mathcal{M}}_n \end{CD} \tag{3.5}$$

and to study the class in the Chow group

$$(\Delta \circ i_m)^!([\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n}]) \in \mathrm{Ch}_0(\Delta^d_{\mathcal{Z}_m})_{\mathbb{Q}}. \tag{3.6}$$

The case $m = 1$ specializes to (3.3). Later we will decompose $\Delta^d_{\mathcal{Z}_m}$ into a disjoint union of connected components, and hence decompose the above zero cycle into summands. When the connected components are proper with support on special fibers, it makes sense to take the degree of the corresponding summand of the zero cycle.

3.6 CM cycles on $\mathcal{A}_g$

In this and the next subsection we come to a key construction in the proof of Theorem 2.3 in [26]. We first introduce the CM cycles on $\mathcal{A}_g$; they are simpler to describe than their analog on $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$.

We consider the "fixed point of the Hecke correspondence $\mathrm{Hk}_{\mathcal{A}_g}$", i.e., the fiber product $\mathcal{A}_g^{\mathrm{Hk}}$ of the Hecke correspondence and the diagonal in $\mathcal{A}_g \times \mathcal{A}_g$:

$$\begin{CD} \mathcal{A}_g^{\mathrm{Hk}} @>>> \mathrm{Hk}_{\mathcal{A}_g} \\ @VVV @VV{(\pi_1,\pi_2)}V \\ \mathcal{A}_g @>{\Delta}>> \mathcal{A}_g \times \mathcal{A}_g. \end{CD}$$

Here, the fiber product $\mathcal{A}_g \times \mathcal{A}_g$ is taken over the base $\mathrm{Spec}\,\mathbb{Z}$. Then the S-points of $\mathcal{A}_g^{\mathrm{Hk}}$ are tuples (A, λ, φ), where $\varphi \in \mathrm{End}^\circ(A)$ such that $\varphi^*(\lambda) = \lambda$. To any geometric point $(A, \lambda, \varphi) \in \mathcal{A}_g^{\mathrm{Hk}}(\kappa)$ for an algebraically closed field κ, we can associate a "characteristic polynomial"

$$\mathrm{char}(A, \lambda, \varphi) = \det(T - \varphi \mid \mathrm{V}_\ell(A)) \in \mathbb{Q}[T]_{\deg=2g},$$

where $\mathrm{V}_\ell(A) = \mathrm{T}_\ell(A) \bigotimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is the rational ℓ-adic Tate module for a prime ℓ invertible in the field κ. The coefficients of the characteristic polynomial lie in $\mathbb{Q}$, because they can be computed as intersection numbers between algebraic cycles (with $\mathbb{Q}$-coefficients) on powers of A. Moreover, since φ preserves a symplectic form on $\mathrm{V}_\ell(A)$, the characteristic polynomial is self-reciprocal.

We obtain a locally constant map (in Zariski topology for the source and the discrete topology for the target)

$$\mathrm{char}\colon \mathcal{A}_g^{\mathrm{Hk}} \to \mathbb{Q}[T]_{\deg=2g}. \tag{3.7}$$

Replacing $\mathrm{Hk}_{\mathcal{A}_g}$ by $\mathrm{Hk}^d_{\mathcal{A}_g}$ (of finite type), we obtain

$$\mathrm{char}_d\colon \mathcal{A}_g^{\mathrm{Hk},d} \to \mathbb{Q}[T]_{\deg=2g},$$

whose image is finite and contained in the smaller ring $\mathbb{Z}[1/d][T]_{\deg=2g}$.

For a self-reciprocal monic polynomial $a \in \mathbb{Q}[T]_{\deg=2g}$, let $\mathcal{A}_g^d(a)$ denote the preimage of a. It is an open-and-closed substack of $\mathcal{A}_g^{\mathrm{Hk},d}$. We have a (finite) disjoint union

$$\mathcal{A}_g^{\mathrm{Hk},d} = \coprod_{a\in \mathrm{Im}(\mathrm{char}_d)} \mathcal{A}_g^d(a). \tag{3.8}$$

Remark 3.3. There is a similar situation on moduli stacks of Shtukas (see [22, Subsection 6.2.7]).

Let a be in the image of the map char. Denote the $\mathbb{Q}$-algebra

$$\mathbb{Q}[a] := \mathbb{Q}[T]/(a).$$

By the self-reciprocality of a, the $\mathbb{Q}$-algebra $\mathbb{Q}[a]$ carries an involution induced by $T \mapsto 1/T$. Let a be regular semisimple in the sense that a has no repeated roots. Then $\mathbb{Q}[a]$ is necessarily a CM algebra (i.e., a product of CM fields) of degree $[\mathbb{Q}[a] : \mathbb{Q}] = 2g$. In this case, we may call $\mathcal{A}_g^d(a)$ the *naive CM cycle* (indexed by a).

Remark 3.4. Fix $d \in \mathbb{N}$. Let a be regular semisimple. Let R_a be the order $\mathbb{Z}[dT]$ in $\mathbb{Q}[a] = \mathbb{Q}[T]/(a)$, which is stable under the involution just defined. Let a_d be the "characteristic polynomial of $d \cdot \varphi$" (i.e., $a_d(T) = a(dT)$). Then $a_d \in \mathbb{Z}[T]$, and $R_a \simeq \mathbb{Z}[T]/(a_d)$. Up to isomorphism, the stack $\mathcal{A}_g^d(a)$ depends only on the order R_a with its involution. In fact, the S-points of the stack $\mathcal{A}_g^d(a)$ can be described as tuples (A, ι, λ), where $\iota : R_a \to \mathrm{End}(A)$ such that Ros_λ induces the given involution on R_a (without the Kottwitz condition). Therefore we may also denote $\mathcal{A}_g^d(a)$ by $\mathcal{A}_{g,R_a}^d$.

The naive CM cycles forget how the two stacks in the fiber product sit above the ambient $\mathcal{A}_g \times \mathcal{A}_g$. Therefore we consider the intersection product, i.e., the Gysin homomorphism with respect to Δ,

$$\Delta^! : \mathrm{Ch}_g(\mathrm{Hk}_{\mathcal{A}_g}^d)_{\mathbb{Q}} \to \mathrm{Ch}_1(\mathcal{A}_g^{\mathrm{Hk},d})_{\mathbb{Q}}.$$

This is well defined since $\mathcal{A}_g$ is smooth over $\mathrm{Spec}\,\mathbb{Z}$, and hence Δ is a regular local immersion [3,5]. We define "the derived fixed point cycle"

$${}^{\mathbb{L}}\mathcal{A}_g^d := \Delta^!([\mathrm{Hk}_{\mathcal{A}_g}^d]) \in \mathrm{Ch}_1(\mathcal{A}_g^{\mathrm{Hk},d})_{\mathbb{Q}}.$$

By (3.8), we have a decomposition

$$\mathrm{Ch}_1(\mathcal{A}_g^{\mathrm{Hk},d})_{\mathbb{Q}} = \bigoplus_{a \in \mathrm{Im}(\mathrm{char}_d)} \mathrm{Ch}_1(\mathcal{A}_g^d(a))_{\mathbb{Q}}.$$

We denote the a-th component by

$${}^{\mathbb{L}}\mathcal{A}_g^d(a) \in \mathrm{Ch}_1(\mathcal{A}_g^d(a))_{\mathbb{Q}}.$$

Remark 3.5. Fix $d \in \mathbb{N}$. Let a be regular semisimple. Unlike the naive CM cycle $\mathcal{A}_g^d(a)$, which depends only on the order R_a with its involution, the derived CM cycle ${}^{\mathbb{L}}\mathcal{A}_g^d(a)$ does depend on more refined invariants than R_a. Informally speaking, it depends on a presentation of R_a as $\mathbb{Z}[T]/(a_d)$.

3.7 CM cycles on $\mathcal{M}_n$

The construction of the (derived) CM cycles on $\mathcal{A}_g$ can be verbatim carried over to $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$. Let us focus on $\widetilde{\mathcal{M}}_n$. We consider the "fixed point of the Hecke correspondence $\mathrm{Hk}_{\widetilde{\mathcal{M}}_n}$":

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_n^{\mathrm{Hk}} & \longrightarrow & \mathrm{Hk}_{\widetilde{\mathcal{M}}_n} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_n & \xrightarrow{\Delta} & \widetilde{\mathcal{M}}_n \times \widetilde{\mathcal{M}}_n. \end{array}$$

Here and henceforth, we work with the base $\mathrm{Spec}\, O_E[1/D]$. Using the action of O_E, we can upgrade the map (3.7) to a locally constant map (in Zariski topology for the source and the discrete topology for the target)

$$\mathrm{char}_E : \widetilde{\mathcal{M}}_n^{\mathrm{Hk}} \to E[T]_{\deg = n},$$

and its restriction to the degree d part

$$\mathrm{char}_{E,d} : \widetilde{\mathcal{M}}_n^{\mathrm{Hk},d} \to E[T]_{\deg = n}. \tag{3.9}$$

We have the analog of (3.8)

$$\widetilde{\mathcal{M}}_n^{\mathrm{Hk},d} = \coprod_{a \in \mathrm{Im}(\mathrm{char}_{E,d})} \widetilde{\mathcal{M}}_n^d(a), \tag{3.10}$$

of the Gysin homomorphism

$$\Delta^!\colon \mathrm{Ch}_n(\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n})_{\mathbb{Q}} \to \mathrm{Ch}_1(\widetilde{\mathcal{M}}_n^{\mathrm{Hk},d})_{\mathbb{Q}},$$

of "the derived fixed point cycle"

$$^{\mathbb{L}}\widetilde{\mathcal{M}}_n^d := \Delta^!([\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n}]) \in \mathrm{Ch}_1(\widetilde{\mathcal{M}}_n^{\mathrm{Hk},d})_{\mathbb{Q}}, \tag{3.11}$$

and the decomposition into a sum

$$^{\mathbb{L}}\widetilde{\mathcal{M}}_n^d = \sum_{a \in \mathrm{Im}(\mathrm{char}_{E,d})} {}^{\mathbb{L}}\widetilde{\mathcal{M}}_n^d(a) \in \bigoplus_{a \in \mathrm{Im}(\mathrm{char}_{E,d})} \mathrm{Ch}_1(\widetilde{\mathcal{M}}_n^d(a))_{\mathbb{Q}}. \tag{3.12}$$

3.8 The third intersection problem

Now we come to the third intersection problem, i.e., to take the fiber product

$$\begin{array}{ccc} \Delta^d_{\mathcal{Z}_m} & \longrightarrow & \widetilde{\mathcal{M}}_n^{\mathrm{Hk},d} \\ \downarrow & & \downarrow \\ \mathcal{Z}_m & \xrightarrow{\ i_m\ } & \widetilde{\mathcal{M}}_n \end{array} \tag{3.13}$$

and to study the class in the Chow group

$$^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m} := (i_m)^!({}^{\mathbb{L}}\widetilde{\mathcal{M}}_n^d) \in \mathrm{Ch}_0(\Delta^d_{\mathcal{Z}_m})_{\mathbb{Q}}, \tag{3.14}$$

under the Gysin homomorphism $(i_m)^!\colon \mathrm{Ch}_1(\widetilde{\mathcal{M}}_n^{\mathrm{Hk},d})_{\mathbb{Q}} \to \mathrm{Ch}_0(\Delta^d_{\mathcal{Z}_m})_{\mathbb{Q}}$.

The composition of the map $\mathrm{char}_{E,d}$ (3.9) with $\Delta^d_{\mathcal{Z}_m} \to \widetilde{\mathcal{M}}_n^{\mathrm{Hk},d}$ gives us a locally constant map

$$\mathrm{char}_{E,d}\colon \Delta^d_{\mathcal{Z}_m} \to E[T]_{\deg = n}, \tag{3.15}$$

and hence induces a decomposition into a disjoint union

$$\Delta^d_{\mathcal{Z}_m} = \coprod_{a \in \mathrm{Im}(\mathrm{char}_{E,d})} \Delta^d_{\mathcal{Z}_m}(a). \tag{3.16}$$

Corresponding to (3.12), the class (3.14) is a sum

$$^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m} = \sum_{a \in \mathrm{Im}(\mathrm{char}_{E,d})} {}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a). \tag{3.17}$$

Remark 3.6. The three global intersection problems (3.3), (3.5) and (3.13) correspond to the local ones (2.1), (2.5) and (2.7), respectively. As mentioned earlier, the third one is an intersection between 1-cycles and divisors, while the first two involve high codimensional cycles. This makes it possible to take advantage of the modular generating series of special divisors (see Remark 3.2). Moreover, the decomposition (3.17) gives us some refined information, for example, for a fixed regular semisimple a, we can consider the generating series (for a suitable constant term c_0)

$$c_0 + \sum_{m \geqslant 1} {}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a)\, q^m,$$

viewing the coefficients as elements in $\mathrm{Ch}_0(\widetilde{\mathcal{M}}_n)_{\mathbb{Q}}$. This generating series itself is not immediately useful. Nevertheless, the coefficients can be upgraded to elements in the (Arakelov) arithmetic Chow group, and the upgraded generating series then plays a key role in [26].

Now we indicate the connection between the three intersection problems. We can break (3.5) into two cartesian diagrams:

$$\begin{array}{ccccc} \Delta^d_{\mathcal{Z}_m} & \longrightarrow & \widetilde{\mathcal{M}}^{\mathrm{Hk},d}_n & \longrightarrow & \mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_m & \xrightarrow{i_m} & \widetilde{\mathcal{M}}_n & \xrightarrow{\Delta} & \widetilde{\mathcal{M}}_n \times \widetilde{\mathcal{M}}_n, \end{array} \tag{3.18}$$

and hence by functoriality of Gysin homomorphisms

$$(\Delta \circ i_m)^! = (i_m)^! \circ \Delta^!.$$

It follows that the class (3.6) in $\mathrm{Ch}_0\left(\Delta^d_{\mathcal{Z}_m}\right)_{\mathbb{Q}}$ can be expressed in terms of the Gysin pull-back of the derived fixed point cycle (3.14):

$$\begin{aligned} (\Delta \circ i_m)^!([\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n}]) &= (i_m)^! \circ \Delta^!([\mathrm{Hk}^d_{\widetilde{\mathcal{M}}_n}]) \\ &= (i_m)^!({}^{\mathbb{L}}\widetilde{\mathcal{M}}^d_n). \end{aligned}$$

Remark 3.7. On $\mathcal{A}_g$, besides the Hodge (line) bundle ω (see [2, p. 24]), there do not exist such a natural collection of special divisors as $\mathcal{Z}_m$ on $\widetilde{\mathcal{M}}_n$. For a regular semi-simple $a \in \mathbb{Q}[T]_{\deg=2g}$, it is an interesting question to study the arithmetic degree of the metrized Hodge bundle $\widehat{\omega}$ along the 1-cycle ${}^{\mathbb{L}}\mathcal{A}^d_g(a)$:

$$\deg_{d,a} \widehat{\omega} := (\widehat{\omega},\ {}^{\mathbb{L}}\mathcal{A}^d_g(a))_{\mathcal{A}_g} \in \mathbb{R}.$$

For the precise meaning of the pairing, see [26, Subsection 9.1]. This should be related to the Faltings height of CM abelian varieties (but contains a little more information due to the contribution of components supported on special fibers).

3.9 From the global to the local

We would like to indicate how to connect the global intersection problem to the local ones in the AFL conjecture.

One can show that, if $a \in E[T]_{\deg=n}$ is irreducible, then $\Delta^d_{\mathcal{Z}_m}(a)$ (3.16) is proper over a finite subscheme of $\mathrm{Spec}\, O_E[1/D]$ (in fact, supported on the supersingular locus of $\widetilde{\mathcal{M}}_n$ above inert primes [16, Theorem 8.5]). We take the arithmetic degree

$$\mathrm{Int}^d_m(a) := \deg({}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a)).$$

Here for an element $\xi \in \mathrm{Ch}_0(X)_{\mathbb{Q}}$ on a DM stack $X = \Delta^d_{\mathcal{Z}_m}(a)$, and $\pi : X \to B$ a proper morphism with finite image in $B = \mathrm{Spec}\, O_E[1/D]$, the push-forward $\pi_*(\xi)$ is a zero cycle on B of the form

$$\sum_{v \in |B|} \xi_v [v], \quad \xi_v \in \mathbb{Q}$$

(a finite sum of prime divisors v of B). Then the arithmetic degree is defined as

$$\deg(\xi) = \sum_{v \in |B|} \xi_v \log q_v \in \mathbb{R},$$

where q_v is the cardinality of the residue field at v.

In fact, there is a refinement of the locally constant map (3.15),

$$\mathrm{inv}_{d,m} \colon \Delta^d_{\mathcal{Z}_m} \to E[T]_{\deg=n} \times E^n \tag{3.19}$$

sending a geometric point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \phi, \varphi) \in \Delta^d_{\mathcal{Z}_m}(\kappa)$ to $(a, b) \in E[T]_{\deg=n} \times E^n$, where

$$a = \mathrm{char}_E(\varphi), \quad \text{and} \quad b = (b_i)_{i=0}^{n-1} \quad \text{with} \quad b_i = \langle \varphi^i \circ \phi, \phi \rangle.$$

Here $\phi \in \mathrm{V}(A_0, A)$ (see (3.4)), and $\varphi \in \mathrm{End}^\circ(A)$ is a quasi-isogeny acting on $\mathrm{V}(A_0, A)$. By definition, we have $b_0 = m$. The map (3.19) is a global analog of (2.8).

Remark 3.8. The invariants (a, b) in (3.19) may be viewed as the number field analog of those invariants appearing in the (relative) "Hitchin moduli space" in Yun's proof of the Jacquet-Rallis fundamental lemma for local fields with positive characteristics [21].

Now we define $\Delta^d_{\mathcal{Z}_m}(a, b)$ to be the preimage $\mathrm{inv}^{-1}_{d,m}(a, b)$, and refine the decomposition (3.17),

$$^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a) = \sum_{b \in E^n} {}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a, b),$$

where the b-th summand in the right-hand side is by definition the b-th component of the left-hand side according to the direct sum

$$\mathrm{Ch}_0(\Delta^d_{\mathcal{Z}_m}(a))_{\mathbb{Q}} = \bigoplus_{b \in E^n} \mathrm{Ch}_0(\Delta^d_{\mathcal{Z}_m}(a, b))_{\mathbb{Q}}.$$

Similar to $\mathrm{Int}^d_m(a)$, we define

$$\mathrm{Int}^d(a, b) := \deg({}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a, b)),$$

where we have dropped m in the notation since $m = b_0$ is determined by $b \in E^n$. If $a \in E[T]_{\deg=n}$ is irreducible, we obtain a decomposition

$$\mathrm{Int}^d_m(a) = \sum_{b=(b_i)_{i=0}^{n-1} \in E^n,\ b_0 = m} \mathrm{Int}^d(a, b).$$

Finally, we are ready to state the global-to-local relation. Fix an irreducible $a \in E[T]_{\deg=n}$, and $b \in E^m$ such that $b_0 \neq 0$. Then one can show that $\Delta^d_{\mathcal{Z}_m}(a, b)$ (if non-empty) must have support in the supersingular locus above a unique place v of $\mathbb{Q}$, and this place is necessarily inert in E (see [16, Theorem 8.5]). Assume that $v \nmid dD$. Then there exists a positive definite n-dimensional $E/\mathbb{Q}$-Hermitian space V that is non-split at v, and a suitable function $\Phi^v_d = \bigotimes_{\nu \neq v,\infty} \Phi_\nu \in \mathscr{C}^\infty_c((\mathrm{U}(V) \times V)(\mathbb{A}^v_f))$ such that the global intersection number decomposes

$$\mathrm{Int}^d(a, b) = \mathrm{Orb}((a, b), \Phi^v_d) \cdot \mathrm{Int}_v(a, b) \log q_v, \tag{3.20}$$

where $\mathrm{Int}_v(a, b)$ is the local intersection number defined by (2.7) and Remark 2.6 (relative to the unramified quadratic extension $E_v/\mathbb{Q}_v$). Here, $\mathbb{A}^v_f$ is the ring of finite adeles of $\mathbb{Q}$ away from v, and the orbital integral $\mathrm{Orb}\,((a, b), \Phi^v_d) = \prod_{\nu \neq v,\infty} \mathrm{Orb}\,((a, b), \Phi_\nu)$ is taken with respect to the action of $\mathrm{U}(V)$ on $\mathrm{U}(V) \times V$ (see Remark 2.6). A precise statement of (3.20) can be found [26, Theorem 9.3], proved along with the same line of [23, Theorem 3.11] and [16, Theorem 8.15].

Remark 3.9. The $E/\mathbb{Q}$-Hermitian space V is uniquely determined by (a, b), and the function $\Phi^v_d = \bigotimes_{\nu \neq v,\infty} \Phi_\nu$ can be made explicit (see Remark 3.1).

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