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General Section

Effective bounds for traces of singular moduli

Havi Ellers[a], Meagan Kenney[b], Riad Masri[c,*], Wei-Lun Tsai[c]

[a] Department of Mathematics, Harvey Mudd College, 301 Platt Boulevard, Claremont, CA 91711-5901, United States of America

[b] Department of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, United States of America

[c] Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843-3368, United States of America

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A B S T R A C T

We give an asymptotic formula for traces of weak Maass forms at CM points with an effective bound on the error term. Upon specializing to the modular j-function, we deduce such a result for traces of singular moduli. Due to work of Zagier, and Bringmann and Ono, these traces of weak Maass forms at CM points appear as Fourier coefficients of half-integral weight weakly holomorphic modular forms. Hence, our results give effective upper bounds for these Fourier coefficients.

1. Introduction and statement of main results

Let $-d < 0$ be an integer with $d \equiv 0, 3 \pmod 4$. Let $\mathcal{Q}_d$ be the set of $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$-reduced, positive definite, integral binary quadratic forms of discriminant $-d$, and let $H(d) = \#\mathcal{Q}_d$ be the Hurwitz-Kronecker class number of $-d$. Given a form $Q(x, y) = a_Q x^2 + b_Q xy + c_Q y^2 \in \mathcal{Q}_d$, define the associated CM point

* Corresponding author.
E-mail addresses: hellers@g.hmc.edu (H. Ellers), kenn0699@umn.edu (M. Kenney), masri@math.tamu.edu (R. Masri), wltsai@math.tamu.edu (W.-L. Tsai).

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$$\tau_Q := \frac{-b_Q + \sqrt{-d}}{2a_Q} \in \mathbb{H}$$

in the complex upper half-plane $\mathbb{H}$. Since Q is reduced, τ_Q lies in the standard fundamental domain $\mathcal{F}$ for Γ.

Given a fundamental discriminant D_1, let χ_{D_1} be the associated genus character for positive definite, integral binary quadratic forms Q whose discriminants are multiples of D_1 given by

$$\begin{aligned}\chi_{D_1}(Q) &= \chi_{D_1}(a_Q, b_Q, c_Q) \\ &:= \begin{cases} 0, & \text{if } (a_Q, b_Q, c_Q, D_1) > 1, \\ \left(\frac{D_1}{r}\right), & \text{if } (a_Q, b_Q, c_Q, D_1) = 1 \text{ and } Q \text{ represents } r \text{ with } (r, D_1) = 1, \end{cases}\end{aligned}$$

where $\left(\frac{D_1}{\cdot}\right)$ is the Kronecker symbol.

Let λ be an integer and D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$. Given a Γ-invariant function $f : \mathbb{H} \to \mathbb{C}$, define the twisted trace of the singular modulus $f(\tau_Q)$ by

$$\operatorname{Tr}(f; D_1, D_2) := \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q},$$

where ω_Q is given by

$$\omega_Q := \begin{cases} 3 & \text{if} \quad \tau_Q = \frac{-1+\sqrt{-3}}{2}, \\ 2 & \text{if} \quad \tau_Q = \sqrt{-1}, \\ 1 & \text{otherwise.} \end{cases}$$

For simplicity, when $D_1 = 1$ and $D_2 = d$ we write

$$\operatorname{Tr}_d(f) := \operatorname{Tr}(f; 1, d) = \sum_{Q \in \mathcal{Q}_d} \frac{f(\tau_Q)}{\omega_Q}.$$

The prototype of such a trace involves CM values of the classical modular j-function, which has the Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + \cdots, \quad q := e(z) = e^{2\pi i z}.$$

The CM values $j(\tau_Q)$ are algebraic integers called *singular moduli*.

Let $J(z) := j(z) - 744$, and define $M_{3/2}^{!,+}(4)$ to be the vector space of weight $3/2$ weakly holomorphic modular forms for $\Gamma_0(4)$ satisfying the Kohnen plus space condition.

In the seminal paper [15], Zagier proved that the generating functions of the traces $\operatorname{Tr}(J; D_1, D_2)$ appear as holomorphic parts of forms in $M_{3/2}^{!,+}(4)$. For example, one has

$$-q^{-1}+2+\sum_{0<d\equiv 0,3 \pmod 4}\mathrm{Tr}_d(J)q^d=-q^{-1}+2-248q^3+492q^4 \\ -4119q^7+\cdots\in M^{!,+}_{3/2}(4).$$

Bringmann and Ono [1] generalized Zagier's results to traces of an important family of weak Maass forms at CM points. To define these functions, for $\nu\in\mathbb{Z}^+$ and $\lambda\in\mathbb{R}$, let

$$F_{\lambda,\nu}(z):=\begin{cases}\pi\nu^{\lambda-\frac{1}{2}}\displaystyle\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\mathrm{Im}\left(\gamma z\right)^{\frac{1}{2}}I_{\lambda-\frac{1}{2}}\left(2\pi\nu\mathrm{Im}\left(\gamma z\right)\right)e\left(-\nu\mathrm{Re}\left(\gamma z\right)\right), & \text{if } \lambda>1,\\ -12+\pi\nu^{\frac{1}{2}}\displaystyle\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\mathrm{Im}\left(\gamma z\right)^{\frac{1}{2}}I_{\frac{1}{2}}\left(2\pi\nu\mathrm{Im}\left(\gamma z\right)\right)e\left(-\nu\mathrm{Re}\left(\gamma z\right)\right), & \text{if } \lambda=1,\end{cases}$$

where $I_{\lambda-1/2}$ is the I-Bessel function of order $\lambda-1/2$.

The function $F_{\lambda,\nu}(z)$ is a Γ-invariant eigenfunction for the weight 0 hyperbolic Laplacian

$$\Delta_0:=-y^2\left(\partial_x^2+\partial_y^2\right)$$

with eigenvalue $\lambda(1-\lambda)$. Moreover, Niebur [12] showed that $F_{\lambda,\nu}(z)$ has an analytic continuation to $\mathrm{Re}(\lambda)>1/2$.

Now, it is known that

$$J(z)=2F_{1,1}(z). \tag{1.1}$$

Therefore, it is natural to ask whether the generating functions of the traces $\mathrm{Tr}(F_{\lambda,\nu};D_1,D_2)$ also appear as holomorphic parts of half-integral weight weakly holomorphic modular forms. This is indeed the case, as was proved in [1].

Let $\lambda\in\mathbb{Z}^+$ and $m\in\mathbb{Z}^+$ be such that $(-1)^{\lambda+1}m$ is a fundamental discriminant (which includes 1). Bringmann and Ono constructed a weight $\lambda+1/2$ Poincaré series $F_\lambda(-m;z)$ for $\Gamma_0(4)$ using Kohnen's projection operator (see [1, Section 2]) and showed that it has the Fourier expansion

$$F_\lambda(-m;z)=q^{-m}+\sum_{\substack{d\geq 0\\ (-1)^\lambda d\equiv 0,1 \pmod 4}}b_\lambda(-m;d)q^d\in M^{!,+}_{\lambda+1/2}(4).$$

They then proved the following result.

Theorem 1.1 *(Bringmann/Ono [1], Theorem 1.2). Let $\lambda\in\mathbb{Z}^+$ and $m\in\mathbb{Z}^+$ be such that $(-1)^{\lambda+1}m$ is a fundamental discriminant. Then for each $d\in\mathbb{Z}^+$ with $(-1)^\lambda d\equiv 0,1 \pmod 4$ we have*

$$b_\lambda(-m;d)=\frac{2(-1)^{\left\lfloor\frac{\lambda+1}{2}\right\rfloor}d^{\frac{\lambda-1}{2}}}{m^{\frac{\lambda}{2}}}\mathrm{Tr}(F_{\lambda,1};(-1)^{\lambda+1}m,d). \tag{1.2}$$

Now, define the twisted exponential sum (see [8, Eq. (18) and Proposition 5])

$$S_{D_1,D_2}(\nu,c) := \sum_{x^2\equiv -|D_1D_2| \pmod{c}} \chi_{D_1}\left(\frac{c}{4}, x, \frac{x^2+|D_1D_2|}{c}\right) e^{\frac{4\pi i\nu x}{c}}.$$

For simplicity, when $D_1 = \nu = 1$ and $D_2 = d$ we write

$$S_d(c) := S_{1,d}(1,c) = \sum_{x^2\equiv -d \pmod{c}} e^{\frac{4\pi i x}{c}}.$$

We will prove the following asymptotic formula for $\mathrm{Tr}(F_{\lambda,\nu}; D_1, D_2)$ with an effective bound on the error term.

Theorem 1.2. *Let D_1 be a fundamental discriminant. Let $\lambda, D_2, \nu \in \mathbb{Z}^+$ be such that $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$. Then we have*

$$\mathrm{Tr}(F_{\lambda,\nu}; D_1, D_2) = \frac{\nu^{\lambda-1}}{4} \sum_{k=0}^{\lambda-1} c_{k,\lambda} \sum_{\substack{0<c<2\sqrt{|D_1D_2|} \\ c\equiv 0 \pmod 4}} S_{D_1,D_2}(\nu,c) \left(\frac{8\pi\nu\sqrt{|D_1D_1|}}{c}\right)^{-k} \times e^{\frac{4\pi\nu\sqrt{|D_1D_2|}}{c}} + R(\lambda,\nu,D_1D_2),$$

where

$$c_{k,\lambda} := (-1)^k \frac{(\lambda-1+k)!}{k!(\lambda-1-k)!} \tag{1.3}$$

and $R(\lambda,\nu,D_1D_2)$ satisfies the bound

$$|R(\lambda,\nu,D_1D_2)| \le C(\lambda,\nu) H(|D_1D_2|),$$

where $C(\lambda,\nu)$ is the constant defined by (3.9). Moreover, we have

$$|\mathrm{Tr}(F_{\lambda,\nu}; D_1, D_2)| \le C'(\lambda,\nu) e^{\pi\nu\sqrt{|D_1D_2|}},$$

where $C'(\lambda,\nu)$ is the constant defined by (3.11).

Remark 1.3. By combining Theorem 1.2 with the identity (1.2), we get the following effective bound for the Fourier coefficients $b_\lambda(-m,d)$ of the Poincaré series $F_\lambda(-m;z) \in M^{!,+}_{\lambda+1/2}(4)$ constructed by Bringmann and Ono:

$$|b_\lambda(-m,d)| \le 2C'(\lambda,1) \frac{d^{\frac{\lambda-1}{2}}}{m^{\frac{\lambda}{2}}} e^{\pi\sqrt{md}}.$$

Table 1
Values of $C(\lambda,\nu)$ and $C'(\lambda,\nu)$.

(λ,ν)	$C(\lambda,\nu)\approx$	$C'(\lambda,\nu)\approx$
(1,1)	194632	58325
(1,2)	1.05501×10^{9}	3.16418×10^{8}
(1,3)	3.32284×10^{12}	9.96830×10^{11}
(2,1)	1.25315×10^{6}	3.75714×10^{5}
(2,2)	2.39544×10^{10}	7.18607×10^{9}
(2,3)	1.61592×10^{14}	4.84774×10^{13}

The `Mathematica` code to compute the constants $C(\lambda,\nu)$ and $C'(\lambda,\nu)$ appearing in Theorem 1.2 is available at [5]. In Table 1, we display some values of these constants.

We will deduce the following special case of Theorem 1.2 for traces of singular moduli.

Corollary 1.4. *For each $d\in\mathbb{Z}^+$ with $-d\equiv 0,1 \pmod 4$, we have*

$$\mathrm{Tr}_d(J)=\frac{1}{2}\sum_{\substack{0<c<2\sqrt{d}\\ c\equiv 0 \pmod 4}} S_d(c)e^{\frac{4\pi\sqrt{d}}{c}}+R(d),$$

where

$$|R(d)|\le(389264)H(d).$$

Moreover, we have

$$|\mathrm{Tr}_d(J)|\le(116650)e^{\pi\sqrt{d}}.$$

Bruinier, Jenkins, and Ono [2, p. 379] proved a Rademacher-type exact formula for traces of singular moduli which can be expressed as

$$\mathrm{Tr}_d(J)=-24H(d)+\frac{1}{2}\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} S_d(c)\left(e^{\frac{4\pi\sqrt{d}}{c}}-e^{-\frac{4\pi\sqrt{d}}{c}}\right). \tag{1.4}$$

The series in (1.4) is only conditionally convergent, hence one cannot simply truncate this series and bound the remainder term-wise to give an asymptotic formula for $\mathrm{Tr}_d(J)$. The method employed in this paper circumvents this difficulty and allows us to give a bound on the error term with an explicit constant.

In [4], Duke proved that if $-d$ is a fundamental discriminant, then

$$\mathrm{Tr}_d(J)=-24H(d)+\frac{1}{2}\sum_{\substack{0<c<2\sqrt{d}\\ c\equiv 0 \pmod 4}} S_d(c)e^{\frac{4\pi\sqrt{d}}{c}}+O(d^{\frac{1}{2}-\delta}) \tag{1.5}$$

for some absolute constant $\delta>0$, verifying a conjecture of Bruinier, Jenkins, and Ono [2] for the limiting distribution of $\mathrm{Tr}_d(J)$ which was based on the formula (1.4). The implied constant in the error term of (1.5) could, in principle, be computed, though it would be difficult to do this because of the methods used to produce a power-saving exponent.

In [6], Folsom and the third author proved an asymptotic formula for $\mathrm{Tr}(F_{\lambda,\nu}; D_1, D_2)$ with a power-saving error term.

Finally, we note that there are many problems in number theory which require explicit constants in error bounds. An example of this can be found in the paper of Locus Dawsey and the third author [10], where methods similar to those employed here are used to give an asymptotic formula for the Andrews smallest parts function $\mathrm{spt}(n)$ with an effective bound on the error term. This bound was used crucially to verify recent conjectures of Chen [3] regarding inequalities satisfied by $\mathrm{spt}(n)$.

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2. Effective bounds for the coefficients of $F_{\lambda,\nu}(z)$

Let $\lambda \geq 1$ and $\nu \in \mathbb{Z}^+$. By [12], we have the Fourier expansion

$$\begin{aligned} F_{\lambda,\nu}(z) = \pi\nu^{\lambda-\frac{1}{2}} y^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi\nu y)e(-\nu x) + c_\lambda + \frac{2\pi^{1+\lambda}\sigma_{2\lambda-1}(\nu)}{(2\lambda-1)\Gamma(\lambda)\zeta(2\lambda)} y^{1-\lambda} \\ + 2\pi\nu^{\lambda-\frac{1}{2}} \sum_{n\neq 0} b(n,\nu;\lambda) y^{\frac{1}{2}} K_{\lambda-\frac{1}{2}}(2\pi\,|n|\,y)e(nx), \end{aligned} \tag{2.1}$$

where

$$c_\lambda := \begin{cases} -12, & \text{if } \lambda = 1, \\ 0, & \text{if } \ \lambda > 1, \end{cases}$$

$$b(n,\nu;\lambda) := \sum_{c>0} \frac{S(n,-\nu;c)}{c} \begin{cases} I_{2\lambda-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right) & \text{if } n > 0, \qquad \text{(a)} \\ J_{2\lambda-1}\left(\frac{4\pi\sqrt{|n|\nu}}{c}\right) & \text{if } n < 0. \qquad \text{(b)} \end{cases} \tag{2.2}$$

Here

$$S(a,b;c) := \sum_{\substack{d \pmod c \\ (c,d)=1}} e\left(\frac{a\overline{d} + bd}{c}\right)$$

is the Kloosterman sum, $\Gamma(\lambda)$ is the Gamma function, $\zeta(\lambda)$ is the Riemann zeta function, $\sigma_{2\lambda-1}(\nu)$ is the divisor function, and I_α, J_α, and K_α are the I, J, and K-Bessel functions, respectively, of order α.

Proposition 2.1. *For $\lambda \geq 1$ and $\nu \in \mathbb{Z}^+$ we have the bounds*

$$|b(n,\nu;\lambda)| \leq \begin{cases} C_1(\lambda,\nu)\,|n|^{\lambda}\,, & \text{if } n<0, \qquad \text{(a)} \\ C_2(\lambda,\nu) n^{\lambda} e^{4\pi\sqrt{n\nu}}, & \text{if } n>0, \qquad \text{(b)} \end{cases} \tag{2.3}$$

where

$$C_1(\lambda,\nu) := (3.2)\pi\sqrt{3\nu} + (16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right) I_{2\lambda-1}(1),$$
$$C_2(\lambda,\nu) := \frac{4\pi\sqrt{6\nu}}{\Gamma(2\lambda)}\left(\lambda-\tfrac{1}{4}\right)^{2\lambda-\frac{1}{2}} + e^{-4\pi\sqrt{\nu}}(16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right) I_{2\lambda-1}(1).$$

Proof. If $0<t<1$, we use the series (see [13, Eq. 10.2.2])

$$J_{2\lambda-1}(t) = \left(\frac{t}{2}\right)^{2\lambda-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(2\lambda+k)k!}\left(\frac{t^2}{4}\right)^k$$

to get the bound

$$|J_{2\lambda-1}(t)| \leq \left(\frac{t}{2}\right)^{2\lambda-1}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2\lambda+k)4^k k!}. \tag{2.4}$$

If $t \geq 1$, then we have the bound (see [9])

$$|J_{2\lambda-1}(t)| \leq \min\left\{(0.7)(2\lambda-1)^{-1/3}, (0.8)t^{-1/3}\right\}. \tag{2.5}$$

Now, we bound $b(n,\nu;\lambda)$ for $n<0$. By using (2.2b), (2.4), (2.5) and the Weil bound

$$|S(n,-\nu;c)| \leq \tau(c)(n,-\nu,c)^{1/2}c^{1/2}, \tag{2.6}$$

we get

$$\begin{aligned} &|b(n,\nu;\lambda)| \\ &\leq |n|^{1/2}\left[(0.8)\sum_{c=1}^{\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\frac{\tau(c)}{c^{1/2}} + (2\pi\sqrt{|n|\,\nu})^{2\lambda-1}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2\lambda+k)4^k k!}\sum_{c>\left\lfloor 4\pi\sqrt{|n|\nu}\right\rfloor}\frac{\tau(c)}{c^{2\lambda-\frac{1}{2}}}\right]. \end{aligned}$$

The divisor function $\tau(c)$ satisfies the bound (see e.g. [10, Section 3])

$$\tau(c) \leq \sqrt{3}c^{1/2}. \tag{2.7}$$

Also, we have

$$\sum_{c>0} \frac{\tau(c)}{c^\lambda} = \zeta^2(\lambda) \tag{2.8}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(2\lambda+k)4^k k!} = 2^{2\lambda-1} I_{2\lambda-1}(1). \tag{2.9}$$

Then inserting (2.7), (2.8) and (2.9) yields

$$\begin{aligned}|b(n,\nu;\lambda)| &\le |n|^{1/2}\left[(3.2)\pi(3\nu)^{1/2}|n|^{1/2} + |n|^{\lambda-\frac{1}{2}}(4\pi\sqrt{\nu})^{2\lambda-1}\zeta^2(2\lambda-\tfrac{1}{2})I_{2\lambda-1}(1)\right] \\ &\le |n|^\lambda\left[(3.2)\pi(3\nu)^{1/2} + (16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right)I_{2\lambda-1}(1)\right].\end{aligned}$$

Thus, (2.3a) follows with

$$C_1(\lambda,\nu) = (3.2)\pi\sqrt{3\nu} + (16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right)I_{2\lambda-1}(1).$$

We next bound $b(n,\nu;\lambda)$ for $n > 0$. If $0 < t < 1$, we use the series (see [13, Eq. 10.25.2])

$$I_{2\lambda-1}(t) = \left(\frac{t}{2}\right)^{2\lambda-1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2\lambda+k)k!}\left(\frac{t^2}{4}\right)^k$$

to get the bound

$$|I_{2\lambda-1}(t)| \le \left(\frac{t}{2}\right)^{2\lambda-1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2\lambda+k)4^k k!}. \tag{2.10}$$

If $t \ge 1$, then we have the bound (see [7, Eq. (2.4)])

$$|I_{2\lambda-1}(t)| < \frac{\sqrt{2}}{\Gamma(2\lambda)}\left(\lambda-\frac{1}{4}\right)^{2\lambda-\frac{1}{2}} \frac{e^t}{\sqrt{t}}. \tag{2.11}$$

Then using (2.2a), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11), we get

$$\begin{aligned}&|b(n,\nu;\lambda)| \\ &\le n^{1/2}\left[\frac{\sqrt{2}}{\Gamma(2\lambda)}\left(\lambda-\tfrac{1}{4}\right)^{2\lambda-\frac{1}{2}} e^{4\pi\sqrt{n\nu}} \sum_{c=1}^{\lfloor 4\pi\sqrt{n\nu}\rfloor} \frac{\tau(c)}{c^{1/2}} + \sum_{k=0}^{\infty}\frac{(2\pi\sqrt{n\nu})^{2\lambda-1}}{\Gamma(2\lambda+k)4^k k!} \sum_{c>\lfloor 4\pi\sqrt{n\nu}\rfloor} \frac{\tau(c)}{c^{2\lambda-\frac{1}{2}}}\right] \\ &\le n^{1/2}\left[\frac{\sqrt{2}}{\Gamma(2\lambda)}\left(\lambda-\tfrac{1}{4}\right)^{2\lambda-\frac{1}{2}} e^{4\pi\sqrt{n\nu}}\sqrt{3}(4\pi\sqrt{n\nu}) + (4\pi\sqrt{n\nu})^{2\lambda-1}\zeta^2\left(2\lambda-\tfrac{1}{2}\right)I_{2\lambda-1}(1)\right] \\ &\le e^{4\pi\sqrt{n\nu}} n^\lambda\left[\frac{4\pi\sqrt{6\nu}}{\Gamma(2\lambda)}\left(\lambda-\tfrac{1}{4}\right)^{2\lambda-\frac{1}{2}} + e^{-4\pi\sqrt{\nu}}(16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right)I_{2\lambda-1}(1)\right].\end{aligned}$$

Thus, (2.3b) follows with

$$C_2(\lambda,\nu) = \frac{4\pi\sqrt{6\nu}}{\Gamma(2\lambda)}\left(\lambda-\tfrac{1}{4}\right)^{2\lambda-\frac{1}{2}} + e^{-4\pi\sqrt{\nu}}(16\pi^2\nu)^{\lambda-\frac{1}{2}}\zeta^2\left(2\lambda-\tfrac{1}{2}\right)I_{2\lambda-1}(1). \qquad \square$$

3. Proof of Theorem 1.2

Inserting the Fourier expansion (2.1) into the definition of the trace yields

$$\mathrm{Tr}(F_{\lambda,\nu}; D_1, D_2) = M(\lambda,\nu,D_1D_2) + E(\lambda,\nu,D_1D_2),$$

where

$$M(\lambda,\nu,D_1D_2) := \pi\nu^{\lambda-\frac{1}{2}} \sum_{Q\in\mathcal{Q}_{|D_1D_2|}} \frac{\chi_{D_1}(Q)\mathrm{Im}(\tau_Q)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))e(-\nu\mathrm{Re}(\tau_Q))}{\omega_Q}$$

and

$$E(\lambda,\nu,D_1D_2) := E_1(\lambda,\nu,D_1D_2) + E_2^+(\lambda,\nu,D_1D_2) + E_2^-(\lambda,\nu,D_1D_2)$$

with

$$E_1(\lambda,\nu,D_1D_2) := \sum_{Q\in\mathcal{Q}_{|D_1D_2|}} \frac{\chi_{D_1}(Q)}{\omega_Q}\left[c_\lambda + \frac{2\pi^{1+\lambda}\sigma_{2\lambda-1}(\nu)}{(2\lambda-1)\Gamma(\lambda)\zeta(2\lambda)}\mathrm{Im}(\tau_Q)^{1-\lambda}\right],$$

$$E_2^{\pm}(\lambda,\nu,D_1D_2) := 2\pi\nu^{\lambda-\frac{1}{2}} \sum_{Q\in\mathcal{Q}_{|D_1D_2|}} \frac{\chi_{D_1}(Q)}{\omega_Q} \times \sum_{\substack{n\in\mathbb{Z}^\pm \\ n\neq 0}} b(n,\nu;\lambda)\mathrm{Im}(\tau_Q)^{\frac{1}{2}} K_{\lambda-\frac{1}{2}}(2\pi\,|n|\,\mathrm{Im}(\tau_Q))e(n\mathrm{Re}(\tau_Q)).$$

We first analyze the main term $M(\lambda,\nu,D_1D_2)$. For $\lambda\in\mathbb{Z}^+$, we have the identity (see [14, Section 3.71])

$$I_{\lambda-\frac{1}{2}}(t) = \frac{1}{\sqrt{2\pi t}}\left[e^t\sum_{k=0}^{\lambda-1}\frac{c_{k,\lambda}}{(2t)^k} + (-1)^\lambda e^{-t}\sum_{k=0}^{\lambda-1}\frac{|c_{k,\lambda}|}{(2t)^k}\right]$$

where

$$c_{k,\lambda} = (-1)^k\frac{(\lambda-1+k)!}{k!(\lambda-1-k)!}.$$

We insert this identity into $M(\lambda,\nu,D_1D_2)$ and write

$$M(\lambda,\nu,D_1D_2) = M_1(\lambda,\nu,D_1D_2) + R(\lambda,\nu,D_1D_2),$$

where

$$R(\lambda,\nu,D_1D_2) := R_1(\lambda,\nu,D_1D_2) + R_2(\lambda,\nu,D_1D_2)$$

with

$$M_1(\lambda,\nu,D_1D_2) := \frac{\nu^{\lambda-1}}{2} \sum_{\substack{Q\in\mathcal{Q}_{|D_1D_2|}\\ \mathrm{Im}(\tau_Q)>1}} \frac{\chi_{D_1}(Q)}{\omega_Q} e^{2\pi\nu\mathrm{Im}(\tau_Q)} \sum_{k=0}^{\lambda-1} \frac{c_{k,\lambda}}{(4\pi\nu\mathrm{Im}(\tau_Q))^k} e(-\nu\mathrm{Re}(\tau_Q))$$

$$R_1(\lambda,\nu,D_1D_2) := \frac{\nu^{\lambda-1}}{2} \sum_{\substack{Q\in\mathcal{Q}_{|D_1D_2|}\\ \mathrm{Im}(\tau_Q)\le 1}} \frac{\chi_{D_1}(Q)}{\omega_Q} e^{2\pi\nu\mathrm{Im}(\tau_Q)} \sum_{k=0}^{\lambda-1} \frac{c_{k,\lambda}}{(4\pi\nu\mathrm{Im}(\tau_Q))^k} e(-\nu\mathrm{Re}(\tau_Q))$$

$$R_2(\lambda,\nu,D_1D_2) := \frac{\nu^{\lambda-1}}{2} \sum_{Q\in\mathcal{Q}_{|D_1D_2|}} \frac{\chi_{D_1}(Q)}{\omega_Q} \left[(-1)^\lambda e^{-2\pi\nu\mathrm{Im}(\tau_Q)} \sum_{k=0}^{\lambda-1} \frac{|c_{k,\lambda}|}{(4\pi\nu\mathrm{Im}(\tau_Q))^k}\right] \times e(-\nu\mathrm{Re}(\tau_Q)).$$

Now, for each $Q \in \mathcal{Q}_{|D_1D_2|}$ and $\mathrm{Im}(\tau_Q) > 1$, we have

$$0 < a_Q < \frac{\sqrt{|D_1D_2|}}{2}, \qquad b_Q^2 - 4a_Qc_Q \equiv b_Q^2 \equiv -|D_1D_2| \pmod{4a_Q}.$$

Hence, letting $c = 4a_Q$ and $x = b_Q$, we get

$$\begin{aligned} M_1(\lambda,\nu,D_1D_2) &= \frac{\nu^{\lambda-1}}{4} \sum_{k=0}^{\lambda-1} c_{k,\lambda} \sum_{\substack{0<c<2\sqrt{|D_1D_2|}\\ c\equiv 0 \pmod 4}} \left(\frac{8\pi\nu\sqrt{|D_1D_2|}}{c}\right)^{-k} e^{\frac{4\pi\nu\sqrt{|D_1D_2|}}{c}} \\ &\quad \times \sum_{x^2\equiv -|D_1D_2| \pmod c} \chi_{D_1}\left(\frac{c}{4}, x, \frac{x^2+|D_1D_2|}{c}\right) e^{\frac{4\pi i\nu x}{c}} \\ &= \frac{\nu^{\lambda-1}}{4} \sum_{k=0}^{\lambda-1} c_{k,\lambda} \sum_{\substack{0<c<2\sqrt{|D_1D_2|}\\ c\equiv 0 \pmod 4}} S_{D_1,D_2}(\nu,c) \left(\frac{8\pi\nu\sqrt{|D_1D_2|}}{c}\right)^{-k} \\ &\quad \times e^{\frac{4\pi\nu\sqrt{|D_1D_2|}}{c}}. \end{aligned} \tag{3.1}$$

Since $Q \in \mathcal{Q}_{|D_1D_2|}$ is reduced, we have $|b_Q| \le a_Q \le c_Q$ and $b_Q \ge 0$ when one of the two inequalities is an equality. It follows that the CM point τ_Q lies in the standard fundamental domain $\mathcal{F}$ for Γ, and thus

$$\mathrm{Im}(\tau_Q) \ge \sqrt{3}/2. \tag{3.2}$$

Then estimating gives

$$\begin{aligned}|R_1(\lambda,\nu,D_1D_2)| &\leq \frac{\nu^{\lambda-1}}{2}\sum_{\substack{Q\in\mathcal{Q}_{|D_1D_2|}\\ \mathrm{Im}(\tau_Q)\leq 1}}\frac{|\chi_{D_1}(Q)|}{\omega_Q}e^{2\pi\nu\mathrm{Im}(\tau_Q)}\sum_{k=0}^{\lambda-1}\frac{|c_{k,\lambda}|}{(4\pi\nu\mathrm{Im}(\tau_Q))^k}|e(-\nu\mathrm{Re}(\tau_Q))|\\ &\leq \frac{\nu^{\lambda-1}}{2}\sum_{k=0}^{\lambda-1}|c_{k,\lambda}|e^{2\pi\nu}\sum_{\substack{Q\in\mathcal{Q}_{|D_1D_2|}\\ \mathrm{Im}(\tau_Q)\leq 1}}\frac{1}{\omega_Q}\\ &\leq \frac{\nu^{\lambda-1}e^{2\pi\nu}}{2}\sum_{k=0}^{\lambda-1}|c_{k,\lambda}|H(|D_1D_2|).\end{aligned}$$

Similarly, we have

$$|R_2(\lambda,\nu,D_1D_2)| \leq \frac{\nu^{\lambda-1}}{2}\sum_{k=0}^{\lambda-1}|c_{k,\lambda}|H(|D_1D_2|).$$

Then, combining these estimates yields

$$\begin{aligned}M(\lambda,\nu,D_1D_2) &= \frac{\nu^{\lambda-1}}{4}\sum_{k=0}^{\lambda-1}c_{k,\lambda}\sum_{\substack{0<c<2\sqrt{|D_1D_2|}\\ c\equiv 0 \pmod 4}}S_{D_1,D_2}(\nu,c)\left(\frac{8\pi\nu\sqrt{|D_1D_1|}}{c}\right)^{-k}e^{\frac{4\pi\nu\sqrt{|D_1D_2|}}{c}}\\ &\quad + R(\lambda,\nu,D_1D_2),\end{aligned}$$

where

$$\begin{aligned}|R(\lambda,\nu,D_1D_2)| &\leq |R_1(\lambda,\nu,D_1D_2)| + |R_2(\lambda,\nu,D_1D_2)|\\ &\leq \frac{\nu^{\lambda-1}(e^{2\pi\nu}+1)}{2}\sum_{k=0}^{\lambda-1}|c_{k,\lambda}|H(|D_1D_2|).\end{aligned}\tag{3.3}$$

To bound the error term $E_1(\lambda,\nu,D_1D_2)$, we use the inequality

$$\mathrm{Im}(\tau_Q)^{1-\lambda} \leq (\sqrt{3}/2)^{1-\lambda} \leq (1.2)^{\lambda-1}$$

to get

$$\begin{aligned}|E_1(\lambda,\nu,D_1D_2)| &\leq \left[c_\lambda + \frac{2(1.2)^{\lambda-1}\pi^{1+\lambda}\sigma_{2\lambda-1}(\nu)}{(2\lambda-1)\Gamma(\lambda)\zeta(2\lambda)}\right]\sum_{Q\in\mathcal{Q}_{|D_1D_2|}}\frac{1}{\omega_Q}\\ &= \left[c_\lambda + \frac{2(1.2)^{\lambda-1}\pi^{1+\lambda}\sigma_{2\lambda-1}(\nu)}{(2\lambda-1)\Gamma(\lambda)\zeta(2\lambda)}\right]H(|D_1D_2|).\end{aligned}$$

We next bound the error terms $E_2^{\pm}(\lambda, \nu, D_1D_2)$.

If $\lambda = 1$, we have the identity

$$K_{\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q)) = \left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)}.$$

On the other hand, if $\lambda > 1$ we have the asymptotic formula (see [11, Eq. (14)])

$$\begin{aligned} K_{\lambda-\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q)) = &\left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)} \\ &\times\left[\sum_{m=0}^{\lambda-1}\frac{a_m(\lambda-\frac{1}{2})}{(2\pi|n|\operatorname{Im}(\tau_Q))^m} + R_\lambda^{(K)}\left(2\pi|n|\operatorname{Im}(\tau_Q), \lambda-\frac{1}{2}\right)\right], \end{aligned}$$

where $R_\lambda^{(K)}$ is a certain remainder term and

$$a_k(\ell) := (-1)^k\frac{\cos(\pi\ell)}{\pi}\frac{\Gamma(k+\frac{1}{2}+\ell)\Gamma(k+\frac{1}{2}-\ell)}{2^k\Gamma(k+1)} \quad \text{for} \quad k\in\mathbb{Z}^+,\ \ell\in\mathbb{R}^+.$$

Using [11, Theorem 1.5 and Proposition B.1], we can bound the remainder term by

$$\left|R_\lambda^{(K)}\left(2\pi|n|\operatorname{Im}(\tau_Q), \lambda-\frac{1}{2}\right)\right| \le \left|a_\lambda\left(\lambda-\tfrac{1}{2}\right)\right|.$$

Hence, for $\lambda\in\mathbb{Z}^+$ we have

$$\left|K_{\lambda-\frac{1}{2}}(2\pi|n|\operatorname{Im}(\tau_Q))\right| \le \delta_\lambda\left(\frac{1}{4|n|\operatorname{Im}(\tau_Q)}\right)^{1/2} e^{-2\pi|n|\operatorname{Im}(\tau_Q)} \tag{3.4}$$

where

$$\delta_\lambda := \begin{cases} 1, & \text{if } \lambda = 1, \\ \sum_{m=0}^{\lambda}\left|a_m(\lambda-\frac{1}{2})\right|, & \text{if } \lambda > 1. \end{cases}$$

Combining (2.3a) with (3.4) gives

$$\left|E_2^-(\lambda, \nu, D_1D_2)\right| \le \pi\nu^{\lambda-\frac{1}{2}}\delta_\lambda C_1(\lambda,\nu)\sum_{n<0}|n|^{\lambda-\frac{1}{2}}\sum_{Q\in\mathcal{Q}_{|D_1D_2|}}\frac{e^{-2\pi|n|\operatorname{Im}(\tau_Q)}}{\omega_Q}.$$

Moreover, by the lower bound (3.2) we have

$$\sum_{Q\in\mathcal{Q}_{|D_1D_2|}}\frac{e^{-2\pi|n|\operatorname{Im}(\tau_Q)}}{\omega_Q} \le H(|D_1D_2|)e^{-\pi\sqrt{3}|n|}.$$

Hence

$$
\begin{aligned}
\left|E_2^-(\lambda,\nu,D_1D_2)\right| &\le \pi\nu^{\lambda-\frac{1}{2}}\delta_\lambda C_1(\lambda,\nu)H(|D_1D_2|)\sum_{n>0} n^{\lambda-\frac{1}{2}}e^{-\pi\sqrt{3}n}\\
&= \pi\nu^{\lambda-\frac{1}{2}}\delta_\lambda C_1(\lambda,\nu)H(|D_1D_2|)\mathrm{Li}_{\frac{1}{2}-\lambda}(e^{-\pi\sqrt{3}})
\end{aligned}
$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function defined by

$$
\mathrm{Li}_s(z) := \sum_{k=1}^{\infty}\frac{z^k}{k^s}, \quad s\in\mathbb{C},\ |z|<1.
$$

Next, using (2.3b), a similar argument yields

$$
\left|E_2^+(\lambda,\nu,D_1D_2)\right| \le \pi\nu^{\lambda-\frac{1}{2}}\delta_\lambda C_2(\lambda,\nu)H(|D_1D_2|)\Sigma
$$

where

$$
\Sigma := \sum_{n>0} n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)}.
$$

Write

$$
4\pi\sqrt{n\nu}-\pi\sqrt{3}n = -n\left(\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}}\right),
$$

and observe that if $n \ge \lceil\frac{32\nu}{3}\rceil$ then

$$
-n\left(\pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{n}}\right) \le -ng(\nu)
$$

where

$$
g(\nu) := \pi\sqrt{3}-\frac{4\pi\sqrt{\nu}}{\sqrt{\lceil\frac{32\nu}{3}\rceil}} > 1.
$$

Hence

$$
\begin{aligned}
\Sigma &\le \sum_{n=1}^{\lfloor\frac{32\nu}{3}\rfloor} n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)} + \sum_{n=\lceil\frac{32\nu}{3}\rceil}^{\infty} n^{\lambda-\frac{1}{2}}e^{-ng(\nu)}\\
&\le \sum_{n=1}^{\lfloor\frac{32\nu}{3}\rfloor} n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)} + \sum_{n=1}^{\infty} n^{\lambda-\frac{1}{2}}e^{-n}\\
&= \sum_{n=1}^{\lfloor\frac{32\nu}{3}\rfloor} n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)} + \mathrm{Li}_{\frac{1}{2}-\lambda}(e^{-1}).
\end{aligned}
$$

Thus we get

$$\begin{aligned}&\left|E_2^+(\lambda,\nu,D_1D_2)\right|\\&\quad\le \pi\nu^{\lambda-\frac{1}{2}}\delta_\lambda C_2(\lambda,\nu)H(|D_1D_2|)\left[\sum_{n=1}^{\lfloor\frac{32\nu}{3}\rfloor} n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)}+\mathrm{Li}_{\frac{1}{2}-\lambda}(e^{-1})\right].\end{aligned}$$

By combining our bounds for $E_1(\lambda,\nu,D_1D_2)$ and $E_2^{\pm}(\lambda,\nu,D_1D_2)$, we get

$$\begin{aligned}|E(\lambda,\nu,D_1D_2)|&\le|E_1(\lambda,\nu,D_1D_2)|+|E_2^-(\lambda,\nu,D_1D_2)|+|E_2^+(\lambda,\nu,D_1D_2)|\\&\le C_E(\lambda,\nu)H(|D_1D_2|)\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}C_E(\lambda,\nu):=&\left[c_\lambda+\frac{2(1.2)^{\lambda-1}\pi^{1+\lambda}\sigma_{2\lambda-1}(\nu)}{(2\lambda-1)\Gamma(\lambda)\zeta(2\lambda)}\right]+\pi\nu^{\lambda-\frac{1}{2}}C_1(\lambda,\nu)\delta_\lambda\mathrm{Li}_{\frac{1}{2}-\lambda}(e^{-\pi\sqrt{3}})\\&+\pi\nu^{\lambda-\frac{1}{2}}C_2(\lambda,\nu)\delta_\lambda\left[\sum_{n=1}^{\lfloor\frac{32\nu}{3}\rfloor}n^{\lambda-\frac{1}{2}}e^{(4\pi\sqrt{n\nu}-\pi\sqrt{3}n)}+\mathrm{Li}_{\frac{1}{2}-\lambda}(e^{-1})\right].\end{aligned}\tag{3.6}$$

Finally, by combining (3.1), (3.3), and (3.5) we have shown

$$\begin{aligned}\mathrm{Tr}(F_{\lambda,\nu};D_1,D_2)=&\frac{\nu^{\lambda-1}}{4}\sum_{k=0}^{\lambda-1}c_{k,\lambda}\\&\times\sum_{\substack{0<c<2\sqrt{|D_1D_2|}\\ c\equiv 0\pmod 4}}S_{D_1,D_2}(\nu,c)\left(\frac{8\pi\nu\sqrt{|D_1D_1|}}{c}\right)^{-k}e^{\frac{4\pi\nu\sqrt{|D_1D_2|}}{c}}\\&+R(\lambda,\nu,D_1D_2)+E(\lambda,\nu,D_1D_2),\end{aligned}\tag{3.7}$$

where

$$|R(\lambda,\nu,D_1D_2)+E(\lambda,\nu,D_1D_2)|\le C(\lambda,\nu)H(|D_1D_2|)\tag{3.8}$$

with

$$C(\lambda,\nu):=C_E(\lambda,\nu)+\frac{\nu^{\lambda-1}(e^{2\pi\nu}+1)}{2}\sum_{k=0}^{\lambda-1}|c_{k,\lambda}|.\tag{3.9}$$

It remains to bound $\mathrm{Tr}(F_{\lambda,\nu};D_1,D_2)$. By the definition of $M(\lambda,\nu,D_1D_2)$, we have

$$|M(\lambda,\nu,D_1D_2)|\le\pi\nu^{\lambda-\frac{1}{2}}\sum_{Q\in\mathcal{Q}_{|D_1D_2|}}\frac{\mathrm{Im}(\tau_Q)^{\frac{1}{2}}\left|I_{\lambda-\frac{1}{2}}(2\pi\nu\mathrm{Im}(\tau_Q))\right|}{\omega_Q}.$$

Then using (2.11), we get

$$|M(\lambda,\nu,D_1D_2)| \le \frac{\sqrt{\pi}\nu^{\lambda-\frac{3}{2}}}{\Gamma(\lambda+\frac{1}{2})}\left(\frac{\lambda}{2}\right)^{\lambda} \sum_{Q\in\mathcal{Q}_{|D_1D_2|}} \frac{e^{2\pi\nu \mathrm{Im}(\tau_Q)}}{\omega_Q}.$$

Now, since

$$\mathrm{Im}(\tau_Q) = \frac{\sqrt{|D_1D_2|}}{2a_Q}$$

we have

$$\mathrm{Im}(\tau_Q) \le \frac{\sqrt{|D_1D_2|}}{2}.$$

The only form $Q\in\mathcal{Q}_{|D_1D_2|}$ with $a_Q=1$ is the principal form, hence

$$|M(\lambda,\nu,D_1D_2)| \le \frac{\sqrt{\pi}\nu^{\lambda-\frac{3}{2}}}{\Gamma(\lambda+\frac{1}{2})}\left(\frac{\lambda}{2}\right)^{\lambda}\left[e^{\pi\nu\sqrt{|D_1D_2|}} + (H(|D_1D_2|)-1)e^{\frac{\pi\nu\sqrt{|D_1D_2|}}{2}}\right].$$

Therefore, a calculation shows

$$\begin{aligned}|\mathrm{Tr}(F_{\lambda,\nu};D_1,D_2)| &\le |M(\lambda,\nu,D_1D_2)| + |E(\lambda,\nu,D_1D_2)| \\ &\le C'(\lambda,\nu,D_1,D_2)e^{\pi\nu\sqrt{|D_1D_2|}},\end{aligned}$$

where

$$\begin{aligned}C'(\lambda,\nu,D_1,D_2) := &\frac{\sqrt{\pi}\nu^{\lambda-\frac{3}{2}}}{\Gamma(\lambda+\frac{1}{2})}\left(\frac{\lambda}{2}\right)^{\lambda}\left[1+(H(|D_1D_2|)-1)e^{-\frac{\pi\nu\sqrt{|D_1D_2|}}{2}}\right] \\ &+ C_E(\lambda,\nu)H(|D_1D_2|)e^{-\pi\nu\sqrt{|D_1D_2|}}.\end{aligned}$$

We have the effective bound (see e.g. [10, Section 3])

$$H(|D_1D_2|) \le 3|D_1D_2|^{5/2}.$$

Hence

$$\begin{aligned}1+(H(|D_1D_2|)-1)e^{-\frac{\pi\nu\sqrt{|D_1D_2|}}{2}} &\le 7.6, \\ H(|D_1D_2|)e^{-\pi\nu\sqrt{|D_1D_2|}} &\le 0.3.\end{aligned}$$

Finally, we get

$$|\mathrm{Tr}(F_{\lambda,\nu};D_1,D_2)| \le C'(\lambda,\nu)e^{\pi\nu\sqrt{|D_1D_2|}}, \tag{3.10}$$

where

$$C'(\lambda,\nu) := (7.6)\frac{\sqrt{\pi}\nu^{\lambda-\frac{3}{2}}}{\Gamma(\lambda+\frac{1}{2})} + (0.3)C_E(\lambda,\nu). \tag{3.11}$$

4. Proof of Corollary 1.4

Using (1.1) and letting $\lambda = \nu = 1$ and $D_1 = 1$ and $D_2 = d$ in (3.7) yields

$$\mathrm{Tr}_d(J) = \frac{1}{2}\sum_{\substack{0<c<2\sqrt{d} \\ c\equiv 0 \pmod 4}} S_d(c)e^{\frac{4\pi\sqrt{d}}{c}} + R_J(d),$$

where

$$R_J(d) := 2[R(1,1,d) + E(1,1,d)].$$

Hence, by (3.8), (3.9) and Table 1, we have

$$|R_J(d)| \leq 2C(1,1)H(d) \leq (389624)H(d).$$

Finally, using (1.1), (3.10), (3.11) and Table 1, we get

$$|\mathrm{Tr}_d(J)| \leq 2C'(1,1)e^{\pi\sqrt{d}} \leq (116650)e^{\pi\sqrt{d}}.$$

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